Affinizations of Extended Affine Lie Algebras

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Extended affine Lie algebras, or EALA's for short, were introduced by Hoegh-Krohn and Torresani in 1990 [9] as natural generalizations of finite dimensional simple Lie algebras and affine Kac-Moody Lie algebras. Many of the basic facts about these algebras were proved in [1]. By definition EALA's are complex Lie algebras that possess Cartan subalgebras and invariant forms, and hence they possess root systems which turn out to be extended affine root systems. Roots of length 0 are called isotropic roots, and they generate a lattice whose rank is referred to as the nullity of the EALA. As has been shown in [3], EALA's of nullity 0 and 1 precisely coincide with finite dimensional simple algebras and affine Kac-Moody algebras respectively. Therefore there has been a lot of interest and activity in the last decade on the study of EALA's of higher rank.

An EALA $\mathcal{L}$ possesses an ideal $\mathcal{L}_c$, called the core of $\mathcal{L}$, which is defined to be the subalgebra of $\mathcal{L}$ generated by the root spaces of $\mathcal{L}$ corresponding to nonisotropic roots. ($\mathcal{L}_c$ is the derived algebra of $\mathcal{L}$ in nullity 0 and 1.) The quotient algebra $\mathcal{L}_{cc} := \mathcal{L}_c/Z(\mathcal{L}_c)$, is called the centreless core of $\mathcal{L}$. Y. Yoshii [14] has recently given an internal characterization of the Lie algebras, called centreless Lie tori, that arise as the centreless core of an EALA. Furthermore, the structure of an EALA is to a large extent governed by the structure of its centreless core. In fact, E. Neher [11] has recently announced a procedure that, given a centreless Lie torus $\mathcal{K}$, describes all EALA's with centreless core $\mathcal{K}$. For this reason, an important equivalence relation for EALA's is isomorphism of their centreless cores.

Many centreless Lie tori, and consequently EALA's, can be constructed using various "matrix" constructions, from coordinate algebras such as the noncommutative quantum tori that generalize Laurent polynomials in several variables. This is a combination of the work of number of authors in the last few years beginning with the paper of Berman, Gao and Krylyuk in [7].

Another approach to the construction of EALA's makes use of loop algebras and affinizations of Lie algebras relative to finite order automorphisms. If $\mathcal{G}$ is a Lie algebra and σ is an automorphism of $\mathcal{G}$ of period m, the loop algebra of $\mathcal{G}$ relative to σ is the algebra $L(\mathcal{G},\sigma)$ of fixed points of the automorphism $x \otimes f(z) \mapsto \sigma(x) \otimes f(\zeta_m^{-1}z)$ of the untwisted loop algebra $\mathcal{G} \otimes S$, where ζ_m is a primitive m^{th} root of unit and S is the ring of Laurent polynomials in the variable z. Further, if $\mathcal{G}$ possesses a nondegenerate invariant symmetric bilinear form that is preserved by σ, one defines the affinization of $\mathcal{G}$ relative to σ to be the Lie algebra $\mathrm{Aff}(\mathcal{G},\sigma)$ obtained from $L(\mathcal{G},\sigma)$ by first forming a 1-dimensional central extension (with cocycle defined as usual using the invariant form) and then adding the 1-dimensional algebra spanned by the degree derivation $z\frac{d}{dz}$.

In his pioneering work on loop algebras in 1969, V. Kac showed that if $\mathcal{G}$ is finite dimensional simple and σ is a finite period automorphism of $\mathcal{G}$ then $\mathrm{Aff}(\mathcal{G},\sigma)$ is an affine Kac-Moody Lie algebra and all such Lie algebras arise in this way. In the language of EALA's this reads as follows: If $\mathcal{G}$ is a EALA of nullity zero then $\mathrm{Aff}(\mathcal{G},\sigma)$ is a EALA of nullity one and moreover, all such algebras arise in this way. When phrased this way it becomes quite natural to ask what happens in the case of EALA's of higher nullity. In our work at

BIRS on this problem, we focused on the case of nullity 2 and we worked at the level of centreless cores.

It is remarkable fact, which follows from a theorem announced recently by Neher in [12] along with the classification theorems for centreless cores of type A [7, 8, 13], that with the exception of one well-understood family, all centreless cores of EALA's are finitely generated as modules over their centroids. (The exceptional family consists of Lie algebras of the form $\mathrm{sl}_{\ell+1}(\mathbb{C}_{\boldsymbol{q}})$, where $\mathbb{C}_{\boldsymbol{q}}$ is the quantum torus determined by a quantum matrix $\boldsymbol{q}$ with at least one entry that is not a root of unity.) For this reason, we concentrated in our work on centreless cores with this additional finiteness property. While at BIRS we were able to complete the proofs of a number of results on this topic.

We showed that every centreless core of an EALA of nullity 2 that is finitely generated over its centroid is isomorphic to a Lie algebra of the form

$$L(\mathcal{G}_{\mathrm{cc}}, \sigma), \tag{1}$$

where $\mathcal{G}$ is an affine Kac-Moody Lie algebras and σ is a diagram automorphism of $\mathcal{G}$. Conversely, we showed that any Lie algebra of the form (1) is isomorphic either to a centreless core of an EALA of nullity 2 (finitely generated over its centroid) or to a Lie algebra of the form $[\mathbb{C}_{\boldsymbol{q}}, \mathbb{C}_{\boldsymbol{q}}]$, where $\boldsymbol{q} = \left(\begin{smallmatrix} 1 & \zeta \\ \zeta^{-1} & 1 \end{smallmatrix}\right)$ and ζ is a root of unity.

The class of Lie algebras of the form (1) is interesting in its own right. We were able to characterize algebras in this class in a number of different ways, including as $\mathbb{Z}^2$-graded-central-simple Lie algebras whose central grading group has finite index in $\mathbb{Z}^2$. We also gave a complete classification of the algebras in this class up to isomorphism. That is, we precisely determined when two algebras of the form (1) are isomorphic.

Precise statements and detailed proofs of the results just mentioned will appear elsewhere. Our proofs make use of techniques and results that we developed recently in a series of papers on EALA's and loop algebras including [4], [5], [6] and our paper [2] with John Faulkner.

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References

[1] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, Extended affine Lie algebras and their root systems, *Mem. Amer. Math. Soc.* **1**26 (603), 1997.

[2] B. Allison, S. Berman, J. Faulkner and A. Pianzola, Graded-simple algebras and multiloop algebras, in preparation.

[3] B. Allison, S. Berman, Y. Gao and A. Pianzola, A characterization of affine Kac Moody Lie algebras, *Comm. Math. Phys.* **1**85 (1997), 671–688.

[4] B. Allison, S. Berman and A. Pianzola, Covering algebras I: Extended affine Lie algebras, *J. Algebra* **2**50 (2002), 485–516.

[5] B. Allison, S. Berman and A. Pianzola, Covering algebras II: Isomorphism of loop algebras, *J. Reine Angew. Math.* **5**71 (2004), 39–71.

[6] B. Allison, S. Berman and A. Pianzola, Iterated loop algebras, *Pacific J. Math.*, to appear.

[7] S. Berman, Y. Gao and Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, *J. Funct. Anal.* **135** (1996), 339–389.

[8] S. Berman, Y. Gao, Y. Krylyuk and E. Neher The alternative torus and the structure of elliptic quasi-simple Lie algebras of type A_2, *Trans. Amer. Math. Soc.* **3**47 (1995), 4315–4363.

[9] R. Høegh-Krohn and B. Torresani, Classification and construction of quasi-simple Lie algebras, *J. Funct. Anal.* **89** (1990), 106–136.

[10] V.G. Kac, Automorphisms of finite order of semi-simple Lie algebras, *Funct. Anal. Appl.* **3** (1969), 252–254.

[11] E. Neher, Extended affine Lie algebras, *Math. Reports Acad. Sci. Canada*, **26** (2004), 84–89.

[12] E. Neher, Lie tori, *Math. Reports Acad. Sci. Canada*, **26** (2004), 90–96.

[13] Y. Yoshii, Coordinate algebras of extended affine Lie algebras of type A_1, *J. of Algebra* **2**34 (2000), 128–168.

[14] Y. Yoshii, Lie tori—A simple characterization of extended affine Lie algebras, preprint 2003.