Cohomogeneity Three Actions on Spheres

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During our stay at BIRS, Dr. McGowan and I were able to modify our original project of classifying cohomogeneity 3 actions on spheres to the following problem: calculate the diameters and q-extents of spherical quotients of irreducible polar actions of cohomgeneities 3 and higher. First let us make the following definition: we call a cohomogeneity k action *classical polar*, when it is a polar action of cohomogeneity k corresponding to a symmetric space G/H where either G or H is a product of classical groups only. Those actions which admit products with classical groups and exceptional groups will be called *exceptional polar*. Note that there is a 1-1 correspondance between polar actions and symmetric spaces [D]. We are interested in this problem given that we have found in joint work with W. Dunbar and S. Greenwald [DGMS], and our own, [MS], when we allow for disconnected groups G to act isometrically on spheres by cohomogeneity 1, 2 or 3 (in the case where the action is classical polar) we obtain the following lower bounds for the diameter:

$$\text{min(diam}(S^n(1)/G) = \begin{cases} \frac{\pi}{12} & \text{for cohomogeneity 1} \\ \frac{\alpha}{2} & \text{for cohomogeneity 2} \\ \beta & \text{for cohomogeneity 3} \end{cases}$$

where $\alpha = arccos(\frac{tan(\frac{3\pi}{10})}{\sqrt{3}})$, and $\beta = arccos(1/\sqrt{40+12\sqrt{2}-8\sqrt{5}-12\sqrt{10}})$.

We note that for these three cohomogeneities the diameter is strictly increasing as the cohomogeneity increases. The conjecture we are then currently trying to verify is: let G be an irreducible polar action of cohomogeneity k on S^n, then the diameter of S^n/G increases to $\frac{\pi}{2}$ as $k \to \infty$. That is, as the cohomogeneity of an irreducible action becomes large, the action "becomes" reducible. We would also like to understand what is going on in terms of the q-extents for these spaces.

We have been able to confirm this conjecture for the classical polar actions of cohomogeneities 3 and higher. The list includes the following groups:

Table 1: Classical Polar Actions of Cohomogeneity $k-1$

Nr.	G	dim(S^m)	Corresponding Symmetric Space
1	$SO(k) \times SO(n)$	$kn-1$	$SO(k+n)/(SO(k) \times SO(n)), k \geq n$
2	$S(U(k) \times U(n))$	$2kn-1$	$SU(k+n)/S(U(k) \times U(n)), k \geq n$
3	$Sp(k) \times Sp(n)$	$4kn-1$	$Sp(k+n)/(Sp(k) \times Sp(n)), k \geq n$
4	$U(2(k))$	$k(k-1)-1$	$SO(4(k))/U(2(k))$
5	$U(2(k)+1)$	$k(k-1)-1$	$SO(4(k)+2)/U(2(k)+1)$
6	$SO(k)$	$\frac{1}{2}(k-1)(k+2)-1$	$SU(k)/SO(k)$
7	$Sp(k)$	$(k-1)(2k+1)-1$	$SU(2(k)/Sp(k)$
8	$SO(2(k))$	$\frac{1}{2}2(k)(2k-1)$	$(SO(2k) \times SO(2k))/SO(2k)$
9	$SO(2k+1)$	$k(2k+1)$	$(SO(2k+1) \times SO(2k+1))/SO(2k+1)$
10	$U(k)$	k^2-1	$(U(k) \times U(k))/U(k)$
11	$Sp(k)$	$2k^2-k-1$	$(Sp(k) \times Sp(k))/Sp(k)$
12	$SU(k)$	k^2-2	$(SU(k) \times SU(k))/SU(k)$

Of the remaining groups, for those whose corresponding symmetric space is of the type $(G \times G)/G$, namely numbers 1, 6, 8 and 10 of Table 2, the result also holds true. During our stay at BIRS we were also working on the remaining groups listed in the following table.

Table 2: Exceptional Polar Actions of Cohomogeneities Greater than ‘2

Nr.	G	dim(S^m)	Corresponding Symmetric Space	Cohomogeneity
1	F_4	51	$(F_4 \times F_4)/F_4$	3
2	$SU(6) \times SU(2)$	39	$E_6/(SU(6) \times SU(2))$	3
3	$SO(12) \times SU(2)$	63	$E_7/(SO(12) \times SU(2)$	3
4	$E_7 \times SU(2)$	111	$E_8/(E_7 \times SU(2))$	3
5	$Sp(3) \times SU(2)$	27	$F_4/(Sp(3) \times SU(2))$	3
6	E_6	77	$(E_6 \times E_6)/E_6$	5
7	$Sp(4)$	41	$E_6/Sp(4)$	5
8	E_7	132	$(E_7 \times E_7)/E_7$	6
9	$SU(8)$	69	$E_7/SU(8)$	6
10	E_8	247	$(E_8 \times E_8)/E_8$	7
11	$SO(16)$	127	$E_8/SO(16)$	7

Since these groups do not admit “easy” matrix expressions, we are using a technique of Hsiang outlined in his book “Cohomology Theory of Topological Transformation Groups” [H] in order to calculate the principal isotropy subgroups of these actions. Once we have computed these subgroups, we then need to find their normalizers so that we may use the technique of G-manifold reductions (cf. [GS]) to compute the quotient space. That is, we must calculate the *core group* ${}_cG = N(H)/H$, where H is the principal isotropy subgroup, and also find the *core* of the corresponding sphere, ${}_cS^n$. Since ${}_cM/{}_cG \simeq M/G$, we may then compute the quotient space.

During our stay, we were able to calculate the connected component of the principal isotropy subgroup for number 2 and we made a fair amount of progress for numbers 3 and 4 (we are completing these calculations now). The only other action with non-trivial principal isotropy is number 9. For the rest of the groups in Table 2, we must use a different technique altogether, which can be found in Straume [S], namely extend the action to a larger dimensional group which will have non-trivial principal isotropy.

We also plan to see how much of Straume’s paper can be extended for polar actions of cohomogeneity 3 and higher.

We would also like to add that while we modified our original proposal for our stay at BIRS, we have by no means abandoned the idea of classifying spherical actions of cohomogeneity 3 and higher. Upon conclusion of this current project, we hope to be able to tackle not only the classification problem, but also to understand how the diameters of spherical quotients of non-polar actions behave in terms of our conjecture.

In conclusion, we would like to add that we feel that our stay at BIRS was incredibly productive for us. This is the first time we have had an entire 2 weeks in which to just concentrate on our research. We are both very happy to have been provided with this opportunity.

References

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