Hyperbolic Geometry and Quasiconformal Mappings

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The interaction between hyperbolic geometry and conformal analysis is a beautiful and fruitful aspect of the fields of analysis and low-dimensional geometry-topology. In particular, the study of hyperbolic geometry intertwines complex analysis, geometric function theory (especially in the guise of the study of quasiconformal mappings), and topology in a way that allows one to study a fixed object from diverse perspectives.

Our specific interest in the interaction between analysis and hyperbolic geometry is in the many uses of quasiconformal mappings to study the geometry. Informally, we recall that a `quasiconformal mapping` is an almost everywhere differentiable mapping on a domain in $\overline{\mathbb{R}^n}$ that has the property that it maps infinitesimal circles to infinitesimal ellipses, so that each ellipse has the property that the ratio of the major to minor axes is uniformly bounded from above. The theory of quasiconformal mappings has proved to be foundational to the modern studies of geometric analysis and low dimensional topology and geometry.

One exciting aspect of this melding of geometry, topology and analysis has been isolated in the form of the quasiconformal homogeneity of a hyperbolic manifold. We recall that a hyperbolic n-manifold M^n is quasiconformally homogeneous that is, for any two points x and y in M^n, there exists a quasiconformal automorphism of M^n that maps x to y. We say M^n is `uniformly quasiconformally homogeneous` if there exists a constant $K > 1$ so that there for any two points $x, y \in M^n$ there exists a K–quasiconformal automorphism of M^n that takes x to y. This notion was originally formulated by F. Gehring and B. Palka in the context of domains in $\mathbb{R}^n$ [3], and was recently extended to hyperbolic manifolds via work joint work of P. Bonfert-Taylor, R. Canary, G. Martin and E. Taylor [2]. It is known that the quasiconformal homogeneity of a hyperbolic manifold is revealing of its geometry and topology. For example, in [2], we show that any uniformly quasiconformally homogeneous hyperbolic manifold has bounded geometry (and that their are uniform bounds which depend on the quasiconformal homogeneity constant.)

As a consequence of McMullen's version [4] of the Sullivan Rigidity Theorem we have shown [2], for all dimensions $n \geq 3$, that an orientable hyperbolic n-manifold is uniformly quasiconformally homogeneous if and only if it is the regular cover of a closed hyperbolic orbifold.

We considered two foundational questions in this theory during our stay. The first foundational question is:

Question 1 *Is there a topological classification of uniformly quasiconformally homogeneous hyperbolic surfaces?*

We note that there are examples of quasiconformally homogeneous surfaces which are not the regular covers of closed hyperbolic 2-orbifolds, and thus any classification in dimension two will be more complex than the higher dimensional classification.

Question 1 was the focus of intense discussion during our stay at BIRS. Even in the setting of planar hyperbolic domains no complete classification is known, however some topological details are known. For instance, F. Gehring and B. Palka [3] have shown that, if D is a multiply connected uniformly quasiconformally homogenous domain in $\hat{\mathbb{C}}$ with more than two boundary points, then every neighborhood of each boundary point of D contains infinitely many components of the complement of D. An example of such a uniformly quasiconformally homogeneous domain is the complement of the Cantor ternary set in $\hat{\mathbb{C}}$ and any quasiconformal image of this domain.

Recall that a domain $D \subset \hat{\mathbb{C}}$ is `uniform` if there exists a constant $c \geq 1$ such that for each pair of distinct points $a, b \in D$, there exists a continuum F containing a and b, so that

$$\bigcup_{z \in F \setminus \{a,b\}} B\left(z, \frac{\min\{|z-a|, |z-b|\}}{c}\right) \subset D.$$

One can show that *any uniform domain whose boundary is uniformly perfect and totally disconnected is uniformly quasiconformally homogenous.*

An important goal for us during our research week at BIRS was to *explore possible classification schemes for both planar domains, and more generally, hyperbolic* 2*-manifolds that are uniformly quasiconformally homogeneous.*

In the planar domain category, we focussed on the following two questions:

1. Which regular sets of Kleinian groups are uniformly quasiconformally homogeneous?

2. Which uniformly quasiconformally homogeneous domains are quasiconformal images of regular sets of Kleinian groups?

We proved the following theorem:

Theorem 2 *Let* $\Delta \subset \Omega(\Gamma)$ *be a component of the regular set of an analytically finite Kleinian group* Γ*. Then* Δ *is uniformly quasiconformally homogeneous if and only if* Δ/Γ_Δ *is compact or* Δ *is simply connected.*

Here, Γ_Δ is the component subgroup of Γ that stabilizes Δ, and we recall that an `analytically finite Kleinian group` Γ is one such that $\Omega(\Gamma)/\Gamma$ is of finite area.

Clearly this result contains geometric information about uniformly quasiconformally homogeneous components of regular sets. However, once again, the general setting is more complicated. We note that, following results in [2], we can observe: *For an arbitrary uniformly quasiconformally homogeneous domain (not simply connected) it is necessary, though not sufficient that its injectivity radius be bounded from above (and below).* We have examples of bounded geometry hyperbolic surfaces, necessarily of infinite area, that are not uniformly quasiconformally homogeneous.

Returning to the category of hyperbolic surfaces, one may initially ask:

Question 3 *Is every uniformly quasiconformally homogeneous surface the quasiconformal image of the regular cover of a closed hyperbolic orbifold?*

If this question has a positive answer then it would follow immediately that every uniformly quasiconformally homogeneous hyperbolic surface admits a co-compact quasiconformal group action. This is a strong requirement of a hyperbolic surface. Nonetheless we discussed at BIRS an interesting geometric question related to Question 3:

Question 4 *Does every uniformly quasiconformally homogeneous hyperbolic surface have a bounded pants decomposition?*

We note that if one could produce an example that shows Question 4 has a negative answer this counterexample would provide tremendous insight into the theory of quasiconformally homogeneous surfaces. In particular such an example would rule out the statement of Question 3 as a topological classification of uniformly quasiconformally homogeneous surfaces.

The second foundational question we considered during our visit is of a more analytical nature. Define

$$K_n = \inf\{K(M^n) : M^n \neq \mathbb{H}^n \text{ is quasiconformally homogeneous}\}$$

Again, as consequence of McMullen's version [4] of Sullivan's Rigidity Theorem, *we have shown [2] in dimension $n \geq 3$ that $K_n > 1$.* As before, the lack of quasiconformal rigidity in dimension 2 does not allow us to transport our methods. We ask:

Question 5 *Is there a constant $K(2) > 1$ such that if $S \neq \mathbb{H}^2$ is a uniformly homogeneous hyperbolic surface then $K(S) \geq K(2)$?*

If we restrict to the class of closed surfaces, we feel confident in conjecturing

Conjecture 6 *There exists a constant $K_2 > 1$ so that if S is a closed hyperbolic surface then $K(S) \geq K_2$.*

Our confidence in the validity of this conjecture follows in part from recent work. We have been able to show ([1]) that a large class of surfaces, each of which admits an automorphism with "many" fixed points, has such a lower bound. Every hyperelliptic surface belongs to this class, for example.

We found a compelling formulation of a potential positive solution to this question in the language of harmonic maps. This new formulation of the problem not only allows for new tools to be applied, but it is also mathematically interesting on its own.

In particular, we ask whether there is a certain "gap" in the energy spectrum of any closed hyperbolic Riemann surface. It would suffice to prove:

Question 7 *Does there exists a number $E_0 > 1$ and a constant $\lambda \geq 84$ such that any closed hyperbolic Riemann surface S of genus $g \geq 2$ satisfies the following: The number of energy-minimizing harmonic maps of S to itself with total energy bounded above by E_0 is at most $\lambda \cdot g$?*

We can show that a positive resolution to Question 7 would imply a positive answer to Conjecture 6, that is the existence of a constant $K_2 > 1$ such that every K-quasiconformally homogeneous closed hyperbolic surface satisfies that $K \geq K_2$. One may also ask whether the existence of a constant K_2 as described in Conjecture 6 would imply that Question 7 has a positive answer.

References

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