Banff International Research Station Proceedings 2023

BIRS

Contents

Five-day Workshop Reports

Chapter 1

The Mathematics of Microbial Evolution: Beyond the Limits of Classical Theory (23w5079)

January 8 - 13, 2023

Organizer(s): Alexander, Helen(University of Edinburgh, United Kingdom), Gandon, Sylvain (CNRS Montpellier, France), Wahl, Lindi (Western University, Canada)

Overview

Microbial populations drive human health and disease, as well as ecosystem function and global biogeochemistry. With large population sizes and relatively short generation times, microbes can evolve rapidly: developing resistance to new antimicrobial drugs, infecting new host species (e.g. SARS-CoV-2), or emerging as highly pathogenic new strains (e.g. pandemic influenza). A deeper understanding of the dynamics of microbial populations, both around and within us, is urgently needed to control the spread of novel pathogens in a globally changing environment.

Historically, mathematical modelling has played a central role in the understanding of population dynamics and evolutionary processes [1, 2, 3, 4]. Models have largely been developed for organisms that reproduce sexually, carrying one maternal and one paternal copy of each gene, in populations of roughly constant size, in environments to which the population is well-adapted. Mathematically, this focus has facilitated progress in understanding evolutionary dynamics through the use of asymptotics and a standard tool-kit of simplifying assumptions. In microbial evolution, however, many of these key assumptions no longer hold. Mutation rates can be high [5] and vary according to time [6] and environment [7]. Single mutations, for instance resistance to lethal antimicrobials, can confer benefits that are orders of magnitude larger than those observed in higher organisms. Microbes reproduce both sexually and asexually, share genes promiscuously across organism and species boundaries [8, 9, 10], and can carry from one up to hundreds of copies of some or all genes [11, 12, 13]. Population sizes are large and highly variable in time. These complexities of microbial evolution not only break the standard asymptotic assumptions of evolutionary models, but demand entirely new approaches: both new models, and new tools for their analysis.

On the empirical side, technological developments (e.g. genetic sequencing, single-cell microscopy, genetic engineering) have facilitated a burgeoning understanding of microbial genetics, physiology, and ‘lifestyle’. Understanding their consequences for microbial population dynamics and evolution calls for theoretical approaches, including development of new models and mathematical methods to address deviations from standard assumptions. However, these cutting-edge biological insights do not always reach mathematical modellers, due to the obstacles of disciplinary divides and specialist terminology. Conversely, experimental approaches provide powerful opportunities to test theoretical predictions.

A key aim of this workshop was therefore to foster a meaningful intersection between applied mathematics and recent experimental discoveries and methodological advances in microbiology. Towards this goal, we invited participants working across the theory-experiment spectrum to share their recent work and open questions. The size and format of the workshop created an excellent venue for exchange between a diverse group of mathematicians/scientists who may not normally meet.

We also invited participants with attention to diversity and with the aim of including early-career researchers. While it can be difficult to identify appropriate ECRs at the proposal stage, one successful approach to expanding our participant list following workshop acceptance was to ask any originally proposed participants who declined the invitation (e.g. due to scheduling conflicts) to nominate a more junior scientist in their place. In the end, our 35 in-person workshop participants included 60% women and 29% graduate students or postdoctoral researchers. Longer talks (see next section) were selected by proposed title, blind to participant name, and turned out to reflect the gender balance of participants as a whole.

Workshop Structure

The workshop started with a series of 22 "lightning talks" (5 minutes) where all the participants not giving a longer talk were invited to introduce themselves and describe a single research topic they were working on. These lightning talks were grouped into three distinct topics, facilitating themed discussions:
(i) measurement and evolutionary consequences of the mutation rate,
(ii) life-history evolution of viruses,
(iii) infectious disease dynamics
plus several diverse talks on microbial evolution falling outside these categories.

By the end of the second day all participants (almost) had had a chance to present themselves and this was a great way to (1) meet everybody and (2) let open problems and topics emerge for the discussion groups (see below).

We also invited 14 longer presentations (30 minutes) that allowed some of the participants to spend more time describing a specific research project. The long talks and Q&A after the long talks were available to the 15 online participants who registered for the workshop.

Group discussions occurred after each group of lightning talks and after each long talk.

Break-out research groups

A unique feature of our meeting was that participants were invited to "pitch" topics and research ideas for break-out discussion groups. After these research questions were pitched to the group, participants chose break-out groups to join and engaged in two discussion sessions. Each group then reported back to the conference as a whole, before a second set of groups was formed (some research topics continued and some new topics were added at this stage).

Many participants commented in the exit survey that a number of new research ideas were formed during these break-out discussions, along with sharing of relevant papers, approaches and potential datasets. A Slack group was created for the workshop participants, with dedicated channels for some discussion groups to facilitate further post-conference interaction and resource-sharing.

The research break-out groups that emerged addressed the following topics at the intersection of microbial evolution and mathematical modelling:

1. Including bacterial recombination in theoretical population genetics
2. Biological differences between exponential growth versus stationary phase in bacteria and their implications for modelling

3. Interpretation of mutation rate measured in the lab versus molecular clock rate estimated by phylogenetic methods

4. How to measure mutation rates

5. Evolutionary consequences of noise versus plasticity in phenotypic trait expression

6. How to measure trait variability

7. Long read sequencing technology for viruses

8. Mobile genetic elements and plasmids

9. *In vivo* estimates versus *in vitro* predictions of antibiotic resistance

This final research question (9) generated sufficient interest that a longer, full-group discussion was devoted to this topic toward the end of the week. Following from this wide-ranging and enthusiastic group discussion, the plan is to write an opinion/perspectives article on the factors driving the probability of treatment failure due to evolution of resistance, discussing whether predictions based on *in vitro* measures align with *in vivo* observations of the emergence of resistance. In brief, theory predicts that the probability of treatment failure P_{TF} can be captured by:

$$P_{\mathrm{TF}} = P_{\mathrm{trans}} + (1 - P_{\mathrm{trans}})\left(P_{\mathrm{before}} + (1 - P_{\mathrm{before}})P_{\mathrm{after}}\right) \tag{1}$$

where P_{trans} is the probability that resistance to treatment is acquired from a transmission event (i.e., superinfection), $P_{\mathrm{before}} = 1 - e^{-N_{\mathrm{before}} P_e}$ is the probability resistance emerges before the implementation of treatment while $P_{\mathrm{after}} = 1 - e^{-N_{\mathrm{after}} P_e}$ is the probability resistance emerges after the implementation of treatment. These expressions depend on N_{before} and N_{after}, the expected number of mutation events producing the resistant strain before and after treatment, respectively. Also, we assume P_e measures the probability of establishment of a resistant strain (which may or not be the same before and after the start of the treatment).

This approach suggests a suite of open questions. Is it possible to estimate the various parameters that appear in the above expression? Does (1) provide accurate prediction *in vitro*? Can these approximations can be used to evaluate the risk *in vivo*? Can this analysis yield robust recommendations for more durable treatment strategies (i.e. treatment strategies with a lower risk of failure due to pathogen evolution)?

The mix of expertise at our workshop (theory/mathematical experts along with experts in bacterial, viral and fungal evolution) allowed us a uniquely broad view of these questions. At the end of the meeting, we sketched out a rough outline for sections of a perspectives paper and identified lead authors who will coordinate the writing of each section.

Feedback from the Meeting

A google form was used as an exit survey and 75% of participants completed the survey. Feedback was overwhelmingly positive. Participants commented on the workshop having led to new collaborations, new mentorship, and new ideas for research projects and/or grant proposals. New research connections between experimentalists and mathematicians were particularly highlighted. Quotations from the anonymous exit survey include:

"I have three specific ... research directions that have come out of conversations in this meeting that I attend to follow up with ... None of these research directions existed before the meeting."

"the main foci of my group's research for at least the next few years will be along the lines set out by this workshop"

"I foresee that I will start collaborations with at least two people and possibly organize a future workshop/conference with one of them on a topic that came up during discussions."

"Invaluable."

"I can't remember the last meeting/week where I feel like I'm walking away with so much new knowledge, paper references to read, and renewed perspectives from different people on some of the things I think about a lot."

"one of the best workshops I participated in recent years (including pre-COVID)."

"Wonderful and impressive leadership from junior and mid-career women throughout the entire conference - this is really unique amongst my experiences at conferences."

Conclusions

We feel that our meeting succeeded in its aim to bring together researchers across the theory-experiment spectrum, and constitutes an important first step towards advancing our understanding of microbial evolution "beyond the limits of classical theory". Some of the discussions that emerged will allow the participants to share their expertise and learn about the theory that has already been carried out on specific questions that emerged during the discussions. Other discussions identified open questions and we expect will contribute to developing new theoretical frameworks to tackle new biological questions. While it is too early to describe the output of these discussions, the format of the conference worked very well. The "mathematics of microbes" is a very active and dynamic field of research but there is little opportunity for interactions between biologists and mathematicians in a relatively small workshop. Given the high interest and enthusiasm for follow-on projects in this fast-developing field, we feel it would be particularly relevant to carry out a similar conference in 4-5 years. In this eventuality, the new organizers could build on the lessons learned about workshop format in this first iteration (including the organizers' observations and participants' feedback in the survey) to repeat what worked well and tweak what might be improved.

As noted in the previous section, the feedback from this meeting was overwhelmingly positive. New research ideas and collaborations emerged for many participants, and a group perspectives paper is in the works. Attending a mathematics conference at which the majority of participants and speakers (60%) were female was a unique experience for many, and was particularly important to our 29% graduate student and post-doctoral fellow participants. The facilities at the Banff Centre were ideal for fostering research interaction and collaboration; we are grateful for the amazing opportunity to host a workshop at BIRS.

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Chapter 2

Computational Modelling of Cancer Biology and Treatments (23w5007)

January 22 - 27, 2023

Organizer(s): Morgan Craig (Sainte-Justine University Hospital Research Centre Universit'e de Montr'eal), Adrianne L. Jenner (Queensland University of Technology)

Overview of the Field

According to the Special Report on Cancer Prevalence published by the Canadian Cancer Society, Statistics Canada, and the Public Health Agency of Canada in November 2022, cancers affect 2 in 5 Canadians, with an estimated 1.5 million Canadians living with and beyond cancer as of 2018 [1]. The prevalence of cancer both globally and in Canada motivates ongoing research to understand how cancers begin, develop, and are treated. Cancer biology and treatment involves complex, dynamic interactions between cancer cells, the tumour microenvironment, and therapeutic molecules. Common standard-of-care therapies generally involve cytotoxic chemotherapies that are hard to tolerate. Thus, much research effort in oncology is focused on the development of improved anti-cancer treatments that will more effectively and more rapidly remove a patient's tumour, while inducing fewer toxic side-effects. At the biological level, many open questions remain about the cells of origin of a variety of tumours, how tumours interact with and adapt to the immune system, and how metastases are seeded.

Mathematical oncology is a young but mature field focused on the development of mathematical models of cancer to respond to gaps in our knowledge of cancer biology and therapy [2]. Since its outset in the 1990s, mathematical oncology has encompassed a wholly interdisciplinary approach through the integration of experimental and clinical data, and in collaboration with researchers in the pharmaceutical industry. Previous research has improved our understanding of a variety of liquid and solid tumours, how they interact within themselves and with the immune system, and the mechanisms and effects of treatment.

The questions related to how cancer begins, develops, and is treated are integral to the drug development process. The drug development pipeline is costly in terms of annualized costs (average 2.7 Billion USD per drug [3]), time, and burden to patients. In oncology, attrition along this pipeline is particularly pronounced. Thus, new solutions to drug development are required. The pharmaceutical industry has increasingly relied on model-informed drug development and quantitative systems pharmacology that integrate mathematical and computational models to facilitate drug discovery and development. Mathematical oncology has evolved in step with this paradigm [4]. Indeed, quantitative approaches combining mechanistic disease modelling and computational strategies are increasingly leveraged to rationalize pre-clinical and clinical studies, and to establish effective treatment strategies. In this way, mathematical approaches lay the foundation for computational "virtual laboratories" that offer fully controlled, and non-invasive conditions in which we can investigate emergent clinical behaviours and interrogate new therapeutic strategies [5].

Among new approaches in this vein are virtual clinical trials that integrate mechanistic mathematical and computational models of cancer development and treatment to predict the effects of therapy on a heterogeneous population of "virtual patients" [6, 7, 8, 9, 10, 11, 12, 13, 14]. Virtual (or *in silico*) clinical trials are useful computational platforms that draw on a range of mathematical techniques and help distinguish mechanisms of therapeutic successes and failures, stratify patient risk classes based on an individual's physiology, and optimize drug-specific parameters. In these platforms, *in silico* patients are generated by drawing from distributions of patient-specific characteristics and used to form virtual clinical trials, in which new treatment strategies can be evaluated prior to human trials. Data fitting, probability theory and optimal control theory are cornerstones of this computational platform and are used to generate realistic virtual patients and evaluate individualized therapies. Such *in silico* clinical trials have been used to understand how to best implement combination therapy, decipher the mechanisms of treatment response, and motivate early phase clinical trials.

Other new approaches in mathematical oncology include agent-based models (ABMs), a computational formalism that describes the way individual agents (e.g., cancer cells) interact through probability distributions based on defined characteristics [17]. ABMs have contributed significant insights into cancer biology at the intra-patient tissue level. In oncology, this technique has been applied to model spatial tumour formation, tumour cell heterogeneity, and the dynamics of treatment in the tumour microenvironment. Modelling individual cells as agents allows for direct translation of biological observation into simulation rules and, like virtual clinical trials, the investigation of new hypotheses and treatment strategies.

This workshop was focused on bringing together researchers working on developing and applying the novel techniques of ABM modelling, virtual clinical trials, and other areas of computational modelling to improve the way we model cancer biology and treatment. Participants were invited with an eye on several factors, including diversity, career stage, career type (e.g., academia and industry), and research focus (e.g., fundamental, methodological, preclinical, and/or clinical). In total, 26 participants attended in-person at BIRS and another 20 joined online. Of these 46 attendees, 20 were women and 26 were men, and 22 were early-career researchers (of which nine were students or postdoctoral researchers).

Workshop Overview

Workshop Objectives

This workshop aimed to provide an overview of cutting-edge research in mathematical oncology. One of the goals was to further a variety of techniques of critical importance to the mathematical oncology community for their continued development. For example, with the increased integration of data-driven and computational approaches in oncology, the technique of virtual clinical trials is to be more readily applied for evaluating model robustness and understanding how heterogeneity impacts on disease trajectories and treatment outcomes. In this vein, ABMs represent an important component of computational modelling in oncology however there is still a lack of consensus on the translation and implementation of basic modelling assumptions in ABMs of tumour growth. In addition to highlighting models using ordinary differential equations and stochastic approaches, we aimed to discuss and explore the advantage of different modelling assumptions with regards to specific goals in oncology, helping to establish commonalities between modelling approaches and advance the field. Lastly, the rate of generation of high-dimensional data requires the development of new mathematical and statistical techniques for their analysis. Thus, a focus of this meeting was to understand cutting-edge bioinformatics techniques and their integration within mathematical modelling in oncology. Importantly, the rapid pace with which new biological insights and treatment modalities are discovered and implemented implies that meetings such as this are necessary for our community.

Overall, the objectives of this workshop were to:

- provide insight into the range of mathematical modelling techniques used to analyze preclinical and clinical data in oncology, including basic tumour biology;
- introduce different techniques for developing *in silico* clinical trials and their ability to account for within- and between-patient heterogeneity;
- review ABM platforms for modelling in cancer biology and treatment;
- discuss the intersection of current modelling approaches to develop improvements to each approach;
- understand state-of-the-art treatment approaches and the ways they can be modelled.

Workshop Structure

We adopted several modalities to address the workshop goals stated above. These include lightning talks of 3-4 minutes delivered by each participant (in-person and virtually) to provide an overview of their work, six plenary (60 minute) talks delivered by experts in the field, and breakout group discussions. Each of these aspects is discussed in the sections below.

Lightning Talks

A total of 35 lightning talks were delivered over the first day of the workshop. Each of these presentations highlighted either a specific aspect of the participant's research (e.g., modelling to improve the treatment of acute myeloid leukemia, insights from mathematical models of gene regulatory networks, treating stem cells with oncolytic viruses etc.) or provided an overview of their research program (e.g., cell fate decision making in stem cells and cancer, real-life tumour simulations as test beds for potential cures etc.). The goal of these talks was to quickly familiarize the group to the research being carried out by attendees, helping to shape the discussion groups that would carry on throughout the week (see below).

Keynote Talks

Six plenary presentations were delivered. Keynote speakers were chosen to highlight research excellence and the diversity of our community. Efforts were made to ensure gender parity in speakers and to ensure that presenters represented a range of research (from fundamental biology to clinical work). A brief description of each talk is provided below.

1. **Mohit Kumar Jolly: "What does not kill cancer cells makes is stronger: Dynamical modeling of drug-induced cell-state switching"** Dr. Jolly presented mathematical models of gene regulatory networks and phenotypic interactions to study heterogenity in cell killing and combination strategies in melanoma and non-small cell lung cancer [15].
2. **Ivana Bozic: "Evolutionary dynamics of tumor progression"** Dr. Bozic's talk centered on chronic lymphoblastic leukemia and colon cancer. In her presentation she described recent work describing the Bayesian classificiation of tumour growth as either 1) logistic, 2) exponential, and 3) indeterminate, and further discussed the dynamics of pre-leukemic expansions [16].
3. **Paul Macklin: "A cell behavior grammar for real-time modeling and knowledge curation"** Dr. Macklin described updates to the PhysiCell (www.physicell.org) agent-based modelling framework that he and his team have developed. These include a GUI and a standard dictionary of cell types and functions to build models interactively, helping to break down barriers between the modelling community and experimentalists/clinicians who may not be familiar with mathematical models [17].

4. **Natalia Komarova: "Evolutionary modeling of cancer treatment"** Dr. Komarova described her work on the stochastic analysis of combinatorial mutation networks during combination therapy with applications to leukemia and colon cancer. She provided an overview of how straightforward models can be used to distinguish mechanisms of drug resistance and treatment success, and discussed minimal data requirements to respond to model parameter non-identifiability [18].

5. **Adam MacLean: "Inference of cancer cell state dynamics in complex tumor microenvironments"** Dr. MacLean spoke about how cancer was a multiscale problem. He described his work on using transcriptomics to understand calcium signalling and cell responses. He also presented recent work on myeloid-derived suppressor cells in the metastatic niche [19]

6. **Renee Brady-Nicholls: "Improving Prostate Cancer Hormone Therapy Through Dynamic Modeling"** Dr. Brady-Nicholls presented her work aimed at reducing racial disparities in metastatic prostate cancer through extensions to adaptive therapy protocols tailored to improve outcomes for underrepresented minorities. She discussed range-bounded adaptive therapy in this context, which is based on the idea that treatment cycling improves the duration of treatment efficacy because it provides sensitive cells the ability to outcompete resistant cells [20].

Breakout Group Discussions

Based on the lightning talks, we met as a group in the afternoon of the first day of the workshop to discuss the elements common to the participants. Several topics were explored, including standardization, what tools do we need to develop?, combination therapies, phenotypic/genetic transitions, parameter identification, cancer immunology and immuno-oncology, multiscale modelling, agent-based models, modelling frameworks, gene regulatory networks, parameter estimation, and virtual clinical trials. We decided to divide into three groups focused on 1) Frameworks (e.g., agent-based models, modelling standards, high performance computing in mathematical oncology, etc.), 2) Treatments (e.g., combination approaches, immunotherapies, etc.), and 3) Biology (e.g., tumour development, cancer stem cells, etc.). Outside of keynote talks, we met in these groups for the rest of the week.

Discussions in the Frameworks group centred on how a standard of modelling in oncology could greatly assist the mathematical oncology. Our group discussed, how with the exponential increase in mathematical modelling works in oncology, we may be losing our ability to validate and reproduce results, which then reduces the reliability of mathematics in cancer research. We discussed how a framework for how a model should be developed and what minimum information was necessary in papers is crucial moving forward in the field. In our discussions, we sketched out what this framework should look like, and hope to share it with the community imminently.

The Treatment group decided to focus on combination therapies, given their importance for anti-cancer therapy. We held extended discussions about a toy model framework that could be developed as a guide for deciding modelling elements to describe combination treatments. The idea behind this basic model was to represent in the most general framework the fundamental building blocks of polytherapy, including sensitive and resistant cells, and the immune response. Conversations also centred on what data we would need to properly parameterize the toy and other more complicated models (e.g., cell counts, cell kinetics etc.) and what data we could expect to be able to access (e.g., pharmacokinetics and pharmacodynamics, etc.) with the goal of providing practical utility and application. In this vein, we performed a literature review of existing models of combination therapies including chemotherapies, immunotherapies, surgeries, radiotherapies, among others. At least one paper is under development as a result of these discussions.

The biology group discussed the problems we encounter when translating cancer biology into mathematical models. These conversations were focused on multiscale models, how to simplify or find the simplest model to describe a biological phenomenon, and the power of the information provided by such simple models. Relatedly, we also wondered whether a complicated model was necessarily best to define the biology for the specific question to answer, i.e., do we care about the output of a biological system or the multiple pathways leading to the response?

As was also discussed in the Frameworks and Treatment groups, a major question we asked ourselves was about the translation between modellers and experimentalists, specifically is it possible to measure the data needed to accurately model the biological question? With respect to model calibration, if we had access to such data, how many data points and how close in time must they be to accurately represent the dynamics of the system in question? This question was answered for simple models describing colorectal cancer and leukemia by Dr. Komarova in her plenary talk later in the week.

Feedback from the Meeting

After the meeting, we received a number of messages and comments from participants about the workshop. A sampling of these is provided below.

"*...The conference was truly remarkable, and I am so grateful for the opportunity to participate. It was great to catch up with everyone on the latest advances in mathematical oncology and it was an awesome learning experience for me. Especially the networking opportunities were invaluable. I had the chance to connect and discuss collaborations with many people. I am sure that some of these would pan out to be exciting new research projects. Once again, thank you for the invitation and for putting together such a wonderful event and at such a beautiful venue. Loved every bit of it!*"

"*...Thanks for organizing and great week at BIRS. I had a great time at the conference and am looking forward to the next one! I was inspired by the keynote speakers and the research of the participants. This workshop stood out from others due to the time devoted to discussion on current themes in computational modeling. I found these discussion sessions immensely valuable as they provided a forum to debate different approaches, share and listen to new ideas, and connect with other researchers.* "

"*Thank you so much...for...making it possible for me to participate online.*"

"*The Computational Modelling of Cancer Biology and Treatments workshop was one of the most important networking events during my training as a PhD student. Talks by keynote speakers opened my eyes on the diversity in the computational approaches used to study the field as well as in the researchers themselves. However, the group discussion sessions were the highlight of the event. They were great opportunities to get to know the scientists in this community, those that can potentially be my next mentors, colleagues, or collaborators. I learned from more experienced researchers about the differences in the restrictions and goals of cancer research for scientists in academia and in industries. We also discussed the challenges of working in collaborations with experimentalists and the multiple problems we encounter when translating tumour biology into mathematical and computational models. Finally, inputs from the more senior researchers influenced me to look at my own research with a new perspective and inspired new ideas to apply in my studies.*"

Conclusions

Mathematical oncology is a significant subfield of mathematical biology. As discussed above, we believe that this workshop achieved its aims, namely to bring together a diverse group of researchers across a swath of research focuses and career-stages to forward computational modelling in cancer biology and treatments. Feedback received from participants was overwhelmingly positive, and talks are ongoing to organize a follow-up event. We wish to thank the Banff International Research Station staff for helping to organize this successful event and hope to be back soon.

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Chapter 3

Women in nonlinear dispersive PDEs (23w5059)

February 5 - 10, 2023

Organizer(s): Mihaela Ifrim (University of Wisconsin Madison), Birgit Schoerkhuber (University of Innsbruck), Katharina Schratz (Sorbonne University)

Overview of the Field

Nonlinear dispersive partial differential equations is one of the vast research areas in the field of partial differential equations. It is perhaps one of the most challenging and exciting directions of scientific pursuit simply because of the complexity of the subject and the endless breadth of applications. Dispersive PDE's encompass nonlinear wave and Schrödinger equations including problems with geometric flavour, classical dispersive problems such as the Korteweg–De Vries (KdV) equation, as well as systems arising in the study of water waves. Such models have been the focus of extensive research over the past decades. This is in part due to their undeniable significance in the physical description of nature ranging from the dynamics of ocean waves to general relativity, but also due to their rich mathematical structure caused by their inherently nonlinear nature. From an analytic point of view, for a large class of problems classical questions concerning local-wellposedness and the behavior of solutions for small initial data are fairly well-understood by now. However, despite recent progress, much less is known concerning *global dynamics of solutions for large initial data*. Central questions include *the formation of singularities* in finite or infinite time, the existence of *multi-solitons* and the famous *soliton resolution conjecture*, as well as the description of *thresholds between different stable dynamical regimes*. Above these questions, there is the pressing and overarching problem of *generic behaviour* which naturally introduces a *probabilistic point of view*. Moreover, in the investigation of global nonlinear dynamics, *reliable numerical simulations* have proven to be indispensable.

In recent years, some major developments in all of these branches have emerged and further ideas are now surfacing. Naturally, this includes many important contributions by female mathematicians. However, as the field of nonlinear dispersive PDEs itself is fragmented into smaller communities, both thematically and geographically, women are usually heavily underrepresented at regular meetings and conferences.

The workshop gathered world-leading female experts and young researchers working - both theoretical and numerically - on the analysis of dispersive PDEs in order to foster scientific exchange on the most important open questions in the field. With this workshop, we hoped we inspired women, non-binary mathematicians and people of other minorities to share their mathematical achievements, as well encouraged an open, welcoming, and inclusive work environment for everybody.

Organization and scope of the workshop

Scientific content of the workshop

The goal of the workshop was to bring together women of all career levels in order to share scientific progress from different sub-areas of the field. Thematically, the topics of the talks roughly can be gathered into four categories:

- **Well-posedness of dispersive PDEs - Low regularity and probabilistic aspects**
- **Fluid dynamics**
- **Global dynamics and singularity formation**
- **Numerical aspects and modelling**

The workshop's schedule reflects our effort to offer a well-balanced mixture of talks by established and early career researchers and to provide a loose structure navigating through the above topics that emerged during the workshop, while meeting at the same time the circumstances of the hybrid character of the event. In addition to regular lectures, Ph.D. students presented their work in a poster session with short talks. Below, we give a detailed summary of the content and the scientific highlights within each sub-area.

Women in ...

In the last decades, at vast effort has been taken to increase the number of women in mathematics and natural sciences and to remove impediments female researchers might face during their career. However, women are still underrepresented in most mathematical fields and despite some progress, the situation has not improved significantly. Today, the topic is still controversial, many questions are hard to address globally due to political and sociological differences and discussions are often driven by particular personal experiences. In order to provide a common ground for discussions, we were happy to have had as an external speaker Prof. Maria Vlasiou (University of Twente), who represents the Netherlands at the European Women in Mathematics and the Committee for Women in Mathematics of the International Mathematical Union, among many other activities. In her inspiring talk on *Women in Academia: challenges and best practices* she provided a quantitative perspective based on data and scientific studies from different countries over the past years. Most interestingly, she showed that for every study that confirms the positive effect of a certain measure (such as the quota), yet another study can be found that demonstrates the contrary. This of course is rather raising questions than giving final answers and triggered many fruitful discussions among participants. However, the only measure that seems to be successful beyond doubt is mentoring of students and young researchers. During the workshop, we were glad to observe that mentoring happened naturally by the interaction of workshop participants of different career levels.

Additional online discussion meetings

In order to guarantee the chance of vivid discussions for the online participants as well (who are unfortunately missing out on discussions within the coffee, lunch and dinner breaks), we organized additional Zoom meetings. Due to the large amount of online participants located in Europe and the rather big time difference, these meetings took place around 12-13 pm Central European Time (CET) on four of the workshop days. The meetings were moderated by the one organizer who was participating remotely. They were structured so that they closely followed the same mathematical discussions as the on-site program. Versions of the talks highlighting possible venues of support for early career researchers were also present in the schedule of these on-line meetings. Specifically, the following discussion sessions were offered in addition to the regular schedule:

- Tuesday, 12-13 pm CET: Meet & Greet

- Wednesday, 12-13 pm CET: Research discussion on bridging the gap: theoretical and numerical analysis of nonlinear dispersive PDEs (what can we learn from PDE theory to improve numerical methods, can numerical simulations help to push forward the theory? What are the open problems, challenges in the various fields?)
- Thursday, 12-13 pm CET: Women in math and career development: advice for (early career) female mathematicians.
- Friday, 12-13 pm CET: Research discussion (continued)

Within these discussions the exchange of ideas between mathematicians in both theoretical and numerical analysis of nonlinear dispersive PDEs lead to highly interesting new directions on the interface of these two areas, such as for instance in the reliable simulation of blow up phenomena with cutting edge PDEs techniques.

Scientific topics and highlights

The workshop featured lectures of one hour within a broad spectrum of topics. All speakers enthusiastically communicated their results. Their talks usually started with a general introduction into the specific field of research. In addition, Ph.D. students had the chance to present their results in a poster session with short talks of 15 min each.

We summarize the content of the lectures (partially based on the abstracts provided by the participants), include references and outline the scientific highlights. Thereby talks can be grouped into four categories that emerged during the workshop. Talks by postdocs are indicated by a bullet (•), poster presentations by Ph.D. students by a start ($*$). The order of the talks within each section follows the schedule of the workshop.

Well-posedness of dispersive PDEs - Low regularity and probabilistic aspects

This portion of the workshop featured talks by Monica Visan, Maria Ntekoume, Hajer Bahouri, Akansha Sanwal, and Katie Marsden. Brief discussion of their work is detailed below.

Monica Visan (University of California) reported on an yet another interesting application of the method *second generation of commuting flows* introduced in joint works with collaborators (B. Harrop-Griffiths, R. Killip, M. Ntekoume), see [8]. The method is meant to provide a recipe for achieving low regularity well-posedness results for complete integrable PDEs. The focus was on the well-posedness of the derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u = -i\partial_x(|u|^2 u)$$

on the line. The speaker gave a broad description of some inherent instabilities which have hindered the study of this equation. The global well-posedness result was discussed in the natural scale-invariant space due to this innovative method.

Hajer Bahouri (Sorbonne University) presented her recent work on the spectral properties of the sublaplacian $-\Delta_G$ on the Engel group, which is the main example of a Carnot group of step 3. The author developed a new approach to the Fourier analysis on the Engel group in terms of a frequency set. This enabled her to give fine estimates on the convolution kernel satisfying $F(-\Delta_G)u = u * k_F$, for suitable scalar functions F, and in turn to obtain proofs of classical functional embeddings, via Fourier techniques. This analysis requires a summability property on the spectrum of the quartic oscillator, which the author obtained by means of semiclassical techniques and which is of independent interest; see [1].

Akansha Sanwal (•) (University of Innsbruck) presented recent result obtained jointly with Robert Schippa, [14] on the well-posedness of the fractional Kadomtsev–Petviashvili equation (fKP-I) equation

$$\partial_t u - D^\alpha \partial_x u - \partial_x^{-1}\partial_y^2 u = u\partial_x u.$$

In the case of strong dispersion, more precisely for $\frac{5}{2} < \alpha < 4$, global well-posedness is obtained in $L^2(\mathbb{R}^2)$ for real valued initial data. This is achieved by exploiting transversality in the resonant case via bilinear Strichartz estimates and the nonlinear Loomis-Whitney inequality. For small dispersion, it was proved that the initial value problem cannot be solved by Picard iteration. However, frequency-dependent time localisation can be used to prove local well-posedness for $2 < \alpha \leq \frac{5}{2}$ in the anisotropic Sobolev space $H^{s,0}(\mathbb{R}^2)$.

Maria Ntekoume (•) (Rice University) continued on the topic of Monica's talk (see [8]) emphasising more properties of the completely integrable model in discussion. Until recently the well-posendess of the equation below $H^{\frac{1}{2}}$ was not known. From the talk we learned that the problem is well-posed in the critical space L^2 on the line. In doing so the speaker highlighted several recent results that led to their result.

Katie Marsden ($*$) (EPFL) presented her current studies on the energy critical nonlinear Schrödinger equation with randomised initial data in dimensions $d > 6$. The result discussed the probabilistic well-posedness theory namely the almost surely globally well-posed with scattering for the randomised super-critical Cauchy problem in the Sobolev space $H^s(\mathbb{R}^d)$ whenever $s > \max\left\{\frac{4d-1}{3(2d-1)}, \frac{d^2+6d-4}{(2d-1)(d+2)}\right\}$; for reference see [12]. The key ingredient was the randomisation which in this case was based on a decomposition of the data in physical space, frequency space and the angular variable. Such result extends previously known results in dimension 4, [15]. The main difficulty in the generalisation to high dimensions was the non-smoothness of the nonlinearity.

Fluid dynamics

This portion of the workshop featured talks by Anna Mazzucato, Sylvie Monniaux, Helena Nussenzveig Lopes, Susanna Haziot, Anne-Laure Dalibard, and Mihaela Ignatova. The talks were about solutions to nonlinear PDEs in various settings where wave interactions take on a very different character on long time scales due to the lack of dispersion. These problems came strongly motivated in part by the fluid dynamics realms of open questions. For brevity we give below an outline of their work.

Anna Mazzucato (Penn State University) discussed about "Irregular transport and loss of regularity for transport equations". The comprehensive picture presented by the speaker included an overview of the broader area in which new recent results in the analysis of irregular transport, with non-Lipschitz vector fields, and its applications to kinetic theory, advection of active and passive scalars, fluid mechanics, and related areas have emerged. The work, which was joint work with collaborators (Gianluca Crippa, Tarek Elgindi, and Gautam Iyer) focused namely on recent results concerning examples of loss of regularity for solutions to linear transport equations with advecting field in Sobolev spaces below the Lipscitz class, see [4]. It was very well described how this loss is generic and can be made instantaneous and total (that is, there exists smooth initial data for which the solution leaves instantaneously any Sobolev space of positive order). Potential applications and limiting properties of these methods were also outlined.

Sylvie Monniaux (Aix-Marseille University) presented her recent work and results obtained with collaborators (Matthias Hieber, Hideo Kozono and Patrick Tolksdorf) about how to construct a solution to the Keller-Segel-Navier-Stokes system in critical spaces via weighted maximal regularity. One of the problems in doing so was reported as being the fact that, in Lipschitz domains (in dimensions 2 or 3), the Stokes operator has regularising properties only in a small range of spaces.

Helena Nussenzveig Lopes (Universidade Federal do Rio de Janeiro) reported on very recent work done with collaborators (Fabian Jin, Samuel Lanthaler, Milton C Lopes Filho and Siddhartha Mishra) that settled the necessary and sufficient conditions on the regularity of the external force for energy balance to hold for weak solutions of the 2D incompressible Euler equations. The problem was motivated by turbulence modeling and the result should be contrasted with the existence of wild solutions in 3D.

Susanna Haziot (•) (Brown University) presented her current findings pertaining to the Muskat equation. This equation models the interaction of two incompressible fluids with equal viscosity propagating in porous medium, governed by Darcy's law. In this talk, the speaker investigated the small data critical regularity theory for this equation, and in particular, the desingularization of interfaces with small moving corners.

Anne-Laure Dalibard (Sorbonne University) gave a talk devoted to the study of the equation

$$uu_x - u_{yy} = f$$

in the domain $(x_0, x_1) \times (-1, 1)$, in the vicinity of the shear flow profile $u(x, y) = y$. This equation serves as a toy model for more complicated fluid equations such as the Prandtl system. The difficulty was reported to lie in the fact that one is, in general, interested in changing sign solutions. Hence the equation is forward parabolic in the region where $u > 0$, and backward parabolic in the region where $u < 0$. The line $u = 0$ is a free boundary and an unknown of the problem. The author together with collaborators (Frédéric Marbach and Jean Rax) proved that even when the data (i.e. the source term f or the boundary data) are smooth, existence of strong solutions of the equation fails in general; more details in [5]. This phenomenon is already present at the linear level, and linked to the existence of singular profiles for the homogeneous linearized equation. In fact, the result obtained proved that strong solutions exist (both for the linearized and for the nonlinear system) if and only if the data satisfy a finite number of orthogonality conditions, whose purpose is to avoid the presence of singular profiles in the solution. A key difficulty in their work was to cope with these orthogonality conditions during the nonlinear fixed-point scheme. In particular, this led to the proof of the stability of these solutions with respect to the underlying base flow.

Mihaela Ignatova (Temple University) reported on her joint work with Jingyang Shu about the analysis of the Boussinesq equations, [10]. These are a member of a family of models of incompressible fluid equations, including the 3D Euler equations, for which the problem of global existence of solutions is open. The Boussinesq equations arise in fluid mechanics, in connection to thermal convection and they are extensively studied in that context. Formation of finite time singularities from smooth initial data in ideal (conservative) 2D Boussinesq equations is an important open problem, related to the blow up of solutions in 3D Euler equations. The Voigt Boussinesq is a conservative approximation of the Boussinesq equations which has certain attractive features, including sharing the same steady solutions with the Boussinesq equations. In her talk which included a brief description of issues of local and global existence, well-posedness and approximation in the incompressible fluids equations, the speaker presented a global regularity result for critical Voigt Boussinesq equations.

Global dynamics and singularity formation

This portion of the workshop featured talks by Svetlana Roudenko, Valeria Banica, Annalaura Stingo, Gabriele Brüll, Susana Guitierrez, and Xueying Yu. The lectures covered recent results on the existence and stability of solitons with a focus on fractional dispersive equations. Furthermore, global regularity and the existence of finite-time blow-up solutions has beed discussed for various different geometric models. We give below an outline of the work discussed.

Svetlana Roudenko (Florida International University) gave a talk on the existence and stability of solitary waves of the fractional Korteweg-de Vries (KdV) family of equations including higher dimensional generalisations. The classical KdV equation is a well-studied dispersive PDE, which originally arises as a one-dimensional model in the description of shallow water waves. Allowing for solitary wave solutions, it gave rise to the soliton resolution conjecture, which now is believed to be a rather general principle in dispersive PDEs. Various generalisations of this equation have been introduced since, including higher dimensional models with various applications. In the recent years, there was rising interest in fractional versions of the KdV equation, where the dispersion operator is replaced by a fractional derivative. After a thorough introduction into this huge field of research, Svetlana addressed, among other things, recent results on the existence and stability/instability of solitary waves fractional models within KdV family in $d \geq 1$, in the subcritical and supercritical regime, obtained in joint work with Oscar

Riaño [13].

Valeria Banica (Sorbonne University)) reported on recent results obtained with collaborators (Renato Lucá, Nikolay Tzvetkov and Luis Vega), see [2]. Their work concerns blow-up for the one-dimensional cubic nonlinear Schrödinger equation (NLS). The global well-posedness of the model is known in $H^s(\mathbb{R})$ for $s > s_c$ with $s_c = -1/2$ denoting the critical regularity (only for $s \geq 0$ the flow map is uniformly continuous on bounded sets of $H^s(\mathbb{R})$). For $s < s_c$ the initial value problem is ill-posed in the Hadamard sense due to norm inflation with loss of regularity. By identifying a particular functional analytic framework from which solutions exit in finite time, blow-up is demonstrated at the boarderline regularity. The proof relies on a reformulation of the problem within the considered class of solutions to the study of large-time solutions of a periodic, non-autonomous cubic Schrödinger equation, by using a pseudo-conformal transformation. The results obtained for the cubic NLWs are applied to obtain a criterion for generating finite-time singularities through the binormal flow, which is a model for one vertex filaments in 3D fluids. The connection between these two models was explained in detail in the talk.

Gabriele Brüll (•) (Lund University) presented recent on traveling waves for the fractional Kadomtsev– Petviashvili equation (fKP or fKP-1)

$$\partial_x(\partial_t u + u\partial_x u - D^\alpha \partial_x u) - \partial_y^2 u = 0,$$

for $u = u(t, x, y)$ with $\frac{1}{3} < \alpha \leq 2$; this work is joint Handan Borluk and Dag Nilsson [3]. This model can be viewed as the two dimensional generalization of the fractional KdV (fKdV) equation. Solitary wave solution of the fKdV solve the fKP equation and are referred to as line solitons. For the classical KP-I equation, it is known that line solitons are unstable. In the talk, linear instability of the line soliton for the fKP-I equation was demonstrated by using a criterion due to F. Rousset and N. Tzvetkov. Furthermore, numerical experiments were shown which support the instability result for the fractional KP-I equation and suggest transverse stability for the fractional KP-II equation (for which the y-derivatives in the above equation come with a positive sign). Moreover, the existence and properties of fully localized solitary solutions for the fractional KP-I equation have been discussed during the talk.

Annalaura Stingo (•) (Ecole Polytechnique) reported on ongoing work with her collaborators (Cecile Huneau and Zoe Wyatt) concerning Kaluza-Klein theories, which represent a classical mathematical approach to the unification of general relativity with electromagnetism . In these theories, general relativity is considered in 1+3+d dimensions with the space-time factorizing as $\mathbb{R}^{1+3} \times K$ with K a compact d-dimensional manifold (the internal space). In the simplest case $K = \mathbb{S}^1$ gravity is compactified on a circle to obtain at low energies a $(3 + 1)$-dimensional Einstein-Maxwell-Scalar systems. The talk addressed the problem of the classical global stability of Kaluza-Klein theories, which is still an open problem in the three dimensional case (despite known positive result for $\mathbb{R}^{1+n}$ with $n \geq 9$). By using wave coordinates, the problem can be reformulated as a system of quasilinear wave equations. By studying toy models within this class, mathematicals mechanisms can be identified underlying the stabilty problem, see also [9].

Susana Guitierrez (University of Birmingham) talked about self-similar solutions of the Landau-Lifshitz-Gilbert (LLG) equation. This system emerges in the description of the dynamical behaviour of spin vectors in ferromagnetic materials and is given by

$$\partial_t m = \beta m \times \Delta m - \alpha m \times (m \times \Delta m)$$

for $m = (m_1, m_2, m_3) : \mathbb{R}^d \times I \to \mathbb{S}^2 \subset \mathbb{R}^3$ and parameters $\alpha \in [0, 1], \beta = \sqrt{1-\alpha^2}$. From a mathematical point of view, the system includes two prominent limiting cases, which is on the one hand the heat flow of harmonic maps into the 2-sphere ($\alpha = 1, \beta = 0$), and on the other hand the Schrödinger map equation (for $\alpha = 0, \beta = 1$). In view of the scale invariance of the LLG equation, it is reasonable to look for self-similar, which can be grouped into expanders and shrinkers, where the latter provide examples for the formation of singularities in finite time. This talk addressed the existence, the properties and the dynamical behaviour of shrinkers in the one-dimensional case, i.e., for $d = 1$. Thereby, a novel geometric a approach was used, which relies on the identification of a self-similar profile with the tangent of a curve in $\mathbb{R}^3$ and the reformulation of the problem in terms of a Serret-Frenet equations, see the joint work [7] with André de Laire for more details. In addition the Cauchy problem for the LLG-equation was discussed including global well-posedness for initial data being small in the BMO norm.

Xueying Yu (•) (University of Washington) talked about uniqueness properties of solutions to the linear generalized fourth-order Schrödinger equations posed in any dimension with bounded real-valued potentials of the following form

$$i\partial_t u + \sum_{j=1}^{d} \partial_{x_j}^4 u = V(x)u.$$

The speaker showed that a solution with fast enough decay in certain Sobolev spaces at two different times has to be trivial. The work presented is jointly with Zachary Lee and can be seen [11].

Numerical aspects and modelling

This portion of the workshop featured talks by Mechthild Thalhammer, Fatima Zohra Goffi, and Karolina Kropielnicka. The first and the third talk addressed numerical issues with broad applications. The second talk added to the workshop the interesting and interdisciplinary problem of mathematical modelling of dispersive phenomena.

Mechthild Thalhammer (University of Innsbruck) discussed the numerical treatment of the Landau equation. The latter is fundamental when applying Hamiltonian operator splitting methods to multi-species Vlasov-Maxwell-Landau systems. The main numerical challenge lies in the computation of the three dimensional case with Coulomb interaction, which is the most relevant case in physics. In the talk, which is based on joint work with J. A. Carrillo, novel approaches for the evaluation of the Landau collision operator based on efficient Fourier techniques were introduced and compared. The new approach in particular allows to greatly reduce computational efforts thanks to pre-computations which are independent of the density function. The new approach allows for mass conserving in the time integration of the Landau equation. Numerical experiments underlined the favorable behaviour of the new method.

Fatima Zohra Goffi (•) (Karlsruhe Institute of Technology) presented results on the mathematical description of wave propagation in meta-materials that have been obtained with her collaborators (Andrii Khrabustovskyi, Ramakrishna Venkitakrishnan, Carsten Rockstuhl and Michael Plum), see [6]. The propagation of electromagnetic waves in meta-materials is described as strong dispersion. This can be seen through the propagation of multiple modes one can observe when taking into consideration a constitutive relation of nonlocal type. This later links the exciting electric field to the electric displacement by considering the effect of the surrounding neighbourhood of the observation point. In the talk it was shown that the spatially nonlocal characterisation of the material law serves for deriving additional effective material parameters. These effective parameters translate the nonlocal effects produced in the response of meta-materials to the exciting electric field.

Karolina Kropielnicka (Polish Academy of Sciences) gave an overarching exposition on numerical methods for highly oscillatory Klein–Gordon type equations. Numerically, the main difficulty thereby stemmed from the time dependent mass term which in physical applications might be highly oscillatory. Karolina showcased the failure of classical methods in this setting: classical splitting and Gautschi type methods can not capture the highly oscillatory nature of the problem which leads to large errors and huge computational costs as the discretisation parameters have to be adapted to the highest frequency. In order to overcome this numerical burden, Karolina introduced a new approach which allows to treat time dependent mass terms reaching from highly oscillatory up to slowly varying regimes. The idea is based on operator splitting methods, integrating the highly oscillatory phases exactly. This leads to a new class of scheme with much higher accuracy than classical schemes. Results of these investigations were obtained with Karolina Lademann, Katharina Schratz, Mrissa Condon and Rafal Perczynski.

Conclusion and Feedback

We share BIRS's dedication to equity, diversity, and inclusiveness. The workshop's organizers carefully selected and recruited a diverse selection of participants, both within the group of workshop members and the group of post-docs and graduate students. The selection of speakers also reflected similar considerations; indeed, early-career mathematicians comprised a substantial portion of the speakers.

The mathematical content of the workshop touched on many nowadays vibrant and hot topics in PDEs, and the breath of novelty was impressive. It was the perfect time to organize this workshop in part due to the high amount of new emerged works which produced substantial progress in a series of conjectures.

As described above, the workshop featured a rather broad spectrum of topics. Thus, during an intensive week of talks and common social activities, experts from different fields had the chance to learn from each other and discuss new scientific directions. The rather small size of the workshop, in addition to the online meetings, provided a great environment for personal interaction between different generations of female mathematicians. Many of the attendees expressed their gratitude for having a well-designed workshop that combined the on-site and online organization in a perfect mathematical and social experience. In comparison with only in-person programs, the new format had the advantage of accommodating more mathematicians, broadening not only the mathematical interactions but also the social interactions. This workshop opened new collaborations among the participants, and we expect to hear soon from them via upcoming publications.

The feedback we, the organizers, received from the participants was beyond our expectations; many of the participants strongly emphasized their gratitude and the positive effects of their participation. In short everyone hopes we will continue this tradition to meet in Banff regularly, every 2-3 years. We are looking forward to have such an amazing ongoing event as we felt that the format and the scientific scope of the workshop met perfectly our goals and the participants' expectations.

Some of the feedback we received is displayed below.

" I didn't get a chance to thank you in person on Friday morning for inviting me to the conference. I think everybody agrees it was a very successful event. I learned a lot, met some interesting people ! "

" I want to thank you for the invitation to give a talk and for a wonderful conference! It was great meeting you and I hope to see you again in Banff ! "

" This is one of the best conferences I have attended after the pandemic ! I have learned a lot from the speakers. It was so easy to talk to all of the participants ! Thank you for the effort you out in organizing this conference; it was just great ! "

" This conference was a fantastic concept! Please hold more of these events. Personally, I felt less frightened than I had in previous conferences, and as a result, I asked more questions than I had ever asked in previous conferences. When I spoke with other participants, I realized that some of them felt the same way. "

Acknowledgement

We have benefited in our attempt to have a successful conference from the experience of BIRS, to which we extend our sincere thanks and deep gratitude. The success of the workshop is in equal part due to them, as they showed amazing generosity and professionalism in all the actions they took, from prompt email responses, to an outstanding on-site technical support, to incredible fitness center, lodging and food amenities, to great outdoor views and helpful staff members.

Participants

On-site

Banica, Valeria (Sorbonne Université)
Byars, Allison (UW Madison)
Goffi Fatima Zohra (Karlsruhe Institute of Technology)
Gutierrez, Susana (University of Birmingham)
Haziot, Susanna (Brown University)
Ifrim, Mihaela (University of Wisconsin Madison) - organizer
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Chapter 4

Non-Markovianity in Open Quantum Systems (23w5083)

February 12 - 17, 2023

Organizer(s): Marco Merkli (Memorial University of Newfoundland), Nicole Yunger Halpern (National Institute of Standards and Technology), Susana Huelga (Ulm University), Kavan Modi (Monash University), Francesco Petruccione (Stellenbosch University)

Overview of the Field

Open systems are subject to the uncontrollable effects of their surrounding environments. As opposed to isolated systems' evolutions, open systems' evolutions are not unitary, but include dissipative components that result from environmental fluctuations. The character of these fluctuations dictates the type of dynamics that the open system will undergo. In their simplest form, environmental fluctuations are merely brief deviations from equilibrium values, and the open system typically loses energy and coherence via a unidirectional flow towards the environment. This is the behavior typically observed in atomic relaxation and in the loss of optical coherence in the presence of the radiation field [13].

The type of master equation or dynamical map that describes open system dynamics in these situations is called *Markovian*, or memoryless. The latter name evokes the lack of any black-flow of information from the environment into the system. When derived *ab initio*, a Markovian evolution typically evinces a clear separation of two time scales: (i) the time scale of the system evolution that results from the coupling to the environment and (ii) the much shorter environmental correlation time. This phenomenology was formulated rigorously in the context of the weak-coupling framework by Davies [12]. Subsequent work by Lindblad [25] and by Gorini, Kossakowsi, and Sudarshan [18] provided a neat mathematical result identifying conditions sufficient for the dynamical generator of open-system dynamics to be completely positive and trace preserving (CPTP). The Davies master equations emerge now as resulting from the canonical structure of the generator of quantum dynamical (Markovian) semigroups. These seminal works can be regarded as the first attempts to formalize the concept of the absence of memory (Markovianity) in quantum evolution. Dynamics that cannot be expressed in this manner would be classified as non-Markovian. In recent years—and perhaps motivated by the community's increasing ability to control, probe, and manipulate quantum systems embedded in soft or solid-state environments—it became timely to fully characterize memory effects in the quantum domain. The aim is to develop comprehensive quantum theory of memory. Rather than resorting to microscopic derivations that may change from platform to platform, the aim is to develop a general mathematical framework for quantum non-Markovianity.

This is the general context of the BIRS workshop, whose main goal has been to bridge two promising approaches to the definition and characterization of non-Markovianity in the quantum realm. The first approach introduces definitions of Markovianity and associated non-Marvovianity measures based upon the structure of CPTP quantum dynamical maps. This includes the definition of Markovianity in terms of CP divisibility [32] and definitions in terms of distance measures that provide contractions for CP maps, like the celebrated trace distance measure [7]. Interestingly, these two definitions are not equivalent [33].

The second approach to quantum non Markovianity aims to extend the theory of classical stochastic processes to the quantum domain. Rather than two-time properties of CPTP generators, the key figure of merit is the multi-time statistics resulting from probing the system at intermediate times. The validity of the quantum regression expression for joint probability distributions provides one possible definition of Markovianity in this setting [36, 28]. A unifying formalism may be facilitated by the recently introduced concept of process tensors [27]. This general description of quantum processes naturally includes the possibility of interrogating the system at multiple times and provides an operational condition for quantum Markov processes, together with associated measures of non-Markovianity [29].

The BIRS workshop brought in researchers exploring these different approaches to non-Markovianity theoretically. We also invited researchers working experimentally and whose observations cannot be explained within the canonical Markovian framework. Our intentions were to foster exchanges and to foment further explorations of the different mechanisms by which memory manifests in the quantum realm and the most general framework able to encompass them.

Emergent, Emerging Threads

In multiple cases, several talks addressed closely related subjects from different perspectives. The progression of related talks built up toward what the group recognized as an opportunity for future research. We highlight two such progressions here, one focused on the Kolmogorov consistency condition and one focused on experiments.

Kolmogorov consistency condition

Four talks referred to the Kolmogorov consistency condition, defined as follows. Consider a classical random variable x that evolves in discrete time, across the instants $t_{1,2,\ldots,f}$. The variable has a joint probability $p(x_0, t_0; x_1, t_1; \ldots; x_f, t_f)$ of assuming the value x_0, then x_1, etc. Suppose that the time-t_j event is skipped. The associated joint distribution, p', follows from summing p over x_j:

$$p'(x_0, t_0; x_1, t_1; \ldots; x_{j-1}, t_{j-1}; x_{j+1}, t_{j+1}; \ldots x_f, t_f) = \sum_{x_j} p(x_0, t_0; x_1, t_1; \ldots; x_f, t_f).$$

This equation is the Kolmogorov consistency condition, an axiom of probability theory. Classical variables obey the equation.

Consider obtaining the sequence $\{x_j\}$ by measuring a quantum system repeatedly. The joint probability distribution over the sequence can violate the Kolmogorov consistency condition. The reason is measurement disturbance: A measurement at t_{j-1} alters the quantum state, affecting the measurement at t_j.

Philipp Strasberg and Andrea Smirne defined classicality of a joint probability distribution as adherence to the Kolmogorov consistency condition. Violations of the condition underlie violations of Leggett-Garg inequalities [24]—Bell-like bounds for correlations across time, rather than across space—as Smirne and Yunger Halpern noted. Rivas pointed out that a closed quantum system can violate the consistency condition despite resembling its classical analogue in being divisible.

Yunger Halpern showed that, if a distribution violates the consistency condition, it equals a linear combination of Kirkwood-Dirac quasiprobabilities (KDQs). For more information about KDQs, see the relevant talk synopsis. This perspective, rooted in experience with KDQs in information scrambling and metrology, contrasted with the other three speakers' perspective, which was rooted in non-Markovianity. How we can leverage known properties of KDQs in non-Markovianity studies was identified as an open question.

Experiments

Many of the talks were (appropriately for BIRS) abstract and mathematical. However, a few reported on experiments. Kade Head-Marsden discussed an experiment close to the abstract: a five-qubit quantum simulation performed on the IBM Quantum Experience [20]. The simulation was precisely of an open quantum system. Similarly, Nicole Yunger Halpern discussed the testing of an entropic uncertainty relation dependent on a weak value [30]. Weak values can be measured via weak measurements, which refrain from disturbing a quantum system much and so help alleviate the measurement-disturbance concerns mentioned in Sec. Kolmogorov consistency condition. Farther afield, Gabriela Schlau-Cohen discussed the engineering of DNA-origami scaffolds that served as environments to influence exciton transport [19]. Avikar Periwal presented recent experimental progress on engineering entangled states of several atoms using photons in a cavity. His work opens up a new means of performing measurement-based quantum computation [9]. Such developments helped bridge the workshop's mathematical and physics contributions. Furthermore, the experimental talks provided inspiration for testing and applying more of the theoretical results in the lab.

Presentation Highlights

Participants presented 24 talks at the workshop. Below are synopses, listed in chronological order of delivery.

Philipp Strasberg (Universitat Autonoma de Barcelona) discussed the foundations of stochastic thermodynamics. Much of our world appears classical, Markovian, and detailed-balanced; but how do these properties emerge from microscopic dynamics? Strasberg offered justifications in the setting of an isolated quantum many-body system [37]. Part serves as the system of interest, and the rest serves as an effective bath. The argument hinged on the eigenstate thermalization hypothesis, an ansatz about the form of the elements of matrices that represent typical local observables relative to the energy eigenbasis. Closely related is the repeated-randomness assumption: The principle of maximum entropy describes a system's state accurately at multiple stages of a thermodynamic system's evolution. Van Kampen believed the repeated-randomness assumption to be reasonable under certain assumptions (if the system is nonintegrable, the observable being monitored is coarse, and the observable is "slow") but had difficulty proving it [38].

Erik Gauger (Heriot-Watt University) talked about modelling non-Markovian dynamics with process tensors. He presented an overview of (analytic and numerical) approaches and equations used in the theory of open quantum systems, as well as what they can deliver. He focused then on the method of matrix product operators, which are process tensors suitable for analyzing complex hybrid systems [10]. The method is mainly numerical. Gauger considered an optically driven GaAs quantum dot, modeled by a driven excitonic two-level system which is linearly coupled to a bath of oscillators. Under a weak laser intensity, the dynamics show non-Markovian features. To set up a theoretical approach allowing for the treatment of such systems, Gauger presented the process-tensor method (used also in Gregory White's talk), based on breaking the evolution into small time steps. He then explained the automatic-compression-of-the-environment (ACE) technique. He illustrated the power and versatility of the method by comparing its predictions with explicit dynamics for (i) a resonant-level model and (ii) one or two quantum dots coupled to two environments.

Andrea Smirne (University of Milan) addressed the question "How can we tell whether a quantum process can be implemented classically?" [28]. That is, consider receiving joint probability distribution

$$p(x_0, t_0; x_1, t_1; \ldots; x_f, t_f)$$

over a time series of data. Can any classical process give rise to that distribution? Smirne defined classicality as the distribution's obeying the Kolmogorov consistency condition. He then connected nonclassicality with coherence [36]. His main result provided a necessary and sufficient condition for the guiding question's answer to be affirmative: Suppose that an observable is nondegenerate, its statistics are Markovian, and a system's initial state is diagonal with respect to the observable's eigenbasis. Classicality of the observable's statistics is equivalent to the dynamics's not generating and detecting coherence.

Nicholas Anto-Sztrikacs (University of Toronto) introduced an effective Hamiltonian for a system S coupled strongly to an environment [2]. Conceptually, one incorporates some degrees of freedom from outside S into the system of interest. Mathematically, one uses a reaction-coordinate map, followed by a polaron transformation. One then truncates the reaction-coordinate Hamiltonian, under an assumption (if the reaction-coordinate frequency is the problem's greatest energy scale). Anto-Sztrikacs illustrated this framework with applications to a spin-boson model and an autonomous quantum refrigerator.

Marlon Brenes (University of Toronto) talked about fluctuations and charge statistics in mesoscale conductors. He began with a model where a central system is coupled to two reservoirs. Then, he introduced the notion of stochastic accumulated charge (integrated over time from an initial to a final instant) and the average current. The noise is defined as the time derivative of the variance of the stochastic accumulated charge. The accumulated charge is additive in time, but its variance is not. In order to be able to treat the strongly coupled regime, Brenes introduced a map effectively introducing several fermionic modes, each one coupled to its own reservoir. This is a mesoscopic reservoir. He then discussed the full counting statistics. He addressed the expressions for currents and noise in Gaussian systems. Brenes reported on results obtained in [6], where an analysis of the current and noise was carried out numerically in strongly driven systems.

Nicole Yunger Halpern (National Institute of Standards and Technology) provided a tutorial about Kirkwood–Dirac quasiprobabilities (KDQs). Quasiprobabilities are quantum generalizations of probabilities, able to break some of Kolmogorov's axioms for probability theory. The best-known quasiprobability—the Wigner function—famously assumes negative values, signaling quantum behaviors under specific circumstances. KDQs can assume not only negative, but also nonreal values, which quantify measurement disturbance [14]. The distribution's complex nature, suitability for continuous and discrete systems, measurability, and other properties underlie KDQs' recent infiltration of diverse subfield of quantum physics. Yunger Halpern presented a few applications to quantum information scrambling [40, 41, 1], metrology [3, 26], quantum Shannon theory [42], and quantum foundations [4]. Two scheduled talks had implicitly featured KDQs, as would a later talk, so the tutorial introduced them explicitly.

Ángel Rivas (Universidad Complutense de Madrid) described the difficulties of defining quantum divisibility. Divisibility helps us answer the question "To predict what will happen next, how far back into the past must we look?" Consider a classical random variable x that evolves in discrete time, across the instants $t_{1,2,\ldots,f}$. What is the joint probability $p(x_0, t_0; x_1, t_1; \ldots; x_f, t_f)$ that x assumes the value x_0, then x_1, etc.? If x is Markovian, then the joint probability factorizes in terms of the initial probability $p(x_0, t_0)$ and conditional probabilities:

$$p(x_0, t_0; x_1, t_1; \ldots; x_f, t_f) = p(x_f, t_f | x_{f-1}, t_{f-1}) \ldots p(x_1, t_1 | x_0, t_0) \, p(x_0, t_0).$$

In a quantum analogue, we ask about the probability that measurements of a quantum system yield the outcomes $x_0, x_1, \ldots, x_f$. If the system is closed, a unitary operator evolves the system between times t_{j-1} and t_j, and the quantum Markov condition implies divisibility. If the system is open, it evolves under a completely positive trace-preserving (CPTP) map. If the map decomposes as a tensor product of positive maps, the dynamics are called *P-divisible*, and one can approximately follow the logic in the closed-system case. Otherwise, P-divisibility must be replaced. Rivas discussed multiple alternatives, including k-positive-divisibility and CP-divisibility [8].

Dominique Spehner (Universidad de Concepción) presented on the Bures geodesic as a non-Markovian physical evolution. He defined the Bures distance between states of a finite quantum system, then reviewed basic facts about metrics and geodesics in Riemannian geometry. Also, he introduced the geodesic as a smooth curve joining two states and minimizing the length locally. Spehner illustrated this in detail on the case of a qubit using the Bloch sphere. He showed in particular that if two density matrices commute then there are infinitely many geodesics linking them. Geodesics find applications in quantum metrology where they can be used to estimate unknown parameters of quantum channels, and they are important in applications to quantum control in order to steer optimally an initial state to a target state. Spehner then showed the following main result. Given a geodesic starting from a given system density matrix, one can always find a Hamiltonian acting on the purification space of the state (a system–ancilla complex) which implements the geodesic as the physical (Schrödinger) evolution of the enlarged system–ancilla complex. This means that geodesics correspond to the physical evolution of the system coupled to an ancilla. Furthermore, if the geodesic passes through a pure state, then the initial system–ancilla state is of a product form. Spehner then analyzed which geodesics lead to the "most" non-Markovian evolutions. He showed that for the qubit case, one can numerically maximize the non-Markovianity.

Stefano Marcantoni (SISSA Trieste) spoke about the mitigation of irreversibility under non-Markovian thermalizing dynamics. He started by defining the notion of stochastic entropy production via two processes: (i) a two-point-measurement (TPM) protocol that yields a joint probability of two measurements and (ii) another TPM protocol, obtained by using the time-reversed dynamics. The stochastic entropy production is then defined as the logarithm of the ratio of the two joint probability distributions. Marcantoni considered dynamics given by Kraus operators satisfying some (nonequilibrium-potential) property and a class of observables. The expressions for the stochastic entropy production and its average value simplify considerably and can be related to relative entropies between final and initial and fixed-point states. Marcantoni proceeded to study the effect of non-Markovianity. He took a model of a qubit dynamics given by a time-dependent Lindbladian, where the non-Markovianity can be tuned by altering the time-dependent jump rates. This model satisfies the assumptions leading to a simpler form of the stochastic entropy production as discussed previously. The point now was to find regimes in which the entropy production *decreases* in time and so does the average of the variance. Marcantoni showed that this is indeed the case when the dynamics is not P-divisible [17]. In this sense, non-Markovianity mitigates irreversibility.

Gerardo Paz Silva (Griffith University) reported about open quantum system control. The control is modeled by time-dependent conrol terms in the system-bath (SB) Hamiltonian. In the interaction picture, this results in time-dependent bath operators. Paz Silva explained that, in this setting, only bath correlations matter, and those can be inferred by system-only measurements (noise spectroscopy). He highlighted that, for Markovian noise (produced by the bath), typical control tasks—such as decoherence suppression—cannot be implemented, while, for non-Markovian noise, they can. Paz Silva presented an iterative procedure to represent the reduced-system dynamics, based on the $B+$ decomposition [31]. In this decomposition, any full SB density matrix is written as a sum of product operators, in which the bath factors are positive operators (but not the system factors, for entangled SB states). Paz Silva then set up a hierarchy of equations governing the evolution of the correlation functions of the bath. Using this correlation information, he developed a "controlled Born approximation," where the bath correlation functions (up to a given order) are approximately constant in time, rather than the bath state's being constant, as in the usual Born approximation. Paz Silva illustrated the accuracy of the approximation by comparing it to the exact solution for an explicitly solvable spin-boson model.

Gniewomir Sarbicki (Nicolaus Copernicus University) talked about the optimization of entanglement witnesses. Starting from the Bell inequalities and explaining the Bell experiment detecting non-classical behaviour, Sarbicki explained the concept of entanglement witnesses—observables which are positive on separable states but not positive overall. Sarbicki explained that such a witness exists for any entangled state, and gave explicit examples for two qubits. He then addressed the optimality of witnesses: How many entangled states can a given witness detect? Graphically, he explained this by a diagram depicting the witness by a line separating the areas of entangled from separable state—the closer the witness lies to the boundary of the two classes of states, the more effective it is. Optimality means that no other witness detects more states than the given one. Sarbicki discussed this explicitly, again for the case of two qubits. Using the concept of spanning subspaces, he presented the "spanning criterion," a sufficient but unnecessary condition for optimality of entanglement witnesses. Sarbicki then showed that using information given by the spanning criterion, non-optimal witnesses can be transformed into optimal ones. He then discussed the realignment criterion for the separability of a density matrix and showed how the latter produces a family of (non-optimal) entanglement witnesses [34].

Alain Joye (Université Grenoble-Alpes) presented results about the time-dependent Wigner–Weisskopf model, where an atom intearacts with a radiation field. A single excitation, initially located on the atom, is eventually emitted into the radiation field. Joye presented recent rigorous results on this model, based on based on the recent work [22], where the atom-field dynamics contains adiabatically varying (slowly in time) Hamiltonian and interaction terms. In particular, Joye analyzed the dynamics of the atom and of the radiation field in the adiabatic and small coupling approximations, in various regimes. He focussed on describing the radiative decay of the atom, but mentioned that analyzing the properties of the emitted excitation (photon) has also been done in this work. Joye's results are as follows: (a) In the weak coupling regime (coupling to reservoir very small compared to adiabatic time scale), the atom evolves purely according to its adiabatic, uncoupled dynamics (no emission) and the effect of the field is negligible. Joye further divided this case into two distinct sub-cases, depending on the the relative size of the two quantities. In the regime (b) where the atom-field interaction strength is of the order of the adiabatic parameter, Joye showed that the probability for the atom to be de-excited into the ground state is not 100%. In the strong-coupling regime (c), where the interaction exceeds the adiabatic parameter, the atom will lose its excitation exponentially quickly, with 100% certainty, within the time span considered. Joye then proceeded to discuss the Markovianity of the atomic dynamics. The generator of the approximating dynamics they found is given by a time-dependent Lindbladian.

Massimo Palma (University of Palermo) spoke about reservoir computing [21]. Reservoir computing involves a neural network that has input, reservoir, and output layers. Only the weights connecting the reservoir to the output are optimized through training. The reservoir's weights are random and fixed. The reservoir remembers the recent past but not the distant past, its present state depending only on the present input. Reservoir computing has applications to classification and to processes extended in time, such as speech recognition; only after listening to a considerable portion of a speech can one understand it. Palma discussed classical and quantum reservoir computing, proposing an implementation of the latter: A (bosonic) squeezed state could form the input, while a fermionic reservoir would experience loss and pumping. The reservoir computer would report whether the input is entangled.

François Danamet (University of Liège) introduced stochastic Schrödinger equations as alternatives to master equations. To massage a master equation into a useful form, one often invokes several approximations. Candidates include the weak-coupling, Born, Markov, secular, and large-detuning approximations. Yet these approximations are often unjustified. For example, non-Markovianity is crucial to the Dicke model's superradiant phase [11]. Hence Danamet introduced the stochastic Schrödinger equation, in Markovian and non-Markovian flavors. The equation presents a quantum system as undergoing a certain trajectory, conditionally on outcomes of measurements of the bath. The equation is solved via the hierarchy-of-pure-states approach. Applications include the Hubbard–Hostein model.

Kade Head-Marsden (Washington University in St. Louis) discussed quantum simulations of open quantum systems. Suppose that we wish to understand how chromaphores transport excitons so efficiently during photosynthesis. Gabriela Schlau-Cohen proposed an experimental approach (discussed in a later synopsis), but a theorist would run a computation—-ideally, several years from now, a quantum computation. Conventional gate-based quantum computations are on closed systems that ideally evolve unitarily. Simulating open quantum systems requires ancilla qubits that serve as the environment. In 2001, Bacon *et al.* showed how to simulate all possible Markovian dynamics of one qubit [5]. This result cracked open the field, which now includes, for example, a simulation of a five-qubit Jaynes–Cummings system on the IBM Quantum Experience [20]. A key open question is "What is the minimal number of ancilla qubits needed to simulate a desired open evolution?"

Bassano Vacchini (University of Milan & INFN) illustrated that quantum-information-theoretic tools can be used to diagnose non-Markovianity [35]. The tools of interest were measures of distance between quantum states. An open quantum state will tend to evolve away from its initial state, but revivals can signal non-Markovianity. Vacchini discussed desirable properties of distance measures (boundedness, normalization, contractivity, and triangle-like inequalities) and several distance measures (the trace distance; the Rényi divergences, including the Kullback-Leibler divergence; and the quantum Jensen–Shanon divergence).

Gregory White (University of Melbourne) introduced the process tensor as a mathematical tool for representing and characterizing general not-Markovian quantum evolutions [39]. A general quantum system evolves under influences controlled by the experimentalist (via "knobs") and uncontrolled influences (noise). The process tensor separates the two sets of influences. A processor tensor is a Choi matrix, a quantum state that represents an evolution. This representation offers the advantage of transforming poorly understood temporal correlations into spatial correlations, which are better understood. White explained how to construct a process tensor mathematically and infer it experimentally. The experimental resources required typically scale exponentially with the system size but can scale polynomially under certain assumptions.

Gabriela Schlau-Cohen (MIT) spoke about engineering a bath to influence a system of interest in a desirable way. Her lab sought to understand how plant chromophores operate so efficiently: Upon receiving a photon from the sun, a chromophore can convert the photon into an exciton and transport the exciton with a quantum efficiency of $\approx 85\%$. Schlau-Cohen's team built analogues of chromophores, using DNA origami [19]. The DNA scaffolds served as baths for the exciton systems of interest. Upon tuning a property of the scaffold (such as rigidity), the experimentalists measured the effect on exciton-transport efficiency. These experiments demonstrate the power of the bath to steer an open quantum system.

Thomas Fay (UC Berkeley) talked about coupled electron-transfer processes and energy-transfer processes in light-harvesting complexes, particularly in photosynthesis. Phenomena such as photoprotection (protecting cells from high intensity light during direct sunlight exposure) cannot be captured by a Lindblad approach. The goal is therefore to develop a more powerful description. Fay explained the difference between excitation energy transfer and electron transfer between a donor (D) and an acceptor (A). The former is mediated by the electrostatic interaction (strength $J_{\rm DA}$) between D and A and happens at a system–bath (protein scaffold) coupling strengths similar to $J_{\rm DA}$. In charge-transfer processes, electrons are transferred between D and A. These processes are mediated by orbital overlaps and alter the charge distribution within molecules. Charge-transfer processes are induced by strong coupling to the reservoir, where the interaction energy is much larger than the system transition matrix elements. Fay then modeled an aggregate of three chlorophyll molecules interacting with each other (via charge and electron exchange) and interacting linearly with a (local, harmonic) protein-scaffold bath. Fay used the HEOM (hierarchical-equations-of-motion) method to describe the dynamics, capturing the non-Markovian nature of the processes. A deep HEOM hierarchy was needed, due to the large energy scale of charge transfer process. This makes the method computationally very costly, if not impossible. Fay presented their ideas [15] to solve this dilemma. They introduced a "quencher" system to which the chlorophyll is strongly coupled, as well as a global polarization bath, coupled to both the chlorophyll and the quencher. This represents a subdivision of the total physical bath into different independent parts. They developed a hybrid HEOM method that separates out the strong-coupling charge-transfer process. The excitation energy transfer was treated exactly with conventional HEOM. Then, the charge transport was tackled with perturbation theory (ultrastrong coupling between the chlorophyll and the quencher). The resulting equations of motion were not of Lindblad form. Numerical simulations of the latter showed the accuracy and computational efficiency. Fay explained how the method also works for LHCII processes, which are much more involved. Detailed non-Markovian effects are revealed by this method, which cannot be captured by the Lindblad approach.

Avikar Periwal (Stanford University) reported on optical-cavity platforms where atoms and cavity modes interact nonlocally, creating highly entangled states such as squeezed states [9]. The atomic interactions are experimentally controllable and non-demolition measurements can be performed on the atoms. Periwal explained the experimental setup and the technical difficulties Monika Schleier-Smith's lab had to overcome. He explained how they can detect and measure entanglement and how they can engineer complex new atomic structures. In particular, Periwal explained how to place atoms in the cavity and to control their positions using lasers. Initially put in the ground state, atoms are excited by cavity modes. Entangled states are created via the absorption and emission of photons. Entanglement can be accessed experimentally by a measurement of the degree of squeezing (of certain observables' enhanced or reduced variances). Periwal then explained how they can engineer atomic entangled states that may be used for quantum computing. One of the advantages of the platform is that it allows one to generate entanglement programmatically over sizeable atomic systems. Throughout the talk, Periwal gave illustrations in diagrams, measurement data, photos of instrumentation, and formulae.

Christoph Simon (University of Calgary) continued the biophysical trend, asking whether quantum phenomena affect brains' functioning. The brain contains photons and spins, two platforms that can store and process quantum information under certain conditions. Focusing first on photons, Simon proposed that axons could serve as waveguides and that opsins (light-sensitive proteins) could serve as photodetectors [23]. Spins feature in the radical-pair mechanism, believed to be responsible for avian navigation. Simon posited, however, that the mechanism could have far more biological applications.

Anton Trushechkin (Steklov Mathematical Institute) discussed the exactly solvable model of pure dephasing of a qubit (S) in an oscillator bath (B). For initially uncorrelated SB state and B in thermal equilibrium, the S populations are constant and there is an explicit expression for the decoherence. Trushechkin presented the following results: For the spectral density $J(\omega) \sim \omega^{\alpha}$ as $\omega \to 0$, the decoherence decays superexponentially for $0 < \alpha < 1$, exponentially for $\alpha = 1$, subexponentially for $1 < \alpha < 2$, as a power for $\alpha = 2$ and for $\alpha > 2$ there is only partial decoherence. Trushechkin then analyzes the validity of the weak coupling, Markovian Davies–van Hove theory for $J(\omega) \sim \omega^2$. On the Davies–van Hove time scale, there is no decay of decoherence, but the exact solution shows that eventually, as $t \to \infty$, decoherence decays (polynomially). This example shows that the weak-coupling limit has to be taken *cum grano salis* and does not apply on all time scales. The *quantum resonance theory*, an extension of the Davies theory and valid for all time scales, holds under the validity of the Fermi Golden Rule Condition (FGRC). The latter is not satisfied in the present model. This analysis shows that one cannot generally draw time-asymptotic information from the Davies theory *unless* the FGRC is satisfied, a "detail" often overlooked in the literature. Trushechkin then addressed the possibility of Markovian embedding, where a non-Markovian system-environment complex can be mapped to a new, enlarged system that interacts in a *Markovian* way with a new reservoir. The reaction-coordinate method is an example of such an embedding. Trushechkin noticed that such an embedding is generally not possible, since it would lead to an exponential decay of observables' expectation values with time. However, the above model shows that power laws or sub- and superperexponential behaviour is also realizable, even at small coupling.

Sergei Filippov (Algorithmiq Ltd, Helsinki) talked about a tensor-network description in Markovian repeated-interaction models. Filippov presented a repeated-interaction model of qubits, a model that allows for an analytic exact form of the dynamics, which exhibit homogenization, dephasing and amplitude damping. He then introduced non-Markovian collision models, arising in particular from situations where the environment ancilla qubits are correlated, which is important, e.g., in quantum optics. Filippov then described the state of such a system using a matrix-product representation. This leads to a tensor-network representation of the repeated-interaction dynamics. He then explained the right-canonical form of matrix-product states and how to use it to obtain a tensor network for the collision model. By changing the point of view and interchanging the horizontal and vertical directions in the tensor network, he obtained a Markovian embedding of the dynamics, the takeaway of a theorem by the speaker [16]. Filippov then illustrated applications where the repeated interaction was with a two-photon wave packet or a photonic cluster state. The effect of the environment correlations on the decoherence of the repeatedly interacting qubit became manifest.

Zoom and social-media engagement

Many participants joined the talks via Zoom. Just as the in-person audience posed many, many questions of the speakers, so did the virtual participants, who chimed in via their audio systems. We suggest that future workshops develop a system for ensuring that Zoom participants always state who they are before posing their questions: The in-person audience—not being able to see them—can't identify them by their faces. Such a system could strengthen connections amongst participants with shared interests. One Europe-based virtual participant tried to follow the workshop live but ended up unable to, due to the infeasibility of staying up through the night. However, multiple participants expressed interest in the recordings of the talks. Also, we know of at least two students, based in North America, who virtually dipped into the workshop for talks relevant to their research.

Beyond the electronic-engagement opportunities provided by BIRS, we brought the workshop to social media. Co-organizer Yunger Halpern posted multiple photos from the workshop on Twitter, Facebook, and LinkedIn. Despite gathering no data rigorously, she registered high engagement with the posts, as measured by "likes," comments, etc. Additionally, Yunger Halpern wrote an article about the workshop for *Quantum Frontiers*, the blog of Caltech's Institute for Quantum Information and Matter [43]. Yunger Halpern blogs for *Quantum Frontiers* every month, and the workshop article served as her post for March 2023. The article introduced non-Markovianity, spotlighted Schlau-Cohen's talk, and referred to the previous two BIRS workshops that she had attended (as well as the blog posts she had written about one of them [44, 45]).

Participant configuration

The total number of participants was 79, and 24 talks took place. The table below summarizes the percentages of different groups, using the abbreviations UG (undergraduate student) M.Sc. (master's student), Ph.D. (doctoral student), PDF (postdoctoral fellow), ECR (early-career researchers, defined as have worked in faculty positions for $\leq$ 6 years), and ER (experienced researchers, defined as having worked in faculty positions for $>$ 6 years).

Total Percentages of: 79 registered participants | 24 delivered talks			
	Onsite | Talks	Online | Talks	**Total | Talks**
UG, M.Sc., Ph.D.	8 | 14	34 | 0	**42 | 14**
PDF	8 | 18	4 | 0	**12 | 18**
ECR	4 | 4	8 | 0	**12 | 4**
ER	11 | 34	23 | 30	**34 | 64**
Total	**31 | 70**	**69 | 30**	**100 | 100**

The high percentage of the UG, M.Sc. Ph.D. and the PDF groups resulted from the concerted effort of the organizers to involve young researchers and to give them a chance to present their work.

Participants

Abbasi, Maryam (Washington University St. Louis)
Albert, Victor (NIST & UMD College Park)
Altherr, Anian (ETH Zürich)
Anindita, Bera (Nicolaus Copernicus University Poland)
Antosztrikacs, Nicholas (U of Toronto)
Bhattacharya, Debankur (U. of Maryland (UMD))
Boyd, Alec (Trinity College Dublin)
Braasch, Billy (NIST)
Brenes, Marlon (University of Toronto)
Burgelman, Michiel A. (Dartmouth College)
Cenatiempo, Serena (Gran Sasso Science Institute)
Chruscinski, Dariusz (Nicolaus Copernicus University)
Chu, Guo (Henan Key Laboratory of Quantum Information and Cryptography)
Ciccarello, Francesco (University of Palermo & NEST, Pisa)
Cohen, Guy (Tel Aviv University)
Cresser, James (University of Exeter)
Damanet, François (Uni Liège)
De Sousa, Guilherme (U. of Maryland (UMD))
Debecker, Baptiste (Université de Liège)

Deffner, Sebastian (UMBC)
Di Meglio, Giovanni (Universität Ulm)
Dowling, Neil (Monash University)
Fay, Thomas (UC Berkeley)
Filippov, Sergei (Steklov Mathematical Institute of Russian Academy of Sciences)
García-Pintos, Luis Pedro (Los Alamos National Lab)
Gatto Lamas, Amanda (U. of Illinois at Urbana-Champaign)
Gauger, Erik (Heriot-Watt University)
Gerry, Matthew (University of Toronto)
Gour, Gilad (University of Calgary)
Gu, Mile (Nanyang Technological University)
Head Marsden, Kade (Washington University in St. Louis)
Huelga, Susana (Ulm University)
Joye, Alain (Univ. Grenoble Alpes)
Kessing, Kevin (University of Ulm)
Krogmeier, Timothy (Washington University St. Louis)
Latune, Camille (ENS Lyon)
Lautenbacher, Lea (Universität Ulm)
Lee, Ahreum (University of Maryland College Park)
Limmer, David (UC Berkeley)
Majidy, Shayan (U. of Waterloo)
Marcantoni, Stefano (SISSA Trieste)
Marín Guzmán, José Antonio (University of Maryland)
Merkli, Marco (Memorial University of Newfoundland)
Milz, Simon (Institute for Quantum Optics and Quantum Information Vienna)
Min, Brett (U of Toronto)
Mitchison, Mark (Trinity College Dublin)
Modi, Kavan (Monash University)
Munson, Anthony (U. of Maryland)
Oruganti, Greeshma Shivali (University of Maryland)
Palma, Massimo (Uni Palermo)
Paternostro, Mauro (Queen's University Belfast)
Paz Silva, Gerardo (Griffith university)
Periwal, Avikar (Stanford University)
Petruccione, Francesco (Stellenbosch University)
Plenio, Martin (Ulm University)
Ringbauer, Martin (University of Innsbruck)
Rivas, Ángel (Universidad Complutense de Madrid)
Sarbicki, Gniewomir (Nicolaus Copernicus University)
Scandolo, Carlo Maria (University of Calgary)
Schlau-Cohen, Gabriela (MIT)
Segal, Dvira (University of Toronto)
Simon, Christoph (University of Calgary)
Sinayskiy, Ilya (University of KwaZulu-Natal)
Smirne, Andrea (University of Milan)
Sorochkin, Timothy (University of Toronto)
Spaventa, Giovanni (Ulm University)
Spehner, Dominique (Universidad de Concepción)
Strasberg, Philipp (Universitat Autonoma de Barcelona)
Taherizadegan, Shahrzad (University of Calgary)
Theurer, Thomas (U Calgary)
Thingna, Juzar (University of Massachusetts Lowell)

Trushechkin, Anton (Steklov Mathematical Institute of Russian Academy of Sciences)
Upadhyaya, Twesh (UMD)
Vacchini, Bassano (University of Milan & INFN)
Viola, Lorenza (Dartmouth College)
White, Gregory (The University of Melbourne)
Yunger Halpern, Nicole (National Institute of Standards and Technology)
Zanoni, Elia (U Calgary)

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Chapter 5

Algebraic Aspects of Matroid Theory (23w5149)

March 12 - 17, 2023

Organizer(s): Matthew Baker (Georgia Institute of Technology), June Huh (Princeton University), Felipe Rincón (Queen Mary University of London), Kris Shaw (University of Oslo)

This workshop brought together international mathematicians working in the areas of matroid theory, algebra, algebraic geometry, and combinatorics at large. The workshop focused on three main areas of research in connection with matroid theory: log-concavity and combinatorial Hodge theory, matroids over hyperfields, and connections to tropical geometry. The scientific activities of the workshop included 10 one hour long talks, 12 talks of 20 minutes by junior participants, as well as open problem, discussion, and Q&A sessions. Here we present an overview of the field, open problems arising from the meeting, as well as a report on workshop outcomes and work in progress.

Overview of the Field

Matroids are beautiful and important objects which lie at the interface of combinatorics and geometry, and in recent years some sophisticated algebra has appeared in connection with matroid theory as well. Originally introduced independently by Hassler Whitney [21] and Takeo Nakasawa [17], matroids are a way of simultaneously axiomatizing the notion of linear independence in vector spaces and the notion of acyclicity in graphs. Here are three topics of current interest which blend algebra and matroid theory in new and exciting ways:

Log-concavity and combinatorial Hodge theory

Read conjectured in 1968 that for any finite graph G, the sequence of absolute values of the coefficients of the chromatic polynomial of G is log-concave (and in particular unimodal). Read's conjecture was proved by June Huh in 2012 using methods from algebraic geometry [13]. Huh's result was subsequently refined and generalized from graphs to matroids in 2015 by Adiprasito, Huh, and Katz, settling a longstanding conjecture of Rota [1]. In the process, these authors found a purely combinatorial incarnation of Hodge theory, establishing analogs in matroid theory of the Hard Lefschetz Theorem and Hodge-Riemann relations in algebraic geometry.

In the last few years, the methods of Huh et al. have been used by several sets of authors to establish log-concavity of various sequences arising naturally in combinatorics and geometry, e.g., certain Littlewood-Richardson coefficients occurring in representation theory and the number of independent subsets of a vector space over a finite field. New methods, closely related to Hodge theory for matroids, have recently been developed for such problems using the theory of Lorentzian polynomials, as introduced by Brändén and Huh [9]. This same class of polynomials was independently studied by Anari, Liu, Oveis Gharan, and Vinzant [3]. Those authors also relate these polynomials to the mixing times of certain Markov chains / high-dimensional random walks and efficient algorithms for approximating the number of bases of a matroid.

Matroids over partial hyperstructures

One of the most important questions combinatorialists ask about a matroid M is over what fields M is representable. More generally, one can ask about representability over partial fields. Pendavingh and van Zwam introduced the universal partial field of a matroid M, which governs the representations of M over all partial fields [18]. Unfortunately, asymptotically 100% of matroids are not representable over any partial field, and in this case, the universal partial field gives no information.

Using the recently introduced theory of matroids over partial hyperstructures developed by Baker and Bowler [6] (itself rooted in the work of Dress and Wenzel), Baker and Lorscheid have introduced a significant generalization of the universal partial field which they call the foundation of a matroid [7]. The foundation of M is a new type of algebraic object which Baker and Lorscheid call a pasture; examples of pastures include both partial fields and hyperfields in the sense of Krasner. Pastures form a natural class of "field-like" objects within Lorscheid's "ordered blueprints" (which were originally introduced in connection with geometry over the "field of one element"), and they have desirable categorical properties (e.g., existence of products and coproducts) that make them an appealing new context in which to study algebraic invariants of matroids. The foundation of M occurs naturally as the "residue pasture" of the point of the moduli space of matroids (an ordered blue scheme constructed by Baker and Lorscheid) corresponding to M.

The foundation of a matroid M represents the functor taking a pasture F to the set of rescaling equivalence classes of F-representations of M; in particular, M is representable over a pasture F if and only if there is a homomorphism from the foundation of M to F. Pastures have nicer categorical properties than partial fields, which allows one to state and prove results such as "the foundation of a direct sum of matroids is the tensor product of the foundations". We expect this formalism to lead to an improved understanding of matroid representations over fields, especially the interaction between such representations and notions such as orientability, unique orientability, and unique representability.

Matroids and tropical geometry

Matroids play an essential role in tropical geometry: for example, they encode tropical linear spaces via the Bergman fan construction [4]. (Technically speaking, a tropical linear space is the Bergman complex associated to a valuated matroid, which incidentally is the same thing as a representation of a matroid over the tropical hyperfield). The tropical varieties which behave most like a smooth complex manifold X are those which look locally like the Bergman fan of a matroid at every point (such tropical varieties are therefore called smooth). For smooth tropical varieties, Itenberg, Katzarkov, Mikhalkin, and Zharkov recently developed theory of tropical cohomology which encodes how Hodge cohomology groups $H^{p,q}(X, \mathbb{C})$ degenerate in one-parameter families of smooth complex varieties [14]. One construction of tropical cohomology is based on the Orlik-Solomon algebra associated to a matroid, which was originally introduced to compute the singular cohomology of the complement of a complex hyperplane arrangement [22]. The development of tropical cohomology opens the door to new applications of tropical geometry to both real and complex geometry.

Bergman fans of matroids are also intimately connected to tropical intersection theory, as shown in the pioneering work of Allerman, Rau, and Shaw [2], [20]. Brugallé and Shaw have provided intersection theoretic obstructions to lifting tropical curves on tropical surfaces to algebraic curves on algebraic surfaces based on positivity properties in the algebraic situation [10]. Although the Hodge index theorem holds on matroids by Adiprasito, Huh, and Katz, it fails on more general smooth tropical surfaces as shown by Shaw [19]. The Hodge index theorem can also fail in the case of non-matroidal fans and has been used by Babaee and Huh to give a counterexample to Demailly's conjectured strengthening of the Hodge Conjecture in algebraic geometry [5].

Additional topics

Other topics of current interest include:

1. Frobenius flocks and algebraic matroids
2. Kazhdan-Lusztig polynomials of matroids
3. Free resolutions of matroid ideals
4. Positively hyperbolic varieties and positroids

Presentation Highlights

The workshop featured the following hour-long talks. They are presented here in sequential order.

Federico Ardila spoke about "Combinatorial Intersection Theory: A Few Examples".
Abstract: Intersection theory studies how subvarieties of an algebraic variety X intersect. Algebraically, this information is encoded in the Chow ring A(X). When X is the toric variety of a simplicial fan, Brion gave a presentation of A(X) in terms of generators and relations, and Fulton and Sturmfels gave a "fan displacement rule" to intersect classes in A(X), which holds more generally in tropical intersection theory. In these settings, intersection theoretic questions translate to algebraic combinatorial computations in one point of view, or to polyhedral combinatorial questions in the other. Both of these paths lead to interesting combinatorial problems, and in some cases, they are important ingredients in the proofs of long-standing conjectural inequalities. This talk will survey a few problems on matroids that arise in combinatorial intersection theory, and a few approaches to solving them. It will feature joint work with Graham Denham, Chris Eur, June Huh, Carly Klivans, and Raúl Penaguião.

Oliver Lorscheid spoke about "Categories of matroids and matroid bundles".
Abstract: Baker and Bowler's theory of matroids with coefficients can be understood as an extension of linear algebra from fields to unwieldier objects such as partial fields and hyperfields. In this talk, we complement this theory with the notion of a morphism of matroids with coefficients, which passes through a subtle process that we call "perfection". Eventually we gain a categorical framework for matroid bundles over F1-schemes. All this stems from joint ideas with Baker, Jarra and Jin. As a sample application, we define the Tutte-Grothendieck ring of an F1-scheme, which can be seen as a "detropicalization" of algebraic K-theory. The Tutte-Grothendieck ring of the moduli space of matroids carries a "universal Tutte class" whose pullback to any matroid is the Tutte polynomial of the matroid. If time allows, we muse about how this might be used to reprove the Fink-Speyer theorem.

Chris Eur spoke about "How or when do matroids behave like positive vector bundles?".
Abstract: Motivated by certain toric vector bundles on a toric variety, we introduce "tautological classes of matroids" as a new geometric model for studying matroids. We describe how it unifies, recovers, and extends various results from previous geometric models of matroids. We then explain how it raises several new questions that probe the boundary between combinatorics and algebraic geometry, and discuss how these new questions relate to older questions in matroid theory.

Shiyue Li spoke about "K-rings of matroids".
Abstract: I will share some discoveries on K-rings of wonderful varieties and matroids. The main result is a Hirzebruch—Riemann—Roch-type theorem. I will also discuss applications to moduli spaces of curves. Joint work with Matt Larson, Sam Payne and Nick Proudfoot.

Nima Anari spoke about "High-dimensional expansion and sampling algorithms: what lies beyond log-concave polynomials and matroids".
Abstract: I will survey high-dimensional expanders (HDX) and the alternative perspective they provide on some of the recent advances in matroid theory concerning log-concave/Lorentzian polynomials. The HDX perspective has been key in solving algorithmic problems concerning sampling and/or counting in combinatorial structures, including matroids and some objects beyond matroids (such as matchings, Eulerians tours, etc.). I will formulate conjectures which, if proven, would generalize parts of the theory that has been developed for

log-concave polynomials/matroids. I will then mention some results concerning fractionally log-concave and sector-stable polynomials, which provide evidence for the general conjectures.

Omid Amini spoke about “Hodge theory for tropical fans”.
Abstract: I will present a proof of the Kähler properties of the Chow ring for a large class of tropical fans based on three basic operations on fans which preserve the balancing condition (orientability). In the case of matroids, this allows to circumvent some of the difficulties arising in the work by Adiprasito, Huh, and Katz. Time permitting, I will discuss generalizations both in the local and global settings, and some applications to geometric questions. Based on joint works with Matthieu Piquerez.

Nick Proudfoot spoke about “Equivariant/Categorical Matroid Invariants”.
Abstract: The characteristic polynomial of a matroid is “categorified” by the Orlik-Solomon algebra, and questions about the characteristic polynomial can be enriched to questions about the Orlik-Solomon algebra, now regarded as a graded representation of the group of symmetries of the matroid. Other polynomial invariants with natural categorifications include the Chow and augmented Chow polynomials, the Kazhdan-Lusztig polynomial, and the Z-polynomial. I will survey various results and conjectures about these categorical invariants, ending with a discussion of what it means for a categorical invariant to be valuative.

Lucía López de Medrano spoke about “Chern classes of tropical manifolds”.
Abstract: In this talk, we will explain the extension of the definitions of Chern-Schwartz-MacPherson (CSM) cycles of matroids to tropical manifolds. With this definition, we will see a correspondence theorems for the CSM classes of tropicalisations of subvarieties of toric varieties, an adjunction formula relating the CSM cycles of a tropical manifold and a codimension-one tropical submanifold and a Noether’s Formula for compact tropical surfaces. Joint work with Felipe Rincón and Kris Shaw.

Alex Fink spoke about “Matrix orbit closures and their classes”.
Abstract: If an ordered point configuration in projective space is represented by a matrix of coordinates, the resulting matrix is determined up to the action of the general linear group on one side and the torus of diagonal matrices on the other. We study orbits of matrices under the action of the product of these groups, as well as their images in quotients of the space of matrices like the Grassmannian. The main question is what properties of closures of these orbits are determined by the matroid of the point configuration; the main result is that their equivariant K-classes are so determined. I will also draw connections to positivity and the work of Berget, Eur, Spink and Tseng. The results of mine featured here are mostly joint with Andy Berget.

Diane Maclagan spoke about “Tropical schemes - problems and progress”.
Abstract: In this talk I will briefly describe the program to develop tropical schemes, with an emphasis on recent progress. Tropical schemes can be described as towers of (valuated) matroids, and I will focus on questions that arise at the inferface between the geometry and matroid theory.

We also had 20-minute talks by Tong Jin, Zach Walsh, Hunter Spink, Jacob Matherne, Benjamin Schroeter, Matt Larson, Colin Crowley, Anastasia Nathanson, Nick Anderson, Chi Ho Yuen, Ahmed Umer, and Tara Fife. In addition, the workshop featured three Open Problem discussions and two Q&A discussions.

Open Problems

Here we compile some open problems that were brought up during the community discussions in order to make this list open and accessible to the research community.

Log-concavity and combinatorial Hodge theory

1. Can one give a purely combinatorial proof of the log-concavity of Kostka numbers in representation theory? What about other instances of Okounkov’s conjectures on Littlewood-Richardson coefficients?

2. Matroids can be thought of as the “Type A” case of the more general notion of Coxeter matroids, due to Gelfand and Serganova. What are the analogs of combinatorial Hodge theory and the theory of Lorentzian polynomials for other “Lie types” (corresponding to other families of finite Coxeter groups)?

3. The well-known "negative correlation" property for spanning trees in graphs does not generalize in a naive way to matroids, as there are subtle counterexamples. However, Huh-Schroter-Wang and Brändén-Huh have proved that "negative correlation" (which is equivalent to "1-Rayleigh") can be replaced by "2-Rayleigh" in the case of matroids, and they conjecture that the optimal Rayleigh constant for matroids is 8/7. Can one further develop the theory of Lorentzian polynomials and combinatorial Hodge theory and in the process hone in on this conjecture?

4. **(Chris Eur)** Consider
$$A^\bullet(X_E)[\delta]/\langle \delta^r + \delta^{r-1}c_1(\mathcal{S}_M) + \cdots + c_r(\mathcal{S}_M)\rangle.$$
The generator of the ideal is called a **Chern polynomial**. If M is realized by a linear space L, then this ring $\simeq A^\bullet(\mathbb{P}(\mathcal{S}_L))$.

   Question: Do Hard Lefschetz and Hodge–Riemann hold for this ring with $l = c\delta + a$, for a ample on X_E?

5. (**Matt Larson**) **Conjecture.** $T_M(x+1, x+1)$ has log-concave coefficients for all matroids M. True for $|E(M)| \leq 9$. Fact:
$$T_M(x+1, x+1) = \sum_{u \in \{0,1\}^n} x^{d(P(M),u)},$$
where d is the lattice distance.

6. **(Chris Eur)**: Given a matroid quotient $M \twoheadrightarrow N$, we have a canonical **Higgs factorization**
$$M \twoheadrightarrow M_1 \twoheadrightarrow M_2 \twoheadrightarrow \cdots \twoheadrightarrow N.$$
Each successive quotient in the factorization comes from a matroid $\hat{M}_i$, where M_{i-1} is a single-element deletion of $\hat{M}_i$ and M_i is a single-element contraction of $\hat{M}_i$. Consider the beta-invariants of the matroids $\hat{M}_i$.

   Question: Is the sequence $\beta(\hat{M}_1), \ldots, \beta(\hat{M}_{r-1})$ log-concave?

7. (**Matt Larson**) Suppose L is a linear subspace of K^n, where K is a field, giving rise to a loopless matroid M of rank r. Let P be a full-dimensional generalized permutohedron in $\mathbb{R}^n$, and define $R^{\cdot}(P, L)$ to be the image of
$$\oplus_{k \geq 0} H^0(X_{A_{n-1}}, \mathcal{O}(kP))$$
in $\mathbb{P}L \cap T$.

   Conjecture 1:
$$\dim R^k(P, L) = \chi(W_L, \mathcal{O}(kP)) = \chi(M, \mathcal{O}(kP)).$$
This would follow if $H^i(W_L, \mathcal{O}(kP)) = 0$ for $i > 0$ and we have surjectivity on H^0.

   Conjecture 2: $R^{\cdot}(P, L)$ is a Cohen-Macaulay ring.

   Conjecture 3:
$$\sum_{k \geq 0} \chi(M, \mathcal{O}(kP)) t^k = \frac{Q(t)}{(1-t)^r},$$
where the coefficients of Q are nonnegative.

   Remarks:

   (a) Conjecture 1 + Conjecture 2 implies Conjecture 3 if M is realizable.

   (b) These are true if $L = K^n$, P is the standard simplex, or P is the negative of the standard simplex.

   (c) If true, these conjectures would show that $[t^r]g_M(t) \geq 0$.

Matroids over partial hyperstructures

1. There are, in fact, two types of representations, called weak and strong, for matroids over pastures; they coincide for perfect pastures (e.g., partial fields and doubly-distributive hyperfields). Is there a purely algebraic characterization of a class of perfect pastures which contains both doubly-distributive hyperfields and partial fields and is closed under taking products?

2. Can the moduli space of matroids be extended to other "types", e.g., to a moduli space of orthogonal or symplectic matroids? (This might be related to the theory of cluster algebras.)

3. Can the moduli space of matroids and the theory of pastures be used to study the (in)famous conjectures of Macpherson et al. regarding the homotopy type of the moduli space of oriented matroids? There is an intriguing analogy between the latter and the Dressian (a kind of "tropical Grassmannnian"), whose "combinatorial skeleton" is somewhat better understood than its oriented counterpart.

4. (**Oliver Lorscheid**) Linear spaces satisfy not just the usual Plücker relations but also **multi-exchange relations**: given bases B, B' and a set $A \subset B \setminus B'$ of size l, there exists a set $A' \subset B' \setminus B$ of size l such that $B \setminus A \cap A'$ and $B' \setminus A' \cap A$ are bases. It is also true that the single exchange relations implies the multi-exchange relations for matroids, i.e. over the Krasner hyperfield $\mathbb{K}$. Is the same true for all idylls? Or at least perfect ones?

Matroids and tropical geometry

1. Valuated matroids arise in the definition of tropical ideals due to Maclagan and Rincón, following the pioneering work on tropical scheme theory due to Giansiracusa and Giansiracusa. Every tropical ideal in the sense of Maclagan-Rincón has an associated tropical variety (a finite polyhedral complex equipped with positive integral weights on its maximal cells), and a basic question is which weighted polyhedral complexes arise in this manner. Using work of Las Vergnas on the non-existence of tensor products of matroids, Draisma and Rincón recently found a matroid whose associated tropical linear space does not come from a tropical ideal. Can one modify the definition of tropical ideal in a way which eliminates such counterexamples while retaining the useful properties (e.g., existence of Hilbert polynomials) of tropical ideals? The alternative theory of tropical schemes proposed by Oliver Lorscheid (based on his theory of ordered blueprints), along with the notion of families of matroids arising in the work of Baker and Lorscheid, could be useful in this context.

2. What is the relationship between tropical cohomology and the Adiprasito-Huh-Katz Chow ring of a matroid?

3. The Riemann-Roch theorem for tropical curves (due to Baker-Norine, Gathmann-Kerber, and Mikhalkin-Zharkov) currently has no cohomological formulation or proof. It seems natural to try to use the theory of ordered blueprints and families of matroids over such objects to develop a suitable cohomology theory in this context.

4. Lucía López de Medrano, Felipe Rincón, and Kris Shaw recently defined tropical Chern-Schwartz-MacPherson (CSM) cycles for an arbitrary matroid M. These are certain balanced weighted fans supported on the corresponding Bergman fan which have nice combinatorial properties, e.g. they "categorify" the reduced characteristic polynomial of M. Can tropical CSM cycles be used to formulate (and perhaps even prove) a higher-dimensional Riemann-Roch theorem in tropical geometry?

5. (**Hunter Spink**) June Huh and Eric Katz proved that $\deg(\Delta_M \cap \beta^{r-1} = T(1,0)$, which counts the number of bases of external activity 0. This says that if we have a linear space $L \subseteq \mathbb{C}^n$ of dimension r and we have an $(r-1)$-dimensional "reciprocal linear space" Λ, then $|L \cap \Lambda| = T_M(1,0)$.

   On the tropical side, there is in fact a natural bijection (Katz, Berget–Spink–Tsend) between the tropical intersection points of Δ_M and $\beta^{r-1} + v$ and bases of external activity zero with respect to the chamber that v lies in. (Note: the multiplicities are always 1.)

Question: What happens when we cross a wall?

More precisely:

Problem: Prove that monodromy is transitive.

Matroid polytopes

(**Chris Eur**) Fact: if P is a lattice generalized permutahedron, then $P \cap ([0,1]^n + v)$ is also, for $v \in \mathbb{Z}^n$. Tile $\mathbb{R}^n$ by cubes; this gives a decomposition of P into translates of matroid polytopes.

Problem: Do this as explicitly as possible for graphical zonotopes,

$$Z_G = \sum_{(v_1,v_2)\in G} \mathrm{Conv}(e_{v_1}, e_{v_2}) \subset \mathbb{R}^{V(G)}.$$

Matroids of other Coxeter types

1. (**Chris Eur**) Let $\pi : \mathbb{R}^{2n} \overset{[I_n\ -I_n]}{\longrightarrow} \mathbb{R}^n$. A delta-matroid D is **envelopable** if there exists a matroid M on $[2n]$ such that $\pi(P(M)) = P(D)$, possibly with scaling depending on conventions. Not all delta-matroids are envelopable.

   Question: Are all even delta-matroids envelopable? Are all delta-matroids with the strong symmetric exchange property envelopable?

2. (**Matt Larson**) If M is a matroid, the rank polynomial $R_M(u,0)$ gives the f-vector of the independence complex of M. Similarly, $R_M(u,-1)$ gives the f-vector of the non-broken circuit complex (or Orlik–Solomon algebra).

   Now let D be a Delta-matroid. In this case, there is again a polynomial $U_D(u,0)$ which gives the f-vector of the independence complex of M.

   Question: What is the meaning of $U_D(u,-1)$?

   Let $U_D(u,-1) = f_0u^n + \cdots + f_n$.

   Problem: Show that $f_i \leq f_{n-i}$ for all $i \leq n/2$ for every Delta-matroid D.

   This would be true if the f_i's were the f-vector of a pure simplicial complex.

3. (**Graham Denham**)

   Problem #1: Develop an activity theory for Delta-matroids.

   Problem #2:Is there an analogue of the Orlik–Solomon algebra for Delta-matroids?

Matroids and representation theory

1. (**Andy Berget**) **Conjecture.** The number of set partitions of $E(M)$ into independent sets of M of sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$, $\lambda \vdash |E(M)|$, is at least the Kostka number K_{λ,ρ^t}, where $\rho = \rho(M) : r_1 \geq r_2 \geq \cdots$ is the **rank partition** of M, determined by the condition that $r_1 + \cdots + r_k =$ size of the largest union of k independent sets of M, i.e. the rank of the k-fold matroid union of M. (Assume M is loopless.)

   Motivation. Pick a realisation $v_1, v_2, \ldots, v_+n \in \mathbb{C}^r$ of M. Form

   $$\mathfrak{S}(v) = \mathrm{span}(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} | \sigma \in \mathfrak{S}_n) \subset (\mathbb{C}^r)^{\otimes n}.$$

   This is an $\mathfrak{S}_n$-representation, so it decomposes into irreducibles, indexed by partitions. It's a consequence of [Berget–Fink] that the multiplicity of each irrep is a valuative matroid invariant.

Theorem. The irrep indexed by λ appears iff $\lambda \trianglerighteq \rho^{\mathrm{t}}$, where $\trianglerighteq$ is dominance order.

Theorem. The multiplicity of $\lambda =$ a hook gives the coefficients of $\overline{\chi}_M$ up to sign.

The **Frobenius character** of a $\mathfrak{S}_n$-representation is its character written as a symmetric function.

Variant conjecture. The Frobenius character of $\mathfrak{S}(V) - e_{\rho^{\mathrm{t}}}$ is Schur-positive. Here $e_{\rho^{\mathrm{t}}}$ is an elementary symmetric function. The Gröbner degeneration $X(v) \rightsquigarrow \mathrm{in}\, X(v)$ from [Berget–Fink] should have a matroidal extension, and the Frobenius character should be computable from it.

2. (**Shiyue Li**) Let $V_\bullet^n$ be the $\mathbb{C}$-vector space spanned by the permutations in S_n. There is a natural S_n action on this space by conjugation. This representation has a natural grading, where the i^{th} graded piece consists of permutations with i cycles.

   Question 1: (Gedeon–Proudfoot–Young): Is $V_\bullet^n$ equivariantly log-concave?

   Note that $\dim V_i^n$ is equal to the unsigned Stirling number $c(n, i)$ of the first kind, and this sequence is known to be log-concave.

   Question 2: Is the Poincaré polynomial of $V_\bullet^n$ "equivariantly real-rooted"?

Combinatorics of matroids

1. (**Johannes Rau**) This question is based on work in progress by Draisma, Pendavingh, Rau, Yuen, and a student of Draisma.

   Given a matroid M, we have inequalities between three numbers:

$$
\begin{aligned}
& d := \mathrm{rk}(M) \\
& \le \min\{2\dim(\Sigma_M + R) - \dim R : R \text{ a rational subspace of } \mathbb{R}^n\} \\
& \le \min\{\textstyle\sum(2\,\mathrm{rk}_M(P_i) - 1) : P_1 \amalg \cdots \amalg P_k = E\}.
\end{aligned}
$$

   The third number is bounded above by $\min\{n, 2d-1\}$. The third number is the second specialized to R being a subspace in the braid arrangement. For M realizable over $\mathbb{C}$ by a subspace V, the second and third agree and both equal $\dim(\mathrm{Log}(V))$.

   Question: Are the second and third always equal?

   Question: Compute these three numbers for the restriction of M to each set $S \subset E(M)$, defining set functions $f_1(S)$, $f_2(S)$, $f_3(S)$. Is f_2 a matroid rank function? f_3?

   Question: Give an interpretation of f_3.

2. (**Federico Ardila**) $T_{K_n}(1, -1) = A_{n-1}$, the number of alternating i.e. up-down permutations of $n-1$. The only proof I know is computing generating functions of both sides.

   Problem: Give a better explanation.

3. (**Oliver Lorscheid**) What is the relationship between CSM-balancing (in the sense of Lopez de Medrano et. al.) and higher balancing (in the sense of Lorscheid et. al.)?

Weak and strong maps

1. (**Alex Fink**) One might naturally ask, if $M \to N$ is a weak map of connected matroids of the same rank on the same ground set (i.e., every basis of N is a basis of M), if there exist a regular matroid polytope subdivision of M of which N is a face. The answer, as shown in a paper of Brandt-Speyer, is in general **no**.

   Question: Can we salvage this by merely asking for a chain of subdivisions

$$M = M_0 \to M_1 \to \cdots \to M_k = N?$$

2. **(June Huh)** Can one prove that certain matroid polynomials (e.g. the Kazhdan–Lusztig polynomial and the Z-polynomial) are (coefficientwise) monotone with respect to weak maps?

3. **(Alex Fink)** Can we model weak maps using matroids over bands, in the sense of Baker–Bowler and Baker–Lorscheid?

   (Alex proposed a specific band which should do the job.)

4. **(Oliver Lorscheid)** If $f : M \to N$ is a strong map of matroids on the same ground set (i.e., N is a quotient of M), there is a factorization theorem which says that f factors as a restriction followed by a contraction. One might wonder if this also holds for B-matroids, where B is an *idyll*. However, the answer is **no**: for example, when B is the sign hyperfield, we are talking about oriented matroids and Richter-Gebert has given a counterexample to the factorization theorem.

   Question For which B is it true?

Scientific Progress Made

Here we survey some of the preprints and collaborations that have already resulted from the 5-day workshop. The BIRS workshop was an excellent meeting ground for established collaborations. For example, Ardila, Dehnam, Eur, and Huh were able to establish a new project combining the stellahedral and the conormal perspectives on matroids. Alex Fink, Kris Shaw, and David Speyer were able to discuss work from a previous Fields institute semester, on the reduction of positivity of Speyer's g-polynomial. Omid Amini, Emanuele Delucchi, Alex Fink, Diane Maclagan, and Nolan Schock are collaborating on Chow rings of compactifications of divisorial arrangement complements. During the workshop, Rudi Pendavingh, Johannes Rau, Chi Ho Yuen were able to finish their ongoing project on amoeba dimensions [11] since three of the authors were present at BIRS. Dave Jensen and Sam Payne had an important breakthrough and were able to update a previous preprint to include a proof that the tropicalization of a linear series on an algebraic curve is finitely generated as a tropical module [15]. This was done using a key fact about valuated matroids.

Following a discussion during a problem session, Matt Baker and Oliver Lorscheid have provided a proof of a theorem often attributed to Lafforgue but missing from the literature [8].

Following conjectures on the vanishing of cohomology of tautological bundles presented by Matt Larson, Andrew Berget, Chris Eur, and Alex Fink have begun working on the analogue of the matroid Schubert variety for pairs of matroids to tackle this problem. Chris Eur's preprint on the cohomology of tautological bundles of matroids also follows up from these problem sessions [12]. Eur and Larson are also working on a project on K-theoretic positivity for matroids.

Matt Baker, June Huh and Oliver Lorscheid discussed applications of F1-geometry to unimodality of the number of matroids of diverse ranks on a fixed ground set. Like many projects mentioned here, this on-going collaboration will most likely pick up momentum during the special year on matroid theory at the IAS in Princeton in 2024.

Many new collaborations were also formed. For example, Nick Proudfoot reported that he also mentioned that, thanks to the hybrid format, Luis Ferroni was able to watch his talk online. This led to very interesting conversations, new conjectures, and counter-examples to old conjectures. Benjamin Schroeter reported that this presentation also led to discussions with Proudfoot and Lorenzo Vecchi, and new work on valuative invariants of (fundamental) transversal matroids. Matt Larson reported that, thanks to conversations during the workshop with Nima Anari and Chris Eur, he was able to obtain some non-trivial inequalities on the number of independent sets of certain delta-matroids, resulting in the preprint [16].

Many participants reported on new collaborations formed between mathematicians in different career stages. For example, Matt Baker and Oliver Lorscheid reported on collaborations with Tong Jin and Zach Walsh. Tong Jin also reported on helpful conversations on orthogonal matroids with Nathan Bowler.

Outcome of the Meeting

The overall feedback from the meetings participants was overwhelmingly positive. On top of the scheduled talks and discussions, the in-person event at BIRS' spectacular location provided many occasions for informal discussions and extra activities.

For many participants, it was one of the first time since before the COVID-19 pandemic that they were among such a large group of researchers in this field. Very importantly, this workshop provided a meeting ground for mathematicians in a a wide cross-section of career stages. Many senior participants expressed the positive outcome of meeting the new up and coming researchers in the field. Moreover, many junior participants expressed gratitude for the opportunity to give talks at the meeting.

There were many invited participants who could not attend in person and expressed interest in participating remotely. However, we unfortunately experienced very low online attendance during the talks and even lower online participation in the discussion sessions.

Another unfortunate event, were participants being forced to cancel their travel due to Canadian visa procedures. Invited participant Manoel Jarra and organiser Felipe Rincón were not able to attend the meeting in person due to visa wait times. This problem disproportionately effects mathematicians who are from underrepresented groups. Though we understand that visa processing times are out of BIRS' control, sending out workshop invitations early and informing future organisers and participants of visa requirements and processing times could help prevent this unfortunate occurrence in the future.

Participants

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Chapter 6

A convergence of computable structure theory, analysis, and randomness (23w5055)

March 19 - 24, 2023

Organizer(s): Johanna N.Y. Franklin (Hofstra University), Timothy H. McNicholl (Iowa State University), Linda Brown Westrick (Pennsylvania State University)

This workshop focused on the newly developing connections between computable structure theory, computable analysis, continuous logic, and algorithmic randomness. While metric structures can be studied through the model-theoretic lens of continuous logic with no additional constraints, computable structure theory has historically been centered around countable algebraic structures such as algebraically closed fields and linear orders. However, with some care, it is possible to study uncountable structures such as Banach spaces and metric spaces in this context and to develop a formal definition for an algorithmically random structure. This workshop brought researchers in these four areas together to build on recent advances in the intersection of these topics and develop new questions in and new approaches to this emerging field of study.

Overview

Computable structure theory

Turing machines provide the standard model of computation. That is, any physical computational device can be simulated by a Turing machine. Turing machines accept as input finite words from a fixed finite alphabet Σ and yield such words as output. However, it is entirely possible that a machine will not halt on some inputs (i.e., computer programs sometime crash). A set S of finite words is *computably enumerable* if it is the halting set of some Turing machine. A fundamental discovery is the existence of a computably enumerable set that is not computable. In other words, there is a Turing machine M so that no Turing machine can determine if M halts on an arbitrary word in Σ^*.

Thus, Turing machines establish a computability theory on the domain Σ^* of all finite words from a fixed finite alphabet Σ. However, mathematical computation takes place in a structure such as the ring of integers or the field of rational numbers. Computation is transferred to these domains by *computable presentations* wherein each element of the structure is labelled with a finite word in such a way that the induced operations and relations (including that of labelling the same element) are computable. The study of such presentations is the heart of computable structure theory. Much of computable structure theory is motivated at some level by the realization that what one can compute on a structure depends on how it is presented. For example, it is easy to show there is a computable presentation of $(\mathbb{N}, <)$ in which the successor relation is not computable. That is, there is no algorithm (Turing machine) that can determine if two words label numbers that differ by 1.

The study of computable presentations of structures is central to much of computability theory. We refer the reader to [1] and [22] for comprehensive treatments of this subject.

Computable analysis

The just-described framework of presentations only applies to countable structures. However, most scientific computation involves continuous data such as real numbers or compact subsets of the plane. Computable analysis bridges the gap between the discrete and continuous by means of computing with approximations. For example, a real number x is said to be *computable* if there is an algorithm (Turing machine) that, given $k \in \mathbb{N}$ as input, produces a $q \in \mathbb{Q}$ so that $|x-q| < 2^k$. A function $f : \mathbb{R} \to \mathbb{R}$ is computable if there is an algorithm P that satisfies the following conditions.

1. Given a rational interval I as input (that is, given the endpoints of I as input), if P halts then it produces a rational interval J that includes $f[I]$.

2. If U is a neighborhood of $f(x)$, then there is a rational interval I so that on input I P computes a rational interval J that is included in U.

Together, these criteria state that for each $x \in \mathbb{R}$, computable or incomputable, P can compute arbitrarily good approximations of $f(x)$ from sufficiently good approximations of x.

As will be seen below, these definitions can be generalized to a wide variety of Polish spaces. The recently released *Handbook of Computability and Complexity in Analysis* describes this development and many other facets of the subject [3].

Algorithmic randomness

Algorithmic randomness begins with the question "When is a sequence (finite or infinite) random?" To a first approximation, one might say "If there is no pattern in the sequence." But there is no discernible pattern in the digits of π, yet as these digits can be algorithmically calculated, one can hardly call them random. Fortunately, computability theory provides a rigorous answer to this question. Let λ denote Lebesgue measure on the Cantor space $\{0,1\}^{\mathbb{N}}$. Informally speaking, $p \in \{0,1\}^{\mathbb{N}}$ is random if it avoids all null sets that are effectively presented. To see what this means, call an open set $S \subseteq \{0,1\}^{\mathbb{N}}$ *c.e. open* if there is a computably enumerable set $U \subseteq \{0,1\}^*$ so that $S = \bigcup_{\sigma \in U} [\sigma]$ (where $[\sigma]$ is the cylinder generated by σ). A *Martin-Löf test* is a sequence $(S_n)_{n\in\mathbb{N}}$ of uniformly c.e. open sets so that $\lambda(S_n) \leq 2^{-n}$. The sequence p avoids such a test if $p \notin \bigcap_n S_n$, and p is *Martin-Löf random* if it avoids all Martin-Löf tests.

There are other notions of randomness such as Schnorr randomness, but most of them are defined similarly and thus our definition above gives a sense of randomness notions in general. The text of Downey and Hirschfeldt provides a very comprehensive treatment of algorithmic randomness [8].

Continuous logic

Continuous logic is the model theory of metric structures. Unlike classical model theory, there are continuum-many truth values; usually the interval $[0,1]$. The value 0 represents truth, and the value 1 represents falsehood. The usual sentential connectives are replaced by three: subtraction from 1, multiplication by $\frac{1}{2}$, and bounded subtraction. The universal and existential quantifiers are replaced by supremum and infimum. The equality sign is replaced by a symbol for the metric.

Continuous logic has undergone considerable development in recent decades. In particular, it has been used to develop the model theory of operator algebras [16].

Connections between these areas

Algorithmic randomness and computable analysis have long been linked. Many theorems from analysis hold on a conull set; researchers in algorithmic randomness define classes of random reals as those that are, in some effective sense, ”large with respect to measure.” The points for which effectivized versions of theorems such as Birkhoff's ergodic theorem and the Lebesgue differentiation hold have been characterized in terms of randomness [2, 4, 10, 14]. Algorithmic randomness and computable structure theory have also been linked, though not for quite as long [13], and the question of defining a random structure has been considered as well [17, 18].

The very natural connection between computable analyis and computable structure theory has only been recognized fairly recently. One merely needs to expand the definition of computable presentation to metric structures. These are structures that consist of a complete metric space together with collections of operations and functionals; for example Banach spaces, C^* algebras, etc. A computable presentation of a metric structure consists of specifying a dense sequence with respect to which the operations and functionals can be computably approximated with arbitrary precision.

Computable structure theory is the foundation for computable model theory. It is hoped to establish effective metric structure theory as the foundation for a computable model theory of metric structures. In fact, computability theory has already been used to produce a negative solution of the Connes Embedding Problem by showing that the universal theory of the hyperfinite II-1 factor is undecidable.

Introductory talks

We began on Monday with an introductory talk in each of the four main areas represented by the participants: continuous model theory, computable structure theory, algorithmic randomness, and computable analysis.

Isaac Goldbring gave us ”A primer in continuous model theory”:

> In this talk, we give an introduction to modern continuous model theory. We use the metric ultraproduct construction to motivate the notion of a continuous language and the appropriate notion of structure for such languages. We give some examples of metric structures of interest in this workshop, such as Banach spaces, Hilbert spaces, and C*-algebras. We then move on to discuss compactness and completeness for continuous logic. Time permitting, we discuss some of the more nuanced aspects of continuous model theory, such as generalized formulas, definable sets, and the metric on type spaces.

Wesley Calvert gave us ”How to think about computable structures”:

> In the very early days of computable structure theory there were only specific algebraic questions about specific structures that were presumed to be well known (e.g. does every explicitly given field have a unique computable algebraic closure). Later, this line of thought was abstracted into computable model theory, which could perhaps have looked something like model theory — except that, as Millar famously remarked, there were ”too many counterexamples.” Over time what took hold was the study of a new category, taking the putative pathologies (e.g. distinctions between classically isomorphic structures) as features of a new and interesting mathematical realm. Apparently niche interests like infinitary logics, admissible set theory, and alpha-jump priority constructions found a natural home in this discipline. This talk will outline the kinds of questions asked in computable structure theory, and the style of arguments used.

Dan Turetsky gave us ”Broad Swathes of Randomness”:

> I will give an overview of algorithmic randomness; this will cover definitions, central and illustrative results, and applications of randomness to other areas of computability theory. My intention is to give a picture of the field and show some of its potential for crossover with other areas.

Finally, Alexander (Sasha) Melnikov gave us ”Computable dualities”:

> I will talk about several recent results that explicitly relate computable algebra with computable topology via various sorts of effective dualities.

Ample time for questions and discussion was provided after each talk.

Group formation and problem selection

After these introductory talks on Monday, we began Tuesday morning with an open problem session. We requested that each problem proposed have two properties:

- it should be related to at least two of the general areas described above, and
- it should be specifically enough stated that a group could reasonably be expected to make progress on it over the course of a week.

The participants identified 12 problems, several with constituent subproblems developed by other participants after a more general problem was stated.

After a tea break, we reconvened to form problem groups that would work together on Tuesday afternoon. We did this "AIM-style": we began with approval voting to identify a short list of problems that the participants would be actively interested in working on and then moved on to choosing our own groups. We began with four problems under consideration.

At the beginnings of Wednesday morning, Thursday morning, and Thursday afternoon, each group reported on their progress to the workshop at large. In addition to reporting on the main ideas, we asked each group to decide whether they would like to continue working on their current problem and, if so, whether they would like to borrow someone with a particular kind of expertise from one of the other groups. We are pleased to say that this "borrowing" was a regular occurrence. If the group had no interest in pursuing their current problem, the participants would either disperse to existing groups or choose a new problem from the list to work on. Occasionally, participants would decide to spend the next session working with a different group while their usual group continued working.

Three problems were discussed intensely over the week; two others were discussed but were found to be trivial upon closer inspection. Each group remaining on Thursday afternoon spent Friday morning writing summaries of their work and planning their next steps. Indeed, there was some friendly competition over which group was going to meet soonest after the workshop concluded.

Final group reports

The following are the reports of the work that was done on the three problems that were discussed most intensely.

Presentations of C^* algebras

The group considered two questions: the computable presentability of uniformly hyperfinite (UHF) algebras and the computability of the Gelfand transform.

The following result is fundamental to understanding UHF algebras.

Fact: *If m, n, t are positive integers so that $n = tm$, then the map*

$$A \mapsto \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}$$

is an isometric embedding of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$.

This leads to the definition of uniformly hyperfinite algebras.

Definition 6.0.1. *A C^*-algebra A is uniformly hyperfinite if there is a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers so that $n_k|n_{k+1}$ for each k and so that A is isomorphic to the direct limit $M_{n_0}(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \to \ldots$.*

UHF algebras are characterized by their supernatural numbers. These are defined as follows. Let $(p_j)_{j\in\mathbb{N}}$ be the increasing enumeration of the prime numbers.

Definition 6.0.2. *Suppose A is a UHF algebra. Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of positive integers so that $n_k|n_{k+1}$ for each k and so that A is isomorphic to the direct limit $M_{n_0}(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \to \ldots$. For every $j \in \mathbb{N}$, let*

$$f(j) = \sup\{m \in \mathbb{N} \ : \ \exists k \ p^m|n_j\}.$$

f is the supernatural number of A.

It is well known that two UHF's are isomorphic if and only if they have the same supernatural number. It is natural to conjecture that a UHF has a computable presentation if and only if it has a computable supernatural number. However, at least one direction of this conjecture is false. In particular, the group demonstrated the following.

Proposition 6.0.3. *There is a computably presentable UHF algebra A so that the supernatural number of A is Turing equivalent to* $\mathbf{Tot}$.

Thus, the computability of the supernatural number is too strong a condition to characterize the computable presentability of a UHF algebra. The following definition from computable analysis provides a possible alternative.

Definition 6.0.4. *Let $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. f is lower semi-computable if there is a computable $g : \mathbb{N}^2 \to \mathbb{N}$ so that for every $n \in \mathbb{N}$, $(g(n, s))_{s\in\mathbb{N}}$ is nondecreasing and $\lim_s g(n, s) = f(n)$.*

The following are fairly straightforward consequences of the definitions.

Proposition 6.0.5. *Suppose A is a UHF algebra. Then the following are equivalent.*

1. *The supernatural number of A is lower semi-computable.*

2. *There is a computable sequence $(n_k)_{k\in\mathbb{N}}$ so that A is isomorphic to the directed limit $M_{n_0}(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \to \ldots$ and $n_k|n_{k+1}$.*

Corollary 6.0.6. *If the supernatural number of a UHF algebra A is lower semi-computable, then A is computably presentable.*

However, we are left with the following.

Question 6.0.7. *If A is a computably presentable UHF algebra, does it follow that the supernatural number of A is lower semi-computable?*

The group then considered the computability of the Gelfand transform. This transform represents a unital commutative C^* algebra A as $C^*(X)$ for a suitably chosen compact metric space X. The group obtained the following, which is currently being prepared for submission.

Theorem 6.0.8. *If an Abelian C^* algebra A is computably presentable, then there is a metric space X with a computably compact presentation so that A is isomorphic to $C^*(X)$.*

The converse of Theorem 6.0.8 is already known to be true [9]. The proof of Theorem 6.0.8 is uniform in the sense that it yields a procedure that, given an index of a presentation of A, yields an index of a computably compact presentation of X.

Random structures

The initial question this group sought to address was this: Can you adapt Gromov-style randomness (for groups) to random Banach spaces? If there is a good notion, would it have to contain a classical sequence space like ℓ^p or C_0 as Banach thought would be the case? The subquestions presented in the open problem session were "What is a random compact Polish space, and what would the right measure be?" and "Is it more appropriate to consider genericity?"

We discussed "Gromov-style" typicality in the sense of Franklin, Ho, and Knight [11] for Banach spaces and concluded that it wasn't a useful approach: the idea of *limiting density* relies on the ability to consider larger and larger proportions of the structures under discussion in a reasonable way based on their presentations, and we did not see a plausible way to carry that out.

We began with the question of what the space of compact Polish spaces should look like and identified it with the space $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ of compact and overt subsets of the Hilbert cube. This approach is equivalent to coding a compact Polish space as the completion of a metric on $\mathbb{N}$ together with a witness of total boundedness.

Avoiding any particular rational hyperplane in $[0,1]^\omega$ is a dense c.e. open property for elements $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$, which implies that any 1-generic compact Polish space is totally disconnected. A 1-generic compact Polish space also has no isolated points, as for any $x \in [0,1]^\omega$ and $0 < r$ it is a dense c.e. open property for $A \in (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ that either $\overline{B}(x,r) \cap A = \emptyset$ or $|B(x,r) \cap A| > 1$.

This reasoning establishes that a 1-generic compact Polish space is homeomorphic to 2^ω. We can even say a bit more and observe that 2^ω is relatively categorical as a compact Polish space, meaning that from a name for some $A \in (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ such that A is homeomorphic to 2^ω we can compute a homeomorphism from 2^ω to A. It is important here that the compact information is provided since 2^ω is not relatively computably categorical as a Polish space.

Being connected is a computably closed property of compact Polish spaces, so the space of connected compact Polish spaces can be identified with a Polish space again. It then makes sense to ask about 1-generic connected compact Polish spaces. It was suggested that such a generic space should be the pseudo-arc.

To define the space of separable Banach spaces, we make use of the fact that any separable Banach space can be obtained by equipping the vector space $c_{00}(\mathbb{Q})$ of finitely supported sequences of rational sequences with a suitable seminorm and then completing it. Being a seminorm is an effectively closed property of some $p \in \mathbb{R}_{\geq 0}^{c_{00}(\mathbb{Q})}$, which gives us a Polish space of representatives for separable Banach spaces.

We then turned our attention to the question of what a *generic Banach space* would look like. One of the options proposed was C[Cantor set], in part because it is universal.

The next day, we were directed to the Gurarij space. The article [15] contains a proof that the Gurarij space is a generic Banach space in the sense of genericity for a forcing condition. Their arguments should translate rather directly into showing that a 1-generic element of the space of separable Banach spaces mentioned above is a Gurarij space.

Supernormal numbers

A well-known result characterizes normal numbers by a condition regarding compressibility by finite-state transducers. In view of this result, we first relativize the notion of normality to a given enumeration and then strengthen the notion of normality by requiring a number to be normal to every enumeration.

Definition 6.0.9. *We define C_r^f and $C_{r,D}^f$:*

$$C_r^f(x) := \min\{|\sigma| : |f(\sigma) - x| < 2^{-r}\}$$

$$C_{r,D}^f(x) := \min\{C_D(\sigma) : |f(\sigma) - x| < 2^{-r}\}$$

Definition 6.0.10. *Given a function f, we say that a real x is strongly f-normal ($x \in \mathrm{SNorm}^f$) if and only if for every finite state machine D there is a constant k such that for all n, we have*

$$C_{n,D}^f(x) \geq n - K(n) - k.$$

Definition 6.0.11. *Given a prefix-free function f, we say that a real x is weakly f-normal $x \in \mathrm{WNorm}^f$) if and only if there is a constant k such that for all n,*

$$C_n^f(x) \geq n - K(n) - k.$$

We proved the following:

Theorem 6.0.12. *The following properties of a real x are equivalent.*

1. *For all upper semi-computable functions f, the real x is strongly f-normal.*
2. *For all upper semi-computable functions f, the real x is weakly f-normal.*
3. *The real x is strongly f-normal for some universal upper semi-computable function f.*
4. *The real x is weakly f-normal for some universal upper semi-computable function f.*

We then considered supernormal reals:

Definition 6.0.13. *We say that a real x is supernormal if and only if it satisfies one (equivalently, all) of the equivalent conditions of Theorem 6.0.12.*

Our results on these reals follow.

Proposition 6.0.14. *If a real A is supernormal, then A is Martin-Löf random.*

Proposition 6.0.15. *Every 2-random real is supernormal.*

Proposition 6.0.16. $\overline{\Omega}$ *is not supernormal.*

Proposition 6.0.17. Ω *is supernormal.*

Organizational matters

Diversity, equity, inclusion, and belonging

The organizers set out from the beginning to create a workshop that not only included participants from underrepresented groups but also actively welcomed them. When we made the ”long list” of participants, we sought balance along many different dimensions, including the participant’s gender, ethnicity, type of employing institution, and geographical location. If we felt we were deficient in one area, we asked an appropriate colleague for suggestions. Then, when we sent invitations, we sought to maintain the balance we had established and had a reasonable amount of success in doing so.

Before the workshop, we established an unofficial mentoring program for the graduate students who would be present. Due to the pandemic, they were less acquainted with the other participants in the workshop than we could have expected five years ago. We introduced them to each other by e-mail, and then, after confirming that this would be acceptable to them, introduced them to some more senior workshop participants by e-mail in advance to give them a greater sense of belonging.

Virtual participation

Based on advance polling of the participants who planned to attend virtually, we decided to establish a group Overleaf project and a Dropbox folder. The Dropbox folder would contain slides from the introductory talks and papers that the participants found helpful, while the progress from the week would be recorded in the Overleaf project. We felt that the presence of Zoom would render tools such as a Slack workspace pointless.

It seems that virtual participation in the working groups worked as well as the schedules of the virtual participants permitted. Indeed, some of the virtual participants mentioned that this was the smoothest integration of virtual participation into a workshop that they had ever experienced; we would particularly like to thank the BIRS staff who provided our tech support. At the beginning of each morning and afternoon working session, each group would leave a Zoom link in the main Overleaf document, and a member of the group would keep the Zoom link open. We would assign groups to different spaces in the Juniper Hotel based on not only the group's size but also the type of AV tech available in each space; the group working on presentations of C^* algebras had the most virtual participants and thus kept the main room that was set up for talks.

It should be noted that this ability to accommodate virtual participation was necessary for the workshop's success. There were 16 participants on site and 11 virtual participants. Of the virtual participants, four initially intended to attend in person but could not. One participant had to change his participation from on site to virtual last minute due to health issues, one due to local responsibilities, and two more due to the unexpected inability to get a Canadian visa in a reasonable amount of time.

Outcome of the workshop

The feedback from the meeting participants was overwhelmingly positive both in terms of the meeting's organization and structure and the scientific content. While all of the participants were well versed in at least one of the four areas represented at this meeting and some were experts in two, none of the participants were familiar with all of them. This resulted in very productive and open conversations.

Furthermore, the structure of the meeting seems to have enabled the groups to work together very effectively. The introductory talks on the first day gave everyone a common framework and provided a familiar setting for the virtual participants to start in. The regular updates and corresponding constant self-evaluations of each of the working groups made it easy for the groups to decide whether they were still being productive, whether they needed to "borrow" another participant, or whether it would make sense for them to disband and work on another topic.

This workshop was beset by practical challenges. A similar proposal by the first two organizers was accepted for a BIRS workshop in May 2020, but the meeting was cancelled due to the pandemic. Later, when they were offered the opportunity to hold the workshop virtually, they declined given the interactive events they anticipated and chose to reapply instead, adding the third organizer. Then, at this March 2023 workshop, two of the four participants who gave introductory talks had to change their participation from on site to virtual, and two of the three organizers developed health issues that precluded them from attending the meeting on site (indeed, one of them had to withdraw from participation entirely). Nonetheless, the meeting was a success.

Participants

Burton, Peter (Iowa State University)
Calvert, Wesley (Southern Illinois University)
Eagle, Christopher (University of Victoria)
Fox, Alec (University of California, Irvine)
Franklin, Johanna (Hofstra University)
Goldbring, Isaac (University of California at Irvine)
Gruner, Emma (Penn State University)
Hanson, James (University of Maryland)
Harrison-Trainor, Matthew (University of Illinois Chicago)
Hart, Bradd (McMaster University)
Hoyrup, Mathieu (Inria)
Marchuk, Margarita (Sobolev Institute of Mathematics)
Mayordomo, Elvira (Universidad de Zaragoza)
McNicholl, Timothy (Iowa State University)
Melnikov, Alexander (Victoria University of Wellington)
Miller, Joseph S. (University of Wisconsin–Madison)
Miller, Russell (City University of New York)
Ng, Keng Meng (Nanyang Technological University)

Noquez, Victoria (St. Mary's College of California)
Pauly, Arno (Swansea University)
Rojas, Diego (University of Dallas)
Selivanova, Svetlana (KAIST)
Slutsky, Konstantin (Iowa State University)
Soskova, Mariya (University of Wisconsin–Madison)
Thewmorakot, Teerawat (University of Connecticut)
Turetsky, Dan (Victoria University of Wellington)
Villano, Java (University of Connecticut)

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Chapter 7

WIN6: Women in Number Theory 6 (23w5175)

March 26 - 31, 2023

Organizer(s): Shabnam Akhtari (Pennsylvania State University), Alina Bucur (University of California San Diego), Jennifer Park (Ohio State University), Renate Scheidler (University of Calgary)

Overview of the Workshop

The Women in Numbers (WIN) network was created in 2008 for the purpose of increasing the number of active women researchers in number theory. For this purpose, WIN sponsors regular conferences, taking place approximately every three years, where women scholars gather to collaborate on cutting-edge research in the field and produce publishable scientific results. The WIN workshops provide an ongoing forum for involving each new generation of junior faculty and graduate students in state-of-the-art research in number theory.

WIN6 workshop was organized in partnership with the Clay Mathematics Institute and held in Banff International Research Station, in Alberta, Canada, from March 26 to March 31, 2023. WIN6 was unique in the sense that the previous scheduled meeting was canceled due to the global pandemic. It was essential to pay careful attention in selecting participants and try to include those who were affected more significantly by travel limitations. In order to help raise the profile of active women researchers in number theory and increase their participation in research activities in the field, this event brought together researchers in different career stages and from broad subfields of number theory for collaboration. Emphasis was placed on on-site collaboration on open research problems as well as training junior number theorists. Both the project leaders and the participants were selected through a competitive application process. These were advertised through the WIN mailing list, as well as various other mailing lists for number theorists. There were 14 applications for project leadership (submitted by two senior mathematicians each) from which 11 groups were selected. Then we solicited applications for the participant pool, for which 71 applications were received. These applicants were asked to specify their preferences, and to explain their mathematical backgrounds. As most of these applicants would have been a great fit for the conference, we prioritized the career stages of the applicants in addition to their mathematical fit – the organizers agreed that postdocs and graduate students towards their final year of PhD are most likely to benefit from this conference, and as such, most of the participants who were invited were within this demographic.

The main goals were

1. To train graduate students and postdocs in number theory and related fields;
2. To highlight research activities of women in number theory;

3. To increase the participation of women in research activities in number theory;
4. To build a research network of potential collaborators in number theory;
5. To enable women faculty at small colleges to help advising graduate students.

We had a total of 42 participants, three of whom participated remotely due to not receiving a Canadian visa in time.

At BIRS, each group gave a brief 20-minute presentation on the topic of their research to facilitate conversations between different groups, and the majority of the remaining time was devoted to working in groups. In the evenings, we had various informal gatherings. Some were more structured in the form of a career panel, and others were more social and spontaneous.

Besides BIRS for providing excellent research facilities, we are grateful to the the following organizations for their support of this workshop: NSF, Number Theory Foundation, PIMS, Journal of Number Theory, and CMI.

Research Projects and Project Groups

In this section, we summarize the research topic and the progress of each project group, as reported by the project leaders of ten participating groups.

Campana points of bounded heights on orbifold stacks

Group members

Shabnam Akhtari, Jennifer Park, Marta Pieropan, Soumya Sankar

Project description

Recent work of Ellenberg, Satriano and Zureick-Brown [ESZ] has stirred a lot of interest towards the problem of counting rational points of bounded height on stacks. Special attention has been given to weighted projective stacks [Dar, BrMa], in particular, seen as moduli spaces of elliptic curves [BrNa, Phi], to modular curves [BoSa] and to stacky curves [NaXi]. The paper [ESZ] defines heights on stacks outlines an explicit connection between Malle's conjecture for number fields of bounded discriminant and Manin's conjecture for rational points of bounded height. Manin's conjecture has been extended to stacks by Darda and Yasuda [DaYa22] who also study Malle's conjecture as counting points of bounded height on classifying stacks [DaYa, DaYa2]. At the same time, there has been a lot of work on asymptotics of Campana points of bounded height. Campana points are rational points on a projective variety with prescribed intersection with a boundary divisor specified by some parameters. As the parameters vary, the sets of Campana points interpolate between the set of rational points and the set of integral points with respect to the same boundary divisor. Manin's conjecture for rational points has been extended to Campana points by Pieropan, Smeets, Tanimoto and Varilly-Álvarado [PSTV], and it has been proven for equivariant compactifications of vector groups [PSTV] and of Heisenberg groups [Xia], for toric varieties [PiSc, San], for various diagonal hypersurfaces in projective spaces [Van, BrVa, BrYa, Shu21, Shu20, BBKOPW] and for some non-strict-normal-crossing boundaries [Str]. This Women in Number Theory 6 project aims to explore the concept of Campana points on stacks. The leading goals are: identify a good notion of Campana points on stacks, compute asymptotic formulas in some examples, extend the Manin-type conjecture for Campana points [PSTV] to stacks, and investigate the relation to Malle's conjecture.

Progress

During the conference we considered two possible definitions of Campana points on stacks and we brainstormed several research directions. As result we collected at least six concrete projects that we can pursue in the future. The most immediate one consists of clarifying the relationship between Malle's conjecture for a finite group G and counting rational points of bounded height on BG. We made explicit calculations in the case where G is a cyclic group and established that the two problems coincide only in the case when the base field contains all $|G|$-roots of unity.

Future plans

We will carry on some of the projects we collected concerning counting Campana points of bounded height on specific stacks. We plan to work in parallel on a shared .tex file and to meet roughly once per month to monitor progress and share further ideas.

Generalized Eckardt points on del Pezzo surfaces of degree 1

Group members

Julie Desjardins, Yu Fu, Kelly Isham, Rosa Winter

Project description

Del Pezzo surfaces can be classified by their degree, which is an integer between 1 and 9. Those of degree 3 (which are smooth cubic surfaces in $\mathbb{P}^3$) famously contain 27 lines (exceptional curves) over an algebraically closed field, and at most 3 lines go through one point, called an Eckardt point. A cubic surface contains at most 45 Eckardt points. Similarly, a del Pezzo surface of degree 2 contains 56 lines over an algebraically closed field, at most 4 lines go through one point, and such a surface contains at most 126 such points. In this project we study the situation for del Pezzo surfaces of degree 1. These contain 240 exceptional curves, and we know that outside characteristics 2 and 3, at most 10 go through one point. There exist a handful of examples of such surfaces with a point contained in the intersection of 10 exceptional curves, but apart from that, very little is known.

Questions we aim to answer are: if 10 exceptional curves intersect in a point, which configurations can these curves have? How many 'Generalized Eckardt Points' (GEPs) can a del Pezzo surface of degree 1 contain? Can we produce examples of surfaces with multiple GEPs?

An important motivation for studying generalized Eckardt points on a del Pezzo surface of degree 1 comes from the search for proof of unirationality of these surfaces – which is a big open problem with only partial results. Del Pezzo surfaces of degree at least 3 are known to be unirational if they contain a rational point, with the extra condition that it is not a generalized Eckardt point for degree 2. Several methods exist for showing the density of the set of rational points on a del Pezzo surface of degree 1 (which is implied by unirationality), given a point which is not contained in many exceptional curves. It is not yet known how to prove similar results starting with generalized Eckardt points.

Progress

During the conference in BIRS we made a great start in classifying the possible configurations in which 10 ecxeptional curves on a del Pezzo surface of degree 1 can intersect. We have a list of 9 potential configurations, two of which are known to be possible, and 1 has been disproved. We are going over the remaining 6 configurations to either find examples that show they are possible or prove that they are not. We have also constructed a whole family of surfaces with 10 exceptional curves intersecting in a point, showing that certain surfaces always have a GEP.

Future plans

In the coming months we will work out the computations that we started with at BIRS. We plan to meet at the end of May to see where we are with the project and make a plan on how to continue working further, with regular meetings online. We want to write a paper for the proceedings, as well as keep working on the project afterwards, since there are still many open questions.

Patching Technique and Local Global Principles

Group members

Parimala Raman, Sujatha Ramdorai, Sarah Dijols, Charlotte Ure

Project description

The classical local global principle (also called Hasse Principle) for quadratic forms asserts that a quadratic form q over a global field has a nontrivial zero if and only if it has a nontrivial zero over all the completions. Such local global principles have subsequently expanded to other contexts:–to name a few, in studying the existence of rational points over varieties defined over global fields, non-triviality of Brauer group elements, nontriviality of torsors of algebraic groups, etc. Around two decades ago, Harbater and Hartmann introduced the patching technique to study algebraic structures over a semiglobal field F, which by definition is the function field $K(C)$ of a smooth integral curve C defined over a complete discretely valued field K. This method has proved fruitful in studying a variety of local global principles over a semi-gloobal field. Recent results of Harbater-Hartmann-Krashen use these methods to study the Galois cohomology groups of semiglobal fields.

Let L be a field and let l be a prime different from the characteristic of L. Let μ_l be the group of l-th roots of unity viewed as a module over the absolute Galois group of L for a field L. We assume that L contains μ_l. Let $\bar{L}$ denote a separable closure of L. Denote the absolute Galois group $\mathrm{G}al(\bar{L}/L)$ by G_L. The Galois cohomology groups $H^i(L, \mu_l) := H^i(G_L, \mu_l)$ for $i \geq 0$ encode important information on the arithmetic of L. A celebrated conjecture of Milnor, settled by Voevodsky, asserts that the groups $H^i(L, \mu_l)$ for $i \geq 2$ are generated by cup-products $(a_1) \cdot (a_2) \cdot \cdots \cdot (a_i)$, where (a_i) denotes the class of an element $a_i \in L^*$ under the Kummer isomorphism $H^1(L, \mu_l) \simeq L^*/L^{*l}$. Elements of the form $(a_1) \cdot (a_2) \cdot \cdots \cdot (a_i)$ are called *symbols*. The symbol length for $H^k(L, \mu_l)$ for $k \geq 2$ is the least number d such that any $\xi \in H^k(L, \mu_l)$ can be expressed as a sum of at most d symbols. For a field L, let $cd(L)$ denote the cohomolgocal dimension of L. The symbol length is intimately connected to other important invariants associated to a field such as the cohomological dimension and the u-invariant.

Conjecture Let F be a semiglobal field. With the notation as above, suppose that the cohomological dimension of k is n. Then every element of $H^{n+2}(F, \mu_l)$ is a symbol.

Goals of the project:

- Investigate symbol lengths for Galois cohomology groups of general semi-global fields
- Investigate applications of Patching Technique to study the conjecture above .

Progress

Some important reductions to the general setting were made. This would enable progress towards understanding symbol lengths.

Future plans

The results are being written up and we plan to continue the project discussion over zoom meetings.

Isogenous Discriminant Twins over Number Fields

Group members

Alyson Deines, Asimina Hamakiotes, Andreea Iorga, Changningphaabi Namoijam, Manami Roy and Lori Watson

Project description

The conductor of an elliptic curve E defined over a number field K is an arithmetic invariant. In particular, the conductor is an integral ideal in K that measures the ramification in the field extensions generated by the torsion points of E. The minimal discriminant is a geometric invariant; it measures the number of irreducible components of E over finite fields. The two invariants are closely related; the primes that divide the conductor are precisely the primes that divide the minimal discriminant. When two elliptic curves have the same conductor and minimal discriminant, they are called discriminant twins. One can ask, how many discriminant twins are there over a number field? In fact, one can start with an easier question, when can two prime-isogenous elliptic curves defined over a number field have the same minimal discriminant? Isogenous elliptic curves already have the same conductor, so this is asking, when can isogenous elliptic curves have the same discriminant?

Over the rationals there are only finitely many semistable isogenous discriminant twins [Dei18]. In particular, isogeny classes of size two with at least one prime with semistable reduction cannot have discriminant twins. In a recent work [BBDHR], the p-isogenous discriminant twins for $p = 3, 5, 7, 13$ over number fields have been classified. The number of j-invariants for the p-isogenous discriminant twins with $p = 3, 5, 7, 13$ over number fields is not necessarily finite. The results relies on studying an explicit parametrized family of elliptic curves for isogeny graphs of rational elliptic curves from [Bar].

Our planned project is to finish categorizing all isogenous discriminant twins over number fields. We can look at a few different questions:

1. Use parametrized family from [Bar] and the technique from [BBDHR] to get explicit result about composite isogeny degree of genus 0.

   i) For degree of isogeny $n = 9, 25$, a similar argument should follow.

   ii) The case when 2 divides the degree of isogeny is more complicated and requires more careful investigation.

2. Classify semistable discriminant twists over number fields.

3. Consider $n = 11, 17, 19, 21$, i.e., when $X_0(n)$ has genus one. Parameterizing the discriminant twins in this case is an open problem that will depend at least on the rank of the elliptic curve, $X_0(N)$, over a given number field.

Progress

During the workshop at Banff we started working on question (1) stated above for $n = 25$ and $n = 2$. We have spend some significant time discussing how to write some of these programs in Sage and checking examples in Sage for arriving at correct conjectures. We have the result for $n = 25$ and some partial results for $n = 2$. We have also discussed about how to approach some other cases like $n = 9, 6$.

Future plans

We have divided into subgroups working on different questions. We are meeting biweekly over Zoom as the whole group. In between subgroups are are meeting as needed. We plan to continue meeting over Zoom. We didn't plan any in-person meetings over Summer but some of us hope to attend the same conference in Summer.

Machine learning and arithmetic

Group members

Kristin Lauter, Cathy Li (participated remotely), Krystal Maughan, Rachel Newton, Megha Srivastava

Project description

We will explore and compare different machine learning approaches to arithmetic problems, some with relevance to cryptography.

Progress

During the WIN6 workshop, we experimented with different machine learning approaches and started building, training and testing a transformer model.

Future plans

We have arranged a regular weekly meeting slot.

Moments of Artin–Schreier covers

Group members

Alexandra Florea, Edna Jones, Mathilde Lalín

Project description

Let q be a power of an odd prime p. An Artin–Schreier cover is given (up to $\mathbb{F}_q$-isomorphism) by an affine model

$$C_f : y^p - y = f(x),$$

where $f(x) \in \mathbb{F}_q(x)$ is a rational function, together with the automorphism $y \mapsto y+1$. Artin–Scherier covers are interesting since they provide extreme cases of the Weil bound [RLW11].

By Weil's conjectures, the zeta function of C_f can be written as

$$Z_{C_f}(u) = \frac{P_{C_f}(u)}{(1-u)(1-qu)},$$

where $P_{C_f}(u)$ is a polynomial of degree $2g = (p-1)(\Delta - 1)$ where Δ codifies the order of the poles of $f(x)$. It can be written using additive characters of $\mathbb{F}_p$ as follows

$$P_{C_f}(u) = \prod_{\psi \neq \psi_0} L(u, f, \psi),$$

where the ψ_k, $k = 0, \dots, p-1$ are the additive characters of $\mathbb{F}_p$ given by

$$\psi_k(a) = e^{2\pi i k a/p}, \quad k = 0, \dots, p-1.$$

The moduli space of Artin–Schreier covers has a stratification given by the cardinality of the p-torsion of the Jacobian. In this context, there are some particular families that have been considered in the literature: $\mathcal{AS}_{g,0}$ corresponding to the case where $f(x)$ is a polynomial; $\mathcal{AS}_{g,p-1}$ corresponding to the case where $f(x)$ is a Laurent polynomial; $\mathcal{AS}_{g,g}$ (appearing when $(p-1) \mid g$) corresponding to the ordinary locus. Another family that has been considered is $\mathcal{AS}_{g,0}^{\text{odd}}$ corresponding to the odd polynomials.

The goal of this project is to study the moments of L-functions attached to Artin–Schreier covers, which can be considered without absolute value

$$\frac{1}{|\mathcal{AS}_{g,*}|} \sum_{f \in AS_{g,*}} L(1/2, f, \psi)^k$$

and with absolute value

$$\frac{1}{|\mathcal{AS}_{g,*}|} \sum_{f \in AS_{g,*}} |L(1/2, f, \psi)|^k,$$

where $\mathcal{AS}_{g,*}$ indicates any of the families described above. A motivation for this is the following: previous studies of mesoscopic statistics of the zeros showed these families to be indistinguishable [BDFL10], while more recent studies on n-level density showed these families to have different behaviour [EP]. One expects the zeros to be modeled by certain ensembles of random matrices depending on the specific subfamily under consideration. One should be able to distinguish the random group by studying moments.

Progress

We employed a description of the L-functions in terms of multiplicative characters due to Entin [Ent12] and we were able to compute the moments

$$\frac{1}{|\mathcal{AS}_{g,0}|} \sum_{f \in \mathcal{AS}_{g,0}} L(1/2, f, \psi)^k.$$

for roughly $k < p + 1$.

Future plans

We are working on the moment

$$\frac{1}{|\mathcal{AS}_{g,0}|} \sum_{f \in \mathcal{AS}_{g,0}} |L(1/2, f, \psi)|^2$$

as well as moments for $\mathcal{AS}_{g,0}^{\text{odd}}$. We are meeting weekly on zoom to continue with the project.

Non-archimedean arithmetic dynamics

Group members

Jacqueline Anderson, Emerald Stacy, Bella Tobin

Project description

The Mandelbrot set is the set of complex numbers c for which the critical orbit (i.e., the orbit of 0) for $f_c(z) = z^2 + c$ is bounded. We'll call such a map *post-critically bounded*, or PCB. A *Misiurewicz point* is a point in the Mandelbrot set for which the orbit of 0 is strictly preperiodic. These points form a dense subset of the boundary of the Mandelbrot set. In 1989, Tan Lei proved that the Mandelbrot set zoomed in on a Misiurewicz point c and the corresponding Julia set for f_c zoomed in on c at the same scale were asymptotically similar about c. Moreover, both are asymptotically self-similar as well.

While the Mandelbrot set and its fractal boundary in $\mathbb{C}$ inspire many questions and reveal detailed dynamical information about quadratic polynomials, the analogous object over the non-Archimedean field $\mathbb{C}_p$ is simply the p-adic unit disk. However, if one generalizes the notion of PCB polynomials to higher degrees d and looks at the case where $p < d$, little is known about the corresponding Mandelbrot set, and what is known about a previously-studied one-parameter family of cubic polynomials over $\mathbb{Q}_2$ suggests it is more akin to the complex setting, with intricate boundaries and self-similarity.

In this project, we aim to develop a stronger understanding of p-adic Mandelbrot sets for degree d polynomials when $p < d$.

Progress

We selected a new cubic family of polynomials to study and worked out many of the basic dynamical properties of this family. We wrote Sage code to explore which maps in this family belong to the Mandelbrot set and which do not, and we proved these results in some cases. We gathered enough evidence to suggest that there are similar properties for this family to the ones we know from the previously-studied family (self-similarity near a Misiurewicz point, for example), but further work needs to be done to understand this family more fully.

We also developed some goals for more general theorems to try to prove regarding Misiurewicz polynomials of any degree d for any prime $p < d$, and we brainstormed some strategies for tackling the proofs.

Future plans

We are having weekly Zoom meetings to continue our work and to prepare our proceedings article. We are currently working to complete our analysis of the cubic family that we studied to better understand the patterns that appear in the Mandelbrot set, and we are also exploring tools to use to prove more general statements about critical orbit behavior of polynomials near Misiurewicz points.

Parametrizing moduli spaces of Artin-Schreier curves

Group members

Juanita Duque Rosero, Heidi Goodson, Elisa Lorenzo García, Beth Malmskog, Renate Scheidler

Project description

Elliptic curve isomorphism classes are given by the their j-invariants. In characteristic different than 2 and 3, an elliptic curve can be written as $y^2 = x^3 + ax + b$ and the j-invariant is given by $j = 1728\frac{4a^3}{16(4a^3+27b^2)}$. In characteristic 2 and 3, the formula looks much more complicated. The isomorphism classes of curves of genus 2 are given by the Igusa invariants $J_2,\ J_4,\ J_6,\ J_8$ and J_{10}. The invariant J_8 is only needed in characteristic 2.

Curves of genus 3 can be hyperelliptic or non-hyperelliptic. For hyperelliptic curves there are in general algorithms for computing their invariants in characteristic 0 by using the ”transvectant construction”. This construction also works in characteristic p for p big enough. In particular, for genus 3 it works for characteristic greater than 7. The remaining cases were treated by Basson in his thesis and some later unpublished work. In characteristic different from 2, the strategy is very similar to the one in characteristic 0, we just need to be careful with the denominators appearing in the computations. The characteristic 2 case needs a completely different approach: in this case these curves are Artin-Schreier curves.

The isomorphism classes of non-hyperelliptic curves of genus 3 are given by the Dixmier-Ohno invariants in characteristic 0. The extra needed invariants for all characeristics except for characteristic 3 are computed in work by Liu-Lercier-Lorenzo García-Ritzenthaler. The problem curves here for the standard method are the Picard curves, given by an equation $y^3 = f(x)$ with $\deg(f) = 4$ in characteristic different from 3 and Artin-Schreier curves in characteristic 3.

For every characteristic except $p = 3$, a stratification by automorphism groups of non-hyperelliptic curves of genus 3 curves is known, see the work of Ritzenthaler and others. Again the obstruction in characteristic 3 is the class of Artin-Schreier curves of genus 3.

In order to deal with these remaining cases we propose to study invariants of Artin-Schreier curves. The first ingredient needed for this is understanding the isomorphisms between Artin-Schreier curves, see Franell thesis.

The proposed project is the following:

1. Describe isomorphism between Artin-Schreier curves.
2. Give an algorithm to compute invariants for a given genus g and characteristic p.
3. Compute the invariants for small values. In particular for $(g, p) = (3, 3)$.

4. Give an algorithm to compute automorphisms of Artin-Schreier curves.
5. Compute the automorphisms for small values. In particular for $(g, p) = (3, 3)$.

Progress

We described the isomorphisms between Artin-Scheier curves, we worked out meaningful examples for the invariants computation ($p = 3$ and $g = 3$, and any p and 4 poles of equal multiplicity to 1) and we sketched a general algorithm to do so.

Future plans

We are meeting every 2 weeks on Zoom. We are very optimistic about getting a paper with all our results ready for the proceedings deadline on 15th January.

Residue fields of algebraic points on curves in a linear system

Group members

Irmak Balçik (participated remotely), Stephanie Chan, Yuan Liu (participated remotely), Bianca Viray

Project description

Let $f\colon Y \to X$ be a finite map of smooth projective geometrically integral curves (not necessarily étale) defined over a number field K, and let d be the degree of the morphism f. Given any $x \in X(K)$ outside of the ramification locus of f we obtain a degree d étale algebra over K by taking the product of residue fields

$$\mathbf{k}(Y_x) := \prod_{y \in Y, f(y)=x} \mathbf{k}(y).$$[1]

Our broad goal is to obtain information about the family of all degree d étale algebras that arise in this way in the case when $X(K)$ is infinite. In the particular case when $X = \mathbb{K}^1$, then Hilbert's irreducibility theorem states that $\mathbf{k}(Y_x)$ will be a degree d field extension for infinitely many points $x \in \mathbb{P}^1(K)$, so a particularly interesting case is to study constraints on the degree d field extensions that arise in this way.

If Y has positive genus and the Jacobian of Y has rank 0, then all but finitely many degree d extensions L/K over which $Y(L) \supsetneq \cup_{L/F/K, L\neq F} Y(F)$ arise as the fiber of such a map. Thus understanding this question gives us complete information about all but finitely many degree d points on Y.

Progress

Over the course of the WIN6 workshop, we completely answered this question for superelliptic curves over a nonarchimedean local field. Namely, if $f\colon Y \to \mathbb{P}^1$ is degree d morphism of curves over a nonarchimedean local field K where Y is given by an equation of the form $y^d = g(x)$ for a separable polynomial g and f, Y and X have good reduction, then we have determined which degree d étale algebras L/K arise from fibers of f. Moreover, we computed the Haar density of the locus in $\mathbb{P}^1(K)$ that gives rise to a specific étale algebra. In addition, we have made progress in extending these results from the case of superelliptic curves to an arbitrary degree d cover of $\mathbb{P}^1$.

Future plans:

We are currently in the process of writing up our results on superelliptic curves. In May, we plan to resume regular meetings to work out the case of arbitrary covers and determine other results that we may wish to pursue.

[1] We can also view $\mathbf{k}(Y_x)$ as the ring of global sections of the sheaf of total quotient rings of Y_x. This is the generalization of "residue field" for a variety that is not itself geometrically integral, but that is a union of geometrically integral varieties.

Large Sums of Fourier Coefficients of Cusp Forms

Group Members

Claire Frechette, Mathilde Gerbelli-Gauthier, Alia Hamieh, and Naomi Tanabe.

Project description

Let χ be a primitive character $\mod q$. An important problem in analytic number theory is establishing the asymptotic $\sum_{n\leq x}\chi(n)=o(x)$ for as wide a range of x as possible. In [GS1], Granville and Soundararajan proved that under the GRH, the character sum $\sum_{n\leq x}\chi(n)=o(x)$, as $\frac{\log x}{\log\log q}\to\infty$. They also proved that this range of x is the best possible. In [Lam], Lamzouri considered the analogous problem for large sums of the normalized Fourier coefficients of a holomorphic Hecke cuspform of weight k, as $k\to\infty$. See [Lam, Corollary 1.2, Corollary 1.4] for the statements of his results. In [GS2], Granville and Soundararajan established more concrete connections between large character sums and zeros of $L(s,\chi)$ which allowed them to establish the following result

Theorem 7.0.1. *Let $\chi \pmod q$ be a primitive character. Let ε and T be real numbers with $1\leq T\leq(\log q)^{\frac{1}{200}}$ and $\varepsilon\geq(\log q)^{-\frac{1}{3}}$. Suppose that for every real τ with $|\tau|\leq T$ the region*

$$\left\{s\,:\,\mathbb{R}(s)\geq\frac{3}{4},|\Im(s)-\tau|\leq\frac{1}{4}\right\}$$

contains no more than $\varepsilon^2(\log q)/1440$ zeroes of $L(s,\chi)$. Then for all $x\geq q^{\varepsilon}$ we have

$$\left|\sum_{n\leq x}\chi(n)\right|\ll\frac{x}{T}.$$

In this project, we aim to establish analogues of Theorem 7.0.1 for sums of Fourier coefficients of an automorphic cusp form π of large analytic conductor $C(\pi)$. We would like to prove that $\sum_{n\leq x}\lambda_\pi(n)=o(x\log x)$ in a wide range of x (perhaps $x>C(\pi)^{\varepsilon}$) under a weaker assumption than the GRH. In this work, we will assume that the family of automorphic forms π satisfies the Generalized Ramanujan Conjecture to ensure that the Fourier coefficients under consideration are bounded by a divisor function.

Progress

The work in [GS2] relies heavily on deep results on mean values of multiplicative functions such as the improved versions of Halász's theorem and the Lipschitz-type estimates established in [GS3]. Such results have been extensively studied in the last 2 decades for 1-bounded multiplicative functions by Granville, Harper, Koukoulopoulos, Matomaki, Radziwill and Soundararajan. However, it was not until recently that generalizations of such results have been pursued for multiplicative functions that are not 1-bounded. Of particular interest to us is the work of Mangerel [Man23] and [GHS]. To establish our goal, we need mean value theorems for multiplicative functions that are bounded by the divisor function. Although [Man23] and [GHS] lay out much of the foundational work in this direction, few results that are crucial to our work are not available in literature in the specific form that we need. During WIN6 workshop, our group worked on collecting and establishing all the preliminary results and lemmas needed to prove our main theorem. In order to gauge the difficulty of the problem, we focused first on Fourier coefficients of a classical cusp form f for $\mathrm{SL}_2(\mathbb{Z})$ of large weight k. Indeed, our efforts were successful, and we managed to obtain all the results needed to establish the desired upper bound for $\sum_{n\leq x}\lambda_f(n)=o(x\log x)$ under certain conditions on the distribution of zeros of the associated L-function $L(s,f)$ in the critical strip.

Future plans

Our team plans to meet biweekly on zoom during the summer. We might plan a hybrid meeting during the summer. Our initial short term goal (i.e. to be accomplished within a month) is to put together all the results that we have established thus far to finalize the proof of our main theorem in the special case of classical cusp forms that are varying in the weight aspect only. In doing so, we follow the general framework employed in [GS2]. The next goal would be to establish these results in the more general setting of cuspidal automorphic forms over $\mathrm{GL}_{\mathrm{d}}(\mathbb{A}_{\mathbb{Q}})$ that are varying in the analytic conductor aspect.

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Chapter 8

Interactions Between Topological Combinatorics and Combinatorial Commutative Algebra (23w5003)

April 2 - 7, 2023

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Overview of the Field

Starting with the pioneering work of Stanley and Hochster in the early seventies, commutative algebra methods have become an essential part of geometric and algebraic combinatorics, and more specifically face enumeration theory. The connecting language between commutative algebra and combinatorics is that of monomial ideals in polynomial rings. For instance, the Stanley–Reisner ideal of a simplicial complex is a monomial ideal generated by monomials representing the non-faces of the complex. The algebraic properties of this ideal are strongly related to the combinatorial and topological invariants of the complex in question. The Stanley–Reisner connection has been used heavily by researchers in combinatorics. In fact, some of the most beautiful results in the theory of face numbers of simplicial (and more general) complexes on one hand and graded Betti numbers of monomial ideals on the other were proved by using this language along with a subtle combination of algebraic and geometric arguments (e.g., Lefschetz elements, generic and not-so-generic initial ideals, local cohomology, rigidity, etc.).

The purpose of this workshop was to bring together two groups of researchers who work on similar problems in this research area from different points of view and often in parallel: combinatorialists who use methods of commutative algebra, and commutative algebraists who use combinatorics. One of the bridges between the two fields is homological algebra — the modern language of commutative algebra; indeed, many homological invariants become computable once tools from topological combinatorics are used. Both fields have developed their own sets of techniques, open problems, and fundamental theorems. What has become apparent over the last few decades is that a result in any of these two fields usually has a major impact on the other, transforming the long term objectives in both fields. Consequently, it is important to have to encourage ongoing communication between the two groups.

This workshop was part of the larger program to bring to allow each side to learn the language and terminology of the other. Learning the language is crucial since an obscure concept in one field often becomes much more transparent when described in the language of the other. One spectacular example is the Cohen–Macaulay property: although this property is very well-behaved in algebra, its algebraic description is sometimes quite technical compared to the description offered by topological combinatorics. In particular, Reisner's theorem [39] that provides a combinatorial characterization of Cohen–Macaulay Stanley–Reisner rings has been a cornerstone of much research in both fields, and as such, greatly illustrates the need for researchers working in these fields to become conversant in both languages.

As we describe in more detail below, our mini-workshop allowed groups of researchers from different areas to work together. Preliminary work during the week at the Juniper Hotel resulted in promising new results about unobstructed simplicial complexes, minimal Cohen-Macaulay simplicial complexes, h-vectors of doubly Cohen-Macaulay simplicial complexes, affine stresses of prime polytopes, and powers of a simplexes. A nice feature of each problem is that they can be attacked using techniques of combinatorial commutative algebra and topological combinatorics.

Recent Developments and Open Problems

The recent years has seen progress and development in a number of problems in topological combinatorics and combinatorial commutative algebra. We describe five broad areas where there has been significant progress. The topics described below are not meant to be comprehensive; however, these topics are closely related to some of the problems and talks at the workshop.

Around the g-conjecture. Applications of Stanley–Reisner rings to the theory of face enumeration is a classical topic in Stanley–Reisner theory. The recent biggest development on this topic is a solution of the algebraic g-conjecture [28, 41, 42], which states that the Stanley–Reisner ring of a triangulation of a sphere has the strong Lefschetz property with respect to a generic linear system of parameters. The solution was first announced by Adiprasito [2], and, two years later, a much shorter proof was found by Papadakis and Petrotou [38]. Moreover, Adiprasito, Papadakis, and Petrotou [3] later announced extensions of these results to a much larger class of simplicial complexes, including triangulated manifolds and doubly Cohen–Macaulay complexes. Their works are closely related to the theory of stress spaces, which is a higher-dimensional analogue of both: the framework rigidity, studied in combinatorics, and Macaulay's inverse systems, studied in commutative algebra.

There is still a number of interesting open problems related to the g-conjecture. One direction is to have more understandings on stress spaces. Papadakis and Petrotou proved that in characteristic 2, the Stanley–Reisner ring of a triangulation of a sphere has a property, which they call anisotropicity. Combinatorial or geometric meaning of anisotropicity is not fully understood, and more applications of this property is expected. Also, a recent study of Novik and Zheng on affine spaces [35, 36, 37] suggests a new connection between commutative algebra and affine stresses. Also, for some beautiful subclasses of simplicial complexes, such as *balanced simplicial complexes* and *centrally symmetric simplicial complexes*, several variations of the g-conjecture positing the strong Lefschetz property w.r.t. certain special linear systems of parameters remain open. For balanced complexes, partial affirmative results were obtained by Cook, Juhnke-Kubitzke, Nevo, and Murai [14] during a BIRS workshop. For centrally symmetric complexes, several new developments are due to Klee, Nevo, Novik and Zheng [29] as well as Novik and Zheng [35]. We expect the solution of the g-conjecture will continue to inspire further interesting research problems. Indeed, while at the workshop, one group discussed h-vectors of doubly Cohen–Macaulay complexes which is closely related to the work of Adiprasito, Papadakis, and Petrotou, and the other group delved deeper into the theory of affine stress spaces. A summary of their new works can find in a later section.

Stanley–Reisner rings and topological invariants. According to a result of Bruns and Gubeladze [11], an isomorphism of two Stanley–Reisner rings as $\mathbf{k}$-algebras implies an isomorphism of the corresponding simplicial complexes, and hence there should be a way to extract various topological invariants of a simplicial complex from its Stanley–Reisner ring. Yet, at present our understanding of the impact of classical topological invariants other than Betti numbers (over a field) on the combinatorics of triangulations is practically nil. In the last few years, Bagchi and Datta [8] proposed new invariants — σ- and μ-numbers that appear to encode more subtle topological/geometric information about the complex in question than the Betti numbers do. For instance, these μ-numbers were recently used by Murai and Novik [31] to prove an inequality on the face numbers of a triangulated manifold in terms of its fundamental group. This seems to indicate that it would be interesting to trace various topological invariants such as the fundamental group, intersection homology, characteristic classes, etc. in the Stanley–Reisner ring of the complex in question. Any progress on finding such connections would be of great significance; in fact, understanding the properties of Stanley-Reisner rings was a central theme for a number of problems discussed at the workshop (see the next sections for more details).

Buchsbaum rings and beyond. A classical way of applying the Stanley–Reisner theory to combinatorics of simplicial complexes is via the use of the Cohen–Macaulay property. Recently, several breakthroughs in topological combinatorics were made by considering the more general Buchsbaum property. For instance, using this approach, Adiprasito and Sanyal [4] solved a long-standing problem on the combinatorial complexity of Minkowski sums of polytopes. Another example is work of Novik and Swartz [33], followed by work of Murai [30], who proved a 30-year old conjecture of Kalai [27] on face numbers of triangulated manifolds. While several deep applications of this approach are known, the theory is still developing and some of its basic properties are not well understood. One interesting question in this area is "what can be said about free resolutions of Buchsbaum complexes and, in particular, homology manifolds?" For homology spheres, the resolution is symmetric by the Gorenstein property, but an analogue of this symmetry for homology manifolds is missing, and an algebraic study of such a problem would be very interesting. Another very tempting direction is to try to generalize known results on face numbers of manifolds to pseudomanifolds. It follows from results of [32, 34] that a certain quotient of an Artinian reduction of the Stanley–Reisner ring of a homology manifold is Gorenstein. An analogous Gorenstein algebra can be defined for pseudomanifolds by using Macaulay's inverse system and Adiprasito-Papadakis-Petrotou [3] recently showed the Lefschetz property for this algebra. However, how to relate this result to combinatorics and topology of pseudomanifolds is still a very big question.

Edge ideals and flag complexes. The study of flag complexes is closely related to both topological combinatorics and commutative algebra. Indeed, on the combinatorial side, it would be of great interest to understand face numbers of flag complexes — a problem that is equivalent to understanding Hilbert functions of edge ideals, which is a classical topic in commutative algebra.

There are many deep open problems on flag complexes. One of them is an unpublished conjecture of Kalai on face numbers of Cohen–Macaulay flag complexes. A recent breakthrough on this combinatorial conjecture is due to Caviglia, Constantinescu, and Varbaro [12]. They used a result of Abedelfatah [1] who gave a partial affirmative answer to the famous Eisenbud–Green–Harris conjecture in commutative algebra. Current developments in the field were also described by in the talk of de Holleben (see more). A number of proposed problems presented at the workshop were also about edge ideals. Moreover, when studying simplicial complexes, the class of edge ideals and flag complexes is a nice subclass to test conjectures and provide intuition since the corresponding monomial ideals are quite simple (i.e., generated by quadratic monomials).

A facet ideal dictionary. Over the past twenty years a substantial amount of work has been done in understanding relations between simplicial complexes and their corresponding facet or edge ideals. This area has been developing parallel to the Stanley–Reisner theory with many overlaps along the way. Resolutions of facet ideals, their Cohen–Macaulay properties, their Rees algebras and much more are constantly under investigation. While Stanley-Reisner theory relies on topological combinatorics, facet ideal theory borrows techniques from hypergraph and matroid theories. What is lacking, is a direct dictionary between the two languages. For example, Reisner's criterion of Cohen–Macaulayness does not have a clear counterpart in facet ideal theory. Researchers have been getting closer to such a characterization via describing the dual notion of linear resolutions [9, 10, 13]. Identifying gaps in the dictionary and working on translations is a key problem going forward. At the workshop, one group worked on improving our understanding of facet ideals and simplicial complexes that support the resolutions of their powers. Additional information about this group is provided below.

Structure of the Workshop

Unlike a more traditional BIRS workshop, the emphasis of our workshop was to encourage new collaborations and to give participants the opportunity to work on new problems. During the morning of the first day, numerous participants described potential research problems (the talks ranged from 5-25 minutes).

Before lunch on the first day , all participants were required to rank which project they would like to work on (some of the problems had been circulated before the start of the program). The organizers then placed the participants into groups of five or six researchers, with the aim of having a balancing of researchers from combinatorial commutative algebra and topological combinatorics. For the remainder of the week, the groups worked together for some of the mornings and most of the afternoons. The Juniper Hotel provided numerous rooms for groups to work together.

At the end of each day, each group would give a 5-10 minute progress report. On the last day, each group gave a slightly longer progress report which included their plans for moving their projects forward.

In addition to the ample time to work on projects, research talks with further questions were presented on Tuesday and Wednesday morning. Like the talks on Monday, these talks were made available to remote participants (they are also available on the BIRS website).

Presentation Highlights

As mentioned above, the focus of our workshop was to encourage new collaborations and work on new problems. Consequently, we encouraged speakers to talk about potential problems and recent developments in one of the two fields to encourage discussion about new projects. Because we wanted to leave a significant amount of time for groups to work together, we had fewer talks than a regular BIRS workshop. Regardless, the talks were well received by all participants (both local and remote) for summarizing recent developments and for promoting possible research questions. Below we highlight the topics of the six talks.

Cotangent cohomology for matroids (Alexandru Constantinescu) In this talk, Constantinescu described a problem related to the cotangent cohomolomogy module. The first cotangent cohomology module T^1 describes the first order deformations of a commutative ring. For Stanley–Reisner rings, this module has a purely combinatorial description: its multigraded components are given as the relative cohomology of some topological spaces associated to the defining simplicial complex. When the Stanley–Reisner ring is associated to a matroid, Constantinescu gave an explicit formula for the dimensions of these components. Furthermore, he showed that T^1 provides a new complete characterization for matroids. This talk was based on a joint work with William Bitsch [7]. Constantinescu lead a working group on problems related to this talk while at the workshop; for progress on this problem, see the next section.

Homological invariants of ternary graphs (Thiago de Holleben) In 2022, Jinha Kim [26] proved a conjecture by Engström [23] that states the independence complex of graphs with no induced cycle of length divisible by three is either contractible or homotopy equivalent to a sphere. These graphs are called ternary. A direct corollary is that the minimal free resolution of the edge ideal of these graphs is characteristic-free. In this talk, Thiago de Holleben showed how to apply this result to give a combinatorial description of projective dimension and depth of the edge ideals of ternary graphs. As a consequence, he was able to give a complete description of the multigraded Betti numbers of edge ideals of ternary graphs in terms of its combinatorial structure and classify ternary graphs whose independence complex is contractible.

Some results and questions in Stanley–Reisner theory motivated by commutative algebra (Hai Long Dao) In this talk, Hai Long Dao discussed several new algebraically-motivated directions in the study of simplicial complexes. They are: minimal Cohen-Macaulay complexes (which include many interesting old and new constructions in combinatorial topology), higher nerve complexes (which capture numerous algebraic invariants) and acyclicity results suggested by Kodaira vanishing. Some of these problems were inspired by Dao's papers [20, 21]. Dao lead a research group at the workshop that looked at some of these problems; the progress of this group is recorded below.

Alexander duals of symmetric simplicial complexes and Stanley–Reisner ideals (Uwe R. Nagel) It is known that any ascending chain $(I_n)_{n\in\mathbb{N}}$ of related squarefree monomial ideals, where I_n is invariant under the action of the symmetric group $Sym(n)$ on n letters, enjoys strong stabilization properties. For example, there are finitely many polynomials whose $Sym(n)$-orbits generate I_n if n is sufficiently large. In this talk Uwe Nagel discussed properties of the corresponding chain of Alexander duals $(I_n^\vee)_{n\in\mathbb{N}}$. It does not have the same stabilization properties. However, it turns out the minimal generating set of $I_n^\vee$ can be described explicitly and that the number of orbit generators is given by a polynomial in n for sufficiently large n. As an application, one obtains that, for each $i \geq 0$, the number of i-dimensional faces of the associated Stanley–Reisner complexes of I_n is also given by a polynomial in n for large n. The needed arguments include a novel combinatorial tool, which is called *avoidance up to symmetry*, and methods from discrete geometry for counting lattice points in polyhedra. This talk was based upon Uwe Nagel's joint work with Ayah Almousa, Kaitlin Bruegge, Martina Juhnke-Kubitzke and Alexandra Pevzner [5].

Garland method, its extensions and potential new applications (Volkmar Welker) In this talk Volkmar Welker introduced participants to the Garland method that comes from geometric group theory and how it could be applied to simplicial complexes Δ. For example, suppose you wish to show that $H_i(\Delta, \mathbb{Q}) = 0$ (this method only works for homology with coefficients in a field of characteristic 0). One approach to this problem is to consider the 1-skeleta (graphs) of the links of all $(i-1)$-simplices in Δ. If the smallest non-zero eigenvalue of the normalized graph Laplacian of all those 1-skeleta is $> i/(i+1)$, then $H_i(\Delta, \mathbb{Q}) = 0$. Welker then discussed his recent and ongoing work with Eric Babson which extends this method to chain complexes satisfying certain rather weak conditions. Welker met with interested participants after the talks to further discuss the Garland method.

Perfect matchings and Alexander duals (Russ Woodroofe) In this talk Russ Woodroofe first reviewed the topological view of the classical Alexander dual of a simplicial complex. He then gave an alternative construction that may be well suited for the independence complex of a graph with a perfect matching. The classical Alexander dual complex can be regarded as the Alexander duality for simplices. Woodroofe suggested to consider the Alexander dual over a cross polytope. The corresponding dual complex is not necessary a simplicial complex, but is a cubical complex since it is a subcomplex of a cube, and has an advantage that it is much smaller than the classical Alexander dual simplicial complex.

Problem Proposals (Various) In addition to the formal talks described above, a number of short presentation were given. These very short presentations described possible research projects. Presentations were given by:

- Christos A. Athanasiadis on h-vectors (this problem is described in more detail below).
- Susan Morey on powers of a simplex (this problem is described in more detail below).
- Sara Faridi on the subadditivity problem.
- Isabella Novik on affine stresses (this problem is described in more detail below).
- Vic Reiner on a question in invariant theory.
- Adam Van Tuyl on Betti splittings.
- Russ Woodroof on Alexander duality.

Although a number of potential problems were discussed, the workshop only focused on five of these problems. For the other problems that were proposed, we have elected not to give further details here since the originators of the problems way wish to pursue them as part of their own research program or give these problems to their own students.

Scientific Progress Made

On the first day of the workshop, five small research groups (5-6 participants in each group) were made to tackle some of the proposed problems. During the work at the Juniper Hotel, each group made progress on their problem. We expect that some, if not all, of these projects will result in a future publications. Below is a summary of each problem and its progress during the five days of the workshop.

Problem: Cotangent Cohomology for simplicial complexes and matroids

Group Participants:

Ayah Almousa (University of Minnesota - Twin Cities), Alexandru Constantinescu (Freie Universität Berlin),

Patricia Klein (Texas A&M University),

Thái Thành Nguyên (McMaster University),

Anurag Singh (Indian Institute of Technology Bhilai),

Lorenzo Venturello (Università di Pisa).

This group investigated the question of when a simplicial complex is unobstructed. A simplicial complex Δ on $[n]$ is *unobstructed* if $T^2(\Delta)$, the second cotangent cohomolgy module of its Stanley-Reisner ring, vanishes. The $\mathbb{Z}^n$-grading of the Stanley-Reisner ring is inherited by T^2, and, due to work of Altmann and Christophersen, understanding $T^2(\Delta)$ boils down to understanding $T^2_{-b}(\operatorname{link}_\Delta A)$ for all $b \in \{0,1\}^n$ and $A \in \Delta$. These vector spaces are described as the relative cohomology of two combinatorially defined topological spaces.

The group started by considering the lowest dimensional cases. For zero-dimensional complexes, being unobstructed is equivalent to having at most 3 vertices. This gives the first of three conditions which characterized unobstructedness in dimension one:

Theorem. *A one-dimensional simplicial complex Δ on $[n]$ is unobstructed if and only if the following three conditions hold:*

(i) Every vertex is contained in at most three edges.

(ii) Every cycle is a dominating set.

(iii) If $\{v, w\} \notin \Delta$, removing v, w and their common neighbors from Δ one gets a connected space.

These are strong restrictions and the group hopes to be able to list all the complexes that fulfill them. However, there are infinitely many unobstructed one-dimensional simplicial complexes, so the general setting in higher dimension was not pursed. In the second part of the workshop, the group focused on matroid complexes. They were able to compute the dimensions of all the multigraded components of T^2 for the uniform matroids $U^r_n = \{F \subseteq [n] \ : \ \#F \leq r\}$ with $r < n$:

$$\dim_{\mathbb{C}} T^2_{-b}(U^r_n) = \begin{cases} 0 & \text{if } \#b \neq 2, \\ r \cdot \binom{n-2}{r} - \binom{n-1}{r} + 1 & \text{if } \#b = 2. \end{cases}$$

This implies that the uniform matroid U^r_n is unobstructed if and only if $r \geq n-2$. They conjecture that if a matroid is unobstructed, then its simplification is a matroid of corank at most 2. They were able to prove this for rank two matroids, and they have some partial results for higher ranks as well.

Problem: Minimal Cohen–Macaulay Simplicial Complexes

Group Participants:

Hai Long Dao (University of Kansas),

Anton Dochtermann (Texas State University),

Jay Schweig (Oklahoma State University),

Adam Van Tuyl (McMaster University),

Russ Woodroofe (University of Primorska).

This group focussed on minimal Cohen–Macaulay complexes. Fix a field k and let Δ be a simplicial complex. We say that Δ is *minimal Cohen–Macaulay* (over k) if it is Cohen–Macaulay and removing any facet from the facet list of Δ results in a complex which is not Cohen–Macaulay. Any Cohen–Macaulay complex can be obtained from a minimal one by shelling moves, thus a systematic study of these objects seem worthwhile. This group obtained some results: Cohen–Macaulay complexes of codimension at most two are not minimal Cohen-Macaulay (unless they are simplices). This complements what we know: the smallest example of a (non-trivial) minimal Cohen–Macaulay complex over $\mathbb{R}$ is the 6 vertex triangulation of the projective plane. The group also gave broad new constructions of minimal Cohen–Macaulay complexes, and used them to show that these complexes can have free faces, might not come from triangulations of manifolds, and might not be closed under barycentric division. The gorup also investigated the general question of what 'nice' properties (e.g. Cohen–Macaulayness, shellability, vertex-decomposability, etc.) of a simplicial complex are destroyed when we remove facets, for instance from the i-skeleton of a simplex. Some interesting patterns emerged, which the group hopes to continue studying after the workshop.

Problem: h-vectors of simplicial complexes

Group Participants:

Christos Athanasiadis (National and Kapodistrian University of Athens),

Susan Cooper (University of Manitoba),

Martina Juhnke-Kubitzke (Universität Osnabrück),

Kazunori Matsuda (Kitami Institute of Technology),

Victor Reiner (University of Minnesota),

Volkmar Welker (Philipps-Universitaet Marburg).

This group worked on the problem to decide whether the inequalities

$$\frac{h_0(\Delta)}{h_d(\Delta)} \le \frac{h_1(\Delta)}{h_{d-1}(\Delta)} \le \cdots \le \frac{h_{d-1}(\Delta)}{h_1(\Delta)} \le \frac{h_d(\Delta)}{h_0(\Delta)} \tag{8.0.1}$$

hold for the h-vector $(h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$ of any $(d-1)$-dimensional doubly Cohen–Macaulay (over some field) simplicial complex Δ (see [40] for basic definitions). This problem was posed in [6], where it was motivated by questions on the real-rootedness of face polynomials of triangulations of simplicial complexes (such as barycentric subdivision). There was little evidence in favor of an affirmative answer prior to the workshop.

During the workshop, the team thoroughly discussed new interesting special cases and was able to either confirm the inequalities, or investigate the problem further and connect it to other topics within enumerative and algebraic combinatorics, or suggest possible generalizations, or provide evidence in favor of an affirmative answer. For instance, it was shown that the inequalities hold if the entries of the h-vector of Δ are replaced by those of the f-vector $(f_{-1}(\Gamma), f_0(\Gamma), \ldots, f_{d-1}(\Gamma))$ of a $(d-1)$-dimensional Cohen–Macaulay simplicial complex Γ, or with the coefficients of a polynomial of degree d having all its complex roots in the interval $[-1, 0]$, and that they are preserved when Δ is replaced by its simplicial join with a zero-dimensional complex, or with the boundary complex of a simplex. Special attention was paid to order complexes (for instance, of Boolean and subspace lattices), balanced complexes and their rank-selected subcomplexes, matroid complexes and complexes with a convex ear decomposition. These special cases connect the problem to topics such as permutation enumeration and the combinatorics of pure O-sequences and allow for natural q-analogues and equivariant versions of the inequalities to be formulated. Some first attempts to prove the inequalities for pure O-sequences (instead of h-vectors of matroid complexes), and thus to generalize the main results of [25], were made. The team intends to continue to investigate some of these aspects of the problem in the future.

Problem: Affine Stresses

Group Participants:

Sankhaneel Bisui (University of Manitoba),

Selvi Kara (University of Utah),

Satoshi Murai (Waseda University),

Uwe Nagel (University of Kentucky),

Isabella Novik (University of Washington),

Jose Samper (Pontificia Universidad Católica de Chile).

This group looked at the following problem:

Problem. *Let $P \subset \mathbb{R}^{2k}$ be a simplicial $2k$-polytope. Is it true that, if the Stanley–Reisner ideal of P has no generators of degree $\geq k+2$, then*

$$\mathrm{Socle}\big(\mathbb{R}[P]/\big((\Theta_P, \ell)\mathbb{R}[P]\big)\big)_{k-1} = 0?$$

Here $\mathbb{R}[P]$ is the Stanley–Reisner ring of P, Θ_P is the sequence of linear forms determined by the coordinates of the vertices of P, ℓ is the sum of variables, and $\mathrm{Socle}\,(-)$ *denotes the socle.*

This problem is motivated by the problem of Gil Kalai asking if the space of affine 2-stresses of a prime d-polytope determines the affine type of P. During the workshop, the group worked on some special cases of the problem. Specifically, the group answered the problem in the affirmative in the following three cases:

- Suspensions of simplicial $(2k-1)$-polytopes;
- k-stacked $2k$-polytopes;
- One-point suspensions of simplicial 3-polytopes.

The group thinks that these results provide nice evidence suggesting that the answer to the question is likely to be "yes". They have a number of interesting questions and possible directions to continue their work on the problem, such as: Can we extend the third result to polytopes obtained by taking simultaneous one-point suspensions? What happens if we take edge subdivisions? Can we extend the second result to i-stacked $2k$-polytopes with $i > k$? Can we solve the general case in dimension 4? In the short term, the group is also planning to write a code to compute the dimension of the socle in the problem and check a number of examples in dimensions 4 and 6 using this code. Some members of the group plan to continue working on the problem.

Problem: Powers of a Simplex

Group Participants:

Trung Chau (University of Utah),

Thiago de Holleben (Dalhousie University),

Art Duval (University of Texas at El Paso),

Sara Faridi (Dalhousie University),

Susan Morey (Texas State University),

Liana Şega (University of Missouri-Kansas City).

The group worked on the following problem.

Problem. *Define a topological power for a q-simplex Δ with the following properties:*

$$\Delta^2 = \mathbb{L}_q^2 \quad \Delta^r \subsetneq \mathbb{L}_q^r \quad \textit{for} \quad r \geq 3,$$

where $\mathbb{L}_q^2$ and $\mathbb{L}_q^r$ are simplicial complexes defined in [19, 16].

The end goal is to have a topologically meaningful definition of Δ^r which supports a minimal free resolution of I^r for some monomial ideal I with q generators (I would be the q-extremal ideal defined in [16]) and for any square-free monomial ideal J with q generators

$$\beta_i(J^r) \leq f_i(\Delta^r)$$

where f_i is the number of i-dimensional faces of a complex.

The group happened to include three people who had previous experience with this specific problem, and could therefore break into subgroups pursuing different angles of approach using the expertise of the newly added members.

One direction was a "bottom to top" approach, which consists of describing products of edges in a way that it is consistent with products of triangles, and then the two would be consistent with products of tetrahedra and so on. The group focused on variations of the well-known Minkowski sum of polyhedra, Cartesion products of simplices and similar products, and moved forward adjusting these definitions to build powers with the required properties.

Another approach was a "top to bottom" approach which starts from a very large simplex (representing the Taylor simplex of a power of an extremal ideal) and deletes faces from it via Morse matchings. This was along the lines of the approach taken in [22], and the team in Banff tried to push it further, with limited success.

The team also found a concrete description of the powers of a simplex as an intersection of hyperplanes in coordinates labeled with the faces of the simplex. This new description will be explored further in subsequent work of the group.

Finally, some team members dedicated extra time to programming in Macaulay2 [24] in order to find a concrete description of the expected faces of the powers of the simplex.

This team has scheduled weekly virtual meetings to continue their work.

Outcome of the Meeting

Overall, it was felt by both the organizers and participants that the workshop was very successful. First, in terms of new research, it is expected that many of the projects that started at our workshop will eventually result in future publications. Indeed, as apparent from the last section, each group made a significant start on their proposed research problem. At the end of the workshop, each group made a plan on how to build upon the momentum of the workshop.

But secondly, and probably more importantly, it was felt that the workshop was very successful in building new bridges and collaborations between the two research groups. Each group had at least one researcher who had never worked with any of the other members on previous projects. We expect these new collaborations will bear fruit over the years to come.

It is also interesting to highlight that for more than half of the participants, this was their first visit to a BIRS workshop in Banff. All participants were very impressed with the facilities and the professionalism of the BIRS staff. As organizers, we heard from many of the participants that this was one of the best workshops that they had attended.

We would like to conclude by thanking the staff of BIRS for their help organizing this conference (a special recognition to Connor and Jake who helped with local details). We would also like to thank the staff at the Juniper Hotel for a wonderful and productive stay.

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Chapter 9

Compensated Compactness and Applications to Materials (23w5018)

April 2 - 7, 2023

Organizer(s): Jean-Francois Babadjian (Université Paris Saclay), Flaviana Iurlano (CNRS and Sorbonne Université), Filip Rindler (University of Warwick)

Overview of the Aims of the Workshop

The workshop "Compensated Compactness and Applications to Materials" brought together experts from the theory of compensated compactness with researchers in material science. The workshop focused on the recent progress in the compensated compactness theory involving measures or L^1-maps. These developments have enabled many new results, for instance in the theories of microstructure, shape optimization, dislocation theory, fracture, and plasticity. The workshop aimed to balance theoretical and more applied talks to give the participants the opportunity to exchange the latest ideas, learn new methods, and start new collaborations.

Over the last years, much progress was made in the quest to understand the structure of L^1-maps or singular measures solving linear partial differential constraints. Deep new theorems have been discovered as a result of novel interactions within the pure mathematics community between PDE theory, geometric measure theory and harmonic analysis. The potential of this approach is far-reaching and has already led to the resolution of several long-standing conjectures as well as opened up new avenues to understand the fine structure of singularities.

The main reason for the added difficulty in the L^1 or measure context is that in addition to dealing with fast oscillation, concentrations also have to be considered. In particular, for vector-valued maps or measures the interaction between oscillations and concentrations causes a whole host of new phenomena. This leads to relaxation results of underlying energies whose explicit expression is very difficult to obtain.

This avenue of research has turned out to be intimately connected to the comprehension of the convergence properties of nonlinear quantities under weak convergence of maps, leading in the 1970s to the notion of polyconvexity by John Ball in [19] and to compensated compactness theory by François Murat and Luc Tartar [49, 46, 45, 47]. However, only after the recent progress has it become possible to take on problems which require a similar theory for "rough" objects like L^1-maps and measures.

The theoretical questions at the core of the emerging field of "linear-growth compensated compactness" are very challenging. Since they promise new insights into the formation of singularities in merely L^1-bounded sequences of maps, some of these questions are being pursued for their own sake. On the other hand, it has become clear by now that the potential impact on more "applied" questions is even more intriguing than the abstract work.

The main objective of the workshop was to bring together specialists of both theoretical and applied communities working on questions related to compensated compactness theory for "rough" objects like L^1-maps and measures. There were three main goals: (a) To take stock of and to disseminate the available results showing an interaction between L^1-bounds, linear partial differential constraints, and compensated compactness, (b) To define questions originating from plasticity, dislocation theory, and shape optimization which require a theory for "rough" objects, and (c) To survey applications of the theory to microstructure, shape optimization, dislocation theory, fracture, and plasticity. The workshop facilitated new collaborations and exchange of ideas.

Presentations

Altogether, 24 leading experts in the field as well as early-career researchers presented research talks. The range of topics was very wide, although accessible for all the audience.

- *Geometric measure theory:* Connor Mooney focused on the Bernstein problem, a reformulation of the problem of minimal surfaces; Giovanni Alberti and Adolfo Arroyo-Rabasa were interested in the development of measure theoretic tools to understand the interaction between oscillation and concentration phenomena, in connection with the vanishing mass conjecture introduced by Guy Bouchitté about 20 years ago in [21].
- *(Multi)scale asymptotic analysis:* Irene Fonseca discussed the effect of oscillating wells in the context of phase transition, Annika Bach presented works related to the interplay between homogenization and phase field models of fracture, and Caterina Zeppieri discussed about the homogenization of variational integrals involving randomly distributed holes. Elisa Davoli presented new results related to the (non) stability through linearization of the noninterpenetration of matter constraint in fractured materials.
- *Nonlinear elasticity*: Duvan Henao discussed the formation of cavitation in hyperelastic media, André Guerra presented new results related to the connections between rank-one convexity and quasi-convexity, while Carolin Kreisbeck extended lower semicontinuity results of quasiconvex integrals to the case of nonlocal energies. Ian Tobasco presented curvature dependent elastic energies to address wrinkling and folding of confined elastic sheets.
- *Plasticity*: Amit Acharya presented a dual variational formulation of dislocation in connection to equations of fluid dynamics, Tom Hudson proposed a model which explains dislocation motions in a nonlinear setting, and Adriana Garroni presented results which explain how a 3D variational model involving incompatible fields can lead to a line tension dislocation model in the mesoscale limit. Elise Bonhomme rather focussed on a macroscale formulation and presented results showing that the presence of irreversibility in a quasistatic evolution setting can lead to non stability phenomenon of Γ-converging damage models.
- *Shape optimization*: Heiner Olberman presented new results on the shape optimization of Willmore type energies under an area constraint expressed through a penalizing Lagrange multiplier. Giuseppe Buttazzo studied shape optimization problems involving the torsional rigidity and the eigenvalue of the Laplace operator and their connection through the Blaschke-Santaló diagram. Mariapia Palombaro constructed optimal microstructures of a 3D polycrystalline material, where the degree of freedom is given by the orientation of the crystal.
- *Optimal transportation*: Guy Bouchitté presented new results about short-range interaction functionals for models of many interacting particles such as in crowd motion.
- *Hyperbolic systems*: Paolo Bonicatto presented a unified measure theoretic framework to address the well-posedness of transport type equations with a Lipschitz vector field where the unknown is a lower dimensional object. Andrew Lorent presented how the use of entropies can lead to rigidity results in the context of the Aviles-Giga functional and differential inclusions. Benoît Merlet spoke about nonoriented Aviles-Giga and rigidity results for small energy configurations. Michaël Goldman discussed about the structure of defect measures associated to energies penalizing oscillations in oblique directions. Finally, Gilles Francfort presented results related to uniqueness issues for non strictly convex linear growth variational problems by the use of conservation laws associated to the Euler-Lagrange equations.

First Day

The first day of the workshop began with Irene Fonseca delivering a talk on ”Phase Separation in Heterogeneous Media.” She discussed how certain modern technologies and biological systems utilize natural heterogeneities of the medium or engineered inclusions to produce composite materials with specific physical properties. She also mentioned that to model such situations, a variational approach based on the gradient theory should be used. In addition, Fonseca explained how different regimes should be considered when the potential and the wells depend on the spatial position, even in a discontinuous way. This was joint work with Riccardo Cristoferi and Likhit Ganedi, based on previous results also obtained with Adrian Hagerty and Cristina Popovici (see [31, 28, 25, 30, 29]).

Amit Acharya’s talk titled ”Field Dislocation Mechanics, Ideal MHD, and a Dual Variational Formulation” reviewed the fully nonlinear system of Field Dislocation Mechanics to establish an exact analogy with the equations of ideal magnetohydrodynamics (ideal MHD) under suitable physically simplifying circumstances. He talked about how weak solutions with various conservation properties have been established for ideal MHD recently by Faraco, Lindberg, and Szekelyhidi (2021) using the techniques of compensated compactness and convex integration. By the established analogy, these results would seem to transfer directly to the idealization of Field Dislocation Mechanics considered. A dual variational principle was also demonstrated for this system of PDE (see [2, 3, 4, 5, 6, 1, 11, 10, 9]).

Elise Bonhomme delivered a talk entitled “Can quasi-static evolutions of perfect plasticity be derived from brittle damage evolutions?”. She addressed the question of the interplay between relaxation and irreversibility through evolution processes in damage mechanics. Bonhomme inquired whether the quasi-static evolution of an elastic material undergoing a process of plastic deformation can be derived as the limit model of a sequence of quasi-static brittle damage evolutions. She talked about how the interplay between relaxation and irreversibility of the damage is not stable through time evolutions and how the obtained effective quasi-static damage evolution may not be of perfect plasticity type, in contrast with the static case considered in [14].

Duvan Henao delivered a talk titled ”Harmonic dipoles in elasticity.” He discussed how whenever the stored energy density of a hyperelastic material has slow growth at infinity, it may undergo cavitation under large hydrostatic tension, constituting a failure of quasiconvexity and a challenge for the existence theory in elastostatics. Henao also talked about how this obstacle has been overcome under certain coercivity hypotheses, which, however, fail to be satisfied by the paradigmatic example in elasticity, that of 3D neo-Hookean materials. He presented a joint work with Marco Barchiesi, Carlos Mora-Corral, and Rémy Rodiac, where this borderline case was solved for hollow axisymmetric domains.

Andrew Lorent delivered a talk on ”Differential inclusions, entropies, and the Aviles Giga functional.” He outlined some elementary questions and theorems about differential inclusions. Lorent also talked about the concept of entropies from scalar conservation laws and the adaptation of this concept to the Aviles Giga functional. He showed how these topics are connected and how the connection has application to both differential inclusions and to the Aviles Giga functional (see [41, 40]).

To close this first day workshop, Caterina Zeppieri discussed the homogenization of vectorial integral functionals with q growth in a bounded domain of $\mathbb{R}^n$, $n > q > 1$, which is perforated by a random number of small spherical holes with random radii and centres. The goal was to show that for a class of stationary short-range correlated processes for the centres and radii of the holes, in the homogenized limit, we obtain a nonlinear averaged analogue of the ”strange term” obtained by Cioranescu and Murat in 1982 (see [27, 26]), in the periodic case. The speaker demonstrated that the clustering holes do not have any impact on the homogenization procedure and the limit functional, despite the fact that there are holes which overlap with probability one. This was achieved by requiring only that the random radii have finite $n - q$-moment, which is the minimal assumption to ensure that the expectation of the nonlinear capacity of the balls is finite. The talk was based on joint work with K. Zemas and L. Scardia.

Second Day

On the second day of the workshop, Elisa Davoli presented her work [8] on the passage from nonlinear to linearized Griffith-fracture theories under non-interpenetration constraints. She characterized sequences of deformations satisfying a Ciarlet-Necas condition in SBV^2 and showed that they admit asymptotic representations in $GSBD^2$

satisfying a suitable contact condition. With an explicit counterexample, she proved that this result fails if convergence of the energies does not hold. She further proved that each limiting displacement satisfying the contact condition can be approximated by an energy-convergent sequence of deformations fulfilling a Ciarlet-Neç as condition. This was joint work with Stefano Almi (Naples) and Manuel Friedrich (Erlangen).

Michael Goldman presented his results on a family of energies penalizing oscillations in oblique directions. These functionals control second order derivatives rather than first order ones and actually have mixed (or oblique) derivatives given by bounded measures. The main focus of his talk was the study of the rectifiability properties of these "defect" measures. He also drew connections with branched transportation, PDE constrained measures, and Aviles-Giga type differential inclusions (see [35, 33, 34]).

Adolfo Arroyo-Rabasa talked about how PDE constraints interact with the formation of mass concentrations (which are necessary for the formation of measure singularities). He discussed how the formation of "strong mass concentrations" along a sequence of PDE-constrained functions is "fully unconstrained" as long as the expectation of its values belongs to the wave cone associated with the PDE. He also explained the "gluing technique" behind the proof as well as some interesting applications.

André Guerra presented his work on quasiconvexity and nonlinear elasticity. Quasiconvexity is the fundamental existence condition for variational problems. He showed that rank-one convexity, a simple necessary condition, implies quasiconvexity in two dimensions in a special class of isotropic energies. He also proved existence theorems for quasiconvex energies in the context of nonlinear elasticity. His proof combined complex analysis with the theory of gradient Young measures. On the way to the main result, he established quasiconvexity inequalities for the Burkholder function, which yielded many sharp higher integrability results. This was joint work with Kari Astala, Daniel Faraco, Aleksis Koski, and Jan Kristensen (see [37, 12]).

Heiner Olbermann discussed the Willmore functional on graphs with an additional penalization of the area where the curvature is non-zero. Sending the penalization parameter to infinity and rescaling suitably, he derived the limit functional in the sense of Γ-convergence.

Finally, Giovanni Alberti presented the "Vanishing Mass Conjecture," which G. Bouchitté formulated about twenty years ago in the context of optimization of light elastic structures (see [15]). He illustrated this conjecture placing the emphasis on its geometric nature and some partial results obtained with Andrea Marchese (University of Trento) and Andrea Merlo (University of the Basque Country).

Third Day

Giuseppe Buttazzo gave a talk on the representation of Blaschke-Santaló diagrams, an important problem that arises in shape optimization. He discussed the case of torsional rigidity and the first eigenvalue of the Laplace operator, although other cases have been considered in the literature. From a numerical perspective, this involves representing the image of a given map $F : X \to \mathbb{R}^k$, where X is a compact metric space and $k = 2$. He also mentioned the interesting case when X is a subset of an Euclidean space $\mathbb{R}^d$ (with d much larger than k), and the suitable use of Voronoi tessellations plays an important role. His last research was in collaboration with Benjamin Bogosel and Edouard Oudet (see [24, 23, 22, 20]).

Benoit Merlet presented a non-oriented version of the Aviles-Giga functional, which serves as a model for pattern formation, particularly striped patterns in 2D. He showed that sequences with uniformly bounded energy as the scale parameter goes to 0 are relatively compact in L^1_{loc}. He also described the limit configurations in the vanishing energy limit case. These results are similar to their counterparts for the classical Aviles-Giga functional, but new phenomena appear in the non-oriented case, and the proofs require new ideas. The work was done in collaboration with Michaël Goldman, Marc Pegon, and Sylvia Serfaty (see [36]).

Carolin Kreisbeck discussed a class of variational problems with integral functionals involving nonlocal gradients, motivated by new nonlocal models in hyperelasticity. She addressed several aspects of the existence theory of these problems and their asymptotic behavior. Her analysis relied on suitable translation operators that allowed her to switch between classical, fractional, and nonlocal gradients. She showed that quasiconvexity characterizes weak lower semicontinuity in the fractional and nonlocal setting, and derived relaxation and homogenization results from a general Γ-convergence statement. The limiting behavior as the fractional order tends to 1 yields localization to a classical model. Her work was done jointly with Javier Cueto and Hidde Schönberger (see [32, 39]).

Ian Tobasco discussed the wrinkling and folding of confined elastic sheets. He presented recent progress towards an effective variational theory for wrinkles and folds. In the first part of his talk, he discussed the wrinkling of curved shallow shells that float on top of a flat water bath. He derived and solved the Γ-limit of a rescaled shallow shell model, which explained the patterns that arise in a given floating shell. He found that the wrinkles of positively and negatively curved shells are linked. In the second part of his talk, he discussed recent work on folds. He used a fully nonlinear model for a plate confined to a planar cavity and laterally squeezed and proved a scaling law and compactness result involving a BV-type energy that arose to control the length of the folds. The limiting mid-plane was shown to deform by a length-preserving map that can change orientation across a singular set containing the folds. His work was done in collaboration with Eleni Katifori, Joey Paulsen, and Samuel Wallace (see [50, 51]).

Fourth Day

On the fourth day of the workshop, Paolo Bonicatto gave a talk entitled "Existence and uniqueness for the Lipschitz transport of normal currents". He discussed efforts to extend classical theory to the case where the unknown is k-currents in $\mathbb{R}^d$, or generalized k-dimensional surfaces, which involve the Lie derivative L_b of currents in direction b and read $\partial_t T_t + L_b T_t = 0$. Bonicatto proved the existence and uniqueness of the equation in the class of normal currents under the natural assumption of Lipschitz regularity of the vector field b. He also talked about the notion of decomposability bundle introduced recently by Alberti and Marchese and how it is crucial to the argument.

Annika Bach presented a talk titled "Γ-convergence and homogenisation of singularly-perturbed elliptic functionals". She studied the asymptotic behavior of a general class of singularly-perturbed elliptic functionals of Ambrosio-Tortorelli type as the perturbation parameter vanishes. Bach showed that the functionals Γ-converge (up to subsequences) to a free-discontinuity functional of brittle type, under mild regularity assumptions and suitable super-linear growth conditions on the integrands. She also provided asymptotic formulas for the limiting volume and surface integrands, which showed that the volume and surface contributions decouple in the limit. Bach's work is based on joint works with T. Esposito, R. Marziani, and C. I. Zeppieri (see [18, 16]).

Tom Hudson gave a talk titled "Elastoplastic evolution of single crystals driven by dislocation flow". He discussed how dislocation motion is a key feature of crystal plasticity at the smallest scales, and how many mathematical challenges must be overcome to establish a well-posed theory that accurately couples dislocation motion and continuum theories in a three-dimensional nonlinear setting. Hudson presented joint work with Filip Rindler [38] on a model that proposes a novel geometric language built on the concepts of space-time currents, or "slip trajectories" and the "crystal scaffold" to describe the movement of discrete dislocations. The model recovers several laws that were known in special cases before, such as the equation for the Peach-Koehler force on a dislocation in a linearized context.

Connor Mooney presented a talk titled "The anisotropic Bernstein problem". He discussed how the Bernstein problem asks whether entire minimal graphs in $\mathbb{R}^{n+1}$ are necessarily hyperplanes, and how the answer is positive if and only if $n < 8$. The anisotropic Bernstein problem asks the same question about minimizers of parametric elliptic functionals, which are natural generalizations of the area functional that both arise in material science and offer important technical challenges. Mooney discussed the recent solution of this problem (the answer is positive if and only if $n < 4$), based on joint work with Y. Yang (see [42, 44, 43]).

Mariapia Palombaro gave a talk titled "Optimal microstructures using infinitely many rotations". She discussed how composite materials display a wide range of conduction properties depending on the geometric configuration of the phases, and how a classical problem is to find the range of the effective conductivity, a constant but in general anisotropic matrix describing the overall electrical behavior of the composite. Palombaro presented optimal bounds for isotropic mixtures and provided new anisotropic optimal microgeometries in the case of a three-dimensional polycrystalline material, where the principal conductivities of the basic crystal are given, but the orientation of the crystal is allowed to change from point to point (see [7]).

Guy Bouchitté conducted research on a class of short-range interaction functionals for a model of many particles interacting in $\mathbb{R}^d$, such as in Density Functional Theory or crowd motion. The energy cost of the particles was usually considered to be repulsive and described by a two-point function, $c_\varepsilon(x, y) = \ell(\frac{|x-y|}{\varepsilon})$, where $\ell : \mathbb{R}_+ \to [0, \infty]$ was decreasing to zero at infinity, and the small parameter $\varepsilon > 0$ scaled the interaction distance. Bouchitté reviewed the link between this model and multimarginal optimal transport before focusing on new results obtained in collaboration with R. Mahadevan from the University of Concepcion, Chile. The study identified the mean-field energy of the model in the short-range regime $\varepsilon \ll 1$, assuming only that $\int_{r_0}^{\infty} \ell(r) r^{d-1}\, dr < +\infty$. This extends and simplifies existing results in the homogeneous case $\ell(r) = r^{-s}$ where $s > d$.

Fifth Day

On the last day of the conference, two talks were presented. In the first talk titled "Hyperbolicity as a possible path to the uniqueness of minimizers for energies with linear growth," Gilles Francfort presented a work in collaboration with Jean-François Babadjian [13] where a functional of the gradient that exhibited linear growth at infinity is investigated. The relaxed functional has BV minimizers. In 2D, their uniqueness was tied to the properties of a spatial continuity equation, for which regular Lagrangian flows did not apply. However, techniques related to the work of Jabin-Otto-Perthame on 2D Ginzburg-Landau models allowed a better understanding of the associated characteristic flow from which uniqueness followed.

In the last talk of the workshop, Adriana Garroni spoke about "Three-dimensional line-tension limits for line singularities and applications". She presented a derivation of a line tension model for dislocations in 3D, starting from a variational model that accounted for the elastic energy induced by incompatible elastic fields. Under a kinematic constraint that forced the dislocations to be diluted on a mesoscopic scale, via Γ-convergence, they deduced energies concentrated on 1-rectifiable lines which could be interpreted as line tension energies for dislocations in a single crystal. The result was based on a recent paper in collaboration with S. Conti and R. Marziani. They treated a quite general framework that included several different regularized variational models present in the literature, ranging from linear elastic energies with core regularization to non-linear elastic energies with sub-quadratic regularization.

Outcome of the Meeting

The final structure of the workshop was a focussed gathering of 27 participants (12 belonging to minorities), among which 8 were online. This small size made it easier to meet all the physically present participants and start mathematical discussions. It was particularly helpful for young researchers to easily meet senior researchers. Out of the 20 physically present participants (10 belonging to minorities), 1 talk was given by a graduate student (Elise Bonhomme), 5 by postdoctoral fellows or assistant professors (Adolfo Arroyo-Rabasa, Annika Bach, Paolo Bonicatto, André Guerra, Ian Tobasco) and 5 by young faculty (Carolin Kreisbeck, Andrew Lorent, Heiner Olbermann, Mariapia Palombaro, Caterina Zeppieri). These were complemented by talks from worldwide experts in different areas of calculus of variations and partial differential equations including Amit Acharya (dislocation mechanics), Giovanni Alberti (geometric measure theory), Guy Bouchitté (shape optimization, optimal transport), Irene Fonseca (Γ-convergence, epitaxial growth), Gilles Francfort (fracture mechanics, plasticity), Adriana Garroni (dislocations).

The presence of so many young participants and the wonderful environment of the Banff International Research Center contributed to a very informal, friendly, and unique atmosphere. We kept the talks to 40 minutes which gave plenty of time for informal discussions and networking. New friendships were formed and collaborations were initiated during the meeting. From the feedback we received, the workshop was very successful and the participants enjoyed the meeting and specifically the pleasant atmosphere of the workshop.

Comments

We attach below some feedbacks we got from several participants:

Acharya: *"I had very nice interactions with many of the participants: Rindler, Alberti, Bonicatto, Bouchitté, Arroyo-Rabassa, and various others. This was a great workshop which I enjoyed very much."*

Alberti: *"I discussed at length with the following people: Amit Acharya (grain boundaries, justifications of the Read–Shockley formula) Adolfo Arroyo-Rabasa (his work related to concentration Young Measures) Paolo Bonicatto (strong locality property of 1st order linear differential operators). I also have brief conversations with other people, including Adriana Garroni e Ian Tobasco. The only real collaboration is with Paolo Bonicatto. I* *have no comment to add, besides the fact that I really enjoyed the workshop, and specifically the atmosphere of the workshop."*

Bach: *"The workshop in Banff was a natural possibility to discuss a bit with Caterina Zeppieri about possible future research questions related to the projects that we have been working on so far. Besides that I am particularly happy that I had the possibility to interact with Carolin Kreisbeck, who some years ago followed the same career path as I am about to do now. It was not only very helpful to learn about her experiences with a tenure-track position in the Netherlands coming from Germany, but we also found the time for a very fruitful discussion about possible future collaborations."*

Fonseca: *"I had very interesting discussions with Gilles Francfort and Guy Bouchitté. The latter is visiting me at CMU for two weeks following the Banff workshop, and we are continuing the project that we initiated while at Banff. This collaboration would have likely not happened if it wasn't for the fact that during one week at BIRS we matured the ideas that we are now pursuing with Leonard Kreutz (postdoc at CMU)."*

Guerra: *"I also had fruitful interactions with several researches in Banff. I would highlight in particular very interesting discussions with Gilles Francfort, Ian Tobasco and Paolo Bonicatto; I initiated new collaborations with the last two researchers. Overall the conference atmosphere was pleasant and prone to interesting mathematical discussions. The relatively small size of the workshop made it easier to meet everyone quickly and to discuss with those with research interests closer to mine."*

Kreisbeck: *"The interactions with the other participants were inspiring and most valuable. Particularly, the week at BIRS has opened up the opportunity to explore a future collaboration with Ian Tobasco on the analysis of high-contrast checkerboard composites and kirigami-type structures. An exchange between our groups is already in planning."*

Lorent: *"I enjoyed it very much and learned a lot. In particular, I valued meeting (or getting to know better) younger mathematicians in the field and reconnecting with old friends and colleagues."*

Palombaro: *"During the workshop I had fruitful discussions about G-closure and quasiconvexity/polyconvexity problems with André Guerra, Gilles Francfort, Carolin Kreisbeck and Caterina Zeppieri, who have worked on closely related problems."*

Zeppieri: *"I had some exchanges with Mariapia Palombaro on certain G-closure problems and optimal bounds."*

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Chapter 10

Interactions between Symplectic and Holomorphic Convexity in 4-dimensions (23w5123)

April 9 – 14, 2023

Organizer(s): Thomas E. Mark (University of Virginia), Laura Starkston (UC Davis), Bülent Tosun (University of Alabama)

Overview of the Field

Convexity and convex function theory are classical topics in geometry and analysis, with variations of these notions appearing in a wide assortment of sub-fields. In complex geometry a vital notion of convexity is holomorphic convexity or, by the solution to the Levi problem, *pseudoconvexity*, while symplectic topology includes the key notion of *symplectic convexity*. It has been known for a long time (since work of Weinstein, Bishop, Eliashberg, Gompf and many others) that these ideas are closely linked, and the interplay has led to fascinating developments over the past decades.

In the last few years, however, there have been new indications of subtle interactions between symplectic and complex convexity, particularly in real dimension 4. Notably, new techniques from the theory of (pseudo-) holomorphic curves, Floer homology and gauge theory have led to fascinating and surprising examples and counterexamples in complex geometry, and ideas and results from complex geometry seem poised to provide new directions for study in symplectic topology. The main goal of the workshop was to introduce researchers on each side—symplectic and smooth low-dimensional topology on one hand, and complex geometry and function theory on the other—to the techniques, ideas, and questions on the other, in hopes of sparking collaborations and new insights as well as attracting new mathematicians to this fascinating field.

Recent Developments and Open Problems

The following is a sample of some of the problems and themes that were highlighted at the workshop.

Convexity and domains in $\mathbb{C}^2$

Compact domains in affine space, particularly domains of holomorphy, are a well-studied and very active area of research in complex geometry; it is understood that domains of holomorphy are the same as *pseudoconvex* domains.

From the point of view of smooth topology, a fundamental open problem is the question of which compact 3-dimensional manifolds admit embeddings in $\mathbb{R}^4$, which amounts to asking whether they arise as boundaries of *smooth* compact domains. Recent research suggests that much progress can be made by combining the points of view, i.e. asking for embeddings of 3-manifolds that satisfy a convexity condition:

- Using techniques from symplectic geometry, Nemirovsky and Siegel [10] gave the first examples of domains in $\mathbb{C}^2$ that are isotopic to pseudoconvex domains, but not to rationally convex domains. Their work uses the close relationship between rational convexity and symplectic convexity: it was shown by Cieliebak-Eliashberg that [2, 3] *rational* convexity, naturally part of holomorphic geometry, is essentially equivalent to the simultaneous requirements of *pseudoconvexity* and *symplectic* convexity.

- Taking different approach, using tools from Floer homology and gauge theory combined with 3-dimensional contact topology, Mark and Tosun [9] recently showed that no 3-dimensional Brieskorn homology sphere can be the boundary of any symplectically convex domain in $\mathbb{C}^2$ (in particular, no rationally convex domain). A Brieskorn sphere is the link of a certain natural complex surface singularity; many of them admit smooth embeddings in $\mathbb{C}^2$ but the full answer to this problem in the smooth category is unknown.

This result concerning Brieskorn spheres mentioned above was proved using techniques from low-dimensional topology and gauge theory (Floer homology), and these methods also give various examples of 3-manifolds that bound pseudoconvex domains but not symplectically convex ones. On the other hand, while Nemirovsky-Siegel's examples of similar phenomena involve somewhat more direct symplectic- and complex-geometric methods, their examples are not accessible to the gauge-theoretic techniques: this already suggests some of the benefits that may be derived from the sort of interactions this workshop was designed to encourage.

The topology of contact type hypersurfaces

A "contact type hypersurface" in $\mathbb{R}^4$ (or $\mathbb{C}^2$) is essentially the same thing as the boundary of a symplectically convex domain. By the work mentioned above, if W is a rationally convex domain in $\mathbb{C}^2$, then its boundary is a contact type hypersurface. Conversely, if it is known for some 3-manifold $Y \subset \mathbb{R}^4$ that Y cannot be made into a contact type hypersurface, then no domain bounded by Y can be rationally (or even symplectically) convex. Contact type hypersurfaces were some of the first examples of contact manifolds for which Weinstein's conjecture concerning periodic orbits of the Reeb vector field was proved.

While many examples exist in higher dimensions, there are few explicit constructions of contact type hypersurfaces in $\mathbb{R}^4$. In fact, both Nemirovsky-Siegel's results and the result concerning Brieskorn spheres can be taken as evidence for the following, appearing in the work of W. Chen [1].
Conjecture: the only contact type hypersurfaces in $\mathbb{R}^4$ having vanishing first Betti number are diffeomorphic to the 3-sphere.

The topology of polynomially convex domains

There are a very limited number of examples of polynomially convex domains in $\mathbb{C}^2$. In fact, Cieliebak and Eliashberg [3] pose the following problem.
Conjecture: A simply connected polynomially convex domain in $\mathbb{C}^n$, $n > 2$, must be *subcritical*, meaning it admits a defining Morse function whose critical points all have index less than n. When $n = 2$, an analogous question would ask whether a polynomially convex domain must be a 1-handlebody.

While it concerns a strictly complex-geometric property (polynomial convexity), this conjecture seems closely tied to subtle issues in symplectic topology: notably the poorly-understood distinction between Liouville and Weinstein symplectic cobordisms. One expects that progress on this problem [4, 5], from either the complex-geometric or symplectic-topological side, will shed light on the other subject.

The topology of Stein boundaries

Many more examples are known, thanks in large part to work of Gompf, of 3-manifolds bounding pseudoconvex domains in $\mathbb{C}^2$, or equivalently Stein domains. In particular, Gompf [7] gave examples of 3-manifolds with vanishing first Betti number that bound Stein domains—even contractible Stein domains. Moreover, Gompf used his

techniques to exhibit a Stein domain in $\mathbb{C}^2$ having the homotopy type of the 2-sphere, disproving a conjecture in complex geometry. However, it is still poorly understood which 3-manifolds can arise as the boundary of pseudoconvex domains in $\mathbb{C}^2$ (even among restricted classes such as Brieskorn spheres). We hope that the sharing of expertise engendered by this workshop will spark progress on this question, particularly the following fascinating conjecture due to Gompf.
Conjecture: no Brieskorn sphere bounds a pseudoconvex domain in $\mathbb{C}^2$.

Convexity in other manifolds

Both symplectic convexity and pseudoconvexity are conditions that can be considered in more general spaces than $\mathbb{C}^n$. A natural next step is to consider domains in complex projective space, where several intriguing avenues are available.

Contact type hypersurfaces in $\mathbb{C}P^2$

Work of Evans and Smith shows that the family of lens spaces $L(p, q)$ that can be found as contact type hypersurfaces in the projective plane is quite constrained: for example, p must satisfy a certain Diophantine equation, and no more than three lens spaces may be embedded disjointly in this way. In fact, Evans-Smith [6] show that their criteria completely determine the lens spaces that arise as contact type hypersurfaces. The results, and particularly the techniques used (which involve pseudoholomorphic curves in orbifolds), are very suggestive of potential further progress. In one possible direction, smooth embeddings of lens spaces and other Seifert 3-manifolds in (orbifold) projective planes is closely related to the longstanding "Montgomery-Yang problem" of classifying smooth circle actions on the 5-sphere: this problem in smooth topology has resisted progress for decades.

Convexity and smooth topology

In some circumstances, a resolution of a problem involving complex or symplectic convexity may actually yield the resolution of a question in *smooth* topology. For example, Weimin Chen observed that obstructing a given 3-manifold from arising as a contact type hypersurface in $\mathbb{C}P^2$ is one step in a possible approach to constructing a smooth 4-manifold homeomorphic but not diffeomorphic to $\mathbb{C}P^2$. The existence, or not, of such an example is one of the major open questions in 4-dimensional topology, and it is natural to hope that Chen's techniques (and those of Evans-Smith above) may have adaptations or extensions that could lead to progress in this direction.

Perhaps the most important outstanding question in low-dimensional topology is the last remaining case of the Poincaré conjecture: that the 4-dimensional sphere has a unique differentiable structure. By the result of Eliashberg that the 3-sphere has a unique Stein filling, this has a reformulation in terms of pseudoconvexity: it is equivalent to the assertion that any compact contractible 4-manifold having boundary diffeomorphic to the 3-sphere admits the structure of a Stein domain [8].

On the other hand, this uniqueness vanishes if one considers open manifolds. Indeed Gompf proves that every open pseudoconvex convex domain contains an uncountable family of other such pseudoconvex domains, all of which are homeomorphic to the original but pairwise non-diffeomorphic. In particular, this yields uncountably many diffeomorphism types of domains of holomorphy in $\mathbb{C}^2$ homeomorphic to $\mathbb{R}^4$.

Working groups and open problems

As mentioned before the most essential purpose of the workshop was to bring together people from different areas (complex, symplectic, contact, and smooth topology/geometry) and initiate conversations to exchange knowledge, ideas, and questions related to different types of convexity and related problems. To contribute this, time was dedicated for discussion groups on Tuesday and Thursday of the conference week, each for at least 1.5 hours. On the first day of the conference, during an organizational meeting with all participants we made the following preliminary list of open problems and research topics.

1. **Gompf's Conjecture.** No Brieskorn sphere admits a pseudoconvex embedding in $\mathbb{C}^2$ with either orientation.

2. Is there a difference between Stein and Weinstein and Liouville cobordisms?
3. In high dimensions, there are infinitely many distinct Weinstein domains which are diffeomorphic to each other. Are there infinitely many diffeomorphic Weinstein structures which are not equivalent? (What if we control the Chern class?)
4. When is a (Lagrangian) cobordism of quasipositive knot (filling of a quasipositive knot) ribbon?
5. Given a Stein manifold, if you remove a holomorphic set, then the complement is Stein. If you remove a pseudoholomorphic set, do we know if the complement is Stein?
6. There are various types of convexity. Can embedded contact homology capacities be used to obstruct the different notions of convexity?
7. Can $\mathbb{R}P^2$ or S^2 be topologically embedded in $\mathbb{C}^2$ as a rationally convex subset?
8. Understand which singular Lagrangians in $\mathbb{C}^2$ have rationally convex neighborhoods, where we allow the singularities to be say cones over Legendrian torus knots.

Presentation Highlights

The first three days of the workshop included plenary lectures intended to introduce audience members to the specialties of the plenary speakers, roughly representing smooth, symplectic, and complex geometric aspects of the topic. The plenary lectures were preceded by introductory lectures by promising junior researchers. Additional lectures were presented by researchers in a variety of specialties, all roughly centered on the theme of convexity.

Introductory and Plenary Talks

- **Smooth topology and complex geometry**

  Kyle Hyden

  Title: The smooth topology of Stein manifolds

  This introductory lecture will explore the basic smooth topology of manifolds admitting Stein structures, with a focus on Stein surfaces (i.e., those of real dimension 4). Guided by the natural Morse functions carried by Stein manifolds, we will unpack Eliashberg's topological characterization of Stein manifolds, Gompf's handlebody construction of Stein surfaces, and the adjunction inequality. We will close with an application to the existence of exotic smooth structures on 4-space.

  Bob Gompf

  Title: Smooth and topological pseudoconvexity in complex surfaces

  Abstract: We will discuss several general tools for finding strictly pseudoconvex subsets of complex surfaces. An open subset U is smoothly isotopic to a Stein open subset if and only if its inherited complex structure is homotopic (through almost-complex structures) to a Stein structure on U. If we allow topological isotopy (homotopy through homeomorphic embeddings with no differentiability assumed), the condition on the complex structure can be dropped, and it is only necessary for U to admit a topological Morse function whose critical points have index at most 2. A deeper version of this shows that every finite 2-complex in a complex surface is topologically isotopic to a Stein compact, in fact, to a nested intersection of uncountably many homeomorphic Stein open subsets. This leads to a notion of pseudoconvexity for unsmoothably embedded 3-manifolds. We discuss examples and applications of such phenomena, with the hope of encouraging further exploration with these tools.

- **Complex geometry and convexity**

  Blake Boudreaux

  Title: Holomorphic convexity in several complex variables

  Abstract: In 1906, F. Hartogs discovered the existence of domains in $\mathbb{C}^n$ for which every holomorphic function can be extended to a larger domain. Domains that do not admit this extension phenomenon satisfy a complex type of convexity, known as pseudoconvexity. This type of convexity can be viewed as convexity "with respect to holomorphic functions", as opposed to geometric convexity, which is convexity "with respect to linear functions". In this talk, we will motivate and define pseudoconvexity. We will also compare and contrast its many equivalent formulations with that of classical convexity. We will also introduce a class of "pseudoconvex" manifolds and discuss their many properties. Notions of convexity with respect to other classes of functions will also be discussed.

  Resul Shafikov

  Title: Polynomial and Rational Convexity

  Abstract: In the first half of the talk I will give an overview of polynomial and rational convexity: I will give basic definitions, examples and outline some fundamental properties of polynomial and rationally convex compacts. In the second half of the talk I will discuss characterization of rational convexity of real submanifolds in complex Euclidean spaces and related problems.

- **Symplectic geometry**

  Joé Brendel

  Title': Toric reduction and applications

  Abstract: In this introductory lecture, we focus on a special case of symplectic reduction, in which the reduction is compatible with a toric group action. We recall the basic notions, discuss an example that will come up in Jonny's lecture and, if time permits, give further applications.

  Jonathan Evans

  Title Open problems around Lagrangian intersections

  Abstract: Let K be a Lagrangian submanifold and L_t be a family of Lagrangian submanifolds. Suppose you can displace K from each L_t. Can you displace K from all L_t simultaneously? If not, from how many L_t can you simultaneously displace K? We will discuss some specific problems which have this flavor and give some small results in this direction.

Additional Research Talks

- Luya Wang

  A connected sum formula of embedded contact homology

Abstract: The contact connected sum is a well-understood operation for contact manifolds. I will focus on the 3-dimensional case and the Weinstein 1-handle model for the contact connected sum. I will discuss how pseudo-holomorphic curves in the symplectization behave under this operation. After reviewing embedded contact homology, we will see how this results in a chain-level description of the embedded contact homology of a connected sum.

- Joseph Breen

  Title: The Giroux correspondence in all dimensions

  Abstract: Twenty years ago, Giroux gave an influential result on the equivalence of contact structures in dimension 3 and open book decompositions up to stabilization. At the time, Giroux and Mohsen also partially extended the correspondence to all dimensions, albeit with different technology. From one point of view, the existence of open book decompositions can be viewed as a convexity statement for contact manifolds, and there are natural connections to symplectic convexity. In this talk, I will describe forthcoming joint work with Ko Honda and Yang Huang on establishing the Giroux correspondence in all dimensions using convex hypersurface theory.

- Purvi Gupta

  Title. Polynomially convex embeddings of compact real manifolds

  Abstract. A compact subset of $\mathbb{C}^n$ is said to be polynomially convex if it is cut out by a family of polynomial inequalities. Polynomial convexity grants certain approximation-theoretic properties to the underlying set. When the set is a real submanifold of $\mathbb{C}^n$, its convexity properties are partly influenced by its topology, and the local and global structure of its CR (complex-real) singularities. The minimum complex dimension into which all compact real manifolds of a fixed dimension admit smooth polynomially convex embeddings is not known (although some bounds can be deduced from the literature). In this talk, we will discuss some recent improvements on the previously known bounds. We will especially focus on the case where the h-principle has proved useful for producing the desired embeddings. This is joint work with R. Shafikov.

- Stefan Nemirovski

  Title: Complex Analysis 2.0

  Abstract: Peculiar features of low-dimensional differential, symplectic/contact topology affect the theory of holomorphic functions of two complex variables. The purpose of the talk will be to illustrate this principle with a few token examples and discuss open problems and possible research directions in this area.

- Marko Slapar

  Title: Representing homology classes of complex hypersurfaces in $\mathbb{C}P^3$

  Abstract: Thom conjecture, proven by Kronheimer and Mrowka in 1994, states that complex curves in $\mathbb{C}P^2$ are genus minimizers in their homology class. We will show that an analogous statement does not hold for complex hypersurfaces in $\mathbb{C}P^3$. This is joint work with Ruberman and Strle.

- Giancarlo Urzua

  Title: Exotic 4-manifolds and KSBA surfaces

Abstract: Although exotic blow-ups of the complex projective plane at n points have been constructed for every $n > 1$, the only examples known by means of rational blowdowns satisfy $n > 4$. It has been an intriguing problem whether it is possible to decrease n. In this talk, I will show how to construct it for $n = 4$ from a configuration of 8 lines and 2 conics in a special position. This is part of a bigger picture to construct exotic $p\mathbb{C}P^2\#q\mathbb{C}\bar{P}^2$ via the construction of particular Kollár–Shepherd-Barron–Alexeev (KSBA) singular surfaces. This is done by explicitly analyzing obstructions coming from configurations of rational curves, and the use of computer searchers. This connection between the geography of configuration of rational curves and exotic 4-manifolds from KSBA surfaces leads to, I believe, a new view on this problem. There is a lot of data out of these searches, showing an intricate picture for KSBA surfaces. I hope to show that in this talk too. This is joint work with Javier Reyes.

- Angela Wu

  Title: On Lagrangian quasi-cobordisms

  Abstract: A Lagrangian cobordism between Legendrian knots is an important notion in symplectic geometry. Many questions, including basic structural questions about these surfaces are yet unanswered. For instance, while it is known that these cobordisms form a preorder, and that they are not symmetric, it is not known if they form a partial order on Legendrian knots. The idea of a Lagrangian quasi-cobordism was first defined by Sabloff, Vela-Vick, and Wong. Roughly, for two Legendrians of the same rotation number, it is the smooth composition of a sequence of alternatingly ascending and descending Lagrangian cobordisms which start at one knot and ends at the other. This forms a metric monoid on Legendrian knots, with distance given by the minimal genus between any two Legendrian knots. In this talk, I will discuss some new results about Lagrangian quasi-cobordisms, based on some work in progress with Sabloff, Vela-Vick, and Wong.

- Kyler Siegel

  Title: On rational curves with cusps and double points

  Abstract: A classic question in algebraic geometry asks what are the possible singularities for a plane curve of a given degree and genus. This is closely related to existence questions for singular Lagrangian surfaces in the affine or projective complex plane, which in turn connect with questions about the topology of rationally convex domains. In this talk I will describe a construction of various new families of rational plane curves with prescribed singularities, and I will wax poetic about how this ties in with the themes of this workshop.

- Morgan Weiler

  Title: ECH spectral invariants for toric contact forms

  Abstract: The embedded contact homology (ECH) chain complex has several natural filtrations, and applications of ECH to symplectic and contact geometry often rely on computing the associated spectral invariants. When the three-manifold is spherical (or more generally, toric), this means there is a precise function from ECH index to the minimal filtration value among cycles representing that index's homology class. ECH practitioners attempt to compute or estimate these functions. We will explain why those attempts are much more successful in the case of convex toric contact forms, including applications of the ECH knot filtration to surface dynamics and the ECH action filtration (aka the ECH spectrum) to symplectic embedding problems. The latter project is based on joint work with several coauthors, in arXiv:2010.08567, arXiv:2203.06453, and arXiv:2210.15069.

- Oliver Edtmair

Title: Convexity, Hamitonian dynamics and symplectic embeddings

Abstract: I will motivate several notions of convexity that play important roles in Hamiltonian dynamics and in the theory of symplectic embeddings. In particular, I will focus on the mysterious role convexity plays in Viterbo's conjecture on the systolic ratio and the symplectic capacities of convex domains in Euclidean space. I will end my talk by reviewing some recent progress towards this conjecture.

- Sümeyra Sakalli

Title: Singular fibers in algebraic fibrations of genus two and their monodromy factorizations

Abstract: Kodaira classified all singular fibers that can arise in algebraic elliptic fibrations. Later, Ogg, Iitaka and then Namikawa and Ueno gave a classification for genus two fibrations. In this work, we split these algebraic genus two fibrations into Lefschetz fibrations and determine the monodromies. More specifically, we look at four families of hypersurface singularities in $\mathbb{C}^3$. Each hypersurface comes equipped with a fibration by genus 2 algebraic curves which degenerate into a single singular fiber. We determine the resolution of each of the singularities in the family and find a flat deformation of the resolution into simpler pieces, resulting in a fibration of Lefschetz type. We then record the description of the Lefschetz fibration as a positive factorization in Dehn twists. This gives us a dictionary between configurations of curves and monodromy factorizations for some singularities of genus 2 fibrations. This is joint work with J. Van Horn-Morris.

Scientific Progress Made

For Tuesday discussion session session we had participants sorted into four groups, each of which had a group leader to facilitate/guide the discussions.

- **Group 1—Hutchings (Room 102)**

  Gupta, Lambert-Cole, Roy, Auyeung, Christian, Capovilla-Searle.

- **Group 2—Siegel (Room 106)**

  Boudreaux, Urzua, Gompf, Brendel, Knavel, Wan, Rodewald.

- **Group 3—Dimitroglou Rizell (Room 107)**

  Shafikov, Min, Ono, Choi, Magill, Nelson, Wang.

- **Group 4—Slapar (Room 202)**

  Wu, Sakalli, Park, Weiler, Lazarev, Breen, Edtmair.

For Thursday meeting we encouraged participants to choose and propose some promising focus topics. This was received well and quickly the following focus groups were formed.

- **Rational convexity in closed symplectic manifolds.**

  Shafikov, Dimitroglou-Rizell, Gupta, Mark, Lazarev, Tosun, Gompf, Boudareaux.

- **Symplectic and Algebro-Geometric approaches to Markov/Unicity Conjecture.**

  Urzua, Brendel, Park, Sakalli, Ono, Capovilla-Searle.

- **ECH contact class.**

  Nelson, Hutchings, Weiler, Magill, Lambert-Cole, Roy, Choi, Min.

Reports from the participants on their experiences and on the progress made were very positive. Particular items mentioned included:

- New examples of polynomially convex domains in $\mathbb{C}^2$ that are not subcritical, resolving a question mentioned in Section 2.
- Development of a strategy to extend results on rationally convex domains in $\mathbb{C}^2$ to other manifolds.
- Deeper understanding and study of the existence problem for rationally convex, topologically embedded 2-spheres in $\mathbb{C}^2$.
- Discussions and interactions among participants from different specialties on wide-ranging topics including Mori theory, exotic 4-manifolds, Floer theory of various sorts, and the subtle relations between Weinstein domains, rational convexity and the Stein condition.

Outcome of the Meeting

Feedback communicated to the organizers by participants was uniformly positive. Participants were particularly appreciative of the broad selection of topics and mathematically diverse group, which was described as unusual, refreshing, and mathematically beneficial.

Participants also praised the format, including the overview lectures and the organization of working groups.

Progress on particular mathematical topics was noted in the previous section; some participants have indicated that at least one preprint is in preparation on work initiated at this meeting.

Participants

Auyeung, Shamuel (Stony Brook University)
Baykur, Inanc (University of Massachusetts Amherst)
Boudreaux, Blake (University of Western Ontario)
Breen, Joseph (University of Iowa)
Brendel, Joé (Tel Aviv University)
Capovilla-Searle, Orsola (uc davis)
Casals, Roger (UC Davis)
Choi, Hakho (QSMS, Seoul National University)
Christian, Austin (Georgia Tech)
Cristofaro-Gardiner, Daniel (University of Maryland)
Dey, Subhankar (University of Alabama)
Dimitroglou Rizell, Georgios (Uppsala University)
Edtmair, Oliver (University of California Berkeley)
Etnyre, John (Georgia Institute of Technology)
Evans, Jonathan (Lancaster)
Golla, Marco (CNRS and University of Nantes)
Gompf, Bob (University of Texas Austin)
Gupta, Purvi (Indian Institute of Science)
Hayden, Kyle (Rutgers University - Newark)
Hutchings, Michael (University of California)
Hyunki, Min (University of California - Los Angeles)
Izzo, Alexander John (Bowling Green State University)
Knavel, Sierra (Georgia Institute of Technology)

Lambert-Cole, Peter (University of Georgia)
Lazarev, Oleg (University of Massachusetts Boston)
Lisi, Samuel (University of Mississippi)
Magill, Nicki (Cornell University)
Mark, Thomas (University of Virginia)
Min, Jie (University of Massachusetts - Amherst)
Nelson, Jo (Rice)
Nemirovski, Stefan (Ruhr-Universität Bochum)
Ono, Kaoru (Kyoto University)
Park, Jongil (Seoul National University)
Pinzon-Caicedo, Juanita (University of Notre Dame)
Rodewald, Thomas (Georgia Institute of Technology)
Roy, Agniva (Georgia Tech)
Sakalli, Sumeyra (University of Arkansas)
Shafikov, Rasul (University of Western Ontario)
Siegel, Kyler (USC)
Slapar, Marko (University of Ljubljana)
Starkston, Laura (University of California, Davis)
Stipsicz, Andras (Renyi Institute)
Tosun, Bulent (University of Alabama)
Urzúa, Giancarlo (Pontificia Universidad Católica de Chile)
Vela-Vick, Shea (Louisiana State University)
Wan, Shunyu (University of Virginia)
Wang, Luya (Stanford University)
Weiler, Morgan (Cornell University)
Wu, Angela (Louisiana State University)

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Chapter 11

Systematic Effects and Nuisance Parameters in Particle Physics Data Analyses (23w5096)

April 23 - 28, 2023

Organizer(s): Olaf Behnke (DESY), Sara Algeri (University of Minnesota), Lydia Brenner (Nikhef), Richard Lockhart (Simon Fraser University), Louis Lyons (Imperial College London and Oxford University)

Introduction to the topic of the workshop

This meeting dealt with systematic uncertainties in Particle Physics analyses. Such analyses are affected by statistical as well as systematic uncertainties. The former arise either from the limited precision of the apparatus and/or observer in making measurements, or from the random fluctuations (usually Poissonian) in counted events. They can be detected by the fact that, if the experiment is repeated several times, the measured physical quantity will vary.

Systematic effects, however, can cause the result to be shifted from its true value, but in a way that does not necessarily change from measurement to measurement. Measurements nearly always have some bias, and the question is by how much are they biased. Results are corrected for known biases, and the uncertainties in these corrections contribute to the total uncertainties. Systematic effects are not easy to detect, and in general much more effort is needed to identify them and to evaluate the corresponding uncertainties. The most worrying sources of systematics are the 'unknown unknowns'.

Typical sources of systematic effects include:

1: Estimated energies of jets of particles.
2: Identification efficiency of electrons.
3: Contamination by the background in selected event sample.
4: Estimates of various theoretical correction factors, and the role of theory in interpreting the data.
5: Total number of colliding beam particles for the selected events.

Such effects are usually parametrised by nuisance parameters.

After identifying possible systematic effects, the next tasks are to estimate their magnitude, and then to incorporate them into the analyses. This meeting focussed on the last aspect. It was considered that this was the topic that would most benefit from interactions between Statisticians and Physicists.

Participants and workshop elements

Some of the organizers of the present workshop are also involved in managing the PHYSTAT [1] event series on statistical issues and methods in particle physics data analyses. A key element of PHYSTAT workshops and seminars is the active involvement of statisticians and we are happy that at the present workshop 13 statisticians attended on-site together with 29 physicists. Regarding *equity, diversity and inclusion* we provide here some benchmark numbers: 13 on-site participants registered themselves as women and 29 as men (compared to the overall average of $\sim 20\%$ of particle physicists being female); 14 as early career researchers, ranging from PhD student to post-doc level, and 28 as seniors. In addition to the 42 on-site participants, 28 persons registered for joining via ZOOM from remote.

A workshop format was chosen for maximizing the amount of discussion. For this purpose, the workshop was structured into ~ 25 topical sessions, listed in Table 11.1, each consisting of an introductory talk of 25 minutes followed by a 35 minutes discussion session. The participants quickly adapted to this scheme and, to the satisfaction of the audience, the allocated time for discussion was fully utilised. Indeed the discussions continued into the coffee and meal breaks, evenings and excursions. For a few sessions, the introductory talks were shared between physicists and statisticians, complementing each other. The workshop was transmitted in ZOOM, and Slack channels were used for intense continued discussions.

Session title	Speaker(s)	Chair
Introductory summaries of virtual Phystat-Systematics 2021 workshop [2]		
Physicist's view	N. Wardle	O. Behnke
Statistician's view	S. Algeri	O. Behnke
Macro Theme 1: Frequentist versus Bayesian		
Marginalizing versus profiling	R. Cousins, A. Davison	G. Cowan
Bayesian approaches in Astrophysics	F. Capel	W. Rolke
Pragmatic versus full Likelihood approaches	D. van Dyk	A. Brazzale
Likelihood-free frequentist inference	A. B. Lee	L. Heinrich
Simulation-based inference of atmospheric Cosmic-ray showers	A. Shen	L. Heinrich
Macro Theme 2: Modeling uncertainties		
Model selection	C. Schafer	N. Wardle
Background model building	L. Kania	N. Wardle
Background and signal shapes at the LHC	N. Morange	S. Williams
Template morphing	L. Brenner	H. Gray
Optimal Transport	P. Winischhofer, T. Manole	R. Lockhart
Systematics in Monte Carlo Simulation	G. Jones	T. Junk
Theory uncertainties	F. Tackmann	I. Volobouev
Macro Theme 3: Nuisance parameters		
Asymmetric uncertainties	R. Barlow	I. Volobouev
Error on error	E. Canonero	N. Wardle
Macro Theme 4: Machine Learning		
Machine Learning	M. Kagan	L. Heinrich
ML for reducing systematic uncertainties	T. Dorigo	H. Gray
Systematics in ML model independent searches	G. Grosso	I. Ochoa
Semi-supervised classifiers	P. Chakravarti	P. Windischhofer
Special experiment specific uncertainties, inverse problems and Banff Challenge		
Systematics in Neutrino physics analyses	E. T. Atkins	R. Cousins
Systematics in selected flavour physics topics	S. Stefkova	K. Tackmann
Systematics in unfolding	M. Stanley, R. Zhu	G. Cowan
Banff Challenge 3	T. Junk	P.-L. Tan
Thoughts on meeting		
Physicist's view	A. David	L. Lyons
Statistician's view	M. Kuusela	L. Lyons

Table 11.1: Topical sessions, speakers and session chairs.

Most of the topical sessions were distributed over four ”Macro themes”, listed in Table 11.1, comprising the topics of frequentist versus Bayesian inference, model uncertainties, nuisance parameters and machine learning. Another session covered topics such as experiment-specific uncertainties and inverse problems (aka unfolding) and one of the participants provided a practical data analysis challenge. The program started with two talks summarizing the outcome of the virtual PHYSTAT-Systematics 2021 workshop [2], which, among other things, served as a preparatory meeting for the present event and gave examples of problems that still need a solution and could be discussed during the meeting. A session with thoughts on the outcome of the current meeting concluded the workshop.

Review of the Meeting

This section contains a brief description of the individual talks; the two ‘Thoughts on the meeting’; and a list of cooperative topics as a result of the Banff meeting.

Presentations

Below are summaries of each of the talks. Each was followed by an extensive discussion period.

PHYSTAT-Systematics 2021: Physicist’s review Nicholas Wardle, Imperial

In this talk, I summarized the recent PHYSTAT-Systematics workshop held virtually in 2021 [2]. The presentation focused on reviewing the three major categories of sources of systematic uncertainties that arise in experimental particle physics; Those that arise from in-house calibrations (largely statistical in nature), those that arise due to model assumptions or using imperfect analysis methods and those that arise due to imprecise theoretical calculations which are extremely difficult to interpret as statistical uncertainties.

The talk discussed the two main methods for including these uncertainties in experimental analyses,

1. Error propagation: By changing some part of the model “one parameter at a time” (OPAT) and recording the change in a measured parameter, the uncertainty is propagated to the final result. Similarly, one can randomly sample from the systematic variations and interpret the “spread” of results as the uncertainty.

2. Nuisance parameters: Representing systematic uncertainties as parameters of the statistical model (potentially with constraints or priors) allows us to include their effect in the final result and often has the added bonus that profiling/marginalising can constrain these parameters further using the data.

Both methods have advantages and disadvantages which are more or less relevant depending on the type of source of systematic uncertainties. Several examples of experimental measurements that made use of methods 1 and 2 above and the various pitfalls and issues they faced were discussed. A list of discussion items or “open questions” to statisticians was provided:

- When reporting uncertainties on more than one parameter of interest via OPAT, is providing covariance enough?

- When modelling systematic uncertainties using nuisance parameters, should we sample many different parameter values to build suitable parameterisation and are there smarter ways (e.g. Gaussian Processes/Machine Learning?) to automate this?

- Are we OK that our fit updates our knowledge of certainty nuisance parameters?

- Is there a better way to include uncertainties due to model choice than taking the difference between (e.g.) simulations? Or approaches such as inflating uncertainty to cover potential bias / discrete profile method [3]?

These were mirrored in a similar summary talk given by statistician Sara Algeri and much of the discussion focused on these points.

PHYSTAT-Systematics 2021: Statistician's review Sara Algeri, UMN

In my talk, I provided a statistical overview of the types of systematic uncertainties that we may encounter in physics data analyses, and how they differ from statistical uncertainties. I also highlighted some open questions and possible points of discussion, which were either raised during our Phystat-Systematics workshop in November 2021 [2] or triggered by some of the talks presented at that meeting. They can be summarized as follows:

- In the context of background mismodelling, can the difference between methods be used to acquire a measure of systematic bias?
- When dealing with nuisance parameters, is it at all possible to reach a consensus on what to do and when? (e.g., marginalizing or profiling)
- Can statisticians effectively access published likelihoods?
- How can we check the validity of regularity conditions required by classical statistics when dealing with complex models?
- What is needed to robustly bridge the gap between the statistics and physics communities?

Marginalising vs Profiling of Nuisance Parameters (Physicist's View) Robert D. Cousins, UCLA

I first spoke generally about the possibilities of eliminating nuisance parameters by using profile likelihood functions (the natural way in the context of a frequentist likelihood-based analyses) and Bayesian-inspired integrated likelihoods, advocated by Berger et al. [4], even when treating the parameter of interest by frequentist techniques.

One must distinguish between two stages in an analysis namely 1) in constructing a test statistic, typically a likelihood ratio, and 2) in using Monte Carlo simulation to generate synthetic data (pseudo-experiments) to obtain the distribution of the test statistic under the null or alternative hypotheses. I focused on the latter. In the fully frequentist parametric bootstrap, one generates the pseudo-experiments by using the profiled values of the nuisance parameters. This is the most commonly used method at the LHC.

I then discussed the simple example of the ratio of Poisson means, which appears in HEP and gamma-ray astronomy [5]. Integrating out the nuisance parameter gives the same numerical p-value as the standard "exact" frequentist solution that conditions on the ancillary statistic (total number of events), while the parametric bootstrap gives a smaller, more extreme p-value. But the discrete nature of the observables means that the exact method is conservative while the parametric bootstrap has excellent coverage. I ended by urging more studies.

Marginalise or profile: A statistician's view Anthony Davison, EPFL

Statistical inferences should be relevant, calibrated and secure. Relevance means that statements of uncertainty that are based on frequentist comparison with a reference set should restrict the latter to be appropriate to the data actually observed, by conditioning on the information content of the latter. Calibrated means that statements of confidence should be accurate, and secure means that they should not be too dependent on secondary details of the problem formulation. Nuisance parameters and discreteness can lead to poorly calibrated statements. In exponential families, the former can often be dealt with by suitable conditioning, and more general models can be approximated locally using the tangent exponential model (TEM) due to Fraser and Reid. Profiling out nuisance parameters typically leads to poor calibration for a scalar interest parameter, but the resulting error can be reduced using TEM approximation or parametric bootstrap simulation. Marginalisation is often associated with the use of a prior density on the nuisance parameters, but it can be regarded as resulting in a Bayesian inference in which a flat prior is placed on interest parameters, in which case the resulting inference may be close to the result of the TEM approximation, though a poorly-chosen prior may lead to a poorly-calibrated inference. Discreteness of responses leads to conservative inferences (confidence intervals are too long on average, leading to coverage exceeding its nominal level).

The subsequent discussion touched on a range of points, including the discreteness, the effect of the prior on nuisance parameters, and the desirability of improved approximations using the TEM. It was very helpful to talk to Bob Cousins.

Bayesian Approaches in Astrophysics Francesca Capel, MPI Munich and TUM

My talk gave an overview of Bayesian approaches to handling systematic uncertainty in astrophysics, which have become increasingly widespread over the past 30 years. Generally speaking, the Bayesian approach to systematics is conceptually straightforward, as systematic uncertainties can be parametrised and handled in the same way as statistical uncertainties. Marginalisation can be used to summarise results in terms of the parameters of interest. However, doing so requires introducing extra free parameters and a choice of priors, and results derived in this way do not guarantee coverage in the frequentist sense. I highlighted a few applications of Bayesian analysis to different types of systematics; those which can be constrained, such as in the case of the background parameter in "on/off" signal measurements of gamma-ray data [6], those which involve uncertain modelling assumptions, such as the discrete models for the Galactic magnetic field in ultra-high-energy cosmic ray studies [7], and those with uncertain theoretical frameworks, such as the shape of the binary black hole mass distribution inferred from gravitational wave data [8]. We discussed that the correct strategy would depend on the details of the problem. Still, workflows including validation against simulated data, model checking through, e.g. posterior predictive checks, or model averaging could be used to reveal and incorporate systematics into a Bayesian model.

Pragmatic vs full likelihood approaches David van Dyk, Imperial

Considering systematic errors that arise from uncertainty in quantities measured in a preliminary experiment that are treated as inputs or nuisance parameters in a primary statistical analysis, we formulate the coherent analysis of systematics in high-energy physics as a multi-stage statistical analysis. Such analyses combine multiple models and data sources into a single omnibus analysis, where outputs from one analysis are the inputs of subsequent analyses. In/Outputs may be high-dimensional with difficult-to-quantify correlations. Researchers involved in such a chain of analyses may work quite independently with different research groups having different areas of expertise and different levels of statistical sophistication. We consider different Bayesian approaches to multi-stage statistical analyses and how they can be applied to handle systematics in high-energy physics. A naïve approach ignores uncertainty in the nuisance parameters, whereas in a fully Bayesian analysis, all data (from both the preliminary and primary measurements) is used to estimate and compute errors for all unknown quantities (including nuisance parameters). In a pragmatic Bayesian analysis, on the other hand, only data from the preliminary analysis is used to estimate and compute errors for the nuisance parameters. We discuss the pros and cons of these approaches using several examples from high-energy astrophysics, including instrument calibration [9, 10] and the disentangling of overlapping sources [11, 12].

Subsequent discussion focused, for example, on the relationship of the pragmatic Bayesian approach to methods for "cutting feedback" or "modularization", i.e. informally limiting the influence of potential misspecification of parts of a multicomponent Bayesian model (see, for example, [13] and references therein). We thank Tom Loredo for pointing out the relevant literature. This work has already led to a publication [14].

Likelihood-Free Frequentist Inference Ann B Lee, CMU

Many areas of science, such as high-energy physics, make extensive use of computer simulators that implicitly encode the likelihood functions of complex systems. Classical statistical methods are poorly suited for these so-called likelihood-free inference (LFI) settings, outside the asymptotic and low-dimensional regimes. Although new machine learning methods, such as normalizing flows, have revolutionized the sample efficiency and capacity of LFI methods, it remains an open question whether they produce confidence sets with correct conditional coverage. In my talk, I summarized recent developments in unifying classical statistics and machine learning methods for (i) constructing Neyman confidence sets with finite-sample guarantees of nominal coverage in the presence of systematic effects, and for (ii) computing diagnostics that estimate conditional coverage over the entire parameter space.

This workshop gave me an excellent overview of the relevance and main challenges of systematic effects in HEP. The workshop also gave me an invaluable opportunity to interact closely with particle physicists for a week. These interactions have, for example, led to a new collaboration with physicist Tommaso Dorigo on optimizing detector design for atmospheric cosmic-ray showers. I am also following up on a connection between calibrated forecasting and optimal transport that I was not aware of until I started discussions at Banff with physicists Michael Kagan, Lukas Heinrich and Philipp Windischhofer. Many congratulations to the Banff organizers for running such a successful and high-quality workshop with both statisticians and physicists as participants!

Likelihood Free Frequentist Inference for Cosmic Ray Reconstruction Alex Shen, CMU

In my talk, I discussed an upcoming project that uses Likelihood Free Frequentist Inference (LF2I) to perform the reconstruction of cosmic rays from secondary footprint data with uncertainty quantification. The LF2I framework is outlined in [15].

Key Points

- Reconstruction is broken down into two steps: determining the identity of the primary cosmic ray (proton or photon), and estimation of key parameters (energy, orientation) for photonic cosmic rays only.
- The first step is complicated by the fact that the prior distributions of cosmic ray identity and energy are not independent. We take a nuisance parameterized LF2I hypothesis testing approach that ensures Type I error control no matter the energy of the primary cosmic ray.
- The second step uses hypothesis test inversion under the LF2I framework to construct confidence sets for parameters of interest for photonic cosmic rays.

Thoughts on model selection Chad Schafer, CMU

The general problem of background model selection in particle physics experiments presents some interesting statistical challenges, chiefly surrounding the issue of the tradeoff between using (1) a sufficiently realistic and flexible background and (2) a model that imposes sufficient structure that there is the power to detect the signal of interest. The *semiparametric* approach to inference seems to be a natural fit in this context. In the classic semiparametric setup, the parameter of interest (in this context, a quantification of the signal strength) is real-valued, while the nonparametric component (i.e, the background) is a nuisance parameter. It could be argued that most particle physics experiments implement a *de facto* semiparametric approach, in that even though parametric background models are utilized, their complexity is increased sufficiently to model the background to achieve an acceptable level of fidelity to the observed data. A well-implemented nonparametric background estimator would behave in a similar, but automated manner. [16] present theory regarding the performance of semiparametric estimators that could be useful in this case. In particular, the authors define versions of the *score function* and *Fisher information* adapted to the case where the nuisance parameter is estimated nonparametrically. The ideas are motivated by considering the *least favorable* background model, i.e., the model that makes the estimation of the parameter of interest the most challenging. If used in conjunction with the *method of sieves* [17] to constrain the space of background models, the result could be consistent estimation of, and more accurate and meaningful measures of uncertainty for, the signal strength.

Uncertainty quantification via influence functions Lucas Kania, CMU

A fundamental problem in High Energy Physics is separating collisions corresponding to known physical phenomena (called background) from those indicating new phenomena (called signal). This is challenging because the signal (if it exists) represents a small fraction of the available data. Current statistical approaches model the data as a mixture of the background and signal, and either (A) require access to an auxiliary sample from the background to estimate it non-parametrically and then attempt signal detection without a signal model [18] or (B) assume parametric background and signal models, and then try to find a signal via the profile likelihood ratio [19].

In this talk, we considered experiments where there is no auxiliary sample, but there is a known signal region. That is, if a signal exists, it must be contained in the signal region. Under this assumption, specific non-parametric background models can be identified, e.g. high-degree polynomials, and their influence functions [20] can be harnessed to perform valid signal detection. In summary, the method allows physicists to perform valid signal detection using complex background models without any auxiliary dataset.

In the discussion that ensued, Robert Cousins recommended analytically comparing our results to the standard profile likelihood machinery [21]. Additionally, Sara Algeri encouraged us to prove non-asymptotic guarantees for our methodology, which relies heavily on asymptotics.

This talk was based on joint work with Larry Wasserman, Mikael Kuusela and Olaf Behnke.

Background and signal shapes Nicolas Morange, Orsay, LAL

This presentation showed why getting accurate signal and background shapes is a major topic of analyses at the LHC, and discussed the main strategies used to increase the precision of the measurements by reducing the systematic uncertainties on the shapes.

With the vast amount of LHC data available, statistical uncertainties in the analyses are getting smaller and smaller. This in turn allows for precision calibration of the detection efficiencies and energy measurements. In the quest for high-precision measurements of physics processes, the uncertainties in the background and signal shapes therefore become more and more relevant.

Getting accurate shapes and reducing the associated uncertainties relies on two main pillars. The first one is using better theoretical predictions and more accurate Monte Carlo samples, taking higher-order effects into account. Discussions highlighted the huge computing effort required to get enough statistics of accurate, modern Monte Carlo samples, and sketched ideas such as multi-dimensional reweighting to alleviate the issue. The second pillar consists in using the data as much as possible to calibrate the Monte Carlo in dedicated signal-depleted regions, or completely replace the Monte Carlo whenever possible, using fully data-driven techniques. Different ad hoc solutions to define systematic uncertainties are used depending on the analyses, and great care is taken to avoid biases and setting too small uncertainties.

Progress on this topic necessitates close collaboration between experimentalists, theorists and statisticians.

Template morphing Lydia Brenner, Nikhef

This talk describes and compares different interpolation/morphing methods that produce signal or signal+background models for fits to data. It is also described in which case different methods might break down. A relatively new method, effective Lagrangian morphing, is described in more detail [22]. The signal model is morphed in a continuous manner through the available multi-dimensional parameter space. An example is given of the calculation leading to the morphing function as well as an example use-case where the morphing function is applied to simulated data of a Higgs boson. The approach described in this talk is expected to substantially enhance the sensitivity to Beyond the Standard Model contributions. Among the considerations discussed are computational costs and uncertainty propagation of systematics.

Optimal transport in high-energy physics Tudor Manole, CMU and Philipp Windischhofer, Uni Chicago

Optimal transport is a branch of mathematics that has recently emerged as a methodological tool for statistics and machine learning. In our survey-style talk, we covered several aspects of data analysis in particle physics where optimal transport could extend, or replace, existing methods.

Optimal transport provides a principled way of defining mappings between probability distributions, which can be leveraged to construct unbinned calibrations of Monte Carlo simulations to data [23], perform data-driven background modelling [18], or build interpolated densities in the context of template morphing. Optimal transport also provides guidance in defining a distance metric on the space of distributions, known as the Wasserstein distance, lending a geometric interpretation to operations such as jet clustering [24]. In the realm of statistics, optimal transport can be used to generalize the notion of a quantile to multivariate distributions, in turn defining a swath of new goodness-of-fit and two-sample tests for multivariate distributions [25].

Recent advances in machine learning have made it possible to solve important classes of optimal-transport problems on large-scale data sets [26], thereby providing the technical framework to study the performance of these methods in real-world analysis settings, a process that has already begun within the experimental collaborations at the Large Hadron Collider.

Systematics in MC Galin Jones, UMN

My talk focused on systematic errors in Monte Carlo, especially in the context of systematics in high energy physics data analyses. The main points of discussion included (i) the effect of model error on Monte Carlo, (ii) the effect of nuisance parameters in Monte Carlo, (iii) Monte Carlo simulations producing erroneous output, (iv) determining Monte Carlo sample size, and (v) the effect of using the Monte Carlo output suboptimally. Some of the specific issues presented included assessing the Monte Carlo error to determine the simulation length, the construction of confidence intervals with convenient marginal interpretations and approaches to maximum likelihood-based inference for simulated likelihoods.

The session prompted substantial follow-up discussion, including after the conclusion of the workshop. Some of the discussion was prompted by concerns about the reproducibility of high-energy physics analyses. Other discussions concerned resolving Monte Carlo errors in neutrino analyses.

Theory uncertainties Frank Tackmann, DESY

Given a measurement of some quantity f, we obtain the parameter of interest p from it by comparing to the expected value $f(p)$ as a function of p. The formula we use for $f(p)$ is the *theory prediction*. The *theory uncertainty* is due to the fact that the formula itself is never really exact but derived in some approximation. To account for the theory uncertainty, we typically quote the prediction with an uncertainty estimate $f(p) \pm \Delta f$. In my talk, I discuss the current prevalent method in high-energy physics for how to estimate Δf, namely scale variations. In perturbative predictions, $f \equiv f(x)$ depends on a small quantity x (the coupling constants), in which we can expand $f(x) = f_0 + f_1 x + f_2 x^2 + ...$ Our approximate prediction comes from truncating this expansion, e.g. at the next-to-leading order (NLO) we would keep the first two terms and drop the quadratic and higher terms. The inexactness and thus theory uncertainty is then due to the dropped terms $f_2 x^2 + ...$ Scale variations essentially amount to performing a variable transformation $\tilde{x}$ such that $x = x(\tilde{x}) = \tilde{x} + b_0 \tilde{x}^2 + ...$, performing the expansion for f in terms of $\tilde{x}$, and using the difference between the expansions in x and $\tilde{x}$ as an estimate of Δf. While this method is convenient, it has many limitations, which I discuss. Another important issue is that of correlations between different predictions. Say we have two predictions $f(x) = f_0 + f_1 x \pm \Delta f$ and $g(x) = g_0 + g_1 x \pm \Delta g$ and we are interested in their ratio $f(x)/g(x)$. We expect or hope that some of the uncertainties would cancel in their ratio, but to quantify this we need to know the precise correlation between Δf and Δg. The perhaps most severe shortcoming of the scale-variation method is that it cannot tell us anything about correlations. The best we can currently do is after estimating Δf and Δg to make an educated guess on how to correlate them.

In the last part of the talk, I discuss the development of a new concept of theory nuisance parameters [27], which promises to overcome the limitations of scale variations. The basic idea is to consider the actual source of uncertainty, i.e. the missing coefficient f_2 in the above example, and parametrize it as $f_2 \equiv f_2(\theta)$, where the *theory nuisance parameters* θ are genuine nuisance parameters that have a true but unknown value. In the simplest case when f_2 is just a number, we could use it itself as the nuisance parameter, $f_2(\theta) = \theta$. This concept turns the theory uncertainty into a truly parametric uncertainty, and much of the following discussion was related to all the resulting advantages. For example, the theory of nuisance parameters could be constrained by auxiliary measurements. In the absence of such constraints, it is also possible to put a constraint on their size from theoretical considerations on the typically expected size of terms in perturbation theory.

Asymmetric Errors Roger Barlow, Huddersfield

In this talk, I presented the development of ideas about the handling of asymmetric errors in particle physics, which are quoted in many results and which are certainly not handled correctly. Preparation for the presentation forced me to develop my original ideas (which had been published only as preprints) and it became clear that the two classes of asymmetric error originally denoted 'systematic' and 'statistical' are, at a more fundamental level, asymmetries in the pdf and in the likelihood. Thus an apparently arbitrary taxonomy has been replaced by a logical division.

That this analysis was accepted by the audience was, for me, very encouraging, and a major paper on the subject is now in preparation. In the discussion session there were helpful suggestions about the use of the Edgeworth expansion and also Azzalini's skew Normal distribution, which I am incorporating in the treatment, in addition to the original two models I had considered.

I also received an offer of assistance with writing the software package(s) that will be needed to get this treatment adopted by practitioners in particle physics experiments. This working collaboration is very encouraging and helps move the project along. The paper and the package will produce a real impact on the treatment of this problem in the community.

Error on Error Enzo Canonero, RHUL

The Gamma Variance Model (GVM) is a statistical model that implements uncertainties in the assignment of systematic errors (informally called *errors-on-errors*), a pressing issue for many LHC analyses, currently or soon-to-be dominated by systematic uncertainties. The GVM treats discrepancies in input data as an indication of additional uncertainty, hence making a negligible impact on results when data are consistent. Conversely, when data are inconsistent, *errors-on-errors* significantly affect the results, thus influencing our final understanding of the parameters we aim to measure, both in terms of their central value and associated confidence intervals. To showcase this property of the model, an investigation of the tension between the latest CDF W-mass measurement, the Standard Model prediction, and the ATLAS measurement was presented, emphasizing how uncertainties in systematic error assignment can substantially affect our understanding of the W-mass.

When *errors-on-errors* are small, one can use standard asymptotic methods to construct confidence intervals for the parameters of interest and compute p-values. This is analogous to the familiar scenario with a large data sample, where all adjustable parameter estimators follow Gaussian distributions. However, if *errors-on-errors* are not small, first-order asymptotic distributions are not an accurate approximation. During the talk, we presented advanced test statistics based on higher-order asymptotics (r^* approximation and Bartlett correction), which are corrections to the first-order statistics so that their asymptotic distributions are a better approximation of the truth even in scenarios with small sample sizes, or equivalently large *errors-on-errors*.

The discussion after the talk focused on how *errors-on-errors* could be assigned in real applications to uncertain systematics, paying particular attention to theory systematics and two-point systematics. The impact of *errors-on-errors* on the W-mass application was also discussed, given the significant and non-trivial effects observed. A comparison was made between the GVM method of combining incompatible measurements and the approach of the Particle Data Group (PDG), along with potential Bayesian approaches to the problem.

Systematics in Neutrino physics data analyses Edward Thomas Atkin, Imperial

Systematic uncertainties in long-baseline neutrino experiments come from the description of the neutrino beam, the near and far detector responses and the modelling of neutrino-nucleus interactions. In particular, systematic uncertainties related to neutrino-nucleus interactions are very important as they change the reconstructed energy spectra of neutrino events which directly impacts the measurement of neutrino oscillation parameters. Unfortunately, neutrino interactions can still have large, often ill-formed, and ad-hoc uncertainties. The near detector is essential in reducing systematic uncertainties and measures the unoscillated neutrino beam. The far detector then measures neutrinos after they have oscillated and gives sensitivity to neutrino oscillation parameters. However, due to limited statistics, the far detector has very little power to constrain systematic uncertainties. Ensuring that the near detector correctly constrains systematic uncertainties is one of the main challenges of long-baseline experiments. If the constraint from the near detector is biased then measurements on oscillation parameters at the far detector will also be biased. This is often a challenge due to near and far detectors being non-identical detectors that span different neutrino fluxes which can also have quite different acceptances.

There were discussions on how neutrino interactions impact the analysis and the checks on the bias that a poor near detector constraint can have on the final contours. In addition, useful advice on how to treat systematics which migrate events between analysis bins was given.

I hope to continue working with Galin Jones on studying some of the statistical problems encountered by neutrino experiments, especially those related to the Markov Chain Monte Carlo analysis which T2K and other experiments use.

Machine Learning Michael Kagan, SLAC

As machine learning (ML) is used more and more in high energy physics (HEP) data analysis, a question that frequently arises is “when do we need uncertainties on the ML model?”. Before answering this question, one must first ask what uncertainties do we mean?

Within the topic of uncertainty quantification (UQ) in ML, uncertainties are typically broken into two categories: aleatoric uncertainties and epistemic uncertainties. Aleatoric uncertainties, often called statistical uncertainties, arise from the inherent randomness in a system, e.g. in stochastic systems, or noisy detectors. For example, a form of aleatoric uncertainty is detector resolution; the same incoming particle can give rise to a distribution of observed measurements. This uncertainty is often described as “irreducible” because no number of measurements will reduce these inherent stochastic effects. As aleatoric uncertainty arises from statistical sources, it is often described using probability distributions. Epistemic uncertainty, often called "model uncertainty", arises from a lack of knowledge about the system, e.g. from missing measurements or limited amounts of data used for fitting a model to data. These uncertainties are often considered reducible, as more data can often help constrain the set of models one would fit to a given data set. While the above-mentioned uncertainty classes assume a training set that is drawn from the same distribution as the data in which the model will be applied, when such distributions differ, uncertainty can arise due to such a domain/distribution shift. These distribution shift uncertainties are akin to Systematic uncertainties in HEP.

Methods to model both Aleatoric and Epistemic uncertainty can be found in the study of UQ in ML. Aleatoric uncertainty, being of a statistical nature, is often described using neural parameterized probability distribution, such as mixture density networks, or generative models such as normalizing flows. Epistemic uncertainty, not being of a statistical nature, often attempts to describe what set of parameters, such as weights in a neural network, could have been fit from a given training set. Attempts to model this include ensembling techniques and Bayesian techniques that attempt to estimate posteriors on weight distributions. As such posteriors can't be calculated analytically, a host of approximation methods have been developed.

When would such epistemic / model uncertainties be needed? In most cases in HEP, such as reconstruction or signal/background classification, the aim of the ML model is to define a good summary statistic for downstream parameter inference. This choice of summary statistic will determine the power of the statistical test, i.e. it will determine the optimality of the statistical test but will not make the model of the data incorrect. In these cases, epistemic uncertainty is likely not needed. However, if an ML model affects the prediction of the rates of process or shapes of distributions given a test statistic, as in background estimation of ML-based simulations, poor predictions will affect the compatibility of the model with real data. As such, epistemic uncertainty estimation is likely needed, or the development of procedures to estimate systematic uncertainties using control data. Ultimately, this question of optimality versus correctness should be addressed for each application of ML in HEP in order to determine if an associated model uncertainty should be estimated.

Using ML to reduce Systematics Tommaso Dorigo, INFN and Uni Padova

A variety of machine learning-powered techniques have been developed over the course of the past decade with the purpose of incorporating the effect of systematic uncertainties in the extraction of statistical inference on parameters of interest. This is particularly useful in multi-dimensional problems, where direct parametrizations are not feasible. Dorigo's presentation briefly touched on the various ideas that have been used to develop models that include a correct treatment of nuisances in typical HEP problems. For a description of these techniques see a recent publication [28].

The realization that the incorporation of nuisance parameters in inference extraction is a holistic optimization problem, where consideration of nuisance-related effects corresponds to a realignment of the statistical procedures with the final goal of the measurement, and the parallel observation that machine learning today offers powerful solutions to such end-to-end optimization tasks, has recently led Dorigo to assemble a group of physicists and computer scientists in order to consider the possibility of modeling together in a single problem not only all aspects of statistical inference, but also the data collection and reduction procedures, and ultimately the data generation procedures. This allows to encode in a global pipeline the full design of an experiment, from the choice of materials and geometry of the detector layout to the pattern recognition and analysis procedures. Such an ambitious goal can today be achieved for systems of moderate complexity through the use of backpropagation of derivatives of a global utility function in a full model of the experiment. The utility must specify in a the faithful way the relative importance of different goals of the experimental endeavour, not dissimilarly to what is done when defining the bandwidth allocated to different triggers in a collider experiment. The MODE collaboration [29], which includes 40 computer scientists and physicists from 24 institutes in Europe, Asia and North America, has thus started a program of investigation to attack simpler end-to-end optimization tasks, to build expertise and a library of solutions and methods to tackle progressively harder optimization problems. A recent publication [30] describes the status of these activities.

Systematics in ML model-independent searches Gaia Grosso, CERN and Uni Padova

In my talk, I presented New Physics Learning Machine (NPLM), a novel machine-learning-based strategy to detect and quantify data departures from a Reference model with no prior bias on the source of discrepancy

[31, 32]. The main idea behind the method is to approximate the log-likelihood-ratio hypothesis test parametrising the data distribution with a universal approximating function, and solving its maximum-likelihood fit as a machine-learning problem with a customised loss function. The method returns a p-value, which measures the compatibility of the data with the Reference model. NPLM has been recently extended in order to deal with the uncertainties affecting the Reference model predictions [33]. The new formulation directly builds on the specific maximum-likelihood-ratio treatment of uncertainties as nuisance parameters, that is routinely employed in high-energy physics for hypothesis testing. The main goal of this presentation was to convey the key steps needed to correctly set up the strategy and validate its robustness in the presence of systematic uncertainties. Some theoretical aspects to be further investigated were pointed out in the conclusions, and a rich discussion emerged from them. In particular, Alessandra Brazzale pointed out that our construction of the family of alternative hypotheses by means of universal approximators (like neural networks) with exponential parametrization could be seen as a generalization of the Exponential family. She also noticed possible connections between our work and the research on nonparametric extensions of the likelihood-ratio-test (see for instance [34]). I am studying the literature and I plan to further discuss this topic with Alessandra. After the presentation, we also discussed the possibility of estimating an error for the outcome of the test by running the algorithm multiple times with different random initializations of the model trainable parameters. We commented on the need to explore smarter regularization schemes to improve the test sensitivity. We discussed the globality of the p-value returned by the algorithm; in particular, we agreed that we should think about how to fairly compare our method with alternative approaches that quote local p-values. Finally, we started discussing similarities and differences between the NPLM approach and the approach presented in the following talk by Purvasha Chakravarti. The two groups are now interacting to work on a comparison of the methods.

Model-Independent Search using Interpretable Semi-Supervised Classifier Tests Purvasha Chakravarti, UCL

In my talk, I introduced some of the recent developments in model-independent searches using classifiers presented in [35]. The aim is to search for new signals that appear as deviations from known Standard Model physics in high-dimensional particle physics data. The key contributions and main discussion points during the talk were:

- Introduced three test statistics using the classifier: an estimated likelihood ratio test (LRT) statistic, an area under the ROC curve (AUC) based test statistic, and a misclassification error (MCE) based test statistic.
- Introduced a method for estimating the signal strength parameter and active subspace methods to interpret the classifier in order to understand the properties of the detected signal.
- Discussed the performance of the methods on a simulated data set related to the search for the Higgs boson at the Large Hadron Collider at CERN and showed that the semi-supervised tests have power competitive with the classical supervised methods for a well-specified signal, but much higher power for an unexpected signal which might be entirely missed by the supervised tests.
- Discussed that the introduced model-independent approaches are sensitive to the background reference data set and the systematic uncertainties in them. However, we can still use the methods to identify and characterize regions of high-dimensional space where the background is mismodelled and/or perform pilot analysis to guide future model-independent searches.
- Discussions included detailed comparisons with the approaches in [33], the first contribution towards dealing with systematic uncertainties in model-independent searches.

Systematics in rare B-decays Slavomira Stefkova, KIT

Rare B-decays are excellent probes of new physics (NP). These are either flavor-changing-neutral-current transitions and/or helicity-suppressed decays and therefore the Standard Model (SM) contribution is small. NP contribution in comparison, if present, could be significant. Two flavour physics experiments, LHCb and Belle II, search for NP in rare B-decays directly (dedicated NP searches) and indirectly (SM precision measurements). In this talk, systematic uncertainties and their treatment in rare B-decays were presented. Given that LHCb and Belle II make use of different collider technologies, not only measurement style but also considered systematic uncertainties are different. Systematic uncertainties are included as nuisance parameters in binned or unbinned likelihoods when building a statistical model. There are three main challenges when building statistical models for rare B-decays. Many analyses find themselves in the low-statistics regime after all the selection and asymptotic approximation may no longer be valid. In addition, the trade-off between smoothness and fit stability, which heavily depends on the included systematic uncertainties, is often an important part of the analysis. The second challenge is especially applicable to LHCb, where all the branching fraction measurements are done relative to another control channel so that the systematic uncertainties due to accelerator and acceptance effects cancel in the ratio. For rare B-decays it is sometimes challenging to find a good channel to normalise to. The final challenge is that backgrounds for rare B-decays could be themselves rare and they may have never been measured or even theoretically predicted. The difficulty with such cases is how to correctly include these rare B-decay backgrounds in the statistical model and assign the appropriate systematic uncertainties to them. Apart from these challenges, the main discussion item evolved around the possibility to make a combination between LHCb and Belle II measurements. At the moment, HFLAV collaboration makes naive combinations but in the future, this can be further refined by publishing the full likelihoods and by correlating the systematic uncertainties, wherever possible, from the two experiments.

Accounting for systematic uncertainties in unfolding uncertainty quantification Michael Stanley, CMU

This talk discussed the computation of bin-count confidence intervals for particle unfolding uncertainty quantification. We first provided the mathematical framework for unfolding and then outlined four system uncertainty sources; regularization bias, wide-bin bias, missing auxiliary variables, and response kernel uncertainty. The first two sources were discussed in some detail in which we identified the primary systematic uncertainty culprit as the Monte Carlo ansatz used in place for the true particle intensity function. As such, our solution aims to reduce the error resulting in this ansatz misspecification by first unfolding to fine bins followed by an adjacent bin aggregation to a desired wide-bin resolution. Further, we show the implementation of two optimization-based statistical methods to directly compute confidence intervals for each desired aggregated wide bin.

This work is thoroughly explored and developed in [36], which is based on the following works [37, 38, 39]. To make these methods accessible to the particle physics community, we next plan to work with Lydia Brenner toward their implementation in RooUnfold.

Systematic uncertainties in unfolding of differential measurements Richard Zhu, CMU

In my talk, I addressed the systematic uncertainties in the forward model for unfolding differential measurements.

The goal of unfolding is to recover the true particle spectrum based on the smeared detector measurement. The forward model (response kernel) models the detector response, which is the conditional probability of the smeared observations given the truth. In practice, the forward model needs to be estimated using detector simulations. The imperfect knowledge of both the detector alignment and calibration as well as the theoretical predictions can affect the forward model in different ways. This leads to systematic uncertainties in the forward model and hence raises nontrivial effects on the unfolding solutions.

To formalize the problem, recall that we have the particle-level spectrum f and detector-level spectrum g related by the Fredholm integral operator $g = \int_T k(y,x)f(x)dx$. The response kernel k inside the integral represents the conditional density of the smeared measurement given the truth, i.e. $k(y,x) = p(y|x)$. The systematic uncertainty in k means that a set of alternative kernels $k_0, k_1, ..., k_m$ might be plausible during detector simulation. The question is how should we account for this systematic uncertainty in the unfolding solution. For simplicity, suppose there are two base kernels k_0, k_1. Then we propose to use optimal transport to morph between k_0 and k_1, which defines a geodesic of kernels $\{k_t : 0 \leq t \leq 1\}$ between the two kernels. Since optimal transport preserves the shape and geometry of the kernel, the geodesic of kernels can be served as the plausible candidate kernels. Then we propagate the geodesic of kernels and unfold with the 'One-at-a-time Strict Bounds' (OSB) intervals proposed by Stanley et al. [36], which results in a collection of confidence sets for λ. The collection of confidence sets has the correct frequentist coverage if the unknown correct kernel lies on the geodesic.

Some open problems and the following discussion include: (1) How can we summarize the collection of confidence sets (dependent on $\mathbf{t}$) into a single confidence set? Can we do profile likelihood? (2) We can view the weight $\mathbf{t}$ as a nuisance parameter. Can we learn $\mathbf{t}$ from the data? (3) How do we apply the method on real HEP and how well does it work?

Banff Challenge 3 Tom Junk, FNAL

Banff Challenge 3 is an exercise designed to test the ability of participants to estimate systematic uncertainties and apply these estimates in the calculation of confidence intervals in a contained environment, without requiring detailed knowledge of elementary particle physics or the accelerators and detectors experimental physicists use. Simulated random data and randomly-sampled model predictions are provided, drawn from two-dimensional probability distributions. The underlying functional forms of the probability distributions, and thus the parameters they depend on, are hidden from challenge participants. The simulated data samples contain mixtures of signal and background data, and the simulated model samples are separated into labeled sets of signal and background data. Participants are to provide confidence intervals on the rate of signal counts in each of 100 simulated data samples. The exercise is patterned after a CDF measurement of the W boson cross section [40], although the probability distributions and parameters are completely artificial. The full problem statement and data files are available at [41].

Thoughts on the meeting session

The concluding talks were given by Physicist Andre David and Statistician Mikael Kuusela. Rather than being required to provide a comprehensive review of the whole meeting, they were asked to give us their thoughts on what they considered were some of the interesting talks and discussions.

Physicist's view Andre David, CERN

Of signals and backgrounds When discussing systematic uncertainties in HEP analyses, one is easily taken to the notion of nuisance parameters and how to deal (usually meant in the "get rid of" sense) with said nuisances. A common trap is to think of the different physics processes involved in ways that are hierarchical, like signal and background, with backgrounds being considered nuisances and signals being of interest.

This is a shortsighted view since in most cases improvements come about by not only a better (more accurate) modelling of the "signal" process but also (if not more) by a better modelling of the "background" processes. There are also cases where the sensitivity to an underlying theory parameter is "spread" among different processes that, in some cases, are not distinguishable at a fundamental, quantum-mechanical, level. A more holistic view when dealing with uncertainties on different processes is expected to become even more important as the field moves towards performing broader combined analyses and interpretation is performed in terms of effective field theories, like the SMEFT, whose parameters have wide-reaching and pervasive effects across whole classes of processes.

One should carefully distinguish and treat the following: processes sensitive to the inference one wants to make, processes that are not sensitive but in many cases may limit the power of said inference, and detector limitations, like noise and finite resolution. The more broadly a result is expected to be used, the more important these matters are, since what is my signal may be your background and vice-versa.

Alternatives and morphing One important argument made during the workshop is that one should be careful to distinguish physical deformations for which the intermediate values have a physical meaning (e.g., the effect of an energy scale uncertainty) and alternatives between differing options, the interpolation of which is devoid of a concrete physical interpretation and meaning. The important distinction (made by Cousins) is that for the latter if at any point the results of the inference end up depending on them, then there will be enough power in the data and the model to likely rule one out as unphysical.

This reasoning removes some of the pressure on how to deal with "pure alternatives" and may point the way forward in establishing procedures for working through them.

Progress in unfolding Unfolding detector-level quantities is an important step to make experimental results more readily used by the broader community. E.g., unfolding allows theorists to compare their results without having to go through the computationally intensive and error-prone task of forward-folding detector effects that are specific to each detector.

In this area, a lot of progress has been made in the recent past thanks to new statistical methods and that effort continues unabated.

My measurement or your calibration? An interesting discussion that arose at the workshop relates to the difference between calibration and measurement and how there are even separate communities in astrophysics.

The interesting point here is how to ensure that the measurement part of the chain does not overstep the stated calibration accuracy and precision and, in case it can legitimately improve the calibration, how to go about that. In the LHC collaborations the "calibrators" and "measurers" are all part of the same community and it was reported that both types of interactions have taken place. I.e., there are cases where a measurement is "blunted" to stay within credible calibration boundaries and cases where a measurement led to improvements in the calibration procedures.

This interplay between measurement and calibration ends up being slightly artificial and two sides of the same coin of what information can be exploited and whether it has been considered in the calibration process or not. In that sense, it is important in the context of systematic uncertainties to keep a two-way road between "calibrators" and "measurers".

The discrete profiling miracle Discrete profiling made a splash in the scene of HEP and after a decade it is not yet clear why it achieves the frequentist properties it seems to have. Perhaps there is hope to understand discrete profiling in the model selection context as discussed at the workshop. The attending question with model selection came as to how would one feel about model averaging being the (weighted) average of estimates across different models and physicists seemed to not be too comfortable with this idea.

There was also a very concrete discussion on the methods that CMS (discrete profiling) and ATLAS (spurious signal) have employed to tackle the lack of knowledge on the background shape and the main point is that both of them use the statistical uncertainty under the signal as their gauge. In that sense, they trace their use to the same underlying quantity and even though they then are used in different ways, can still be expected to have similar impacts.

Unleash the tails The workshop saw a very interesting discussion on how certain one is of a given systematic uncertainty by allowing for longer and heavier tails, therefore allowing for the corresponding nuisance parameters to be pulled farther than one would otherwise consider.

The technique seems to be very well suited for application to theory uncertainties, not the least because it could be interesting what experimentalists from experiment i would say about how certain they are of the systematic uncertainties of experiment k, especially when $i = k$.

"Last mile" corrections The workshop saw a large number of machine learning topics discussed and among the most interesting are those related to optimal transport in the context of fine corrections. In a typical experiment, there will be the main, general-purpose, corrections to observables that are called calibrations, and then there is a second (or third) level of corrections that are small and dedicated to particular corners of the phase-space of reconstructed properties, e.g., the electromagnetic shower shapes of low-energy photons.

"Last mile" corrections can be cast as finding (learning) awhat multidimensional function that takes MC-simulated events and matches actual data in a control region. This function is then used to "correct" MC events in the region of interest when performing inference.

Depending on the level of discrepancy, it may be that optimal transport methods presented at the workshop can become a new tool in performing this kind of "last mile" correction that will become ever more important in precision analyses.

ML for actual intelligence The rise of machine learning is reshaping the way we think about analyses and the use of information. Many applications are possible and almost all are being pursued: detector operation, construct (optimal) observables, design (optimal) detectors, model-independent methods (vs MC simulation), sample sizes, skirt systematically-affected phase spaces, etc. Now that creativity in the applications is partly exhausted, we need to become proficient at writing loss functions that allow these algorithms to explore outside the box. Only then will these algorithms become effective support systems for AI (actual intelligence, as defined by Cousins) i.e., us.

That all said, it is good to see that there is now a general consensus that a bad ML algorithm in constructing an observable will not lead to a wrong result, just a sub-optimal one. Of course, this cannot be said of all ML algorithms, especially generative ones, where inaccuracies can have dire consequences.

Statisticians view Mikael Kuusela, CMU

In my talk, I highlighted selected thoughts I had about the meeting topics and discussions from the perspective of a statistician who actively collaborates with high-energy physicists. The talk had six key messages:

1. **Beyond profiling/marginalization:** We heard a lot about profiling and marginalization for handling nuisance parameters during the meeting. But are there other ways of handling nuisance parameters? Specifically, in the setting of Gaussian linear models, we know how to get exact confidence intervals for individual model parameters despite the presence of nuisance parameters. How do we extend this to more general settings?

2. **Which confidence set to report?** High-energy physicists usually report ellipsoidal confidence sets but, in some settings, it could also make sense to report hyperrectangle confidence sets, as was illustrated in Galin Jones's talk. Similarly, many talks and discussions during the meeting raised the question of whether high-energy physicists should consider reporting simultaneous confidence intervals instead of one-at-a-time confidence intervals.

3. **Systematics in other fields:** Handling systematic uncertainties is a widespread challenge across the physical sciences. I showed an illustrative example of this from a recent intercomparison of ocean heat content estimates. The error-on-error model presented by Enzo Canonero could prove useful in these other fields.

4. **How to best learn likelihood ratios?** We heard from Gaia Grosso and Purvasha Chakravarti about model-independent searches of new physics using likelihood ratio tests where the test statistic is learned from the data using machine learning. In the case of Grosso, the alternative hypothesis is modeled using a neural network while Chakravarti trains a classifier to learn the likelihood ratio. This leads to two different ways of learning the likelihood ratio so I posed the important open question about the pros and cons of these two ways of performing the test.

5. **Model discrepancy:** Instead of using nuisance parameters, one can handle systematic uncertainties by writing down a stochastic model to capture the model discrepancy. This is a common strategy in computer model calibration. I briefly explained the popular Kennedy–O'Hagan method [42] for modeling model discrepancy as an additive Gaussian process.

6. **Interdisciplinary collaboration:** How do we sustain and expand the interdisciplinary interaction between physicists and statisticians beyond workshops like this? How do we achieve active in-depth collaborations between the two fields? I suggested co-supervision of Ph.D. students, where a physicist is a co-advisor of a statistics student and a statistician is a co-advisor of a physics student, as a potential avenue for this. This is a model that has worked successfully for us at Carnegie Mellon in several interdisciplinary projects.

Connections

The meeting benefitted from the unique atmosphere at Banff and was characterised by vigorous and productive discussions, especially between Physicists and Statisticians. From a subjective physicist's standpoint these exchanges can be roughly categorized in:

- Statisticians *agreeing* with what physicists do, e.g. the asymmetric uncertainty procedures presented by Roger Barlow.
- Statisticians *questioning* what physicists do, e.g. Sara Algeri about checking the validity of regularity conditions required by classical statistics when dealing with complex models.
- Statisticians *suggesting* physicists what to do, e.g. Anthony Davison on making use of the tangent exponential model for improved statistical inference.
- Statisticians *competing* with the physicists, e.g. on Machine learning algorithms for anomaly detection, as presented by Gaia Grosso (P) and Purvasha Chakravarti (S); the symbol S denotes a Statistician and P is for Physicist.

The organizers encouraged participants to continue and expand the established contacts, which could be very useful. Some of these are listed below:

- Galin Jones (S) and Ed Atkin (P): Implementing MCMC methods in neutrino analyses.
- Lydia Brenner (P) and Michael Stanley (S): Incorporating unfolding procedures in Particle Physics libraries.
- Roger Barlow (P) and Igor Volobouev (P): Methods for dealing with asymmetric uncertainties.
- Gaia Grosso (P) and Purvasha Chakravarti (S): Procedures for model-independent searches for New Physics.
- Tudor Manole (S) and Philipp Windishhofer (P): Optimal Transport.
- Tommaso Dorigo (P) and Ann Lee (S): End-to-end experimental design.
- Ann Lee (S) and Lukas Heinrich (P), Michael Kagan (P) and Philipp Windischhofer (P): Calibrated forecasting and Optimal Transport.
- Alessandra Brazzale (S), Purvasha Chakravarti (S) and Gaia Grosso (P): Properties of ML methods.
- Sara Algeri (S) and Lydia Brenner (P): Format for future interaction between Physicists and Statisticians.

Conclusions

We are very appreciative of BIRS having provided us with the opportunity of having this meeting on the important role of systematics in Particle Physics analyses. The ambiance of the Banff Centre was ideal for fostering dialogue, and we made use of this by having more time for discussions than for talks during the formal sessions. In addition, many of the interesting interactions took place during the coffee breaks, meals, pub visits, outdoor excursions, and on our dedicated Slack channel.

The presence of a good number of Statisticians encouraged the idea of incorporating them in Physics analyses, for the mutual benefit of Physicists and Statisticians. We are pleased that, as a result of this Workshop, there appear to be several examples in which collaborative work is already happening. We hope this will significantly improve the treatment of systematic uncertainties in our data analyses.

Participants

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Chapter 12

Applications of Stochastic Control to Finance and Economics (23w5011)

April 30 - May 5, 2023

Organizer(s): Jakša Cvitanić (California Institute of Technology), George Georgiadis (Northwestern University), Dylan Possamaï (ETH Zürich), Nizar Touzi (École Polytechnique)

Overview of the Field

A distinguished feature of the last few decades is the increasing availability of technological innovations to an everyday broader spectrum of society. This phenomenon has rapidly paved the way for the development of new economies such as e-commerce, sharing economies and online advertising. These activities are characterised by, for instance, a higher number of agents from both the supply and demand side, and an unprecedentedly large menu of tailor-made and personalised services. Naturally, this has led to more intricate behaviours and interactions at all levels of the economy. Consequently, there is a pressing need for the creation of economic theories that are able to explain the rationale of the participants, the way their actions are aggregated in society, and which are compatible with documented evidence.

As such, stochastic optimal control theory applied to problems in economics and finance has recently evolved into one of the most active and exciting areas of contemporary applied mathematics. It provides the mathematical community with a rich source of challenges, most of which stem from the deep insights of economists, which, in turn, enrich society's understanding of the complex dynamics which underline its functioning, helping all involved actors to make better and more informed choices. The proximity of these theories to practice, in addition to their purely mathematical appeal, make this field especially attractive to young mathematicians. Moreover, it provides for wider employment opportunities both within the academic world and outside of it.

One of the most challenging aspects has nonetheless remained, in the sense that the communication of recent mathematical results to the economics community has been facing a certain number of hurdles, as the language used is often not the same, making the interactions unfortunately limited. This workshop was aimed at fostering the interactions between researchers from both the theoretical and practical side, and at being one of the limited spaces where a significant number of representatives of these two communities, which do not often meet, can dedicate themselves to the dissemination of new theoretical results and their implementation in the design of new economic models.

Presentation Highlights

Beatrice Acciaio

'Equilibria as solutions to Schrödinger problems'

ABSTRACT. I will present a connection between solutions to equilibrium problems in markets with asymmetric information, and solutions of Schrödinger problems. In particular, we will see how in some cases the former ones can be found as limit of the latter ones. Based on joint ongoing work with Umut Çetin.

René Aïd

'A stationary mean-field equilibrium model of irreversible investment'

ABSTRACT. We develop a stationary mean-field model of singular control irreversible investment where production capacity is subject to random gaussian fluctuations while prices are affected by macroeconomic shocks following a two-state Markov chain (boom/burst episodes). We prove existence and uniqueness of the mean-field stationary equilibrium and we characterise it through a system of nonlinear equations. We provide several insights on the effects of crisis on the equilibrium, in particular on the firms size distribution and on the firms value at equilibrium. Joint work with Giorgio Ferrari and Matteo Basei.

Alexander Bloedel

'Persistent private information revisited'

ABSTRACT. This paper revisits [9] (henceforth PPI's) continuous-time principal-agent model of optimal dynamic insurance with persistent private information. We identify three independent issues in PPI that implicate its characterisations of incentive compatible and optimal contracts: (i) the agent cannot over-report increments of his type, a constraint that does not follow from the common assumption that the agent cannot over-report his type; (ii) the agent's feasible set of reporting strategies does not include standard 'no Ponzi' constraints, without which PPI's main analysis of infinite-horizon incentive compatibility is incomplete; and (iii) most importantly, in PPI's main application, which concerns hidden endowments, the contract identified as optimal is generically strictly suboptimal. For this application, we address the first two issues and elucidate the third one by analysing a class of 'self-insurance contracts' that can be implemented as consumption–saving problems for the agent, and which includes the contract derived in PPI as a particular case. We characterise the optimal self-insurance contract and show that, generically, it strictly dominates PPI's. Our analysis does not support PPI's main economic finding that immiseration generally fails or its attribution of this failure to continuous time and persistence. (Joint work with R. Vijay Krishna and Bruno Strulovici).

Umut Çetin

'Order routing and market quality: Who benefits from internalisation?'

ABSTRACT. The practice of retail internalisation has been a controversial topic since the late 1990s. The crux of this debate is whether this practice benefits, via the price improvement relative to exchange, or disadvantages, via the reduced liquidity on exchange, retail traders. To answer this question we set two models of market design that differ in their mode of liquidity provision: in the model capturing retail order internalisation the liquidity is provided by market makers (representing wholesalers) competing for the retail order flow in a Bertrand fashion, whereas in the model characterising the open exchange the price-taking competitive agents act as liquidity providers. We discover that, when liquidity providers in both market designs are risk averse, routing of the marketable orders to the wholesalers is preferred by retail traders: informed, uninformed and noisy. In addition to addressing optimal order routing problem, we identify a universal parameter that allows comparison of market liquidity, profit and value of information across different markets and demonstrate that the risk aversion of liquidity providers fundamentally changes market outcomes. In particular, we observe mean reverting inventories, price reversal, and lower market depth as the result of retail investor (informed or not) absorbing large shocks in their inventory to compensate for the unwillingness of liquidity providers to bear risk.

Tahir Choulli

'Optimal stopping and reflected BSDEs for models under arbitrary random horizon'

ABSTRACT. We consider the setting described by $(\Omega, \mathcal{G}, \mathbb{F}, \tau, \mathbb{P})$. Here $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space, called initial model hereafter, in which $\mathbb{F}$ represents the flow of information available to all agents throughout time. τ is an arbitrary random time, which might not be observable via $\mathbb{F}$, and represents default time of a firm in credit risk theory, or death time of an insured in life insurance, and/or the occurrence time of any event that might impact the market somehow. In this framework of informational model, various valuation approaches and various hedging methods —which we are developing— point to the following family of reflected backward stochastic differential equations (RBSDE hereafter)

$$\begin{cases} \mathrm{d}Y_t = f(t, Y_t, Z_t)\mathrm{d}(t \wedge \tau) + g(t, Y_t, Z_t)\mathrm{d}U_t^\tau + Z_t \mathrm{d}W_t^\tau + \mathrm{d}M_t - \mathrm{d}K_t,\ Y_\tau = \xi, \\ Y \geq S \text{ on } [0, \tau],\ \displaystyle\int_0^\tau \big(Y_{s-} - S_{s-}\big)\mathrm{d}K_s = 0, \mathbb{P}\text{–a.s.} \end{cases}$$

Here W is an $\mathbb{F}$–Brownian motion, and (f, g, ξ, S, U) is the data-quintuplet in which U is an RCLL nondecreasing and $\mathbb{F}$-adapted process, f and g are the drivers functionals, ξ is the terminal condition and the barrier process S is RCLL $\mathbb{F}$-adapted. We allow the initial model $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ to be chosen as ""'nice enough"" as possible if needed, while τ is let to be as arbitrary general as possible. This will allow to measure adequately the impact of τ on the initial model in various aspects without interferences. In my talk, I will single out the challenges arising from allowing τ to be arbitrary general, and I will detail our approaches for these challenges and the intermediate problems arising from them as well. Among these, we address the impact of τ on the mathematical structures of these RBSDEs, which yields to the mathematical structures in the optimal stopping problem under random horizon τ. Besides being intimately related to RBSDEs and vital in addressing the mathematical structures, the optimal stopping problem is interesting on its own and is useful in solving other financial problems. This talk is based on joint works with Safa Alsheyab.

Théo Durandard

'Under pressure: comparative statics for optimal stopping problems in non-stationary environments'

ABSTRACT. We formulate a general optimal stopping problem that can represent a wide variety of non-stationary environments, *e.g.*, where the decision maker's patience, time pressure, and learning speed can change gradually and abruptly over time. We show that, under some mild regularity conditions, this problem has a well-defined solution. Furthermore, we characterise the shape of the stopping region in a large class of "monotone"

environments. As a result, we obtain comparative statics on the timing and quality of decisions for many sequential sampling problems *à la* Wald. For example, we show that accuracy is increasing (decreasing) over time when (i) learning speed increases (decreases) in time, or (ii) the discount rate decreases (increases) over time (i.e., the decision maker values the future more over time), or (iii) time pressure decreases (increases) over time. Since our main comparative static results hold locally, we can also capture non-monotone relations between time and accuracy that consistently arise in both perceptual and cognitive testing.

Christoph Frei

'Principal trading arrangements: optimality under temporary and permanent price impact'

ABSTRACT. We study the optimal execution problem in a principal-agent setting. A client (for example, a pension fund, endowment, or other institution) contracts to purchase a large position from a dealer at a future point in time. In the interim, the dealer acquires the position from the market, choosing how to divide his trading across time. Price impact may have temporary and permanent components. There is hidden action in that the client cannot directly dictate the dealer's trades. Rather, she chooses a contract with the goal of minimizing her expected payment, given the price process and an understanding of the dealer's incentives. Many contracts used in practice prescribe a payment equal to some weighted average of the market prices within the execution window. We explicitly characterise the optimal such weights: they are symmetric and generally U-shaped over time. The talk is based on joint work with Markus Baldauf (University of British Columbia) and Joshua Mollner (Northwestern University).

Paolo Guasoni

'Holding stocks, trading bonds'

ABSTRACT. In an economy with random growth, several long-lived agents with heterogeneous risk aversions, time-preferences, and personal income streams make consumption and investment decisions, trading stocks and a consol bond, and borrowing from and lending to each other. We find in closed form equilibrium stock prices, interest rates, consumption, and trading policies. Agents do not trade stocks, although their returns are time-varying and predictable. Agents dynamically trade the consol bond in response to growth shocks, as to hedge their effect on interest rates, dividends, and personal incomes. Static fund separation holds if agents have also access to a linear bond and two additional hedges for dividend and growth shocks. Such additional assets can be dynamically replicated with the stock and the consol bond. No representative agent exists.

Nicolás Hernández Santibáñez

'A continuous-time model of self-protection'

ABSTRACT. We present an optimal linear insurance demand model, where the protection buyer can also exert a time-dynamic costly prevention effort to reduce her risk exposure. This is expressed as a stochastic control problem that consists in maximising an exponential utility of a terminal wealth. We assume that the effort reduces the intensity of the jump arrival process and interpret this as dynamic self-protection. We solve the problem by using a dynamic programming approach, and we provide a representation of the certainty equivalent of the buyer as the solution to a backward stochastic differential equation (BSDE). Using this representation, we prove that an exponential utility maximiser has an incentive to modify her effort dynamically only in the presence of a terminal reimbursement in the contract. Otherwise, the dynamic effort is actually constant, for a class of compound Poisson

loss processes. If there is no terminal reimbursement, we solve the problem explicitly and identify the dynamic certainty equivalent of the protection buyer. This shows in particular that the Lévy property of the loss process is preserved under exponential utility maximisation. We also characterise the constant effort as the unique minimiser of an explicit Hamiltonian, from which we can determine the optimal effort in particular cases. Finally, after studying the dependence of the BSDE associated to the insurance buyer on the linear insurance contract parameter, we prove the existence of an optimal linear cover that is not necessarily zero or full insurance.

Camilo Hernández

'Propagation of chaos for Schrödinger problems with interacting particles'

ABSTRACT. In this work, we study the mean field Schrödinger problem from a purely probabilistic point of view by exploiting its connection to stochastic control theory for McKean–Vlasov diffusions. Our main result shows that the mean field Schrödinger problem arises as the limit of 'standard' Schrödinger problems over interacting particles. Due to the stochastic maximum principle and a suitable penalisation procedure, the result follows as a consequence of a novel quantitative propagation of chaos result for forward–backward particle systems. Our stochastic control technique further allows us to solve the mean field Schrödinger problem and characterise its solution, the mean field Schrödinger bridge, by a forward–backward planning equation. The approach described in the paper seems flexible enough to address other questions in the theory. For instance, it allows us to study zero noise limits for the Schrödinger problem.

Julien Hugonnier

'Asset pricing with costly short sales'

ABSTRACT. We study a dynamic general equilibrium model with costly-to-short stocks and heterogeneous beliefs. The closed-form solution to the model shows that costly short sales drive a wedge between the valuation of assets that promise identical cash flows but are subject to different trading arrangements. Specifically, we show that the price of an asset is given by the risk-adjusted present value of future cash flows which include both dividends and an endogenous lending yield. This formula implies that returns satisfy a modified CAPM and sheds light on recent findings about the explanatory power of lending fees in the cross-section of returns. In particular, we show that once returns are appropriately adjusted for lending fees, stocks with low and high shorting costs offer similar risk-return tradeoffs.

Kostas Kardaras

'Portfolio choice under taxation and expected market time constraint'

ABSTRACT. We consider the problem of choosing an investment strategy that will maximise utility over distributions, under capital gains tax and constraints on the expected liquidation date. We show that the problem can be decomposed in two separate ones. The first involves choosing an optimal target distribution, while the second involves optimally realising this distribution via an investment strategy and stopping time. The latter step may be regarded as a variant of the Skorokhod embedding problem. A solution is given very precisely in terms of the first time that the wealth of the growth optimal portfolio, properly taxed, crosses a moving stochastic (depending on its minimum-to-date) level. The suggested solution has the additional optimality property of stochastically minimising maximal losses over the investment period.

Nabil Kazi-Tani

'The role of correlation in diffusion control ranking games.'

ABSTRACT. In this talk, we will study Nash equilibria in two player continuous time stochastic differential games with diffusion control, and where the Brownian motions driving the state processes are correlated. We consider zero-sum ranking games, in the sense that the criteria to optimise only depends on the difference of the two players' state processes. We explicitly compute the players' equilibrium strategies, depending on the correlation of the Brownian motions driving the two state equations: in particular, if the correlation coefficient is smaller than some explicit threshold, then the equilibrium strategies consist of strong controls, whereas if the correlation exceeds the threshold, then the equilibrium controls are mixed strategies. To characterise these equilibria, we rely on a relaxed formulation of the game based on solutions to martingale problems, allowing the players to randomise their actions. The talk is based on a joint work with Stefan Ankirchner (University of Jena) and Julian Wendt (University of Jena).

Daniel Kršek

'Relaxed principal–agent problem'

ABSTRACT. We study a principal–agent problem with a lump-sum payment on a finite-time horizon. Extending the dynamic programming approach in [3] we consider problems involving constraints on the optimal contract. We introduce a framework, in which the agent is allowed to randomise his actions. This in turn gives more freedom to the principal, when she chooses the contract, and allows us to show existence of an optimal contract in problems with fairly general constraints, for which the standard PDE approach ceased to be tractable.

Ali Lazrak

'Democratic policy decisions with decentralised promises contingent on vote outcome'

ABSTRACT. We study how decentralised utility transfer promises affect collective decision-making by voting. Committee members with varying levels of support and opposition for an efficient reform can make enforceable promises before voting. An equilibrium requires stability and minimal promises. Equilibrium promises exist and are indeterminate, but do share several key characteristics. Equilibria require transfer promises from high to low intensity members and result in enacting the reform. When reform supporters lack sufficient voting power, promises must reach across the aisle. Even if the coalition of reform supporters is decisive, promises must preclude the least enthusiastic supporters of the reform from being enticed to overturn the decision. In that case, equilibrium promises do not need to reach across the aisle. We also discuss a finite sequence of promises that achieve an equilibrium.

Marcel Nutz

'Unwinding stochastic order flow in a central risk book'

ABSTRACT. We study the optimal execution problem for the Central Risk Book (CRB), a centralised trading unit recently established in many large banks and trading companies. The CRB aggregates orders from the other business units within the organisation in real time, netting opposite orders and executing outstanding orders such as to minimise transaction costs. Thus, the in-flow orders of the CRB are a stochastic process. We introduce a tractable model for the price impact and spread cost paid by the out-flow orders and find the optimal execution strategy for a general class of in-flow processes. The strategy highlights how future in-flows are taken into account to determine the optimal trade-off between trading speed and transaction costs. (Joint work with Kevin Webster and Long Zhao.)

Huyên Pham

'Generative modelling for time series via Schrödinger bridge '

ABSTRACT. We propose a novel generative model for time series based on Schrödinger bridge (SB) approach. This consists in the entropic interpolation via optimal transport between a reference probability measure on path space and a target measure consistent with the joint data distribution of the time series. The solution is characterised by a stochastic differential equation on finite horizon with a path-dependent drift function, hence capturing the temporal dynamics of the time series distribution. We can estimate the drift function from data samples either by kernel regression methods or with LSTM neural networks, and the simulation of the SB diffusion yields new synthetic data samples of the time series. The performance of our generative model is evaluated through a series of numerical experiments. First, we test with a toy autoregressive model, a GARCH Model, and the example of fractional Brownian motion, and measure the accuracy of our algorithm with marginal and temporal dependencies metrics. Next, we use our SB generated synthetic samples for the application to deep hedging on real-data sets. Finally, we illustrate the SB approach for generating sequence of images. Based on joint work with M. Hamdouche and P. Henry-Labordère

Zhenjie Ren

'Uniform-in-time propagation of chaos for mean-field Langevin dynamics'

ABSTRACT. In recent times, there has been a growing interest in the study of mean-field Langevin (MFL) dynamics, primarily due to its natural application in training two-layer neural networks. To simulate the MFL dynamics' invariant distribution, one relies on the corresponding N-particle system, hoping that the error between the particle system and the mean-field dynamics remains small over an extended period. Our recent research focuses on the uniform-in-time propagation of chaos for the MFL dynamics with convex mean-field potential, motivated by this observation. We establish that this holds true for the $\mathbb{L}_2$-Wasserstein distance and relative entropy under mild conditions.

Alejandro Rivera

'Contracting with a present-biased agent: Sannikov meets Laibson'

ABSTRACT. This paper develops a methodology to solve dynamic principal–agent problems in which the agent features present-biased time preferences and naive beliefs. There are three insights. First, the problem has a recursive representation using the agent's perceived continuation value as a state variable (i.e., the remaining value the agent (wrongly) anticipates getting from the contract). Second, incentive compatibility corresponds to a volatility constraint on the agent's perceived continuation value. Finally, due to the agent's naiveté, a perceived

action constraint needs to be satisfied. This constraint is accommodated by linking the agent's perceived effort policy and the volatility of his perceived continuation value. Novel economic insights regarding optimal time-varying incentives and the term-structure of compensation are also explored.

Marco Rodrigues

'BSDEs and reflected BSDEs driven by general martingales'

ABSTRACT. We study the well-posedness of BSDEs and reflected BSDEs on random time horizons with stochastic Lipschitz constants and driven by general (possibly pure-jump) martingales. We employ a fixed-point theorem and provide various conditions, depending on the complexity of the generator, that ensure the existence and uniqueness within a class of processes.

Chiara Rossato

'Sannikov's principal–agent problem with jumps'

ABSTRACT. Based on recent work by [7] we consider an extension of Sannikov's principal–agent problem by letting the agent control the drift and jump intensity of the output process. We investigate whether the problem exhibits a golden parachute, that is, whether there is a scenario in which the agent retires and receives a continuous stream of payments or the agent receives a lump-sum payment as compensation for the termination of the contract by the principal. With the introduction of the face-lifted utility, we can study the two cases simultaneously in a different principal–agent problem that we reduce to a standard mixed control–stopping problem.

Halil Mete Soner

'Viscosity solutions for the mean-field control'

ABSTRACT. McKean–Vlasov or mean-field control problems are very closely related to the mean-field games. Dynamic programming for these problems results in a nonlinear partial differential equation on the space of probability measures. These equations require the value function to be not only differentiable but also its derivatives (which are in the dual of the set of measures, hence continuous functions) to be twice differentiable. Despite these difficulties, several approaches to characterise the value function as the unique appropriate weak solutions have been developed. We employ the classical viscosity solutions using the intrinsic linear derivative. We obtain a comparison result between the Lipschitz viscosity sub- and super-solutions under a structural assumption on the control set together with the given functions. roof follows the classical variable doubling argument of Crandall & Lions by using the Fourier characterisation of the dual Sobolev norm. The value function is also shown to be Lipschitz continuous with respect to this metric. This is joint work with Qinxin Yan of Princeton University.

Bruno Strulovici

'Smoothness of value functions in general control–stopping diffusion problems'

ABSTRACT. We study the properties of value functions in joint optimal control and stopping problems where (i) the state variable may be multi-dimensional, (ii) the domain may be unbounded, and (iii) the primitives may be time-inhomogeneous. We show that the value function is (i) the unique $\mathbb{L}^p$-solution of the Hamilton–Jacobi–Bellman equation, (ii) twice parabolically differentiable a.e., and (iii) continuously differentiable in the non-time variables, under general conditions relevant for most economic applications. In particular, we show that the smooth-pasting property holds everywhere with respect to all space variables. We also derive sufficient conditions under which smooth pasting also holds with respect to time. Our results imply that numerical solutions obtained by standard methods converge uniformly to the value function.

Mehdi Talbi

'Mean field games of optimal stopping'

ABSTRACT. We are interested in the study of stochastic games for which each player faces an optimal stopping problem. In our setting, the players may interact through the criterion to optimise as well as through their dynamics. After briefly discussing the N-players game, we formulate the corresponding mean field problem. In particular, we introduce a weak formulation of the game for which we are able to prove existence of Nash equilibria for a large class of criteria. We also prove that equilibria for the mean field problem provide approximated Nash equilibria for the N-players game, and we formally derive the master equation associated with our mean field game. Joint work with Dylan Possamaï.

Ludovic Tangpi

'Forward–backward propagation of chaos via displacement monotonicity'

ABSTRACT. In this talk I will present quantitative convergence results for a class of mean field games with common noise and controlled volatility. The basic strategy we employ is the one introduced recently by Laurière and myself—roughly speaking, we use a synchronous coupling argument to prove a 'forward–backward propagation of chaos' result for the FBSDEs which characterise the (open-loop) equilibria of the N-player and mean field games. Unlike in earlier works which have adopted this strategy, we do not require smallness conditions, and instead rely on monotonicity. In particular, (displacement) monotonicity of the Hamiltonian and the terminal cost allow us to establish a (uniform in N) stability estimate for the N-player FBSDEs, which implies the convergence result. The arguments are relatively simple, and flexible enough to yield similar results in the setting of mean field control and infinite horizon (discounted) mean field games.

Stéphane Villeneuve

'Money implements optimal contract'

ABSTRACT. We study a mean-field principal–agent model where interactions take place through the allocation of capital to agents. We analyse the impact of information asymmetry on risk-sharing and investment. The optimal contract is explicitly characterised and implemented with money and taxes.

Hao Xing

'Reward and monitoring in dynamic contracts'

ABSTRACT. We develop a dynamic principal–agent model that examines the intricate balance between monitoring and rewards in executive compensation. In the optimal contract, monitoring and success reward dynamically substitute each other, exhibiting two main characteristics: (i) a reduction in contract sensitivity to monitoring and a rise in the success reward when positive evidence of the agent's effort accumulates, and (ii) a larger success reward when the signal about the agent's effort is more noisy, particularly when negative evidence accumulates. Using the changes in the availability of direct flights for board directors to the firm's headquarters as an exogenous proxy of signal quality, we present empirical evidence that supports our model's predictions. This is a joint work with Kerry Back, Clint Hamilton, and Ali Kakhbod.

Mihail Zervos

'Market equilibrium under proportional transaction costs in a stochastic factor model'

ABSTRACT. We consider an economy with two agents. Each of the two agents receives a random endowment flow. We model this cumulative flow as the stochastic integral of a deterministic function of the economy's state, which we model by means of a general Ito diffusion. Each of the two agents has mean-variance preferences with different risk-aversion coefficients. The two agents can also trade a risky asset. We determine the agents' optimal equilibrium trading strategies in the presence of proportional transaction costs. In particular, we derive a new free-boundary problem that provides the solution to the agents' optimal equilibrium problem. Furthermore, we derive the explicit solution to this free-boundary problem when the problem data is such that the frictionless optimiser is a strictly increasing or a strictly increasing and then strictly decreasing function of the economy's state.

Jianfeng Zhang

'Viscosity solutions for fully nonlinear path dependent HJB equations on the Wasserstein space'

ABSTRACT. In this talk we investigate path dependent mean field optimal control problems with both drift and volatility controls. The value function is characterised by a fully nonlinear path dependent HJB equation on the Wasserstein space of probability measures. By lifting to the space of processes, we introduce a new notion of viscosity solutions and establish both existence and comparison principle. The main feature of our notion is that the test function consists of an extra component, besides the standard smooth part, which helps to get around of some major difficulty arising from the volatility control. We shall use the doubling variable arguments, combined with the Borwein–Preiss variational principle in order to overcome the non-compactness of the state space. A smooth gauge-type function on the path space is crucial for our estimates. The talk is based on an ongoing joint work with Nizar Touzi and Jianjun Zhou.

Scientific Progress Made

Contract theory. A large number of researchers shared their recent contributions on moral hazard problems in frameworks unavailable until very recent developments, which started from the seminal contribution of Sannikov [8], and culminated more recently in a series of papers by Possamaï, Cvitanić and Touzi [2, 3]. Hence Alexander Bloedel discussed deep new insights on a continuous-time principal–agent model of optimal dynamic insurance with persistent private information, while Christoph Frei and Nicolás Hernández Santibáẽz respectively addressed an optimal execution problem, and a model of self-protection in an insurance context. Daniel Kršek presented completely new results on the general problem of existence of randomised optimal contracts, and Alejandro Rivera tackeled a new class of moral hazard problems with present-bias. In the same vein, Chiara Rossato offered an analysis of extension of the model of [8] to a setting incorporating potential sudden accidents, Stéphane Villeneuve considered a mean-field principal–agent model where interactions take place through the allocation of capital to agents, and Hao Xing examined the intricate balance between monitoring and rewards in executive compensation.

Mean-field games and mean-field Schrödinger problems. A strong emphasis was also put in studying state-of-the-art problems in mean-field game theory and their applications, which evolved from the seminal contributions by Lasry and Lions [6] on the one hand, and Huang, Caines and Malhamé [5] on the other hand. Hence, René Aïd talked about a stationary mean-field equilibrium model of irreversible investment, Zhenjie Ren presented a study of uniform-in-time propagation of chaos for mean-field Langevin dynamics, Mehdi Talbi concentrated on mean-field games of optimal stopping, while Ludovic Tangpi commented on very recent results for forward–backward propagation of chaos via displacement monotonicity. In a related fashion, several researchers presented progress on both the Schrödinger problem in the framework of [4], as well as the very recent mean-field Shrödinger problem, coined by Backhoff-Veraguas, Conforti, Gentil and Léonard [1]. In particular, Beatrice Acciaio explored how solutions to equilibrium problems in markets with asymmetric information could be connected to solutions of Schrödinger problems, while Huyên Pham showed us how one could design generative modelling for time series via Schrödinger bridges, and Camilo Hernández explored the limit theory for for Schrödinger problems with interacting particles.

Optimal control theory. Several very interesting contributions related to theoretical issues stemming from control problems, and how these could find applications for a deeper understanding of the solutions to certain economics problems were also presented. As such, both Tahir Choulli and Marco Rodrigues were concerned with different classes of backward stochastic differential equations and their applications for optimal control problems and optimal stopping problems. We also had the pleasure to listen to Théo Durandard and Bruno Strulovici who presented respectively deep results in optimal control problem allowing to obtain comparative statics on the timing and quality of decisions for many sequential sampling problems *à la* Wald, and new regularity results for Hamilton–Jacobi–Bellman equations. There were as well two talks, by Halil Mete Soner and Jianfeng Zhang, presenting two approaches to notions of viscosity solutions on the Wasserstein space of probability measures, which is an extremely recent and active research area, finding applications both for mean-field games and contract theory as described above. Let us also mention the talk by Nabil Kazi-Tani exploring the role of correlation in diffusion control ranking games.

Equilibrium theory and mathematical finance. The topic of competitive equilibria—traditionally housed in economics—has found its way to the mainstream of mathematical research in quantitative finance thanks to the versatility and mathematical richness of the stochastic models it supports. We thus had the pleasure to listen on this topic to Paolo Guasoni, Julien Hugonnier and Mihail Zervos. Other talks on recent takes on timely problems in mathematical finance were presented by Umut Çetin, on order routing and market quality, Kostas Kardaras on portfolio choice under taxation and expected market time constraint, as well as Marcel Nutz on the optimal execution problem for the central risk book.

Finally, let us mention the talk by Ali Lazrak, which outlined a whole new domain of research at the interface of financial mathematics and political economics, studying how decentralised utility transfer promises affect collective decision-making by voting.

Participants

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Aïd, René (Université Paris Dauphine–PSL)
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Bloedel, Alexander (University of California Los Angeles)
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Çetin, Umut (London School of Economics)
Choulli, Tahir (University of Alberta)
Cvitanić, Jakša (Caltech)
Durandard, Theo (Northwestern University)
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Chapter 13

Recent Advances in Banach lattices (23w5115)

May 7 - 12, 2023

Organizer(s): Vladimir Troitsky (University of Alberta), Marcel de Jeu (Leiden University), Pedro Tradacete (Consejo Superior de Investigaciones Cientificas), Niushan Gao (Ryerson University)

This workshop focused on recent developments in the area of Banach lattices. The goal of the workshop was to bring together leading experts and active young researchers to discuss the current and future directions of these developments and to identify potential applications and the main open problems. We planned to understand the "big picture" of connections between these developments and other areas of Functional Analysis.

Overview of the Field

Various spaces of functions are ubiquitous in Analysis and Applications. Function spaces are the central objects of study of Functional Analysis. A *Banach space* is an abstraction of a function space, hence Banach space theory has been central to Functional Analysis. However, while Banach space theory captures the algebraic and the metric structures of function spaces, it "does not see" their order structures. Most classical function spaces have natural orders in the sense that for two functions f and g, one considers $f \leq g$ when $f(t) \leq g(t)$ for all t or for "almost all" t, depending on the context. Moreover, in most function spaces, this order is a lattice order. This naturally brings us to Banach lattices. Just as a *Banach space*, a *Banach lattice* is an abstraction of a function space, but it also takes its order into account. So a Banach lattice is a Banach space equipped with a lattice order, satisfying appropriate compatibility axioms. Most of the classical function spaces that appear in modern Functional Analysis are Banach lattices.

Banach lattice theory is a synthesis of Banach space theory and vector lattice theory. A *vector lattice* (Riesz spaces) is a vector space which is also a lattice, such that the two structures are compatible in a certain natural way. Vector lattices themselves form a rich and interesting category. Understanding of vector lattices is critical for Banach lattice theory.

The unofficial beginning of this area was the address of F. Riesz at the ICM in 1928, followed by works of H. Freudenthal and L. Kantorovich on vector lattices. In its early days (around the 1950s), most activities were devoted to vector lattices and were concentrated in several major schools: the Soviet school (L. Kantorovich and the Krein brothers), the Japanese school (H. Nakano and T. Ando), the German school (H.H. Schaefer), and the Dutch school (A.C. Zaanen and W. Luxemburg). Since then, the area has expanded and has developed connections with many other areas of mathematics, including Banach spaces, which led to the rise of Banach lattices. Several major advances in Banach lattices were made around the 1970s by H.H. Schaefer, P. Meyer-Nieberg, J. Lindenstrauss, L. Tsafriri, W. Johnson, and N. Ghoussoub. To mention a few other important connections (which were not addressed at the workshop): Spectral Theory of positive operators, Perron-Frobenius Theory of non-negative matrices, Ergodic Theory, Operator Semigroups, Convex Analysis, Inequalities, etc. There have been numerous applications of vector and Banach lattice to Math Economics and Math Finance. These applications go back to works of L. Kantorovich and, more recently, of C. Aliprantis and I. Polyrakis. In these applications, vector and Banach lattices are used to model markets and the theory of positivity is used in the search for a market equilibrium.

The subject area of vector and Banach lattices (which is nowadays often referred to as "*Positivity*") is old and well established, and many parts of it have become classical. Members of the Positivity community have been distinguished in various ways. L. Kantorovich received the Nobel Prize in Economics for applications of Positivity in Economics. H. Schaefer was a member of the Mathematics and Natural Sciences Class of the Heidelberg Academy of Sciences and of the Academy of Sciences in Zaragoza. C. Aliprantis was a distinguished professor of Economics and Mathematics at Purdue University; he was also the founding editor of the journals Economic Theory and Annals of Finance. W. Luxemburg was a Fellow of the American Mathematical Society; he, A. Zaanen, and H. Freudenthal were members of the Royal Netherlands Academy of Arts and Sciences. In 2000, the International Commission on Mathematical Instruction of the International Mathematical Union instituted the Hans Freudenthal Medal. B. Johnson was awarded a Stefan Banach Medal in 2007. G. Curbera was a Curator at the International Mathematical Union over the period of 2011–2014.

There have been many regular and incidental conferences and workshops. In the 70s–80s, there was a series of workshops on Riesz spaces in Oberwolfach: June 24–30 1973, June 22–28 1975, June 19–25 1977, June 24–30 1979, June 27 – July 3 1982, July 1–5 1985, and April 23–29 Apr 1989. Currently, *Positivity* conferences are normally held every second year; they are the main events for the Positivity community. Below we list the locations of past Positivity conferences.

- Ankara, Turkey, 1999
- Nijmegen, Netherlands, 2001
- Rhodos, Greece, 2003
- Dresden, Germany, 2005
- Belfast, UK, 2007
- Madrid, Spain, 2009
- Leiden, Netherlands, 2013
- Chengdu, China, 2015
- Edmonton, Canada, 2017
- Pretoria, South Africa, 2019
- Ljubljana, Slovenia, 2023 (forthcoming)

Here we list a few recent "incidental" conferences and workshops:

- "Positivity in Functional Analysis and Applications" session at the 2006 Summer Meeting of the Canadian Mathematical Society, Calgary, Alberta.
- Symposium on Positivity and Its Applications in Science and Economics, Bolu, Turkey, September 2008.
- Workshop on Nonnegative Matrix Theory, American Institute of Mathematics, Palo Alto, California, December 2008.
- Ordered Spaces and Applications Conference, National Technical University of Athens, Greece, November 2011.
- Ordered Banach Algebras Workshop, Lorentz Center, Leiden, Netherlands, July 2014.
- Workshop on Operators and Banach lattices I and II, Universidad Complutense de Madrid, October 2012 and November 2016.

- Workshop on Recent Advances in Banach lattices, BIRS, Oaxaca, April 2018.
- Workshop on Banach spaces and Banach lattices I and II, ICMAT, Spain, September 2019 and May 2022.
- Conference on Ordered Structures and Applications, Tozeur, Tunisia, January/February 2023,
- Workshop on Ordered Vector Spaces and Positive Operators, Bergische Universität Wuppertal, Germany, March/April 2023

For further information about the field, we refer the reader to the following classical monographs: [1, 2, 23, 25, 26, 30].

Presentation Highlights — Recent developments

All talks at the workshop were devoted to important recent discoveries in the area of Banach lattices. Most of the talks were grouped by subject.

Free Banach lattices

A vector lattice X is a *free vector lattice* over a set A if $A \subseteq X$ and every map from A to an arbitrary vector lattice Y extends uniquely to a vector lattice homomorphism from X to Y. While free vector lattices have been known for a long time, see, e.g., [7], this area came to prominence thanks to a a recent paper [27], where B. de Pagter and A. Wickstead proved the existence of free Banach lattices over sets and characterized some of their properties. Then in [3], A. Avilés, J. Rodríguez, and P. Tradacete came up with the concept of a free Banach lattice over a Banach space. A Banach lattice X is a free Banach lattice over a Banach space E if E is a closed sublattice of X and every bounded operator T from E to an arbitrary Banach lattice Y extends uniquely to a lattice homomorphism from X to Y with the same norm. Not only did the authors of [3] prove that such an X exists, but they also found a representation of it as a function space, as a sublattice of the space of weak*-continuous functions on the unit ball of E^*. This was a major impetus to the area. This approach created a framework to study subspaces of Banach lattices as Banach spaces. From a categorical point of view, this can be seen as a functor from the category of Banach spaces and bounded linear operators into the subcategory of Banach lattices and lattice homomorphisms. It has become clear that understanding this functor is a key to properly understand the interplay between Banach space and Banach lattice properties, a goal that has been pursued ever since the first developments of these theories.

[3] led to numerous recent results, including [21], where free Banach lattices were constructed in several other categories, [29], where various properties of there spaces were established, [20], where complex free Banach lattices were constructed, and [12], where free Banach lattices were constructed in the category of dual spaces.

The workshop had three talks on free Banach lattices. The first talk, delivered by M. de Jeu, presented a historic overview of free objects in the framework of universal algebras, including free vector lattices and free vector lattice algebras. It followed by a talk by T. Oikhberg, outlining some of the results of [29], with a focus on connections between lattice homomorphisms between free Banach lattices and positively homogeneous weak*-continuous maps between the dual unit balls of the underlying spaces. Finally, D. Leung presented a very recent construction of free objects in the category of Banach lattices with upper p-estimates over Banach spaces.

Stable phase retrieval

Let X be a subspace of a Banach lattice E. Suppose that for every $f \in X$, one can recover f from $|f|$, up to a scalar multiple λ with $|\lambda| = 1$. We then say that X has a *phase retrieval* property. This has numerous applications as it means that one can recover a signal by only measuring its magnitude. Notable examples in physics and engineering which require phase retrieval include X-ray crystallography, electron microscopy, quantum state tomography, and cepstrum analysis in speech recognition.

In practice, we often have to recover f from $|Tf|$, where T is a linear transformation, e.g., the Fourier transform. As any application of phase retrieval would involve error, it is of fundamental importance that the recovery of f from $|Tf|$ not only be possible, but also be stable. If, moreover, we can do the retrieval in a Lipschitz-continuous way, we say that the subspace has *stable phase retrieval*.

The issue of phase retrieval has existed in applied mathematics (in particular, in frame theory) for a long time; see, e.g., [8]. Only recently it was recognized that the problem may often be solved using Banach lattice techniques. In the last few years, several papers and preprints have appeared, solving the stable phase retrieval problem for certain classes of subspaces of Banach lattices: [9, 10]. It has been discovered that stable phase retrieval problem is closely related to of asymptotically disjoint sequences explored by V. Kadec and O. Pełczyński.

At the workshop, M. Taylor and D. Freeman presented an overview of the subject and recent advances. They also proposed several open questions. This area of research clearly has a great potential.

Convergence structures

It has been known for a long time that several important convergences of sequences or nets in the theory of vector and Banach lattices are not topological. In particular, order convergence, unbounded order convergence, and relative uniform convergence are not induced by any topologies. These are not some exotic convergences: as a special case, almost everywhere convergence of sequences of measurable functions is not topological. These convergence came to the forefront a few years ago, when unbounded order convergence found applications in Math Economics, [13, 14, 15, 16, 17, 18].

On the other hand, there has long existed a theory of *convergence structures*, see, e.g., [6]. However, this theory was formulated in terms of filters, which made it less suitable for applications in analysis. It was only recently realized that the theory of convergence structures might equivalently be reformulated in terms of nets. This allows one to study convergences in vector and Banach lattices using the framework of convergence structures, see [28].

At the workshop, J.H. van der Walt presented an overview of the theory of convergence structures and showed how it could be applied to convergences in vector lattices. In particular, he introduced locally solid convergent structures, that naturally generalize locally solid topologies. E. Bilokopytov then presented recent applications of this theory to various problems in Banach lattices. In particular, he discussed relationship between adherences (or closures) of a sublattice with respect to various convergence structures.

Linear versus lattice embeddings

If a Banach lattice X has a closed subspace isomorphic to c_0, then X contains a closed sublattice that is lattice isomorphic to c_0. This result is one of the gems of Banach lattice theory; it goes back to P. Meyer-Nieberg in the 80s. It has been a long-standing open question whether the same is true for $C[0,1]$; some partial results were obtained in [24, 19]. The question was answered in the affirmative in [5]. Moreover, the authors characterized all Banach lattices with this property. At the workshop, G. Martinez-Cervantes presented a clear overview of these results and their proofs.

A motivation and a key ingredient in the proof is a recent result by D.H. Leung, L. Li, T. Oikhberg and M.A. Tursi that every separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1)$, that was presented during the previous BIRS workshop in 2018. Another key ingredient is the recent concept of projectivity for Banach lattices introduced by B. de Pagter and A.W. Wickstead in their free Banach lattice paper [27]

Complemented subspace problem

It has been a long-standing open problem whether every complemented subspace of a Banach lattice is itself isomorphic to a Banach lattice. Recently, this problem was solved in the non-separable case. This result was presented at the workshop by D. de Hevia.

Building bridges between areas

An important feature of this meeting was building bridges between areas. This was reflected in the choice of participants and in the subjects of the presentations.

Since the theory of Banach lattices rests on two pillars, vector lattices and Banach spaces, special care was taken to bring together experts on these two topics. Several experts on Banach space theory were invited and connections between Banach lattices and Banach spaces were discussed at length. In particular, a considerable portion of the workshop was devoted to free Banach lattices over Banach spaces; in this subject the relationship between Banach spaces and Banach lattices is very important, so having experts in Banach spaces was invaluable.

There are also clear connections between free Banach lattices and Lipschitz-free metric spaces. To every metric space M, it is possible to assign a Banach space $F(M)$ generated by M in such a way that Lipschitz maps between metric spaces are converted into bounded linear operators between the corresponding Banach spaces. We call Banach space $F(M)$ the Lipschitz-free space over M (also known as the Arens-Eells space or Transportation Cost Space). The above universal property makes Lipschitz-free spaces an important tool in functional analysis because it allows the application of linear methods to nonlinear problems. While Lipschitz-free spaces are not Banach lattices, they are Banach spaces and vector lattices. The workshop was a natural place to explore connections between these structures. This connection was presented at the workshop by E. Pernecká.

Many problems in Banach lattice theory may be reduced to questions about spaces of continuous functions on topological spaces. This justified inviting several experts on such spaces. In particular, the solution to the Complemented Subspace problem, presented at the workshop, was based on a recent result about spaces of continuous functions [4].

P. Tradacete delivered a talk on *valuations* in Banach lattices. This connects Banach lattice theory with various applications, including measures, volumes, and finite-dimensional convex geometry.

Recent advances in convergence structures in vector lattices naturally suggest a link to convergence structures in general lattices and in even more general ordered sets. In the last few years, several results about order and unbounded order convergences in vector lattices have been extended to general lattices. It was therefore helpful to have several experts on this subject at the workshop.

Outcome of the Meeting

The meeting reached two main goals.

First, it exposed some very recent important advances (many of them not yet even published) to the community. So it contributed to dissemination of results, techniques, and ideas.

Second, it provided an environment for informal discussions. During the workshop, participants would often split into small groups exchanging ideas, making connections, and starting new collaborations and new projects. We expect that this workshop will result in several new projects.

In addition, it was helpful in bringing young mathematicians into the area. Several graduate students and postdocs participated in the workshop. The workshop provided them with opportunities to join the cutting edge research in this area, both by attending talks and by talking to other participants.

Participants

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Chapter 14

Mathematical methods in cancer biology, evolution and therapy (23w5084)

May 14 - 19, 2023

Organizer(s): Peter Van Loo (MD Anderson Cancer Center), Wenyi Wang (MD Anderson Cancer Center), Ronglai Shen (Memorial Sloan-Kettering Cancer Center), Quaid Morris (Memorial Sloan-Kettering Cancer Center)

Overview

The unifying theme of the our workshop was the development of creative mathematical and statistical models for analyzing patient samples, with a particular focus on tumor evolution and immuno-oncology, two of the most timely and important foci of cancer research. There is an exciting growth of research in these foci fueled by the increasing availability of whole-genome and whole-transcriptome next-generation sequencing data in a large number of tumor samples, as well as in single-cell and cell-free DNA settings. It is now well understood that most patient samples are comprised of a mixture of cell types, including multiple genetically distinct populations of cancer cells (i.e., 'subclones') and a variety of normal cell types (e.g., normal stromal tissue cells, fibroblasts, and diverse immune cells). Effective use of these samples to extract relevant inference on tumor biology requires information from individual cell types to be deconvolved from the 'bulk' data collected. This workshop provided the opportunity to identify synergies in mathematical deconvolution techniques developed independently in different cancer subfields.

Our workshop brought together leading experts from diverse geographic regions – Canada, US, UK, Australia, Spain, Singapore and others – and a wide range of scientific backgrounds – computational biology, mathematics and statistics, computer science, and various biomedical fields – and a variety of career stages including trainees, early stage researchers and senior faculty. It provided a forum for cross-disciplinary learning, discussion, collaboration, and mentoring. We organized the talks in the workshop into five focus areas: (1) intra-tumor heterogeneity and subclonal reconstruction, (2) copy number analysis of cancer genomes, (3) signatures of mutational processes in cancer, (4) deconvolution of tumor, normal and immune transcriptomes and epigenomes, and (5) immunogenomics and immuno-oncology.

Background and Open Problems in the Five Focus Areas

1. **Intra-tumor heterogeneity and subclonal reconstruction algorithms.** Intra-tumor heterogeneity, i.e., the presence of a multitude of genetically and phenotypically distinct cancer cell populations in a tumor, is a key challenge in cancer medicine. These intra-cellular differences can include somatic point mutations, chromosomal

aberrations and epigenetic changes. Heterogeneity introduces genetic and phenotypic diversity that fuels ongoing tumor evolution and resistance to therapy, ultimately contributing to disease progression and metastasis.

The reconstruction of subclonal populations and their evolutionary relationships from sequencing data thus represents an important but extremely challenging deconvolution problem. Ideally subclonal reconstruction algorithms would distinguish and characterize populations of normal cells with no somatic aberrations and multiple populations of tumor cells with both shared and unique somatic changes. Multiple bespoke approaches such as non-parametric, tree-based Dirichlet processes and Hidden Markov Models have been developed, including many prominent works from our invitees. While this is a thriving field, multiple opportunities for further mathematical and statistical method development remain, including time-efficient and optimal phylogenetic inference, integration of information across multiple time points and multi-regional samples and across bulk and single-cell sequencing data, and integration across different classes of somatic variants (e.g. subclonal copy number changes and subclonal point mutations).

2. **Mathematical and statistical approaches for copy number analysis of cancer genomes.** Chromosome instability is a hallmark of cancer, and is associated with poor prognosis and therapeutic resistance. These chromosomal aberrations can range from small-scale deletions and duplications, through gains and losses of entire chromosomes, to even duplication of the entire genome. They play an important role in cancer development, and in addition are a confounder that needs to be accounted for in many genomics analyses (including intra-tumor heterogeneity, subclonal reconstruction and in immunogenomics analysis). Cancer genomics data such as SNP arrays and exome or whole-genome sequencing can be used to infer genome-wide copy number profiles of cancers. However large-scale aneuploidy and the admixture of normal cells are confounders that need to be addressed. While important strides have been made in tackling this deconvolution problem over the past decade, several mathematical challenges remain, including: (i) ambiguity in whole-genome duplication inference in 10-20% of cancers, (ii) cancer samples with high normal cell admixture, and (iii) accurate inference of subclonal copy number changes. These are issues also relevant in applying copy number analysis in clinical settings. In addition, there are several unfulfilled opportunities to infer copy number profiles from other -omics data, including transcriptomic and epigenomic.

3. **Deconvolution of signatures of mutational processes in cancer.** The somatic changes in cancer genomes are caused by the interplay of DNA damage and repair. From patterns of co-occurence of different types of somatic mutations across cancers, signatures of mutational processes can be identified. For example, tobacco mutagens cause C->A mutations, UV radiation causes CC->TT mutations. These signatures can help understand the causes and progression of cancer, as well as helping to identify treatment strategies that target deficient DNA damage repair unique to cancer cells. Mutational signature analysis is a type of "mixed membership" statistical problem that is often approached as a (non-negative) matrix factorization problem where each tumor's mutational repertoire is modelled as a combination of core mutational 'signatures' (representing the recurrent mutational processes active across tumors), each with appropriate 'weights' (representing the different activities of these mutational processes in the given tumor). While different methods have been developed for point mutations leading to more than 70 signatures of mutational processes with identfied etiologies (e.g. tobacco mutagens, UV light exposure, exposure to aristolochic acid), there is yet no complete consensus on the best deconvolution approaches; and current approaches lack sensitivity, leaving many mutational processes unidentified. Also, promising approaches to identify mutational signatures of structural variants and copy number changes are starting to emerge, opening up new potential avenues of discovery and treatment optimization.

4. **Mathematical and computational approaches to deconvolve tumor, normal and immune signals from transcriptomic and epigenomic data.** The clinical analysis of tumor samples is complicated by the tumor-stroma-immune interaction. The number and types of tumor-infiltrating immune cells in these samples predicts clinical outcome, and can affect treatment decisions. Decomposing tumor samples into their constituent parts in the lab is expensive, technically challenging and time-consuming, making it difficult to discern the relevant immune signals and motivating computational approaches to integrate the estimation of cell type-specific expression profiles, and epigenetic profiles, for tumor cells, immune cells, and the tumor microenvironment. Most deconvolution methods assume that malignant tumor tissue consists of two distinct components, epithelium-derived tumor cells and surrounding stromal cells, and are thus unsuitable for characterising immune populations. Other deconvolution methods for more than two compartments require list of immune cell-type-specific reference gene lists, thus re-

stricting the resolution, robustness, and scope of these analyses. Fortunately, this is a fast-growing field, where many research groups are developing models that relax these restrictive assumptions and input data requirements. However, key questions remain unsolved, such as dissecting expression signals from individual cell types for all 20,000 genes at once. A promising direction of inquiry is using matching bulk and single cell RNAseq data to borrow strength across data types, initial results suggests that this leads to better deconvolution of signals. Nonetheless, significant opportunities remain for theoretical and methodological research in this complex and high-dimensional deconvolution problem.

5. **Immunogenomics.** The immune system plays a critical role in the body's defense against cancer, and the recent development of cancer immunotherapies has clearly demonstrated the importance of host immune cells in cancer treatment. The immune response to cancer is activated by 'neoantigens', novel, tumor-specific peptides generated by missense mutations. These neoantigens appear on the cancer cell surface bound by the patient's HLA molecules and specific T-cells identify and respond to these neoantigens by initiating an immune response. The study of this process, immunogenomics, has benefitted from new sequencing technologies and associated computational data analysis methods, including the deconvolution of the appropriate signals from bulk sequencing data. In individual cancers, immunogenomics has improved the prediction of neoantigens for prognostic purposes or to inform immunotherapeutic interventions. New methods have also been developed to study HLA sequences, to investigate changes in the T cell repertoire, to characterize the gene expression signatures of the immune cell types present in the tumor mass (as described above), and to design personalized vaccines or adoptive cell transfer (ACT) therapies. The new immunogenomic methods have the promise of being widely used in clinical applications. However, this is an emerging field and various analytical challenges remain to be resolved, including the prediction of MHC presentations, an integrative analysis of somatic mutations and the immune repertoire, and the prediction of response to immunotherapies, both in terms of tumor killing effects and autoimmune side effects.

Presentation Highlights

The talks over the five-day workshop covered dynamic discussions on the past lessons, current state-of-art methods and future directions in which the field will develop.

1. **Intra-tumor heterogeneity and subclonal reconstruction algorithms.** Speakers in this focus area included Sohrab Shah, Nicholas McGranahan, Tony Papenfuss, Ben Raphael, Mohammed El-Kebir, Paul Boutros, Quaid Morris, Cenk Sahinalp, Gryte Satas, Yuchao Jiang, Sitara Persad, Leah Weber, Ethan Kulman, Chay Paterson and Adam Olshen. Several speakers presented on phylogeny analysis for the reconstruction of tumor evolution from single-cell and bulk sequencing. Such study typically involves multi-region tumor sequencing or sequencing of autopsy samples where multiple tumor samples are included. Phylogeny construction is a combinatorial problem that are often computationally intractable. Parsimony is a key principle in phylogeny analysis, for example, by limiting the transformation from one state to the other with minimal number of events. Ben Raphael discussed models for cancer evolution and lineage tracing, including the constrained k-dollo phylogeny model and dynamic lineage tracing using CRISPR-induced mutations. Cenk Sahinalp focused on the inference of mutational progression history and subclonal composition of tumor samples, highlighting the CITUP and MQIP methods for constructing mutation trees. Mohammed El-Kebir presented MACHINA, a tool for reconstructing metastatic migration histories, and discussed the identification of temporally consistent co-migrations using MACH2. Quaid Morris discussed clone tree reconstruction and introduced SubMARine, a noise-free phylogeny method that improves tree construction performance, and Pairtree, a scalable and fast bulk sample reconstruction method. Leah Weber introduced Phertilizer, a method for building clonal trees from ultra-low coverage single-cell DNA sequencing data, addressing the tradeoff between depth and uniformity and assessing clone trees based on a probabilistic model and fit to observed data. Ethan Kulman presented Orchard, a method for reconstructing mutation trees from bulk DNA data using a combinatorial search approach and the Gumbel Max Trick, allowing a ten-fold increase in the size of the cancer trees that could be reconstructed. Chay Paterson discussed multi-stage clonal expansion models for simulating pre-invasive cell populations and estimating cancer risk, with applications to colorectal adenocarcinoma and vestibular schwannoma. Paul Boutros discussed research on the prostate cancer epitranscriptome, exploring the clonality of m6A peaks, germline m6A interactions. Additionally, he presented work on the impact of host factors such as hypoxia, sex and ancestry on tumor evolution.

Several speakers presented their work on using single cell sequencing technologies to study intra-tumor heterogeneity. Gryte Satas presented ArtiCull, a feature-based classifier for removing variant calling artifacts from DLP+ single-cell WGS DNA sequencing data, and discussed its applications in improving concordance with bulk data and identifying subclones. Yuchao Jiang presented work on scRNA+ATAC multiomic single cell. Leah Weber presented work on addressing sparsity of SNV signal in ultra low coverage DNA-sequencing (0.01x). Adam Olshen presented an integrative clustering method for Dab-seq with joint DNA and cell surface protein sequencing and future directions in combining Dab-seq with other single cell sequencing modalities such as CITE-seq. Tony Papenfuss presented an ensemble learning approach to integrate structural variant calls from whole-genome sequencing. Sohrab Shah identified distinct categories of high-grade serous ovarian cancer (HGSOC) based on genomic features and developed TreeAlign for single-cell integration to understand phenotypic consequences. Sitata Persad presented SEACELL, which leverages single-cell genomics data to study biological heterogeneity, including inferring epigenetic regulation and studying gene accessibility dynamics.

2. **Mathematical and statistical approaches for copy number analysis of cancer genomes** Speakers in this focus area included Peter Van Loo, Nana Mensah, Carla Castignani, Barbara Hernando, Geoff Macintyre, Henri Schmidt, and Kristiana Grigoriadis. Peter Van Loo discussed the analysis of copy number intratumor heterogeneity, inferring clonality from point mutations, constructing phylogenetic trees from mutation clusters and DNA methylation deconvolution using CAMDAC. Nana Mensah presented CAMDAC-WGBS for allele-specific methylation analysis in metastatic cancer, including the distinction between matched normal or panel methods. Carla Castignani introduced CREDAC, a tool for copy number-based expression deconvolution analysis of cancers, which models pure tumor expression profiles form bulk tumor expression alone. Barbara Hernando introduced a framework for modelling chromosomal instability (CIN) using copy number mutational signatures, identifying 17 CIN signatures in TCGA data and exploring integration with additional covariates. Geoff Macintyre highlighted the impact of chromosomal instability (CIN) in precision medicine, correlating CIN signatures with treatment response and exploring the translational potential of CIN signatures for ovarian cancer. Henri Schmidt presented Lazac, a zero-agnostic model for copy number evolution in cancer, addressing challenges in constructing evolutionary histories and proposing a model for efficient solution of the small-parsimony problem. Kristiana Grigoriadis presented ALPACA, a tool for inferring subclonal copy number states based on SNV-derived phylogenetic trees, considering evolutionary constraints.

3. **Deconvolution of signatures of mutational processes in cancer.** Speakers included Steve Rosen, Bin Zhu, Teresa Przytycka, and Caitlin Harrigan. Steve Rosen presented a hierarchical Dirichlet process for mutation signature decomposition with a prior on the shape of single base substitution (SBS) signature. Flat signatures (e.g., SBS3 vs SBS40) are hard to discriminate and additional information (indels with microhomology) is needed to refine the separation. Bin Zhu presented a non-negative matrix decomposition for mutation signature analysis of targeted sequencing with very sparse mutation counts. SBS count is biased for targeted panel (and whole-exomes) as coding regions are more enriched for certain trinucleotide changes. He presented a normalization of the SBS counts with respect to the reference genome. He presented analysis of the AACR-GENIE consortium data and simulation studies to investigate the sensitivity and specificity of detecting signatures with certain prevalence and flatness (using Shannon equitability index) of shape. He showed 100,000 samples are needed to detect flat signatures (e.g., SBS3, SBS5) with low frequency. Teresa Przytycka presented computational approaches to study mutagenic signatures, including EcoSigClust for identifying signature etiology. Caitlin Harrigan presented DAMUTA, a Bayesian method for identifying damage and misrepair mutation signatures, extracting 18 damage and 6 misrepair signatures, and assessing their association with gene mutations.

4. **Mathematical and computational approaches to deconvolve tumor, normal and immune signals from transcriptomic and epigenomic data.** Speakers included Wenyi Wang, Aaron Newman, Francesca Petralia, Venkatraman Seshan, Carla Castignani, Kyle Coleman, and Yaoyi Dai. The Wenyi Wang lab aims to understand variations in both the transcriptome and genotype and integrate deconvolution models to define plasticity features in human tissues mathematically. They developed a new metric, total mRNA expression per haploid genome in tumor cells (TmS), and found that it can serve as a marker for worse prognosis and potentially predict response to therapy. Yaoyi Dai demonstrated that TmS is a pan-cancer prognostic marker in triple-negative breast cancer (TNBC) and correlated with TNBC subtypes, immune activation, and differential gene expression pathways.

Carla Castignani introduced CREDAC, a tool for copy number-based expression deconvolution analysis of cancers, which models pure tumor expression profiles from bulk tumor expression alone. Aaron Newman discussed the challenges of profiling cell heterogeneity from bulk RNA-seq data and introduced CIBERSORTx as a method to improve accurate profiling of cell subsets. Francesca Petralia presented BayesDeBulk, a reference-free method for quantifying cell types in the tumor microenvironment using bulk data, highlighting its improved association with overall survival. Kyle Coleman introduced MISO, a multi-modal spatial omics method that integrates data from different modalities and aligns well with pathologist annotations. Venkatraman Seshan explored the application of James-Stein shrinkage estimation in principal components analysis (PCA), revealing overestimation in PCA and the potential application of shrinkage to these estimates.

5. **Immunogenomics.** Speakers included Wei Sun, Nicholas McGranahan, Vicky Yao and Ronglai Shen. Wei Sun presented his work on identifying co-occurrence of HLA allele and TCR clonal-type and predicting surface neoantigen generated by somatic mutations. Nicholas McGranahan presented work built on the TRACERx project. He discussed the role of HLA genes in tumor evolution and immune evasion. They found association of loss-of-heterozygosity (LOH) of HLA allele with elevated tumor mutation burden, and that alternate splicing of HLA molecule could change its function. Exon skipping in HLA domains is observed in 30% of lung adenocarcinomas. Ronglai Shen presented work on using Latent Dirichlet Allocation (LDA) models for flow cytometry analysis of cancer patients' blood samples to infer the immune contexture and pharmacodynamics upon immunotherapy exposure. She further presented a spatial LDA approach to delineate tumor microenvironment from multiplex imaging data. Vicky Yao explored the cancer-associated microbiome using semi-supervised NMF, identifying microbial signatures associated with survival.

Outcome of the Meeting

The 5-day workshop sponsored by BIRS included 38 participants attended in person and several virtual attendees with a mix of established and early-career investigators and 14 postdoc or graduate student trainees. The dynamic discussions and interactions led to several outcomes: dissemination of research results, techniques, and ideas; new connections, collaborations, and projects being formed; networking opportunities for trainees (postdocs and graduate students). The success of the workshop would be impossible to achieve without the support of BIRS.

Participants

Boutros, Paul (University of California Los Angeles)
Castignani, Carla (The Francis Crick Institute)
Coleman, Kyle (University of Pennsylvania)
Dai, Yaoyi (The University of Texas MD Anderson Cancer Center)
Dannenfelser, Ruth (Rice University)
Demeulemeester, Jonas (VIB – KU Leuven)
Edrisi, Mohammadamin (Rice University)
El-Kebir, Mohamed (University of Illinois Urbana-Champaign)
Grigoriadis, Kristiana (The Francis Crick Institute)
Gundem, Gunes (Memorial Sloan Kettering Cancer Center)
Hari, Ananth (NIH)
Harrigan, Cait (University of Toronto)
Hernando, Barbara (CNIO - Spanish National Cancer Research Center)
Hovens, Christopher (UNIVERSITY OF MELBOURNE)
Huang, Xiru (Rice University)
Jiang, Yuchao (University of North Carolina at Chapel Hill)
Kulman, Ethan (University of Minnesota)
Kumar, Sanjeev (Dr. Bhimrao Ambedkar University)
Lai, Siqi (The University of Texas MD Anderson Cancer Center)
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Li, Mingyao (University of Pennsylvania)
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Newman, Aaron (Stanford University)
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Pan, Yidan (Merck)
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Parmentier, Mathieu (Université Libre de Bruxelles)
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Tarabichi, Maxime (IRIBHM)
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Weber, Leah (University of Illinois at Urbana-Champaign)
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Zhu, Bin (National Cancer Institute)

Chapter 15

Joint Spectra and related Topics in Complex Dynamics and Representation Theory (23w5033)

May 21 - 26, 2023

Organizer(s): Rongwei Yang (University at Albany, the State University of New York), Raúl Curto (University of Iowa), Nguyen-Bac Dang (Université Paris Saclay), Rostislav Grigorchuk (Texas A&M University)

Overview of the Field

Mathematics has a wide range of different disciplines, and new discoveries are made daily in every corner. Some discoveries, however, are able to link several disciplines together and open new fields of interplay. Two of such recent discoveries are self-similar group representations and projective spectrum in Banach algebras. This workshop, which took place at Banff International Research Station from May 22 to 26, 2023, brought together scholars in the fields of spectral theory, representation theory, geometric group theory, complex dynamics and Lie algebras to examine this development. In the overview, we briefly recall several notions of joint spectrum, first for commuting operators and then for noncommuting ones. The theory is vastly different for the two settings.

The Commuting Case

Consider an abelian unital Banach algebra $\mathcal{B}$. The idea of joint spectrum for several elements $A_1, ..., A_n \in \mathcal{B}$ goes back to the 1960s. The following notion was studied in [33].

Definition 15.0.1. *For a tuple $A = (A_1, ..., A_n)$ of elements in $\mathcal{B}$, the joint spectrum* $\mathrm{Sp}(A)$ *is the collection of $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ such that the ideal generated by $A_1 - \lambda_1 I, ..., A_n - \lambda_n I$ is proper in $\mathcal{B}$.*

In other words, λ is not in $\mathrm{Sp}(A)$ if and only if there are elements $B_1, ..., B_n$ in $\mathcal{B}$ such that $(A_1 - \lambda_1 I)B_1 + \cdots + (A_n - \lambda_n I)B_n = I$.

A more popular notion of joint spectrum was defined by Taylor in the early 70's using Koszul complex [44, 45]. Let $\oplus_{0\le p\le n}\Lambda^p(V)$ be the exterior algebra on a vector space V of dimension n. Given a Hilbert space $\mathcal{H}$, we let $B(\mathcal{H})$ be the C^*-algebra of bounded linear operators on $\mathcal{H}$. A tuple A of n commuting operators in $B(\mathcal{H})$ gives rise to the following Koszul complex $E(\mathcal{H}, A)$ of cochains:

$$0 \xrightarrow{d_{-1}} \mathcal{H}\otimes\Lambda^0 \xrightarrow{d_0} \mathcal{H}\otimes\Lambda^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathcal{H}\otimes\Lambda^n \xrightarrow{d_n} 0, \tag{1}$$

where $d_{-1} = d_n = 0$, and $d_p : \mathcal{H}\otimes\Lambda^p(V) \to \mathcal{H}\otimes\Lambda^{p+1}(V), 0 \le p \le n-1$, are the linear maps defined by

$$d_p(x\otimes\omega) = \sum_{i=1}^{n} A_i x \otimes (e_i \wedge \omega), \quad x \in \mathcal{H},\ \omega \in \Lambda^p(V).$$

The reader shall gain some insight by looking at the special cases $n = 1$ and 2. Direct computation can verify that $d_{p+1}d_p = 0$ for each p. The complex (1) is said to be exact if $\operatorname{Ker} d_{p+1} = \operatorname{ran} d_p$ for each $0 \le p \le n$. For a vector $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$, we let $A - \lambda$ stand for $(A_1 - \lambda_1 I, ..., A_n - \lambda_n I)$.

Definition 15.0.2. *The Taylor spectrum of a commuting tuple A is defined as*

$$\sigma_T(A) = \{\lambda \in \mathbb{C}^n \mid E(\mathcal{H}, A-\lambda) \text{ is not exact}\}.$$

The following theorem has many applications.

Theorem 15.0.3. *For a commuting tuple of operators $A = (A_1, ..., A_n)$, we have*

a) $\sigma_T(A)$ is a nontrivial compact subset of $\mathbb{C}^n$;

b) if a complex domain Ω contains $\sigma_T(A)$ and $f : \Omega \to \mathbb{C}^m$ is holomorphic, then $\sigma_T(f(A)) = f(\sigma_T(A))$.

Part b) is known as the spectral mapping theorem. One observes that both $\operatorname{Sp}(A)$ and $\sigma_T(A)$ are natural generalizations of the classical spectrum. Moreover, if $f(z) = z_j$ is the projection to the jth coordinate, then b) implies the projection property of Taylor spectrum, namely, the projection of $\sigma_T(A)$ onto any axis z_j equals $\sigma(A_j)$. It is known that $\sigma_T(A) \subset \operatorname{Sp}(A)$, and the inclusion can be proper. Some other notions of joint spectrum have also been investigated in the past half century. Curto's mini-course gave an in-depth survey on this subject.

Non-Commuting Operators

Unfortunately, the two notions of joint spectrum above do not have a good generalization to non-commuting operators. A drastically different approach was taken. The notion of projective spectrum for general (possibly non-commuting) operators came independently from two directions: the spectral theory of self-similar groups and the theory of nonabelian Banach algebras.

Given several matrices $A_1, ..., A_n$ of equal size, the determinant of the linear pencil $A(z) := z_1A_1 + \cdots + z_nA_n$ was studied as early as the late 19th century and early 20th century. Let λ stand for the regular representation of a finite group $G = \{g_1, ..., g_n\}$. Starting in 1896 [12], Frobenius studied the factorization of $\det(z_1\lambda(g_1) + \cdots + z_n\lambda(g_n))$ in a series of papers. Indeed, this work is the birth place of group representation theory [7]. In 1900 [9], Dixon considered the problem whether a homogeneous polynomial is of the determinantal form $\det A(z)$. The linear pencil related to infinite groups was only seriously considered more than half century later. Kesten [37, 38], published in 1957, studied random walks on groups and discovered a probabilistic criterion of amenability. In the sequel, we let group G be generated by a finite set $\{g_1, ..., g_n\}$. The Markov operator M_λ is the average of $\lambda(g_1), ..., \lambda(g_n)$, namely, $M_\lambda = \frac{1}{n}(\lambda(g_1) + \cdots + \lambda(g_n))$. A weaker version of Kesten's theorem is as follows.

Theorem 15.0.4. *The following are equivalent for a finitely generated group G:*

a) G is amenable;

b) $1 \in \sigma(M_\lambda)$;

c) the spectral radius of M_λ is 1.

The theorem also holds for the weighted average $M = x_1\lambda(g_1) + \cdots + x_n\lambda(g_n)$, where x_i are nonnegative real numbers such that $x_1 + \cdots + x_n = 1$.

Self-Similar Groups. The discovery of the Grigorchuk group $\mathcal{G}$ in 1980 [14] inspired greater interest in the spectral properties of groups. The group $\mathcal{G}$ is a finitely generated infinite torsion group that possesses obvious self-similarity features. Its construction is given in Example 15.0.12. Such torsion groups are also known as the Burnside groups, and they are related to one of the most famous problems in algebra posted by Burnside in 1902. In 1983, Grigorchuk discovered that $\mathcal{G}$ has other remarkable properties [15].

Theorem 15.0.5. *The following hold for the group $\mathcal{G}$.*

a) It has intermediate (between polynomial and exponential) growth.

b) It is amenable but not elementary amenable.

Part a) settles a question of Milnor from 1967 [40], and part b) answers a question of Day from 1957 [8]. Moreover, the construction of $\mathcal{G}$ motivated the definition of self-similar group representation.

Definition 15.0.6. *Given an integer $d \geq 2$, a unitary representationr $(\pi, \mathcal{H})$ of a group G is said to be d-similar if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}^d$ such that for every $g \in G$ the $d \times d$ block matrix $\hat{\pi}(g) = U\pi(g)U^*$ has all of its entries either equal to 0 or of the form $\pi(x), x \in G$.*

It is clear that $\hat{\pi}$ is a unitary representation of G on $\mathcal{H}^d$. Since each of its nonzero block entries are unitary operators on $\mathcal{H}$, every row or column of $\hat{\pi}(g)$ has precisely one nonzero entry. This is manifested clearly in Examples 15.0.11 and 15.0.12 in the next section.

As more examples of self-similar groups are discovered, for instance see Grigorchuk and Gupta-Sidki (1983), efforts have been made to generalize them to the Grigorchuk-Gupta-Sidki (GGS) groups [2], spine groups, and other classes of groups. The idea of self-similarity was thus introduced prominently into group theory, opening a new direction in mathematics that has numerous applications in algebra, dynamical systems, random walks, operator algebras, geometry, computer science, cryptography, and mathematical physics, etc.

In the pioneer paper [4] published in 2000, Bartholdi and Grigorchuk studied the spectra of Schreier graphs associated with self-similar groups. In particular, they considered the invertibility of the pencil

$$A_\pi(z) := z_1\pi(g_1) + \cdots + z_n\pi(g_n), \tag{2}$$

where π is any unitary representation of G and z_i are complex coefficients. This idea has motivated a great amount of subsequent work in geometric group theory. In a broader context, a big part of mathematics is devoted to the study of spectral properties of graphs (finite and infinite), which has applications not only in many areas of mathematics but also in technology, for instance in communication systems via the notion of expanding graphs. Among infinite graphs, a special subclass constitutes graphs with uniformly bounded degree and regular graphs, in which the degrees of all vertices are equal. Such graphs usually have a realization as the Schreier graphs of groups, such as free groups. Schreier graph generalizes the notion of Cayley graph, and it enables much wider applications of group theory methods. The study of Schreier graph is a fast developing frontier in mathematics. Part of this development was presented in the mini-course by Grigorchuk at the workshop.

Projective Spectrum in Banach Algebras. The lack of a proper notion of joint spectrum for non-commuting operators to a large extent confined multivariable operator theory to commuting settings. Attempting to address this issue, and unaware of the early works on group determinant and self-similar groups, Yang defined the notion of projective joint spectrum in 2009 [49].

Definition 15.0.7. *Given elements $A_1, ..., A_n$ in a Banach algebra $\mathcal{B}$, their projective spectrum is defined as $p(A) = \{z \in \mathbb{P}^{n-1} \mid A(z)$ is not invertible$\}$.*

Here, $\mathbb{P}^{n-1}$ stands for the complex projective space of dimension $n - 1$, hence the terminology. This definition has the following obvious features:

1) It is valid for all elements in $\mathcal{B}$, commuting or not.

2) It lets go of the formulation $A - \lambda I$ and treats all elements in a symmetric way.

3) Last but not least, it is easy to compute in many examples.
Projective spectrum has the following general properties.

Theorem 15.0.8. *For $n \geq 2$, the following holds for any elements $A_1, ..., A_n \in \mathcal{B}$.*

a) $p(A)$ is a nonempty compact subset of $\mathbb{P}^{n-1}$.

b) The complement $\mathbb{P}^{n-1} \setminus p(A)$ is a Stein domain.

c) If the operators are commuting, then $p(A)$ is a union of projective hyperplanes.

This result put the study of projective spectrum on a solid base. Many examples were computed following the definition, for example tuples of compact operators or projections, Cuntz algebra, irrational rotation algebra, free group von Neumann algebra, Coxeter groups, etc. A comprehensive treatment of this subject is presented in the upcoming book [50].

In the case $\dim \mathcal{H} < \infty$, the projective spectrum led naturally to the definition of multivariable characteristic polynomial.

Definition 15.0.9. *Given square matrices $A_1, ..., A_n$ of equal size, their characteristic polynomial is defined as*

$$Q_A(z) := \det(z_0 I + z_1 A_1 + \cdots + z_n A_n), \quad z = (z_0, ..., z_n) \in \mathbb{C}^{n+1}.$$

The idea is quickly applied to the study of Lie algebras, leading to a new classification of finite dimensional simple Lie algebras [13] as well as new invariants for the classification of solvable Lie algebras (an unsolved problem) [1].

The Origin of the Workshop. The afore mentioned two fronts of study met in fall 2015 when Yang spent a semester of sabbatical leave at Texas A&M University. In collaboration, Grigorchuk and Yang [30] obtained the following result regarding the infinite dihedral group $D_\infty = \langle a, t \mid a^2 = t^2 = 1 \rangle$.

Theorem 15.0.10. *For the pencil $A_\lambda(z) = z_0 I + z_1 \lambda(a) + z_2 \lambda(t)$, where λ is the regular representation of D_∞, we have*

$$p(A_\lambda) = \bigcup_{-1 \leq \xi \leq 1} \{z \in \mathbb{P}^2 \mid z_0^2 - z_1^2 - z_2^2 - 2z_1 z_2 \xi = 0\}.$$

This theorem found applications to C^*-algebras, Maurer-Cartan differential forms, and Fuglede-Kadison determinant. In particular, it led to a rational map linking self-similarity to complex dynamics and Julia set, see Example 15.0.11 and Goldberg and Yang [32]. Moreover, due to the connection between D_∞ and $\mathcal{G}$, this theorem sheds new light on the spectral picture of $\mathcal{G}$. This collaboration is the original motivation of this Banff workshop.

Recent Developments and Open Problems

The focus of this workshop is on the connection between self-similarity, projective spectrum, and complex dynamics. Two important groups, D_∞ and $\mathcal{G}$, underpined the development that led to this workshop. Both groups have self-similar representations on $L^2[0, 1]$. We provide some details here for clarity. Express the numbers in $[0, 1]$ by a binary sequence, omitting the dot, i.e., a sequence (denoted by w in the sequel) of 0s and 1s. To make this expression unique, we assume that a rational number ends with a sequence of 0s instead of 1s. For example, 0.5 is expressed as $100\cdots$ rather than $0111\cdots$. Any measure-preserving action g on $[0, 1]$ lifts canonically to a unitary map $\pi(g) : L^2[0, 1] \to L^2[0, 1]$ such that $\pi(g)f(x) = f(g^{-1}x)$.

Two Examples. The following two examples serve to illuminate the subsequent report.

Example 15.0.11. *The self-similar representation π of $D_\infty = \langle a, t \mid a^2 = t^2 = 1 \rangle$ on $L^2[0, 1]$ is realized by the following measure preserving action on the interval $[0, 1]$:*

$$a(0w) = 1w, \;\; a(1w) = 0w; \quad t(0w) = 0a(w), \;\; t(1w) = 1t(w).$$

In fact, this action maps dyatic subintervals to dyatic subintervals. Clearly, $L^2[0,1] = L^2[0,1/2] \oplus L^2[1/2,1]$, and we can identify both summands with $L^2[0,1]$ by giving a weight 2 to the Lebesgue measure on the two subintervals. This produces a unitary operator $U : L^2[0,1] \to L^2[0,1] \oplus L^2[0,1]$. Then in light of Definition 15.0.6, we have

$$\hat{\pi}(a) \cong \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \hat{\pi}(t) \cong \begin{bmatrix} \pi(a) & 0 \\ 0 & \pi(t) \end{bmatrix}. \tag{3}$$

Example 15.0.12. *The Grigorchuk group $\mathcal{G}$ is generated by the following four actions on $[0,1]$:*

$$\begin{array}{llll} a(0w) = 1w, & a(1w) = 0w; & b(0w) = 0a(w), & b(1w) = 1c(w); \\ c(0w) = 0a(w), & c(1w) = 1d(w); & d(0w) = 0w, & d(1w) = 1b(w). \end{array}$$

Using the same identification U as that in the previous example, we have

$$\begin{aligned} \hat{\pi}(a) &\cong \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & \hat{\pi}(b) &\cong \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(c) \end{pmatrix}, \\ \hat{\pi}(c) &\cong \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(d) \end{pmatrix}, & \hat{\pi}(d) &\cong \begin{pmatrix} I & 0 \\ 0 & \pi(b) \end{pmatrix}. \end{aligned} \tag{4}$$

Two of $\mathcal{G}$'s outstanding properties are described in Theorem 15.0.12. But two related problems are unsolved.

Problem 1. Is there a finitely generated group whose growth is strictly less than that of the Grigorchuk group $\mathcal{G}$ but faster than polynomials?

Given a representation π of a group G, we denote by $C^*_\pi(G)$ the C^*- algebra generated by the set $\pi(G)$.

Problem 2. Is $C^*_\pi(\mathcal{G})$ just-infinite, meaning that every proper quotient is finite dimensional?

Renormalization Maps. When the invertibility of the pencils $A_\pi(z) = z_0 I + z_1\pi(a) + z_2\pi(t)$ for D_∞ and $B_\pi(z) = z_0 I + z_1\pi(a) + z_2\pi(b) + z_3\pi(c) + z_d\pi(d)$ for $\mathcal{G}$ is considered, the Schur complement leads to the following two rational maps (called the renormalization maps):

$$\begin{aligned} F_A(z) &= \left[z_0(z_0^2 - z_1^2 - z_2^2) : z_1^2 z_2, z_2(z_0^2 - z_2^2)\right], \quad z \in \mathbb{P}^2, \\ F_B(z) &= \left[z_0\alpha - z_1^2(z_0 + z_4) : z_1^2(z_2 + z_3) : z_4\alpha, z_2\alpha, z_3\alpha\right], \quad z \in \mathbb{P}^4, \end{aligned}$$

where $\alpha(z) = (z_0 + z_4)^2 - (z_2 + z_3)^2$.

The connection shown in the two examples above merged three areas of mathematics: spectral theory, holomorphic dynamics, and group theory. Such rational mappings exist for other well-known examples of self-similar groups, such as the Lamplighter group $\mathcal{L}$, the Hanoi towers group $\mathcal{H}^3$, the Basilica group, the iterated monodromy group IMG$(z^2 + i)$. These maps allow an in-depth analysis of the invariant sets and their dynamical behavior. On the other hand, the Basilica group produces a simple 2-dimensional map that is quite hard for investigation. The paper of Dang, Grigorchuk and Lyubich [10] contains a comprehensive study of maps associated with the groups $\mathcal{G}$, $\mathcal{L}$ and $\mathcal{H}^3$. It provides a base for further work. The work in progress on the Basilica group by the same co-authors and Bedford is in a final stage. This part of investigation was presented in the mini-course given by Dang. A plethora of examples of multidimensional rational maps and associated dynamical figures are presented in Grigorchuk and Samarakoon [20]. An important question thus arises.

Problem 3. Can the rational maps, such as F_A and F_B in the examples above, help us understand the spectral property of the groups?

Problem 3 for D_∞ was solved with satisfaction (Theorem 15.0.10). For $\mathcal{G}$, although only partial results are known, the map F_B, in its various forms, indeed helps, see Grigorchuk and Nekrashevych [19]. For many other groups, such as the Basilica group, the Lamplighter group, the Gupta-Sidki 3-group, the overgroup $\tilde{\mathcal{G}}$, just to name a few, computation of their corresponding joint spectrum sheds light on the spectrum of their associated Markov operator [4]. For example, the following facts have been discovered.

a) For $\mathcal{G}$, the spectrum of the Schreier graph coming from the representation π in Example 15.0.12 is a union of two intervals.

b) For the Gupta-Sidki 3-group, the corresponding spectrum is a Cantor set.

c) For the Gupta-Fanrikovsky group, the spectrum is a Cantor set union with a countable set of isolated points accumulating to it.

These are new phenomena in the spectral theory of regular graphs. Indeed, the connection between spectral theory, self-similarity and complex dynamics has led to a number of remarkable results concerning the spectral theory of groups and graphs.

d) The self-similar realization of the Lamplighter $\mathcal{L}$ by automaton of Mealy type over binary alphabet found in [21] led to a proof that $\mathcal{L}$ has a system of generators that produces a Cayley graph with a pure point spectrum. This was the first example of this sort, and it was immediately used in [22] to answer Atiyah's question on the existence of closed Riemannian manifolds with non-integer L^2-Betti number.

e) Further, in [23], the result from [21] was generalized to show that the spectrum of a Cayley graph could have infinitely many gaps.

Schreier Graphs. An open problem related to the e) above is as follows.

Problem 4. Is there a finitely generated group whose associated Cayley graph has a Cantor set spectrum?

There is a plenty of examples of Schreier graphs with such property, see b) above and Grigorchuk, Nagnibeda and Perez [24]. Moreover, in some important cases, for instance $\mathcal{G}$, the spectrum of the Schreier graphs coincides with the spectrum of the Cayley graph [11]. Another important discovery is the link between the spectral problem for the Schreier graphs of linear type and that for the random Schroedinger operators, Grigorchuk, Lenz and Nagnibeda [25]. These results were presented in Nagnibeda's talk. The spectral theory of Schreier graphs of self-similar groups is closely related to the spectral theory of Koopman representation π in the Hilbert space $L^2(\partial T, \mu)$, where ∂T is a boundary of the rooted tree T on which the group acts and μ is a uniform Bernoulli measure on ∂T. It is also related to the quasi-regular representations of the type λ_{G/G_x}, where G_x is a stabilizer of a point x of the boundary. It is a miracle that despite the representation π being a direct sum of finite dimensional subrepresentations, while $\lambda_{G/G_x}, x \in \partial T$, are infinite dimensional irreducible representations in the space $l^2(G/G_x)$, the spectrum of all of them is the same and does not depend on x. Even more interesting are the C^*-algebras generated by these representations. In particular, they are related to the just-infinite trichotomy for C^*-algebras [27]. The technique of joint spectrum allows one to compute the spectra of elements in these algebras, providing important information about their properties. The Hulanicki type theorem for graphs mentioned earlier is a combinatorial analogue of the Hulanicki criterion of amenability: a group is amenable if and only if the regular representation weakly contains every unitary representation of the group.

Problem 5. Find the spectrum of the dendrite type Schreier graphs associated with IMG$(z^2 + i)$.

Observe that for the latter, a 3-dimensional renormalization map was found. Further work is needed to describe the joint spectrum and make conclusion about the spectral properties of of Laplacian on Schreier graphs.

While the interest in group theory focuses on the spectrum, which can be obtained as the limiting set of the zeros of characteristic polynomials associated with the sequences of adjacency matrices, there are other natural polynomials and functions associated with graphs, such as the chromatic polynomials and, more generally, the partition functions. Given a sequence of diamond shaped graphs (called hierarchical), Roeder, in joint work with Bleher and Lyubich, has presented how techniques from holomorphic dynamics can be used to understand the zeros of the partition function associated to a given Ising model. In the same spirit, Peters explained in his talk (based on a joint work with de Boer, Buys and Regts) how one can study zeros of the independence polynomials for sequences of lattices. The main idea is to relate the independence polynomial to the free energy associated with certain quantum field theory. Although these are different polynomials, the nature of the questions are very similar: what is the distribution of the zeros of these polynomials? The method to tackle them involves very recent techniques from holomorphic dynamics in several complex variables.

The characteristic polynomial of Lie algebras. Given a Lie algebra $\mathcal{L} = \operatorname{span}\{x_1, ..., x_n\}$, its characteristic polynomial is defined as

$$Q_{\mathcal{L}}(z) = \det(z_0 I + z_1 \operatorname{ad} x_1 + \cdots + z_n \operatorname{ad} x_n),$$

where ad stands for the adjoint representation of $\mathcal{L}$. It is shown in Hu and Zhang [34] that $\mathcal{L}$ is solvable if and only $Q_{\mathcal{L}}$ is a product of linear factors. Since simple Lie algebras have been fully characterized, the following interesting question awaits an answer.

Problem 6. For a simple Lie algebra $\mathcal{L}$, what are the irreducible factors of $Q_{\mathcal{L}}(z)$?

Presentation Highlights

Mini-courses. An important feature of this workshop is the presentation of mini-courses by the organizers. For a meeting that involves a wide range of fields, it is necessary to lay a common ground and instigate common interest for the participants. The mini-courses by the four organizers served this purpose very well.

Curto's mini-course discussed the foundational aspects of multivariable spectral theory and provided some applications. It started with a description of the algebraic and spatial spectral theory for several commuting operators, with an emphasis on the axiomatic approach to spatial spectra. The he proved the spectral mapping theorem for spatial spectra, assuming that the projection property holds. Next, the analytic functional calculus was defined for the Taylor spectrum, and the connections with the Bochner-Martinelli kernel was mentioned for the case when the operators belong to a C^*-algebra. The Fredholm theory was also mentioned and applied to subnormal n–tuples, Bergman n–tuples, and the n–tuple M_z of multiplications by the coordinate functions acting on the Bergman space of a Reinhardt domain in $\mathbb{C}^n$.

Yang's mini-course first surveyed some recent work on the projective joint spectrum for linear operators. In finite dimension, the notion in part motivated the definition of joint characteristic polynomial for several matrices. His first lecture presented some examples and described an application of the idea to the classification of finite dimensional Lie algebras. The second lecture examined the connection between self-similarity and holomorphic dynamics, focusing particularly on the case of the dihedral group D_∞. He showed that the Julia set of the "normalized" map F_A turns out to coincide with the projective spectrum $p(A_\lambda)$ in Theorem 15.0.10.

Certain problems related to the computation of spectrum and joint spectrum of specific self-similar groups can be translated into a study of the dynamics of very particular transformations. In Dang's mini-course, he introduced the main techniques from complex geometry for studying the iterates of rational transformations. He explained how and when one can define some invariant currents by a given rational transformation and then explained in which situation iterative preimages of a hypersurface/subvarieties can converge to these currents.

In his mini-course, Grigorchuk introduced a general framework of the self-similar groups, explained how self-similarity and a classical tool known as the Schur complement led in some cases to either a complete solution of the spectral problem or to a reduction to a multidimensional dynamical problem: computing iterations of a rational function in $\mathbb{C}^n$ (or $\mathbb{R}^n, n \geq 2$), finding invariant subsets and attractors, etc. The process was demonstrated by examples, including the group $\mathcal{G}$, the Lamplighter group, and the Hanoi Towers groups. The lectures gave a ground for understanding the lectures by Dang and some other talks of the workshop.

On the Theme of Self-Similarity. Amenable groups (under different name) were defined by von Neumann in 1929 as a result of his study on a mysterious phenomenon in mathematics known as the Banach-Tarski Paradox. Independently they were discovered in 1939 by Bogolyubov as a class of groups whose action on a compact space always has an invariant probability measure – a remarkable property that generalizes a powerful theorem of Krylov-Bogolyubov in dynamical systems. The notion of amenability has numerous interpretations: in terms of unitary representations, random walks, combinatorics, dynamics, and joint spectrum, etc. A spectral approach to the amenability of self-similar groups led to the discovery of the "Munchhausen trick", Bartholdi and Virag [3], Kaimanovich [36], which was later used to prove the amenability of the Basilica and many other groups. The method also has a renormalization feature and allows one to associate with any self-similar group a continuous self map $\mathcal{K}$ (the Kaimanovich map) of the simplex $\Delta(G)$ of probability measures on the group. Finding $\mathcal{K}$-invariant finite dimensional non-degenerate subsimplexes of $\Delta(G)$ is a challenging problem. Study of the limit behaviour of random walks on self-similar groups is related to the study of the (covering) random walks on the covering group $\tilde{G}$ which could be a free group or a free product of finite groups. The wreath recursions, which equip G with a self-similar structure, lead to the study of random walks and their limit behavior on the direct products of copies of $\tilde{G}$. This is another challenging problem presented in the talk by Kaimanovich.

The talk by Nekrashevych was dedicated to the relation of random walks on self-similar groups and conformal dimension. Random walks on self-similar groups induce random walks on the (Schreier) graphs of their action on the levels of the tree. If the group is contracting, then the graphs converge in a certain sense to the limit space of the group. The geometry of the limit space can be used to study the random walks on the groups. The conformal dimension of the limit space and the critical exponent of contraction play important role in this study. Since the original approach to these questions involves a map equivalent to the Schur complement, there must be a non-trivial connection of the geometry of the limit space and its dimension with the spectral properties. This workshop was a good opportunity to explore it. The talk presented by Nekrashevych was based on joint work with Matte Bon and Zheng, and it presented new developments in that direction.

The talk by Sunic, "On the Schreier spectra of iterated monodromy groups of critically-fixed polynomials," also discussed the spectra of the Schreier graph of iterated monodromy groups associated with a special type of polynomials. Every self-similar group G of d-ary tree automorphisms induces a sequence of finite Schreier graphs X_n of the action of G on the level n of the tree, along with a sequence of d-to-1 coverings $X_{n+1} \to X_n$. There are interesting examples of self-similar groups for which the spectra of the corresponding Schreier graphs are described by backward iterations of polynomials of degree 2 (the first Grigorchuk group, the Hanoi Towers group, the IMG of the first Julia set, ...). In his talk, for every $r > 1$, Sunic provided examples of self-similar groups for which the spectra of the Schreier graphs are described by backward iterations of polynomials of degree r. The examples come from the world of iterated monodromy groups of critically-fixed polynomials. A critically-fixed polynomial is a polynomial that fixes all of its critical points, and such polynomials are clearly post-critically finite. In general, if we start with any post-critically finite rational map f of degree d on the Riemann sphere, the iterated monodromy group of f (due to Nekrashevych) is a self-similar group acting on the d-ary rooted tree by automorphisms in such a way that the corresponding sequence of Schreier graphs approximates the Julia set of f and the coverings approximate the action of f on the Julia set. In the examples considered, the degree r of the polynomial that describes the spectra of the Schreier graphs coincides with the maximal local degree of f at the critical points.

As iterated monodromy groups can be constructed out of post-critically finite rational maps f, their structure is read out by the action on a certain covering of the punctured sphere. This topological viewpoint has a natural counterpart in arithmetic where a particular Galois group acts on the prime on a field extension. This new direction of research is particularly active in the last decades and many cases of Odoni's conjecture were established. This was the subject of Juul's talk.

Self-similar groups are related to many areas of mathematics, in particular, with the theory of Totally Disconnected Locally Compact Groups (TDLC) discovered recently by Willis [6, 48]. An important subclass of the class of self-similar groups constitutes the self-replicating groups. Scale TDLC groups are closed subgroups of the group $Aut(\tilde{T}_{d+1})$ of automorphisms of the $d+1$-regular tree $\tilde{T}_{d+1}, d \geq 2$ acting vertex transitive and fixing the end of the tree. The stabilizers of vertices in scale groups projected on d-regular rooted tree are self-replication groups. Grigorchuk and Savchuk introduced the class of lifting groups and showed that the closures of such groups embeds into scale groups. Moreover, the closures of the associated ascending HNN-extensions embed as scale groups. They showed for instance that the group $\mathcal{G}$ produces scale group that acts 2-transitively on the punctured boundary of the tree (boundary with one end deleted. Meanwhile it is also shown that groups of isometries of the ring $\mathbb{Z}_p$ and field $\mathbb{Q}_p$ of p-adics are isomorphic to important groups of tree automorphisms. Also the group of dilations of $\mathbb{Q}_p$ is identified. These results were presented in the talk by Savchuk.

In the last two decades, new directions in group theory emerged: dynamically defined groups and measurable group theory, and they are related. A recent book of Nekrashevych [42] (as well as his previous book "Self-similar groups" [41]) contains the main features of these two directions. An important new class of groups of dynamical origin is the class of Topological Full Groups (TFGs) introduced by Giordano, Putnam and Skau [29] as complete algebraic invariant of flip conjugacy for the minimal Cantor systems. They where studied by Matui [39] and other mathematicians. The conjecture of Grigorchuk and Medynets claiming that the TFGs of minimal Cantor systems are amenable was confirmed by Juschenko and Monod [35]. More interesting results about the TFGs are obtained in [39], where it is shown in particular that the commutators of these groups are finitely generated and simple. They are infinitely presented but the presentations by generators and realtors could be written using the Kakutani-Rohlin towers. Also they can be factored into a product of two locally finite groups [26]. The talk of Medynets gave an account of the results on these topics.

The theory of self-similar groups is closely related to the theory of groups generated by finite automata of Mealy type which can be synchronous or asynchronous. Asynchronous invertible automata could generate groups such as the Thompson's group F. The isomorphism type of the group $\mathcal{R}$ of invertible asynchronous automata, introduced in Grigorchuk, Nekrashevych and Suschanskii [28] under the name "group of Rational Homeomorphisms of a Cantor Set", does not depends on the cardinality of the involved finite alphabet $X = \{x_1, \ldots, x_d\}, d \geq 2$. It attracted much attention during the last decade, and a number of results about embedding various types of groups into $\mathcal{R}$ were obtained in the works of Bleak, Belk, Matucci and others.

There is also a way to generalize groups generated by finite synchronous automata, making their higher dimensional analogous. This approach was described in the talk by Vdovina " Higher structures in mathematics: buildings, k-graphs and C^*-algebras." She presented buildings as universal covers of certain infinite families of CW-complexes of arbitrary dimension. Then she showed several different constructions of new families of k-rank graphs and C^*-algebras based on these complexes, for arbitrary k. The underlying building structure allows explicit computation of the K-theory as well as the explicit spectra computation for the k-graphs. Also, she suggested a new definition of higher-dimensional automata motivated by cocompact quotients of buildings, which allows one to construct infinite series of such automata and produce very explicit constructions of Ramanujan higher-dimensional graphs. The talk was based on joint results with Bondarenko, Grigorchuk and Stix.

On the Theme of Spectral Theory. The workshop also consists of inspiring talks in the broader context of spectral theory.

Klep reviewed their solution of the two-sided version of the 2003 conjecture by Hadwin and Larson concerning linear pencils of the form $L = T_0 + x_1 T_1 + \cdots + x_m T_m$, where $x_1, ..., x_m$ are matrices. He showed that ranks of such pencils constitute a collection of separating invariants for simultaneous similarity of matrix tuples.

A (non-selfadjoint) operator algebra is said to be residually finite dimensional (RFD) if it embeds into a product of matrix algebras. A theorem of Exel and Loring characterizes RFD C^*-algebras in terms of the state space and in terms of a finite-dimensional approximation property for representations. Hartz talked about a non-selfadjoint version of the Exel-Loring theorem.

Describing the joint invariant subsapces of a tuple A is in general an insurmountable challenge. The situation is more tractible for multiplication operators on holomorphic function spaces. Misra's talk considered unitary representations of the Möbius group acting on the Hilbert space $\mathcal{H}^{(\lambda)} \otimes V$, where $H^{(\lambda)}$ is a weighted Bergman space and V is a finite dimensional Hilbert space. He showed a one-to-one correspondence between such representations and tuples of subnormal homogeneous operators.

Kuhlmann and Infusino's talks provided an extension of the classical truncated moment problem. They showed a criterion for the existence of a representing Radon measure for linear functionals defined on a unital commutative real algebra. This allows one to extend these existence results to infinite dimensional instances, for example the case when the generating vector space is endowed with a nuclear topology. This allows them to prove a Riesz-Haviland type theorem, extending some classical results in the moment problems.

Given a tuple $A = (A_1, \ldots, A_n)$ of operators or matrices, how much does the projective spectrum $p(A)$ determine the tuple? The tuple is said to be spectrally rigid if any other tuple (probably with additional conditions) that has the same projective spectrum is equivalent to $(A_1, \ldots, A_n)$. Results addressing this question are called spectral rigidity theorems. In his talk, Stessin surveyed some surprising rigidity theorems related to representations of

a) Coxeter groups;

b) certain subgroups of permutation groups which are related to Hadamard matrices of Fourier type;

c) Lie algebra $\mathfrak{sl}(2)$;

4) infinitesimal generators of quantum $SU(2)$ groups.

In addition, Kinser gave an expository introduction to quiver representations and their associated rank schemes. Like joint spectra, they are also defined by determinantal equations. Schiffler's talk aims to link cluster algebras with joint spectrum, and it provided some intriguing examples. Reinke's talk provided a model for random automorphisms of spherically homogeneous rooted trees such that the action on the ends has high emergence almost surely.

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Chapter 16

Spinorial and Octonionic Aspects of G_2 and $\mathrm{Spin}(7)$ Geometry" Final Workshop Report (23w5006)

May 28 - June 2, 2023

Organizer(s): Ilka Agricola (Philipps-Universität Marburg), Shubham Dwivedi (Humboldt-Universität zu Berlin), Sergey Grigorian (University of Texas Rio Grande Valley), Spiro Karigiannis (University of Waterloo), Jason D. Lotay (University of Oxford)

Overview of the Field

The *holonomy* of an oriented Riemannian manifold (M, g) of dimension n is a compact Lie subgroup of $\mathrm{SO}(n)$, which is a global invariant that is intimately related to the Riemann curvature tensor of g, via the Ambrose-Singer theorem. More precisely, its Lie algebra is generated by the Riemann curvature tensor of the metric. Because of this, metrics with reduced holonomy (a proper subgroup of $\mathrm{SO}(n)$) have restrictions on their curvature, which makes them interesting solutions to certain prescribed curvature equations. Note that the holonomy condition is actually a *first order* condition on the metric, which automatically implies a second order condition (a curvature constraint). In 1955, Marcel Berger classified the possible Riemannian holonomy groups that can occur. In the case that M is not locally reducible and not locally symmetric, he found that only seven possible holonomy groups could occur. These groups are summarized in Table 16.1.

In particular, the last two examples are called the *exceptional* holonomy groups, as they occur in particular dimensions and are related to exceptional structures in algebra, the octonions, which was one of the themes of the workshop. It was initially thought that, although Berger could not exclude these possibilities, they would not actually occur. This was proved to not be the case, as Bryant found the first local examples in the 1980's, followed by complete non-compact examples by Bryant–Salamon and independently by groups of physicists, and later compact examples by Joyce in 1994. The last five holonomies in Table 16.1 (all but the generic and Kähler holonomies) are often called *special* holonomies. They are also characterized by the fact that they admit parallel or Killing spinors, which are important ingredients in theories of physics that incorporate supersymmetry. As a result, such metrics have long been of intense interest in physics. All metrics with reduced holonomy come equipped with

Holonomy group	n	Name	Remarks
$\mathrm{SO}(n)$	n	oriented Riemannian	generic
$\mathrm{U}(m)$	$2m$	Kähler	complex and symplectic
$\mathrm{SU}(m)$	$2m$	Calabi-Yau	Ricci flat and Kähler
$\mathrm{Sp}(m)$	$4m$	hyperKähler	Calabi-Yau (Ricci flat) in an S^2 family of ways
$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$	$4m$	quaternionic-Kähler	positive Einstein (but *not* Kähler)
G_2	7	G_2 manifolds	Ricci-flat, related to octonion algebra
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$ manifolds	Ricci-flat, related to octonion algebra

Table 16.1: The possible Riemannian holonomy groups

one or more differential forms which are parallel with respect to the Levi-Civita connection.

G_2 manifolds

After Bryant first proved the local existence of metrics with holonomy G_2 in 1985, Bryant and Salamon soon constructed the first examples of complete metrics with holonomy G_2: these metrics are asymptotically conical and play a crucial role in the field. These examples justified the notion of G_2 manifold: a manifold endowed with a Riemannian metric whose holonomy is contained in G_2.

Then in 1994, Joyce constructed the first compact examples of holonomy G_2 manifolds, which was a fundamental breakthrough in the field, and the analytic theory developed by Joyce underpins all known methods to construct compact G_2 manifolds. In 2003 Kovalev gave a new construction for compact holonomy G_2 manifolds, based on a idea of Donaldson; this construction was later extended by Corti–Haskins–Nordström–Pacini. Based on these constructions, there are now known to be many examples of compact G_2 manifolds.

G_2-structures

The key to understanding and constructing G_2 manifolds goes via G_2-structures: 3-forms on 7-manifolds satisfying a certain positivity condition. A G_2-structure determines a metric and an orientation on a 7-manifold, and the condition for the G_2-structure to define a metric with holonomy contained in G_2 is the so-called *torsion-free* condition: namely that the 3-form is parallel for the Levi-Civita connection of the metric it defines or, equivalently, that it is closed and co-closed (again, using the metric and orientation that it defines). The torsion-free condition can thus be viewed as a system of (nonlinear) partial differential equations for the 3-form.

Although the main interest is in torsion-free G_2-structures, one can also consider splitting the torsion-free condition into two sub-cases: those which are closed and those which are co-closed. In fact, the co-closed condition is essentially vacuous: on any 7-manifold (compact or otherwise), a G_2-structure can be deformed to a co-closed one by the h-principle. By contrast, the closed condition is vital for all known constructions of compact G_2-manifolds, and yet is poorly understood.

Gauge theory and calibrated geometry

Donaldson–Thomas and Donaldson–Segal pioneered the notion of gauge theory in higher dimensions, and in particular in the setting of G_2 geometry. In particular, they defined G_2 instantons, which are connections generalising the more familiar anti-self-dual instantons from 4-dimensional geometry. Specifically, G_2 instantons are connections whose curvature satisfies the condition that its 2-form part lies pointwise in the Lie algebra of G_2, viewed as a subspace of the 2-forms. On G_2 manifolds, G_2 instantons are automatically Yang–Mills connections: that is, they are critical points of the Yang–Mills functional. The proposal is to try to build enumerative invariants for compact G_2 manifolds by "counting" G_2 instantons.

There is a close relationship between G_2 gauge theory and a "dual" theory of certain submanifolds. On a G_2 manifold, the G_2-structure and its Hodge dual are *calibrations*; that is, they are closed differential forms with comass one. The submanifolds calibrated by these calibrations (those submanifolds on which the forms restrict to be the volume form) are called *associative* and *coassociative* submanifolds, and they are automatically homologically volume-minimizing. There are also conjectures suggesting that one can build enumerative invariants using calibrated submanifolds.

Related geometries

There are two close cousins to G_2 geometry: $\mathrm{SU}(3)$ geometry in 6 dimensions and $\mathrm{Spin}(7)$ geometry in 8 dimensions.

Of particular relevance in 6 dimensions are *Calabi–Yau 3-folds* which have metrics with holonomy $\mathrm{SU}(3)$, and *nearly Kähler* 6-manifolds which have the property that the Riemannian cone on them has a torsion-free G_2-structure. In these contexts one has associated problems in gauge theory, namely (pseudo-)Hermitian–Yang–Mills connections, and in calibrated geometry, namely (pseudo-)holomorphic curves and special Lagrangian submanifolds.

In 8 dimensions the most important geometry comes from metrics with holonomy $\mathrm{Spin}(7)$, giving $\mathrm{Spin}(7)$ manifolds: these include Calabi–Yau 4-folds and hyperkähler 8-folds as special cases. This yields some corresponding geometries in 7 dimensions: for example, nearly parallel G_2 manifolds, which have a co-closed G_2-structure (called nearly parallel) with the property that the Riemannian cone on them has holonomy contained in $\mathrm{Spin}(7)$; 3-Sasakian 7-manifolds and Sasaki–Einstein 7-manifolds, where the Riemannian cone on them is hyperkähler and Calabi–Yau respectively. More generally, one can try to understand classes of $\mathrm{Spin}(7)$-structures, which are defined by a certain very restricted type of nondegenerate 4-form on an 8-manifold. In particular, closed $\mathrm{Spin}(7)$-structures are necessarily torsion-free and so define a metric with holonomy contained in $\mathrm{Spin}(7)$.

Physics

Another key direction of interest in G_2 geometry comes from theoretical physics. Compact G_2 manifolds, and compact 7-manifolds with other types of G_2-structures, appear when compactifying String Theory and M-Theory, as well as in the study of anomaly cancellation in heterotic String Theory. In this context, G_2 instantons on compact G_2 manifolds are important because they minimize the Yang–Mills action, and calibrated submanifolds play a crucial role because they minimize volume.

There are several groups of researchers in theoretical physics actively pursuing G_2 geometry, and the physics perspective motivates multiple research directions in G_2 geometry for pure mathematics. In particular, the physics viewpoint leads to various predictions which remain conjectural mathematically.

Another important aspect is the role of spinors and Dirac operators in these settings. Spin geometry seems to be natural for describing many of these structures. For example, we have already mentioned that the Ricci-flat manifolds that have special holonomy admit parallel spinors. Work of Harvey and others shows that calibrations can be obtained as the "square" of a spinor.

Objectives

The first, and primary, objective of our workshop was to bring together individuals with very specific skill sets and knowledge that have not generally interacted sufficiently with each other in the past. In fact, we envisioned three such groups: (i) Riemannian geometers studying G_2 and Spin(7)-structures, (ii) spin geometers, and (iii) non-associative algebraists.

While octonions have been understood by algebraists for quite some time, geometers are not as familiar with all of the peculiarities of octonion algebra and its implications. An increased proficiency with octonion algebra would very likely be enormously beneficial to geometers. Therefore, our plan was to invite a few algebraists who are intimately knowledgeable about octonions in particular or non-assocative algebras in general, but know very little about their applications to Riemannian geometry. Again, such cross-fertilization promised to be extremely fruitful at least for the geometers, and hopefully to the algebraists as well.

Bringing together participants from three different research groups, especially (iii) above, also contributed to the secondary objective of our workshop, which was to increase the diversity of researchers working on exceptional Riemannian geometry, to initiate new connections and new collaborations, and to expose young researchers, who have not yet had the opportunity to sufficiently expand their mathematical breadth, to a wider and more diverse group of mathematicians. Such exposure to different viewpoints, ideas, and techniques, served to improve the abilities and achievements of all participants.

It was very important for us that the event would allow for a significant number of collaborations. To this end, we arranged the schedule so that there would be a fair number of lectures, but also a large amount of open time, with the specific goal of encouraging informal discussions during this time. Fortunately, there have been a large number of PhD students, postdoctoral researchers and other early career researchers who have joined the field in recent years. As a consequence, we deliberately made the meeting a forum for researchers at an early career stage: the vast majority of the speakers and participants were PhD students, with the rest of the speakers either postdocs or researchers who had recently been postdocs. We ran two successful Open Problem sessions where the participants described the problems and ideas in detail. More specifically, the Open Problem sessions identified several interesting research problems that the participants considered worth pursuing, which we describe in the ”Outcome of the Meeting” section.

Recent Developments and Open Problems

G_2 and Spin(7)-manifolds

Recently, there have been various successful generalisations of the known constructions of compact G_2-manifolds which could lead to further examples, including by Joyce–Karigiannis (who extend the Joyce construction) and Nordström (who extends the Kovalev construction). One key problem is to have similar constructions for holonomy Spin(7)-metrics.

In another direction, there has been progress in the rigorous construction of complete non-compact G_2-manifolds which had been predicted by physicists. This work by Foscolo–Haskins–Nordström produces infinitely many cohomogeneity one examples which are asymptotically conical and *asymptotically locally conical*: the latter are asymptotic to a circle bundle over a Calabi–Yau cone. Foscolo–Haskins–Nordström have also produced infinitely many asymptotically locally conical G_2-manifolds which have at most an $\mathcal{S}^1$-symmetry. In general, this is contrary to predictions from physics.

The key problem in the study of holonomy G_2 and $\mathrm{Spin}(7)-$metrics remains open:

- which compact 7-manifolds (8-manifolds) admit holonomy G_2 metrics (Spin(7)-metrics)?

We are far from having even a plausible statment of a Calabi–Yau type theorem for such manifolds, although there were many discussions about “gerbes” and their possible applications for a sufficient condition for existence of holonomy G_2-metrics.

Our understanding of this problem is incredibly limited, but there has been some progress on defining topological and analytic invariants of G_2 structures by Crowley–Goette–Nordström.

Geometric flows of G_2 and Spin(7)-structures

The question of existence of torsion-free G_2 and Spin(7)-structures on a manifold is a challenging problem. Geometric flows are a powerful tool to tackle such questions and one hopes that a suitable flow of such structures with torsion might help in proving the existence of corresponding torsion-free structures. There has been a significant amount of work in this direction, with a notable increase in activity in recent years.

The Laplacian flow for closed G_2-structures and the (modified) Laplacian co-flow for co-closed G_2-structures have been very successful with several analytic and geometric results established for them by many researchers. One particular aspect of these flows is that dimensional reductions of them in 3, 4 and 6-dimensions have been very useful for studying not only the induced flows in these lower dimensions, but they have also shed new light on the 7-dimensional situation as well. The works by Fine–Yao, Lambert–Lotay and Picard–Suan, the latter of which was also presented in the workshop, are some examples of this.

Recently, there has been a lot of activity on the harmonic flow of geometric-structures from both the general point of view in works of Sá Earp–Loubeau and Sá Earp–Fadel–Loubeau–Moreno (this was presented at the meeting) and in particular contexts of G_2, Spin(7) and Sp(2)Sp(1)-structures (this was also presented at the meeting). Apart from analytic interests in these flows themselves, one hope is that these flows can be coupled with other flows of metrics (like the Ricci flow) and the coupled flow might have "nice" properties.

Another type of geometric flow for G_2 and Spin(7)-structures which has been investigated and those which arise as the negative gradient flow of natural energy type functionals associated to these structures. We heard about the gradient flow of Spin(7)-structures in the workshop.

The key issue for the Laplacian flow of closed G_2-structures is that one needs a closed G_2-structure to start the flow and so a major open problem is:

- which compact 7-manifolds admit closed G_2-structures?

A related natural problem, which is central to the field, is:

- can a compact 7-manifold admit an *exact* G_2-structure? For example, does the 7-sphere admit a closed (and hence exact) G_2-structure?

For the harmonic flow of geometric structures, some important issues are:

- understand the singular set and singularity models for such flows, which can help us analyze the long-time behaviour of the flow;
- understand the analytic behaviour of the harmonic flow coupled with an appropriate flow of metrics.

G_2-instantons

An area where there has been a large amount of activity and recent progress is in the study and construction of G_2-instantons.

Building on the earlier gluing results of Walpuski, Sá Earp, and Sá Earp–Walpuski for G_2-instantons on the Joyce and Kovalev examples of compact G_2-manifolds, there has been a great deal of study of the relationship between G_2-instantons and associative 3-folds, and the Seiberg–Witten equations with multiple spinors on 3-manifolds. In particular, there have been significant results by Haydys, Walpuski, Haydys–Walpuski, and Doan–Walpuski.

In another direction, Oliveira, Clarke, and Lotay–Oliveira have constructed new examples and have studied the moduli space of cohomogeneity one G_2-instantons on cohomogeneity one G_2-manifolds, including the Bryant–Salamon G_2-manifolds and some examples of asymptotically locally conical G_2-manifolds. Moreover, Ball–Oliveira have constructed homogeneous G_2-instantons on Aloff–Wallach spaces (which are nearly G_2-manifolds), and have used them to distinguish between nearly parallel G_2-structures on the same Aloff–Wallach space.

In general, the key open problem in the field of G_2-instantons, aside from the many analytic issues, is:

- can G_2-instantons be used to distinguish between compact G_2-manifolds? For example, can they be so used for the known compact G_2-manifolds?

One can also ask similar questions for Spin(7)-instantons, where much less is known, and yet one expects to find many analogous results as in the G_2 setting.

An additional recent research direction has been the study of so-called deformed G_2-instantons (also known as deformed Donaldson–Thomas connections), which are "mirror" to calibrated submanifolds in a similar way to deformed Hermitian–Yang–Mills connections on Calabi–Yau manifolds. The study of these connections (and their Spin(7) analogues) is very much in its infancy, but there are many intriguing problems to explore in this setting in parallel to the more "classical" instantons.

Presentation Highlights

The presentations in the meeting were divided into two types of talks: 1) Longer talks of 50 minutes duration. 2) "Lightning talks" of 15 minutes duration. The latter type of talks were decided so as to have more opportunities for researchers to present work, particularly PhD students who are only partway through their studies. The lightning talks were very well received by the participants and were one of the highlights of the meeting. The research presented at the meeting can be broadly be described using 5 main interrelated themes, together with a pair of introductory talks.

- Introductory talks on octonionic and spinorial aspects of G_2 and $\mathrm{Spin}(7)$-geometry
- Instantons
- Spinors
- Special Structures
- Geometric Flows
- Calibrated Submanifolds

Many of the results discussed touched on more than one of these themes. We mention the talks below in detail with the Lightning talks marked by a $*$.

Introductory talks on octonionic and spinorial aspects of G_2 and Spin(7)-geometry

Speaker: **John Huerta**
Title: Octonions and spinors.
Abstract: The octonions are an eight-dimensional analogue of the complex numbers, formed by adjoining seven square roots of -1 to the real numbers, instead of just one. They are nonassociative, and thus fall outside the scope of much of the usual theory of algebras and their modules that we learn in school. Nevertheless, this strange algebra turns up in surprising corners of mathematics, essentially whenever "exceptional" structures appear. This includes the G_2 and Spin(7) manifolds that are our focus in this workshop. To get started with these geometric structures, I will introduce the octonions, and show how they naturally encode spinors in seven and eight dimensions.

Speaker: **Cristina Draper**
Title: The Killing's gift.
Abstract: When, in 1887, Wilhelm Killing unexpectedly found a new family of complex simple Lie groups, he gave the scientific community a precious gift: a group which can always be studied and continued to be amazing. Of course, we are talking about G_2, the group of the thousand facets.

Instantons

Speaker: **Daniel Platt**
Title: An example of a G_2-instanton on a resolution of $K3 \times T^3/\mathbb{Z}_2$ coming from a stable bundle.
Abstract: I will begin with a brief explanation of what G_2-instantons and G_2-manifolds are. There is a general construction by Joyce–Karigiannis for G_2-manifolds. Ignoring all analysis, I will explain one example of their construction. The example is the resolution of $K3 \times T^3/\mathbb{Z}_2$ for a very explicit $K3$ surface. Furthermore, there is a construction method for G_2-instantons on Joyce–Karigiannis G_2-manifolds. I will explain the ingredients needed for the construction, say nothing about the proof, and then explain one example of the ingredients.

Speaker: **Alfred Holmes***
Title: Spin(7) instantons and the ADHM Construction.
Abstract: In this talk I'll give an overview of a potential way to construct Spin(7) instantons from solutions to the ADHM–Seiberg–Witten equations.

Speaker: **Mario Garcia-Fernandez***
Title: Instantons from the Hull-Strominger system.
Abstract: I will explain how to construct an instanton on a real orthogonal bundle, from a solution of the Hull-Strominger system on a (possibly non-Kähler) Calabi–Yau manifold. If time allows, I will comment on how this basic principle leads to obstructions to the existence of solutions and also on conjectural extensions to the G_2 and Spin(7) heterotic systems. Based on joint work with Raúl Gonzalez Molina, in arXiv:2303.05274 and arXiv:2301.08236.

Speaker: **Sergey Grigorian**
Title: Non-associative gauge theory.
Abstract: In this talk, we generalize some results from standard gauge theory to a non-associative setting. Non-associative gauge theory is based on smooth loops, which are the non-associative analogues of Lie groups. The main components of this theory include a finite-dimensional smooth loop $\mathbb{L}$, together with its tangent algebra and pseudoautomorphism group Ψ, and a smooth manifold with a principal Ψ-bundle $\mathcal{P}$. A configuration in this theory is defined as a pair (s, ω), where s is an $\mathbb{L}$-valued section and ω is a connection on $\mathcal{P}$. Each such pair determines the torsion, which is a key object in the theory. Given a fixed connection, we prove existence of configurations with divergence-free torsion, given a sufficiently small torsion in a Sobolev norm. We will also show how these results apply to G_2-geometry on 7-dimensional manifolds.

Speaker: **Izar Alonso Lorenzo**
Title: New examples of $\mathrm{SU}(2)^2$-invariant G_2-instantons.
Abstract: G_2-instantons are a special kind of connections on a Riemannian 7-manifold, analogues of anti-self-dual connections in 4 dimensions. I will start this talk by giving an overview of why we are interested in them and known examples. Then, I will explain how we construct G_2-instantons in $\mathrm{SU}(2)^2$-invariant cohomogeneity one manifolds and give new examples of G_2-instantons on $\mathbb{R}^4 \times \mathbb{S}^3$ and $\mathbb{S}^4 \times \mathbb{S}^3$. I will then discuss the bubbling behaviour of sequences of G_2-instantons found.

Speaker: **Leander Stecker**
Title: Reducible G_2-structures and solutions to the heterotic G_2 system.
Abstract: We discuss reducible G_2-structures, more precisely G-structures with $G \subsetneq \mathrm{G}_2$ admitting a characteristic connection with parallel skew-torsion. We investigate how these structures can simplify the so-called heterotic G_2 system. Our study focuses on a 1-parameter deformation of the characteristic connection. We find this family to contain two G_2-instantons on 3-(α, δ)-Sasaki manifolds and a new solution of the heterotic G_2 system for arbitrary string parameter α' in the degenerate case. For further geometries we obtain approximate solutions. Joint work with Mateo Galdeano.

Spinors

Speaker: **Diego Artacho***
Title: Generalised Spin^r Structures on Homogeneous Spaces.
Abstract: Spinorial methods have proven to be a powerful tool to study geometric properties of Spin manifolds. The idea is to make accessible the power of Spin geometry to manifolds which are not necessarily Spin. The concept of Spin^c and Spin^h structures provide examples of work in that direction. In this talk, I will present a generalisation of these structures and comment on what these structures look like on homogeneous spaces, particularly on spheres.

Speaker: **Guilia Dileo***
Title: Generalized Killing spinors on 3-(α, δ)-Sasaki manifolds.
Abstract: 3-(α, δ)-Sasaki manifolds are a special class of Riemannian manifolds generalizing 3-Sasaki manifolds, and admitting a canonical metric connection with totally skew-symmetric torsion. In the present talk I will show that every 7-dimensional 3-(α, δ)-Sasaki manifold admits a canonical G_2-structure, which determines four generalized Killing spinors. The corresponding generalized Killing numbers are explicitly obtained, providing characterization of the cases where they coincide. This is part of a joint work with Ilka Agricola.

Speaker: **Markus Upmeier***
Title: Spinors, calibrated submanifolds, and instantons.

Abstract: In the context of enumerative geometry for manifolds of special holonomy there is a deep connection between calibrated submanifolds and instantons through 'bubbling'. During the talk, I will use spinors and Dirac operators to discuss an interesting link between the (linearized) deformation theories of calibrated submanifolds and instantons. The applications of the main result include a solution to the open problem of constructing orientation data in DT-theory for Calabi–Yau 4-folds.

Speaker: **Michael Albanese**
Title: Spin^h and further generalisations of spin.
Abstract: The question of which manifolds are spin or spin^c has a simple and complete answer. In this talk we address the same question for the lesser known spin^h manifolds which have appeared in geometry and physics in recent decades. We determine the first obstruction to being spin^h and use this to provide an example of an orientable manifold which is not spin^h. The existence of such an example leads us to consider an infinite sequence of generalised spin structures. In doing so, we determine an answer to the following question: is there an integer k such that every manifold embeds in a spin manifold with codimension at most k? This is joint work with Aleksandar Milivojevic.

Special Structures

Speaker: **Henrik Naujoks***
Title: Geometry and Spectral Properties of Aloff–Wallach Manifolds (Part I).
Abstract: The focus of our attention will be the Aloff–Wallach manifolds $\mathrm{SU}(3)/S^1{}_{k,l}$. The family of manifolds depending on the embedding parameters k, l will each be equipped with a metric depending on four additional parameters. These six parameters in total lead to various interesting structures (K-contact as well as Sasakian structures, Einstein metrics, etc.) on this set of Riemannian manifolds. The interplay of these structures will be discussed. Furthermore, we investigate the spectrum of the Laplace operator: The metrics on the Aloff–Wallach manifolds $\mathrm{SU}(3)/S^1{}_{k,l}$ are not normal, but for $k = l = 1$ some of them are isometric to a normal homogeneous space. For the latter, the spectrum of the Laplace operator can be explicitly computed using methods of representation theory.

Speaker: **Jonas Henkel***
Title: Geometry and Spectral Properties of Aloff–Wallach Manifolds (Part II).
Abstract: The focus of our attention will be the Aloff–Wallach manifolds $\mathrm{SU}(3)/S^1{}_{k,l}$. The family of manifolds depending on the embedding parameters k, l will each be equipped with a metric depending on four additional parameters. These six parameters in total lead to various interesting structures (K-contact as well as Sasakian structures, Einstein metrics, etc.) on this set of Riemannian manifolds. The interplay of these structures will be discussed. Furthermore, we investigate the spectrum of the Laplace operator: The metrics on the Aloff–Wallach manifolds $\mathrm{SU}(3)/S^1{}_{k,l}$ are not normal, but for $k = l = 1$ some of them are isometric to a normal homogeneous space. For the latter, the spectrum of the Laplace operator can be explicitly computed using methods of representation theory.

Speaker: **Christina Tonnesen-Friedman***
Title: Sasakian geometry on certain fiber joins.
Abstract: This presentation will be based primarily on past and ongoing work with Charles P. Boyer. We will discuss the Sasakian geometry of certain 7-manifolds constructed by the so-called fiber join construction for K-contact manifolds, introduced by T. Yamazaki around the turn of the century. This construction can be adapted to the Sasaki case and produces some interesting examples. We will talk about some of these examples and also discuss some limitations of the construction.

Speaker: **Lucia Martin-Merchan**
Title: Topological properties of closed G_2 manifolds through compact quotients of Lie groups.
Abstract: In this talk, we discuss two problems where compact quotients of Lie groups are useful for understanding topological properties of compact closed G_2 manifolds that don't admit any torsion-free G_2 structure. These problems are related to the questions: Are simply connected compact closed G_2 manifolds formal? Could a compact closed G_2 manifold have third Betti number $b_3 = 0$?

Using compact quotients of Lie groups, we first outline the construction of a manifold admitting a closed G_2 structure that is not formal and has first Betti number $b_1 = 1$. Later, we show that compact quotients of Lie groups do not have any invariant G_2 structure. The last result is joint work with Anna Fino and Alberto Raffero.

Speaker: **Anton Iliashenko**
Title: Betti numbers of nearly G_2 and nearly Kähler manifolds with Weyl curvature bounds.
Abstract: We use the Weitzenböck formulas to get information about the Betti numbers of nearly G_2 and nearly Kähler manifolds. First, we establish estimates on two curvature-type self adjoint operators on particular spaces assuming bounds on the sectional curvature. Then using the Weitzenböck formulas on harmonic forms, we get results of the form: if certain lower bounds hold for these curvature operators then certain Betti numbers are zero. Finally, we combine both steps above to get sufficient conditions of vanishing of certain Betti numbers based on the bounds on the sectional curvature.

Speaker: **Gavin Ball**
Title: Irreducible SO(3)-geometry in dimension 5.
Abstract: The action of $\mathrm{SO}(3)$ by conjugation on the space of symmetric traceless matrices gives an embedding of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$. A 5-manifold whose structure group reduces to this copy of $\mathrm{SO}(3)$ is said to carry an $\mathrm{SO}(3)$-structure. The integrable examples of these structures are the symmetric spaces $\mathbb{R}^5$, $\mathrm{SU}(3)/\mathrm{SO}(3)$ and $\mathrm{SL}(3)/\mathrm{SO}(3)$, and general $\mathrm{SO}(3)$-structures may be thought of as non-integrable analogues of these spaces. In my talk, I will describe work in progress on the local geometry of a subclass of $\mathrm{SO}(3)$-structures called the nearly integrable $\mathrm{SO}(3)$-structures. The nearly integrable condition was introduced by Bobienski and Nurowski as an analogue of the nearly Kähler condition in almost Hermitian geometry. However, despite the similarity of the definitions, it turns out that the local geometry of nearly integrable $\mathrm{SO}(3)$-structures is significantly more restricted compared to the nearly Kähler case. The rigid nature of the local geometry suggests the possibility of giving a global classification of nearly integrable $\mathrm{SO}(3)$-structures and I will sketch out such a program. If time permits, I will describe relations with G_2-geometry.

Speaker: **Fabian Lehmann**
Title: Closed 3-forms in dimension 5.
Abstract: There is a notion of non-degenerate 3-form in six and seven dimensions which are the pointwise model for G_2- and $\mathrm{SL}(3,\mathbb{C})$-structures, respectively. These are directly related, as the restriction of a 3-form which defines a G_2-structure on a 7-manifold to a real hypersurface induces an $\mathrm{SL}(3,\mathbb{C})$-structure. I will describe the geometric structure induced on a real hypersurface inside a 6-manifold with an $\mathrm{SL}(3,\mathbb{C})$-structure under a certain convexity condition. This is based on joint work with S. Donaldson.

Geometric Flows

Speaker: **Udhav Fowdar**
Title: On the harmonic flow of Sp(2)Sp(1)-structures on 8-manifolds.
Abstract: The harmonic flow of an H-structure (aka the isometric flow) is the gradient flow of the energy functional for the intrinsic torsion. In recent years the cases when $H = \mathrm{U(n)}, \mathrm{G}_2$ and Spin(7) have been studied in great detail mainly due to their relation with special holonomy. In this talk I will discuss the case when $H =$ Sp(2)Sp(1) (i.e. the quaternionic Kähler case) and shed some light into how the representation theory of H allows for a more unified approach. I will also discuss the cases when $H =$ Sp(1) and Sp(2) to illustrate certain similarities and differences. Aside from analytical aspects of the flow, I will also describe explicit examples of non-trivial harmonic H-structures and as well as soliton solutions to the flow. This is a joint work with Henrique Sá Earp.

Speaker: **Gonçalo Oliveira**
Title: Lagrangian mean curvature flow and the Gibbons-Hawking ansatz.
Abstract: In this talk, I will report on joint work with Jason Lotay on which we prove versions of the Thomas and Thomas-Yau conjectures regarding the existence of special Lagrangian submanifolds and the role of Lagrangian mean curvature flow as a way to find them. I will also report on some more recent work towards proving more recent conjectures due to Joyce.

Speaker: **Henrique Sá Earp**
Title: Flows of geometric structures.

Abstract: We develop an abstract theory of flows of geometric H-structures, i.e., flows of tensor fields defining H-reductions of the frame bundle, for a closed and connected subgroup $H \subset \mathrm{SO}(n)$, on any connected and oriented n-manifold with sufficient topology to admit such structures.

The first part of the talk sets up a unifying theoretical framework for deformations of such H-structures, by way of the natural infinitesimal action of $\mathrm{GL}(n, \mathbb{R})$ on tensors combined with various bundle decompositions induced by H-structures. We compute evolution equations for the intrinsic torsion under general flows of H-structures and, as applications, we obtain general Bianchi-type identities for H-structures, and, for closed manifolds, a general first variation formula for the L^2-Dirichlet energy functional $\mathcal{E}$ on the space of H-structures.

We then specialise the theory to the negative gradient flow of $\mathcal{E}$ over isometric H-structures, i.e., their harmonic flow. The core result is an almost monotonocity formula along the flow for a scale-invariant localised energy, similar to the classical formulae by Chen–Struwe for the harmonic map heat flow. This yields an ε-regularity theorem and an energy gap result for harmonic structures, as well as long-time existence for the flow under small initial energy, with respect to the L^∞-norm of initial torsion, in the spirit of Chen–Ding. Moreover, below a certain energy level, the absence of a torsion-free isometric H-structure in the initial homotopy class imposes the formation of finite-time singularities. These seemingly contrasting statements are illustrated by examples on flat n-tori, so long as $\pi_n(\mathrm{SO}(n)/H) \neq \{1\}$; e.g. when $n = 7$ and $H = \mathrm{G}_2$, or $n = 8$ and $H = \mathrm{Spin}(7)$.

Speaker: **Caleb Suan**
Title: Flows of G_2-structures associated to Calabi–Yau manifolds.
Abstract: The Laplacian flow and coflow are two of the most studied flows in G_2 geometry. We will establish a correspondence between parabolic complex Monge–Ampère equations and these flows for initial data on a torus bundle over a complex Calabi–Yau 2- or 3-fold given from a Kähler metric. We will use estimates for these complex Monge–Ampère flows to show that both the Laplacian flow and coflow exist for all time and converge to a torsion-free G_2 structure induced by a Ricci-flat Kähler metric. This is joint work with Sébastien Picard.

Speaker: **Shubham Dwivedi***
Title: A gradient flow of Spin(7)-structures.
Abstract: We will introduce a geometric flow of Spin(7)-structures which is the negative gradient flow of a natural energy functional on the space of Spin(7)-structures. We will evaluate the evolution of the Riemannian metric and show that the flow exists for a short time.

Calibrated geometry

Speaker: **Jesse Madnick**
Title: Harmonic Spinors and Associative 3-folds.
Abstract: There are several relationships between (twisted) harmonic spinors and associative submanifolds. For example, the Dirac operator appears in the PDE for associative graphs, in the deformation theory for associative submanifolds, and in the second variation formula for volume.

In the first part of this talk (joint work with Gavin Ball), we provide another relationship. If a 7-manifold M has a closed or nearly-parallel G_2-structure, we show that the second fundamental form of an associative can be viewed as a twisted spinor. Moreover, if M has constant curvature (e.g., if $M = \mathbb{R}^7, \mathbb{S}^7$, or $\mathbb{T}^7$), then this twisted spinor is harmonic. Intuitively, this is a spin-geometric analogue of the classical Hopf differential for 2-dimensional surfaces.

In the second part, we elaborate on this theme, highlighting the many analogies between harmonic spinors and holomorphic objects. In particular, we provide a "taxonomy" of Dirac equations that have arisen in the literature, which in turn suggests several avenues for further work.

Speaker: **Da Rong Cheng**
Title: A variational characterization of calibrated submanifolds.
Abstract: I will report on recent joint work with Spiro Karigiannis and Jesse Madnick where we discover, for a number of different calibrations, a characterization of calibrated submanifolds in terms of the first variation of the volume functional with respect to a special set of deformations of the ambient metric determined by the calibration form. Generalizing earlier such results due to Arezzo and Sun for complex submanifolds, we obtain variational characterizations for associative 3-folds and coassociative 4-folds in manifolds with G_2-structures, as well as for Cayley 4-folds in manifolds with Spin(7)-structures.

Speaker: **Federico Trinca**
Title: Calibrated geometry in G_2-manifolds with cohomogeneity two symmetry.
Abstract: Constructing associative and coassociative submanifolds of a G_2-manifold is, in general, a difficult task. However, when the ambient manifold admits symmetries, finding cohomogeneity one calibrated submanifolds is more tractable. In this talk, I will discuss joint work with B. Aslan regarding the geometry of such calibrated submanifolds in G_2-manifolds with a non-abelian cohomogeneity two symmetry. Afterwards, I will explain how to apply these results to describe new large families of complete associatives in the Bryant–Salamon manifold of topology $S^3 \times \mathbb{R}^4$ and in the manifolds recently constructed by Foscolo–Haskins–Nördstrom.

Scientific Progress Made

We summarize the scientific progress made in each of the main themes highlighted in the previous section.

Instantons

There has clearly been a significant increase in our understanding of higher-dimensional gauge theory beyond the established settings of compact Calabi–Yau, G_2, and Spin(7)-manifolds. There have been extensions to non-integrable structures, such as almost complex 6-manifolds, nearly parallel G_2-manifolds, and Sasaki–Einstein 7-manifolds, and in the study of the non-compact setting of asymptotically conical G_2-manifolds. In particular, we have seen classification and deformation theory results. Moreover, there is interest in a non-associative gauge theory on manifolds with G_2-structure.

In the compact and non-compact G_2-manifold setting, which holds the greatest interest in G_2 geometry, there has been exciting progress towards potentially constructing a large number of G_2-instantons on the new examples of G_2-manifolds due to Joyce–Karigiannis as well as instantons on non-compact examples of G_2-manifolds due to Bryant–Salamon and Foscolo–Haskins–Nordström.

A new avenue has been study of the Hull–Strominger system and solutions to the heterotic G_2-system, as well as their relation to the study of moduli spaces of G_2-instantons.

Spinors

A lot of progress has been made in the study of special structures on manifolds arising from spinors. There have been spinorial classifications of manifolds with G_2 and Spin(7)-structures. There have also been many results on properties of the Dirac operator, in particular, in studying the spectrum of the Dirac operator and applications to understanding the geometric properties of manifolds. There have been many significant results on the relationship between spinors and calibrated submanifolds of manifolds with special holonomy, and spinors have also been used to study properties such as the orientation of the moduli spaces of instantons. Further generalisations of a spin structure like the Spin^r and Spin^h-structures have also been studied.

Special Structures

Special structures have received relatively little detailed attention and are generally quite poorly understood. The results presented in the meeting clearly show a marked improvement in our ability to study and understand these structures. For example, we learned about advances in our knowledge of the spectral properties of a class of nearly parallel G_2-manifolds. and saw very interesting results on topology of manifolds with closed G_2-structures and nearly parallel G_2 manifolds. We also saw the geometry of nearly integrable $\mathrm{SO}(3)$-structures in dimension 5 and their relationship to G_2-geometry. In similar themes, there were exciting results about the geometry induced by closed 3-forms in dimension 5. The techniques described in these talks will definitely have many application in related problems.

Geometric Flows

There were some interesting results concerning the dimensional reduction of the G_2-Laplacian flow and the G_2-Laplacian co-flow and its applications, building on the general theory developed in recent years. Specifically there were some impressive long-time existence and convergence results in the setting of reducing the flow to dimensions 6 and 4 and understanding the relationship with the more well-understood Kähler–Ricci flow and the so called Monge–Ampère flow. Both of these results certainly merit further examination and reveal exciting future research avenues for investigation.

The analytic foundations were developed for the harmonic flow of geometric structures which has G_2, $\mathrm{Spin}(7)$ and Sp(2)Sp(1) as special cases. This is a new research topic that has links to several research groups in geometry of manifolds with special holonomy, and so will certainly continue to be studied.

A new flow of Spin(7)-structures was introduced and analytic properties of the flow were discussed. Since flows of Spin(7)-structures have not been studied in much detail before it is expected that the flow discussed during the meeting will be studied in much more detail in the future.

There were exciting results on the Lagrangian mean curvature flow and its applications to prove versions of conjectures due to Thomas, Thomas–Yau and Joyce. The discussion emphasized the strength of geometric flows techniques to prove hard conjectures on the existence of calibrated submanifolds inside Calabi–Yau manifolds.

Calibrated Submanifolds

There was notable progress made in the study of calibrated submanifolds both in special holonomy manifolds and outside of the setting of well-known areas of manifolds with special holonomy equipped with their usual calibrations. We saw a variational characterisation of calibrated submanifolds of manifolds with exceptional holonomy. Symmetry methods are a powerful tool in constructing new examples of geometric objects and we had a discussion on cohomogeneity two associatives in noncompact G_2 manifolds constructed by Bryant–Salamon and Foscolo–Haskins–Nördstrom. There was a very nice discussion on spinors and the second fundamental form of associatives in nearly G_2 manifolds. In particular, the techniques developed in these works are certainly to yield further results in related areas.

Outcome of the Meeting

The key outcome of the meeting was the increase in communication and collaboration between researchers in G_2 and Spin(7)-geometry who work on seemingly different aspects of these structures, which has and will continue to lead to exciting new research directions and results. It is particularly worth emphasizing the positive outcome of the meeting for early career researchers present, mainly for PhD students but also some postdocs and other participants, who unanimously expressed how enjoyable and productive the meeting was for them. Senior researchers also remarked on how refreshing it was to have so many early career researchers interacting significantly with them and each other, which provided a unique opportunity to learn about and to offer input towards the research avenues pursued by the next generation of researchers in the field.

More specifically, the Open Problem sessions identified several interesting research problems that the participants considered worth pursuing, which we describe below.

1. **(Jason Lotay)** Recall the notion of "triality" from John Huerta's talk. We know about triality at the algebraic level: it is a symmetry between vectors and spinors for the normed division algebra $\mathbb{O}$. Suppose we have (M^8, Φ) which is an 8-dimensional manifold with a torsion-free $\mathrm{Spin}(7)$-structure Φ. Is there any geometric meaning (as opposed to the purely algebraic structure at the level of Clifford algebras and $\mathbb{O}$) of triality in this setting?

   Does it make sense to define the notion of "mirror triality" for a triple of objects similar to the notion of mirror manifolds and mirror symmetry?

Some ideas related to the first question were suggested by **Gavin Ball:** If we look at $Gr_{\text{Cayley}}(4,8)$, the Grassmannian of Cayley 4-planes in $\mathbb{R}^8$ and the Grassmannian of 3-planes in $\mathbb{R}^7$ then $Gr_{\text{Cayley}}(4,8) \cong Gr(3,7)$.

If we look at the space of curvature tensors of a Spin(7)-manifold, i.e., (M^8, Φ) with a torsion-free $\mathrm{Spin}(7)$-structure Φ then that as an irreducible $\mathrm{Spin}(7)$-representation is isomorphic to $V_{0,2,0}$ which is also isomorphic to the space of curvature tensors of Ricci-flat 7-manifolds. This might give a hint for the geometric implication of triality for $\mathrm{Spin}(7)$-manifolds.

In fact, an analogous question would be that if we have two $\mathrm{Spin}(7)$-manifolds M^8_+, M^8_- then do they relate to a Ricci-flat 7-manifold if we have a geometric notion of triality?

2. **(Spiro Karigiannis)** If b^2, b^3 denote the 2nd and 3rd Betti numbers of a G_2-manifold then b^2+b^3 is invariant under "mirror symmetry" for G_2-manifolds, i.e, they remain the same for the mirror manifolds. There is a notion of conifold transition in Calabi–Yau geometry and an analogous idea of G_2 conifold transitions has been given by Atiyah–Witten [1]. Recall that a G_2-manifold M^7 is called **semi-flat** if M is a coassociative fibration and the fibers are flat tori T^4. What can be said about the G_2 conifold transitions in the semi-flat case?

3. **(Jesse Madnick)** Construct non-trivial compact associative submanifolds in the Aloff–Wallach spaces $N_{k,l}$ with $(k,l \neq (1,1))$ where $N_{k,l} = \dfrac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,l}}$ with its homogeneous nearly parallel G_2-structure.

   What can be said about the conformal structure of associatives $\Sigma^3 \subset M^7$? An idea would be to use harmonic spinors just like one uses holomorphic sections to study conformal structures for holomorphic curves.

   Can an **open** Riemann surface be conformally embedded in S^6? It's a theorem due to Robert Bryant that closed Riemann surfaces can be conformally embedded in S^6.

4. **(Sergey Grigorian, Spiro Karigiannis, John Huerta.)** Consider **gerbes** on G_2-manifolds. Suppose we have a manifold with a closed G_2-structure, i.e., (M^7, φ) with $d\varphi = 0$ and $[\varphi] \in H^3(M, \mathbb{Z})$. Is there any relation between $\mathrm{U}(1)$-gerbes on (M^7, φ) and $[\varphi]$. Or, consider $d * \varphi = 0$ and $[*\varphi] \in H^4(M, \mathbb{Z})$. Does there exist a relation between a 2-gerbe over M^7 and $[*\varphi]$?

   A motivation to study these questions come from Kähler geometry and to try to come up with a "G_2-Calabi–Yau theorem". A more precise but still vague question is the following: Recall that if we have a Kähler manifold (M^{2m}, g, J, ω) and we consider the canonical bundle $K = \Lambda^{m,0}(T^*M)$, then it is a line bundle over M, and Yau's proof of the Calabi conjecture states that $c_1(K) = 0 \iff$ there exists a Ricci-flat metric with its Ricci form in $[\omega]$. Here $c_1(K)$ is the first Chern class of K.

   So now suppose we have a manifold with a G_2-structure (M^7, φ, g). Does there exist some "canonical gerbe K" on M such that $c_1(K) = 0 \in H^3(M, \mathbb{Z}) = 0 \iff$ there exist a torsion-free G_2-structure in $[\varphi]$? Here $c_1(K)$ is the "first Chern class of the gerbe K" or more precisely the Dixmier–Douady class. Note that the question as stated is particularly vague because we still do not understand the actual notion of gerbes on G_2-manifolds and their relation to the torsion-freeness of the G_2-structure.

   More information about gerbes can be found in [2, 3, 4].

5. **(Mario Garcia-Fernandez)** Consider the heterotic G_2-system: that is, we have a compact (M^7, φ), $P \to M$ is a principle G-bundle with compact G and A a connection on P, and let $\alpha' \in \mathbb{R}$ be such that $dH = \alpha' \langle F_A \wedge F_A \rangle$, where $H \in \Omega^3(M)$.

   (a) Is there a spinorial interpretation for the cases when the torsion component $\tau_0 \neq 0$?

   (b) There have been both exact and approximate solutions of the heterotic G_2-system. Construct large classes of solutions and maybe solutions with large volume?

   (c) Is there a geometric flow to study the heterotic G_2-system?

   (d) From considerations in physics, α' is hoped to be "small". Do the solutions proposed in Leander Stecker's talk in conference (based on his work with Mateo Galdeano) have small α'?

(e) Consider a sequence of heterotic G_2-systems $\{(M_{\alpha'_n}, \varphi_{\alpha'_n}, A_{\alpha'_n})$ where α'_n is a sequence in $\mathbb{R}$ and suppose that $\alpha'_n \to 0$. What can be said about the limit?

(f) In the case of part (e), suppose that $M_{\alpha'_n} = M_{\alpha'} = M$ and that it admits a torsion-free G_2-structure φ. Do we have $\varphi_{\alpha'_n} \to \varphi$ and $A_{\alpha'_n} \to A$ with A a G_2-instanton?

(g) If $M_{\alpha'}$ does not have a torsion-free limit then what happens to the limit? Does the limit collapse? Is the limit a soliton? (There is a notion of a heterotic G_2-system being a soliton)

(h) Can we construct solutions of the heterotic G_2-system with $\alpha' \neq 0$ from a limit?

6. **(Henrique Sà Earp)** Consider instantons of Sasakian 7-manifolds, i.e., we take $\sigma \in \Omega^3(M)$ with $\sigma = \eta \wedge d\eta$ with η the contact 1-form to define the notion of instanton. For the Sasakian case (which could be viewed as transverse-Kähler geometry) we have $d\eta = \omega$ and hence $\sigma = \eta \wedge \omega$. The space of 2-forms decompose further with
$$\Omega^2 = \Omega^2_V \oplus \Omega^2_H$$
and furthermore
$$\Omega^2_H = \Omega^2_8 \oplus \Omega^2_6 \oplus \Omega^2_1.$$
Instantons A with $F_A \in \Omega^2_8$ are self-dual contact instantons. If we look at the moduli space of self-dual contact instantons $\mathcal{M}_{SDCI}$ then one can show that dim $\mathcal{M}_{SDCI}$ = ind $\not{D}$ and $\mathcal{M}_{SDCI}$ is Kähler on its smooth locus.

(a) Can we define an orientation on $\mathcal{M}_{SDCI}$?

(b) What happens to the blow-ups, bubbling, and compactifications of $\mathcal{M}_{SDCI}$?

(c) Suppose we consider the 3-Sasakian case. Can we prove that $\mathcal{M}_{SDCI}$ is hyperKähler?

7. **(Jesse Madnick)** Can we say something about the non-zero torsion classes of a G_2-structure, the appearance of which will be a necessary and sufficient condition for every G_2-instanton being a Yang–Mills connection?

8. **(Gonçalo Oliveira)** There is a result of Derdzinsky from the 80s which says that "(M^4, g, ω) extremal (i.e., ∇S is a holomorphic vector field with S the scalar curvature) and g Bach-flat $\implies$ $(M^4, S^{-2}g)$ is Einstein."

Can we find conditions on (N^5, g, η, Φ) which is a Sasakian 5-manifold and is extremal, analogous to Bach-flatness in the 4-dimensional case, which would imply the existence of a conformal metric which is Einstein?

9. **(Spiro Karigiannis)** Let α be a calibration k-form on $\mathbb{R}^n$ equipped with the standard metric and orientation. (That is, α has constant coefficients and comass one.) Let $G = \mathrm{Stab}_{\mathrm{O}(n)}\alpha$. There are several properties that α may or may not have. These are the following:

(a) G acts transitively on the unit sphere S^{n-1} in $\mathbb{R}^n$.

(b) G acts transitively on the Stiefel manifold $V_{r,n}$ of r-tuples of orthonormal vectors in $\mathbb{R}^n$ for $1 \leq r \leq k-1$. (Note that (a) is just (b) for $r = 1$.)

(c) G acts transitively on the Grassmanian Gr_α of α-calibrated k-planes in $\mathbb{R}^n$

(d) Let W be an α-calibrated k-plane in $\mathbb{R}^n$, and let $H = \{P \in G : P(W) = W\}$ be the stabilizer in G of W. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G, H respectively. Then we can write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp_\mathfrak{g}}$. We always have the equality $\mathfrak{h} = \Lambda^2(W) \oplus \Lambda^2(W^\perp)$ and the inclusion $\mathfrak{h} \supseteq \mathfrak{g} \cap (W \otimes W^\perp)$. Property (d) is that the inclusion is an equality. We say such an α is *compliant*. If property (c) holds, then property (d) is independent of the choice of $W \in \mathrm{Gr}_\alpha$.

(e) Suppose that (c) holds. Let $W \in \mathrm{Gr}_\alpha$. Let $e_1, \ldots, e_k$ be an oriented orthonormal basis of W and let $\nu_1, \ldots, \nu_{n-k}$ be an oriented orthonormal basis of $W^\perp$. Then in terms of the decomoposition $\Lambda^k(\mathbb{R}^n) = \Lambda^k(W \oplus W^\perp) = \oplus^k_{p+q} \Lambda^p(W) \otimes \Lambda^q(W^\perp)$, we can write $\alpha = \sum_{p+q=k} \alpha_{p,q}$. Property (e) is that only even values of q occur in this decomposition. One can show that (e) implies (d).

(f) For any $v \in S^{n-1}$, both $v \lrcorner \alpha$ and $v \lrcorner \star \alpha$ have comass one. This is equivalent to the fact that any unit vector v lies in an α-calibrated k-plane and also lies in a $(\star\alpha)$-calibrated $(n-k)$-plane. We say that such an α is *rich*.

(g) For $v \in S^{n-1}$, let $L_v = \mathrm{Span}\{v\}$, so $\mathbb{R}^n = L_v \oplus L_v^\perp$. Write $\alpha = v \wedge \beta_v + \gamma_v$ where $v \lrcorner \beta_v = v \lrcorner \gamma_v = 0$, so $\beta_v \in \Lambda^{k-1}(L_v^\perp)$ and $\gamma_v \in \Lambda^k(L_v^\perp)$. Property (g) is that $\langle \beta_v, w \lrcorner \gamma_v \rangle = 0$ for all $w \in \mathbb{R}^n$ and all $v \in S^{n-1}$.

For some mysterious reason, every single one of the above properties is satisfied by the interesting geometric calibration forms (Kähler, special Lagrangian, associative, coassociative, Cayley). Several of these properties are in some sense quantifying that there are many α-calibrated k-planes. Is there a single property that a calibration α could have which implies all of these? If so, what is the geometric significance of such a property?

10. **(Daniel Platt)** Let $s : \mathbb{T}^3 \to X^{4k}$ (hyperKähler) and let $\{x_1, x_2, x_3\}$ be coordinates on $\mathbb{T}^3$ and I_1, I_2, I_3 be the triple of complex structures on X^{4k}. The Fueter operator on s is $Fs = \sum_{i=1}^3 I_i \left(ds(\frac{\partial}{\partial x_i}) \right)$. If $Fs = 0$ then s is called a Fueter section. A known fact about F is that it is an index 0 operator and hence the expectation is that Fueter sections are rigid. However, all know examples of X^{4k} where s is explicit have moduli. So can we have Fueter sections which do not have moduli, i.e., that are rigid?

11. **(Jason Lotay)** Suppose $E \to (M^7, \varphi)$ with M being a G_2-manifold. Suppose A is a G_2-instanton on E. What does it tell us about E? The situation we have in mind is that of bundles over Kähler manifolds, where existence of Hermitian–Yang–Mills connection $\iff$ the bundle is stable due to Donaldson–Uhlenbeck–Yau theorem or the Kobayashi–Hitchin correspondence. So the questions are 1) Is there a notion of stability of bundles E over a G_2-manifold? 2) Is there a Donaldson–Uhlenbeck–Yau/Kobayashi–Hitchin correspondence type theorem?

    The loop space of a G_2 manifold is a Calabi–Yau manifold. Can we use this information and point of view to get a notion of stability and answer above questions?

    Is there a Geometric Invariant theory, moment map and/or symplectic reduction picture associated with the not-yet-defined notion of stability?

12. **(Spiro Karigiannis)** In the Kähler case, $\Omega^2 = \Omega^{2,0} \oplus \Omega^{0,2} \oplus C^\infty\omega \oplus \Omega_0^{1,1}$ and a connection A is Hermitian–Yang–Mills $\iff$ $F_A \in \Omega_0^{1,1}$. Now consider a gerbe over (M^7, φ) and let A be a connection on the gerbe. Then F_A is a 3-form on M. Is it true that F_A is "Hermitian–Yang–Mills" $\iff$ $F_A \in \Omega^3_{27}$?

13. **(Gavin Ball)** Consider the standard G_2-structure φ on $\mathbb{R}^7$ and let S be the set of degenerate 3-forms on $\mathbb{R}^7$. S is singular. What can we say about $\mathrm{dist}(\varphi, S)$, the distance between φ and S? We know that $\mathrm{dist}(\varphi, S) \leq 1$ but can it be smaller?

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Chapter 17

Random Growth Models and KPZ Universality (23w5075)

May 28 - June 2, 2023

Organizer(s): Ivan Corwin, Jessica Lin, Jeremy Quastel, Firas Rassoul-Agha, Benedek Valkó

Overview of the Field

Irregular growth is a ubiquitous phenomenon in nature, from the growth of tumors, crystals, and bacterial colonies to the propagation of forest fires and the spread of water through a porous medium. Mathematical models of random growth have been a driving force in probability theory over the last sixty years and a rich source of important ideas [2].

The analysis of random growth models began in the early 1960s with the introduction of the *Eden model* by Eden [42] and *first-passage percolation* (FPP) by Hammersley and Welsh [51]. About two decades later, in 1979, early forms of a directed variant of FPP, *directed last-passage percolation* (LPP), appeared in a paper by Muth [66] in connection with *series of queues in tandem*. Soon after, Rost [71] introduced a random growth model, now known as the *corner growth model* (CGM), in connection with the *totally asymmetric simple exclusion process* (TASEP), a model of interacting particles first introduced by Spitzer [80]. In 1985 Huse and Henley [56] introduced the *directed polymer with bulk disorder* to model the domain wall in the *ferromagnetic Ising model* with random impurities.

In 1985 Kardar, Parisi, and Zhang [61] introduced the *KPZ equation*, an ill-defined stochastic PDE, as a proposed limit for the evolution of fluctuations of growing interfaces. Using non-rigorous renormalization group arguments, they predicted that scaling exponents of 1/3 and 2/3 should describe the fluctuations and correlations for a large class of systems, including the *asymmetric simple exclusion process* (ASEP) and the directed polymer model. These exponents were also predicted by van Beijeren, Kutner and Spohn in [86].

The next decade brought numerous mathematically rigorous results in the study of scaling limits of interacting particle systems [63, 70, 68, 74, 62]. The CGM arose naturally from LPP in queueing theory in the work of Szczotka and Kelly [81] and Glynn and Whitt [47]. Seppäläinen [74] connected the CGM and LPP to Hamilton-Jacobi equations and Hopf-Lax-Oleinik semigroups and Bertini and Giacomin [15] gave a rigorous derivation of the KPZ equation as the scaling limit of the *weakly asymmetric simple exclusion process*.

Also in the 1990s, Newman and coauthors [67, 64, 54, 55] pioneered the study of the geometric structure of the semi-infinite paths (called geodesics) that optimize the Hopf-Lax-Oleinik variational formula, in the context of

FPP. In particular, borrowing ideas from classical metric geometry, [67] introduced the tool of *Busemann functions* into the field. Later, Newman's program was implemented in LPP models [27, 28, 29, 44, 87] and certain stochastic Hamilton-Jacobi equations [7, 4, 5].

In 1999 the breakthrough of Baik, Deift, and Johansson [3] showed that the fluctuations of the Poissonian LPP have the same limit as the fluctuations of the largest eigenvalue of the *Gaussian unitary ensemble* derived by Tracy and Widom in [82]. This result was extended to the *exactly solvable* versions of the CGM by Johansson in [60]. The CGM and the related LPP and TASEP models were thus marked as members of the KPZ universality class.

These striking results caused a flurry of activity surrounding random growth models and new analytical tools from representation theory and combinatorics were developed. The subject of *integrable probability* was born. See the reviews [21, 23].

Over the last decade, ASEP, directed polymers, and the KPZ equation itself were shown to also belong to the KPZ universality class, along with many more newly discovered random growth models. See [12, 84, 1, 20, 72, 18, 17, 40, 16, 33, 19] (using integrable probability), [75, 10, 30, 31] (via coupling methods), and [50] (using Hairer's regularity structures [49]). The subject started moving at a high pace and the intensity of breakthroughs increased in other directions as well. We give some highlights of these recent developments in the next section.

Recap of objectives

The study of random growth models connects to a large number of areas in probability theory such as integrable probability, homogenization, percolation, disordered systems, interacting particle systems, random matrices, SPDEs, random polymer measures, random dynamical systems, and random walk in random environment.

The last two decades have seen rapid advances in all of these directions, with a significant acceleration in progress in several of these subfields recently, including solutions of several long-open problems. This is an exciting time for the subject, with new possibilities in extending universality, new geometric approaches, and more. **The main objective** of this workshop was to bring together a number of top experts on these various subfields to disseminate these recent developments and exchange ideas that will fertilize the ground for yet another leap forward. We give below a partial list of key topics. We also used this opportunity to celebrate the work of Timo Seppäläinen in the field.

Busemann functions

Following Newman's introduction of Busemann functions [67] and the subsequent seminal work of Hoffman [53], these objects became a principal tool for studying semi-infinite geodesics and generating stationary distributions for growth models. The existence of these objects came originally as a consequence of geodesics coalescence and relied on strong hypotheses on the limit shape. More recently, generalized Busemann functions were constructed without assumptions on the limit shape [34, 35, 46, 45, 58]. One consequence of this construction is a connection between analytic properties of the Busemann functions and regularity properties of the limiting shape function. This provides an avenue for tackling the long-standing open questions of differentiability and strict concavity of the limiting shape.

The existence of Busemann functions in the case of the KPZ equation itself is also an open problem that now seems accessible using the recent techniques mentioned above. Busemann functions for the *KPZ fixed point* (a central object in the KPZ universality class, constructed recently by [65]) may also be accessible by working with the LPP-like description, made possible by the recent construction of the *directed landscape* [36, 37].

Integrable probability

The new field of *integrable probability* has been extremely effective in proving scaling limits for various random growth models and interacting systems. The basic idea is that for certain models due to some hidden algebraic or combinatorial structures one can compute the expected values of certain observables (often in the form of complex contour integrals) [16, 83, 1, 23, 21, 24]. By accessing a sufficient number of these expected observables this information can be used to derive and identify the scaling limit. This method has been successfully applied to a number of classical models and led to the discovery of new families of models in the KPZ universality class. It has also led to new results about the limiting objects; see the recent [65] on the KPZ fixed point and [36] on the directed landscape. This is a dynamically evolving field with a lot of activity and many more directions to explore.

Coupling techniques

The early use of coupling techniques in the study of random growth models was to prove hydrodynamic limits and large deviation results [71, 70, 43, 73]. The more recent uses of these techniques relied on various generalizations of Burke's property (a result from queuing theory) by finding generalized versions of systems of queues in certain random models. Although these techniques so far could not produce fluctuation limit theorems, they have successfully identified and bounded scaling exponents [26, 9, 11, 75, 76]. In the recent [8, 25] coupling techniques were used to prove the non-existence of bi-infinite geodesics in the exponential LPP and of bi-infinite Gibbs measures in the log-gamma polymer model.

Geometry of random geodesics

The last couple of years brought a deeper understanding of the geometry of random geodesics in random growth models. [13] and [8] proved (with two different approaches) that bi-infinite geodesics do not exist in the exponential LPP. Moreover, [59] gave the first complete description of the structure of semi-infinite geodesics in the LPP model with exponential weights. The analogous questions for models with general weight distributions are very much open, but the techniques developed for the recent results could provide a possible route to attack these problems.

Random matrices

Random matrix theory is closely connected to the study of random growth models. These connections can be observed both at the level of finite models and the limiting objects, and can be used to study these objects using the tools of random matrices. A successful example is the introduction and study of the *Airy line ensemble* using *Dyson's Brownian motion* [32], which was used in the recent work [36] in the study of the scaling limit of the Brownian LPP. [85] studied large deviation properties of the KPZ equation by connecting it to the edge point process limit of the Gaussian unitary ensembles via a distributional identity of [22]. The main tool is a description of the edge process as the spectrum of a random differential operator given in [69].

Random dynamical systems

The connection with the Hamilton-Jacobi equation allows one to view random growth models as random dynamical systems. This approach was initiated by Sinai [77], followed by the various extensions [48, 39, 79, 41, 52, 57]. These results are in direct relation with the more recent ones [5, 4, 7, 58, 34, 35, 27, 28] on existence of Busemann functions and the geometric properties of semi-infinite geodesics, mentioned above.

It is believed [6] that the statistics and structure of the shocks formed by the solutions of the associated stochastic Burgers equation (studied in [78, 41, 14, 4, 59]) are closely related to the KPZ universality phenomenon. This is yet another widely open direction in the field. A promising starting point is offered by the following idea: The shocks in the KPZ equation should be in duality with the geodesics coming from the aforementioned, recently constructed, directed landscape [36, 37, 38].

Presentation Highlights

The workshop centered around 29 30-minute talks that covered all the different approaches that have been successful in the subject. The broad spectrum of topics occurred organically; participants were invited to voluntarily give a talk on a topic of their choice, and we naturally found cohesion within the scientific program. Here we provide an overview of each, arranged loosely according to topic.

Interacting particle systems

Márton Balázs explained how the two-parameter stationary last passage percolation picture and some planarity tricks can be used to establish stabilisation of the point-to-point geodesic tree to the semi-infinite one.

Eric Cator introduced a periodic version of the polynuclear growth model (PNG) and showed that it is a solvable model. He described the stationary measures at a fixed time and the distribution of the space-time paths.

Ivan Corwin introduced a new method to construct the stationary measure for open boundary KPZ models, focusing on geometric LPP, the log-gamma polymer, and the KPZ equation. These stationary measures are realized as marginals of two-layer Gibbs measures by constructing local Markov dynamics that preserve this class of measures and project on the top layer to the KPZ models.

Pablo Ferrari spoke about the KMP model where each vertex of a finite graph is assigned a random nonnegative energy value and then when a Poisson clock rings at an edge, the energies of the two vertices are randomly redistributed. The main result described the invariant measure of the process. Applications included hydrodynamic limits.

In his talk, Nicos Georgiou introduced a new totally asymmetric exclusion process on a rooted Galton-Watson tree without leaves and then focus on the aggregated current of particles across generations. He showed how one can use a coupling with last passage percolation to obtain upper and lower bounds for the aggregated current of particles across generations, depending on the tree structure and jump rates on each node. As a consequence, one can construct time intervals where the current across a fixed generation jumps from 0 to linear order.

Vadim Gorin described his work on the six-vertex model in a large square with Domain Wall Boundary Conditions where he found that the local description of the model near a straight segment of the boundary depends on the value of a single parameter Δ, with a single limiting object for all and a richer class of limits at $\Delta > 1$.

Sunder Sethuraman considered the one dimensional asymmetric simple exclusion process (ASEP), starting from a stationary state, and studied the typical behavior of a tagged particle, conditioned to deviate to an atypical position at a fixed time. Among their results is an upper tail large deviations bound for the position of the tagged particle.

Random geometry

Duncan Dauvergne talked about the directed landscape model which is the scaling limit for models in the KPZ universality class. He gave a full description of the exceptional pairs of points on the plane between which there are multiple paths that maximize the model's action functional.

Bálint Virág talked about several new results for models in the KPZ universality class. He explained how the Wick-ordered stochastic heat equation with planar white noise can be defined as the free energy of a certain undirected polymer in a random environment, and how one can recover one-dimensional KPZ fluctuations from this model in an appropriate limit. He also presented results about the Brownian web distance, an integer-valued scale invariant random metric with scaling exponents 0:1:2 – how this metric can be obtained from coalescing simple random walks on the plane and how in the shear limit one can recover the Airy process.

Joseph Yukich showed that the rescaled maximal radial and longitudinal fluctuations of the boundary of the convex hull of n i.i.d. random points uniformly distributed on a smooth convex set asymptotically follow the Gumbel law as n tends to infinity and, in $d = 2$, they asymptotically exhibit $(1/3, 2/3)$ scaling, with precise logarithmic corrections. When the convex set is the unit disc, the radial fluctuations satisfy process level convergence. He also introduced a dual space-time two parameter growth process, which for $t = 1$ coincides with the support function of the convex hull, displays 1:2:3 scaling, and converges to a two-parameter limit process given by the Hopf-Lax formula.

Limit shapes, Busemann functions, and directed polymers

Yuri Bakhtin considered a class of continuous-space polymer models in positive and zero temperature for which he showed that the shape function is differentiable.

Erik Bates delivered a presentation on the Busemann process for directed polymers. He revealed certain parallels with the zero-temperature last-passage percolation model and, at the same time, intriguing distinctions. For instance, within the solvable realm of the inverse-gamma model, the Busemann process's countably dense set of discontinuities remains the same across all edges of the square lattice. This stands in contrast to the exponential last-passage percolation model, where an edge's set of discontinuities is countable but nowhere dense, while the union of all discontinuities across edges is dense.

Ofer Busani presented recent progress in the understanding of the multi-type stationary measures of the KPZ fixed-point as well as the scaling limit of multi-type stationary measures of two families of models in the KPZ class: metric-like models (e.g. last passage percolation) and particle systems (e.g. exclusion process).

Ofer Zeitouni described the evolution of high moments of the partition function of two-dimensional directed polymers in the weak disorder regime.

Fluctuation bounds and scaling limits

Elnur Emrah talked about the limiting boundary fluctuations of the eigenvalue minor process of a unitarily invariant random Hermitian matrix with eigenvalue sequence a_n. He presented a classification theorem that identifies five fluctuation regimes in terms of the parameters a_n and describes the corresponding limit processes.

Milind Hegde gave a proof of a recent conjecture stating that the geodesic in the directed landscape between $(0, 0)$ and $(0, 1)$, conditioned on the event that its weight is at least some large value L, and suitably rescaled, converges as $L \to \infty$ to a Brownian bridge.

Xiao Shen gave upper and lower bounds for the correlation between the free energies of two inverse-gamma polymers with endpoints in close proximity or far apart.

Philippe Sosoe presented a general methodology, based on a formula of Rains and Emrah-Janjigian-Seppalainen, to obtain scaling and tail bounds in several KPZ models, including some that are not known or expected to be integrable.

Xuan Wu proved the convergence of the KPZ equation to the directed landscape.

Algebraic methods and integrable probability

In the heart of the construction of the KPZ fixed point (by Matetski-Quastel-Remenik) lies the solution of TASEP with general initial conditions in terms of a random walk hitting representation. Nikos Zygouras presented a step-by-step derivation of this via principles of the RSK correspondence.

Homogenization and Hamilton-Jacobi equations

Jessica Lin presented quantitative estimates on the the parabolic Green function and the stationary invariant measure in the context of stochastic homogenization of elliptic equations in nondivergence form. Then, she discussed implications of these homogenization results, such as a quenched, local CLT for the corresponding diffusion process and a quantitative ergodicity estimate for the environmental process.

Hamilton-Jacobi PDEs with random stationary Hamiltonian functions are popular toy models for studying the dynamics of interfaces in various phenomena in physics and biology. There is a one-to-one correspondence between piecewise smooth solutions and marked tessellations. Fraydoun Rezakhanlou explained how a kinetic theory can be developed to describe the dynamics of such tessellations. As an application, he derived kinetically describable Gibbsian solutions for such PDEs.

Atilla Yilmaz considered viscous Hamilton-Jacobi equations with a general (nonconvex) continuous superlinear momentum and a bounded Lipschitz continuous time-independent potential. He showed that under a certain "hill and valley" condition on the diffusivity-potential pair, the equation homogenizes and there are sublinear correctors outside of the linear segments of the effective Hamiltonian.

Random matrices

The complex eigenvalues of non-Hermitian random matrices have attracted much research interest due to their relevance to several branches of theoretical physics, and in particular to the study of scattering chaotic systems. In her talk, Tatyana Shcherbina discussed a certain form of universality appearing in the distribution of the "resonance widths" in the case of a few classical ensembles of random matrices.

SPDE, KPZ, SHE

Chirs Janjigian discussed some recent results on the ergodicity properties of the Brownian motion stationary distributions of the KPZ equation through the lens of the synchronization by noise phenomenon. He showed that the solution to the equation started in the distant past from an initial condition with a given slope will converge almost surely to a Brownian motion with that same drift for all but a random countable set of slopes.

Hao Shen consider the Langevin dynamics of a large class of lattice gauge theories on the 2D torus and proved that these discrete dynamics all converge to the Markov process induced by the local solution to the stochastic Yang-Mills equation on the torus. Using this universality result for the dynamics, he showed that the Yang-Mills measure on the 2D torus is the universal limit for these lattice gauge theories. He also proved that the Yang-Mills measure is invariant under the dynamics. thod for invariant measures.

Evan Sorensen showed the existence of a random, countably infinite dense set of directions for which the Busemann process for the KPZ equation is discontinuous. This demonstrated the failure of the One Force–One Solution principle in those exceptional directions and resolved a recent conjecture of Janjigian, Rassoul-Agha, and Seppäläinen.

Li-Cheng Tsai considered the n-point, fixed-time large deviations of the KPZ equation with the narrow wedge initial condition. He explained how to analyze the multipoint moments of the Stochastic Heat Equation via a system of attractive Brownian particles and how to use the moments to obtain the n-point Large Deviation Principle and spacetime limit shape.

Random turn games

Many combinatorial games, such as chess, Go and Hex, are zero-sum games in which two players alternate in making moves. In a random turn variant, each player wins the right to move at any given turn according to the flip of a fair coin. In 2007, Peres, Schramm, Sheffield and Wilson [PSSW] found explicit optimal strategies for a broad class of random-turn games, including Hex. The gameplay in random turn Hex is a novel random growth process related to planar critical percolation; other random turn games offer growth processes related to other universality classes.

In his talk, Alan Hammond discussed stake-governed random-turn games and showed how to evaluate the relative strategic importance of intermediate positions in multi-turn games.

Open problems session

Yuri Bakhtin posed the problem of pushing the work he presented in his talk to try and get strict convexity of the limiting free energy.

Ivan Corwin asked about stationary distributions for the KPZ equation on $[0, \infty)$ with initial condition $h(0, x) = vx$ and Neumann boundary condition $\partial_x h(t, 0) = u$ (for given parameters $u, v \in \mathbb{R}$).

Alan Hammond considered the planar stochastic process that is rotating at angular speed n and whose radius is a standard Brownian motion, conditioned on staying positive. He asked about the scaling limit (as $n \to \infty$) of this process, when conditioned on not intersecting itself.

In the standard first-passage percolation model, Chris Hoffman asked about the shape of the boundary of the (random) ball of radius t near its right-most point.

Xuan Wu noted that in the exponential last passage percolation model, the eigenvalues of the Laguerre unitary ensemble (LUE) and the last passage times are equal in distribution. On the LUE side, one may introduce its dynamical version by replacing the Gaussian entries with Brownian motions, resulting in an evolving system with rich integrable structures. The open problem is then to figure out how to introduce a dynamical last passage percolation model to lift the identification to a dynamical version.

Mentoring Program

The conference had a mentoring program associated to it. All participants were invited to participate as both mentors and mentees within the program. Small mentor groups of 4 members (2 mentors and 2 mentees) were formed. The matchings were made based on (i) requests of the participants (ii) pairing disjoint groups of people together (i.e. no mentor/mentees were supervisors/trainees, nor collaborators). In total, 4 groups of 4 were formed (constituting 16 participants total). Participants were encouraged to sit together for at least one meal and exchange contact info in order to find out about social activities. Moreover, mentees were encouraged to solicit feedback on their talks from mentors; this involved some preparatory feedback on slides and feedback after the talk took place.

Demographics

21% of the invitees self-identified as being from an underrepresented group. 16% of the invitees were females. 16% of the invitees were postdoctoral fellows and 14% were tenure-track assistant professors.

18% of the participants self-identified as being from an underrepresented group. 13% of the participants were females. 18% of the participants were postdoctoral fellows and 16% were tenure-track assistant professors.

Conclusion

The workshop was a resounding success in many ways. The research presentations shared cutting-edge developments in the study of random growth models and adjacent, related topics. The infrastructure provided by BIRS created a welcoming environment to promote interaction and discussions. The mentoring program encouraged engagement between participants of all levels, and it was greatly appreciated by junior participants who are at early stages in their careers.

Participants

Alberts, Thomas (University of Utah)
Assiotis, Theo (University of Edinburgh)
Bakhtin, Yuri (New York University)
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Bates, Erik (University of Wisconsin - Madison)
Busani, Ofer (Universität Bonn)
Cator, Eric (Radboud University, Nijmegen)
Corwin, Ivan (Columbia University)
Dauvergne, Duncan (University of Toronto)
Dow, Douglas (New York University)
Emrah, Elnur (University of Bristol)
Ferrari, Pablo (Universidad de Buenos Aires)
Georgiou, Nicos (University of Sussex)
Gorin, Vadim (UC Berkeley)
Groathouse, Sean (University of Utah)
Hammond, Alan (University of California - Berkeley)
Hegde, Milind (Columbia University)
Hoffman, Christopher (University of Washington)
Janjigian, Christopher (Purdue University)
Khoshnevisan, Davar (The University of Utah)
Kosygina, Elena (Baruch College and the CUNY Graduate Center)

Krishnan, Arjun (University of Rochester)
Kumar, Rohini (Wayne State University)
Lin, Jessica (McGill University)
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Quastel, Jeremy (University of Toronto)
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Rezakhanlou, Fraydoun (University of California - Berkeley)
Rider, Brian (Temple University)
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Shcherbina, Tatyana (University of Wisconsin - Madison)
Shen, Hao (University of Wisconsin - Madison)
Shen, Xiao (University of Utah)
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Sosoe, Philippe (Cornell University)
Sweeney, Mikhail (University of Utah)
Toth, Balint (University of Bristol & Renyi Institute of Mathematics Budapest)
Tsai, Li-Cheng (University of Utah)
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Virag, Balint (University of Toronto)
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Chapter 18

Quantum Information Theory in Quantum Field Theory and Cosmology (23w5092)

June 4 - 9 2023

Organizer(s): Arpan Bhattacharyya (Indian Institute of Technology Gandhinagar), Saurya Das (University of Lethbridge), Shajid Haque (University of Cape Town)

Overview of the Field

*I*n recent years, quantum information theory (QIT) has become a melting pot between disparate branches of physics. Within the last decade, tools and techniques from QIT are bringing new perspectives into fields such as quantum field theory (QFT) and cosmology, and are inspiring new questions and research directions.

Quantum many-body systems and quantum field theories, often found in condensed matter and particle physics, are essential for our understanding of modern physical phenomena, but can be challenging to quantitatively study and simulate, both from a theoretical and experimental point of view. Because the size of the Hilbert space grows exponentially with the number of degrees of freedom, brute force techniques for studying these systems often fail. In this context, recent progress in QIT, such as quantum computation and computational complexity, can provide important insights.

One area that has seen significant recent progress is the field of *quantum complexity*, an extension of the concept of computational complexity from classical computer science to quantum mechanics and quantum computing. Interestingly, quantum complexity can provide important insights into the simulation of a class of quantum-many body systems [1]. Since John Preskill et al. have demonstrated that quantum computers can in fact simulate certain aspects of quantum field theory exponentially faster than the classical non-perturbative algorithms [2], this naturally raises several questions: Do insights acquired from quantum complexity help simulate QFT more efficiently? What novel features of QFT can we learn from complexity? With the possibility of a working, large qubit quantum computer in the near future, techniques in QIT that may be useful in a simulation of a QFT on a quantum computer take on even greater practical importance.

Simultaneously with these recent advances in QIT, there has been considerable theoretical progress in the field of holographic duality, also known as the AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence. AdS/CFT, as a duality between a strongly coupled quantum field theory and a weakly coupled gravity theory, provides an alternative tool for studying the behavior of QFTs in regimes where the usual perturbative tools do not apply. Recent proposals in holography have synergistically combined with advances in QIT, linking apparently disparate systems such as black holes and conformal field theories.

In one such example, geometric quantities that probe the physics beyond the horizon of a black hole in AdS spacetime are identified with the *quantum circuit complexity* of a holographically dual conformal field theory. More generally, studies of entanglement entropy, scrambling, quantum chaos, and other information-theoretic tools and their holographic duals have suggested possible limits on the dynamics of interacting systems, and have initiated several interesting new lines of research [3].

Cosmology is another arena where QIT has recently provided new insights and connections. Cosmic inflation, in which quantum vacuum fluctuations are stretched to cosmological scales by a period of rapid near-de Sitter (dS) expansion, is a leading paradigm for the dynamics of the early Universe. As with its AdS counterpart, the study of dS space with information-theoretic tools can potentially uncover new phenomena and lead to a better theoretical understanding of this important cosmological background from the perspective of quantum gravity. Recent progress, including the application of QIT concepts such as complexity, entanglement entropy, and scrambling, among others, to cosmological backgrounds, hint at potential insights for holographic descriptions of dS as well as potential fundamental limits on the duration of cosmological inflation.

Recent Developments and Open Problems

While progress in the different fields of QIT, QFT, AdS/CFT and cosmology has been ongoing for many years, it is only in the last few years that there has been a sustained interest in the connections between these fields. Many of these ideas have the potential to fundamentally shift our thinking about complex quantum systems to a perspective in which information and information-theoretic concepts play a primary and essential role. Below, we will highlight some recent progress made by several leading researchers working in this field. Note that it is not exhaustive and we apologise for any inadvertent omissions. We will highlight some of the works which have played an important role in the context of this workshop.

Complexity plays an important role in the link between QIT, QFT, and their applications. The properties of complexity in QFTs have been studied by several groups, e.g., Prof. Robert Myers and his group at the Perimeter Institute of Theoretical Physics in Waterloo, Canada, [4] in the context of holography, and the groups of Prof. Johanna Erdmenger of Würzburg University, and Prof. Michal Heller of the Max Planck Institute of Gravitational Physics, who have made significant complementary progress in understanding complexity in the context of Conformal Field Theory [5].

Besides complexity, various other QIT measures, such as entanglement entropy, have provided key insights for some of the longstanding problems in theoretical physics. One example is the semi-classical computation of the Page curve, which plays an important role in the context of black hole information paradox [6]. Significant contribution in this field came from the groups of e.g., Prof. Juan Maldacena, and Prof. Edward Witten of IAS Princeton, Prof. Raphael Bousso of U.C Berkeley, Prof. Aron Wall of Cambridge University, Prof. Geoff Pennington of UC-Berkeley, California, Prof. Thomas Hartman of Cornell University, and Prof. Vijay Balasubramanian of the University of Pennsylvania, among others.

Along with the high energy physics community, the condensed matter physics community has also been using QIT techniques for various interesting physical systems. Prof. Shinsei Ryu of University of Chicago, and Prof. Koji Hashimoto of Osaka University, Japan have recently made important contributions in understanding chaotic systems [7]. Note that there are a plethora of groups working in this direction and made several important contributions. Two of the organizers (Arpan Bhattacharyya and S. Shajidul Haque), along with their collaborators, have also proposed complexity as a new diagnostic for quantum chaotic systems [8].

In the past few years, QIT concepts and tools and are also enriching cosmology. Recent works by Prof. Gary Shiu of University of Wisconsin-Madison, Prof. Robert Brandenberger of McGill University, Canada, and the organizers, have described perturbations in cosmological spacetimes in the language of entanglement entropy, scrambling, complexity, and chaos. The techniques of QIT have uncovered bounds on the growth of chaos and potential limits on the amount of expansion of the Universe, suggesting that QIT has new things to teach us about the structure of our Universe on the largest scales.

Some of the significant open problems that one would like to answer in this context are the following: (1) To what extent is the effective field theory description of cosmology valid? (2) Investigate Krylov complexity for interacting quantum fields and what it teaches us about holography. (3) Do tools from holography provide any insight into the Hamiltonian simulation problem, which is important for the simulation of quantum many-body physics via state-of-the-art quantum computers? (3) Does the island proposal resolve the black hole information paradox?

Presentation Highlights

The 4.5 day workshop was well-attended and most participants gave talks. The only participants who did not make oral presentations were some graduate students, many of whom made poster presentations. In addition to a large number of excellent talks (6-8 per day), there were two discussion sessions spread out throughout the conference, which also saw strong participation and lively discussions. We also organized two EDI sessions, which were well-attended and led to excellent discussions. Although most of the talks were in-person, due to visa and travel issues, a few speakers (and participants) were compelled to participate online. However, due to the state-of-the-art facilities at BIRS, the online talks (total 11 in number, including the 2 EDI talks) felt no different from the ones in-person and the switch from online to hybrid and vice-versa was seamless. Presentation highlights include the following (speaker and talk):

- Cliff Burgess: *Primordial Decoherence & Reliable Late-time Predictions*
- Anatoly Dymarsky: *Chaos and complexity through the lens of dynamics in Krylov space*
- Janet Hung: *Constructing lattice integrable models from topological theories in one higher dimensions and their applications*
- Bartek Czech: *Everything Everywhere All at Once: Holographic Entropy Cone, Entanglement Wedge Nesting, Differential Entropy, Black Holes, Modular Chern Numbers, Toric Code, Cubohemioctahedron…*
- *Javier Magan: Long times, chaos, and spread complexity*
- Claire Zukowski: *State-Changing Modular Berry Phases*
- **EDI talk1:** Martha Mathurin-Moe, Vice-Provost, EDI, University of Lethbridge.
- **EDI talk2:** Glenda Bonifacio, Professor of Women and Gender Studies and EDI Scholar, University of Lethbridge. In the EDI talks, the EDI experts (Mathurin-Moe and Bonifacio) touched upon various aspects of EDI in the academia, with data, information, examples and an interactive session. The talks were attended by most of the conference participants and participation was spontaneous.

Scientific Progress Made

- We had arranged the daily talks based on a common theme. Each day, we started with a review talk, followed by more specialized talks. We had ample amount of time for discussion during and after the talks. We also had dedicated discussion sessions. These discussions generated many new ideas, especially in the areas of application of QIT in cosmology, effective field theory and topological theory of gravity.
- Motivated by the talk presented by Prof. Cliff Burgess on the models of the cosmological effective field theory, it was felt that it will be interesting to study them using QIT. This will help one to bring new perspective in the field of decoherence for cosmological perturbation.
- Motivated by several talks on Krylov complexity, we learned that behaviour of the Krylov complexity at late time is dictated by the density of states of the systems. Hence it will be interesting to understand it for various other cases e.g., in presence of dissipation to make this in more concrete footing.

- Prof. Hung discussed connection between tensor network and topological field theory. In light of these progress, it is reasonable to compute some QIT, e.g., Spectral Form Factor for topological gravity theory and see what happens. Do we get the same dip-ramp-plateau structure? What does it teach us about quantum gravity? One interesting thing that emerges is that, we can do these computation for 3D gravity concretely and match with tensor network predictions. This we hope to accomplish in near future.
- Lastly we clarified the importance of von-Neumann algebra in the context of cosmological spacetime and inperplay between symmetry, charged moments and entanglement for quantum many-body systems.

Outcome of the Meeting

- Due to the space left for for ample discussions in the workshop schedule, junior students/postdocs had the opportunity to interact with experts. We believe this benefits their future job application. In fact, students from cosmology background got interested in quantum information and vice versa.
- We list some publications as well as pre-prints which resulted form the discussion that took place at this workshop. This workshop has been acknowledged in each of these papers (names of the participants are highlighted):

    1. S. Nandy, B. Mukherjee, **Arpan Bhattacharyya** and A.Banerjee,“Quantum state complexity meets many-body scars,” J. Phys. Condens. Matter **36** (2024) no.15, 155601, doi:10.1088/1361-648X/ad1a7b
    2. **Arpan Bhattacharyya**, D.Ghosh and **Poulami Nandi**, “Operator growth and Krylov complexity in Bose-Hubbard model,”JHEP **12** (2023), 112, doi:10.1007/JHEP12(2023)112
    3. **Arpan Bhattacharyya**, **S.S.Haque**, G.Jafari, J.Murugan and D.Rapotu, “Krylov complexity and spectral form factor for noisy random matrix models,” JHEP **10** (2023), 157, doi:10.1007/JHEP10(2023)157.
    4. **Arpan Bhattacharyya**, M. Dogra and **Shubho R.Roy**, “CFT reconstruction of local bulk operators in half-Minkowski space”, [arXiv:2308.08547 [hep-th]].
    5. **A. Bhattacharyya,** S.Ghosh and S.Pal,“ deformed 2D topological gravity : from partition function to late-time SFF”, [arXiv:2309.16658 [hep-th]].

- Motivated by the talks on application of effective field theory in cosmology that were held at the workshop, we have started new projects with Dr. Suddhasattwa Brahma, University of Edinburgh, a former postdoc of Robert Brandenberger, in the area of deoherence of cosmological perturbation and complexity. We are in the final stage of the project and preparing the manuscript. We hope to post in on arXiv soon.
- To follow up on the discussion on Krylov complexity, **S.S.Haque** and **Arpan Bhattacharyya** gave talks on workshop (lonk of which is given here: `https://sites.google.com/view/holography2023/talks?authuser=0`) organized after our workshop by Prof. Keun-Young Kim from Gwangju Institute of Science and Technology at APCTP, South Korea. Prof. Keun-Young Kim was an invited speaker for the BIRS workshop.
- Furthermore, the success of this workshop and the positive response that we got, combined with the fact that this is a rapidly growing field, motivates us to propose another workshop with a bigger goal of keeping track of recent developments. Two of the organizers, Arpan Bhattacharyya and Saurya Das, along with Rusa Mandal, have proposed a sequel to this workshop in 2024 at the BIRS station located at CMI, Chennai, India, with another workshop titled *Quantum Gravity and Information Theory: Modern Developments*, no. 25w5386. We hope that our proposal will be accepted.

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Chapter 19

Scientific Machine Learning (23w5129)

June 18 - 23, 2023

Organizer(s): Brendan Keith (Brown University), Tom O'Leary-Roseberry (University of Texas at Austin), Lu Lu (University of Pennsylvania), Siddhartha Mishra (ETH Zürich), Zhiping Mao (Xiamen University)

This report documents a 2023 Banff International Research Station for Mathematical Innovation and Discovery (BIRS) workshop that united international experts from academia, industry, and national laboratories in the field of Scientific Machine Learning (SciML). The workshop successfully fostered collaboration among researchers from diverse backgrounds, including applied mathematics, myriad engineering disciplines, scientific computing, optimization, and machine learning. The event endeavored to stimulate innovation in the field through scientific talks and open interactions aimed at various scientific and engineering applications.

Introduction

Scientific Machine Learning (SciML) is an evolving discipline that melds machine learning (ML) principles with scientific computing, paving the way for groundbreaking innovations in science and engineering. Recognizing the immense potential and rapid developments in this field, the 2023 BIRS Workshop on Scientific Machine Learning aimed to provide a fertile ground for collaboration and knowledge exchange.

Overview of the Field

SciML is a burgeoning interdisciplinary field that marries traditional scientific computing with advanced ML techniques. At its core, SciML seeks to harness the power of ML models, such as deep neural networks, to accelerate and enhance the simulation, modeling, and analysis processes inherent in scientific disciplines. This fusion allows for more sophisticated handling of complex systems and phenomena, from predicting turbulent flows in fluid dynamics to optimizing molecular configurations in material science. Through leveraging data-driven ML methodologies alongside traditional physics-based models, SciML is redefining the boundaries of computational capabilities and the accuracy of predictions, making it possible to tackle previously intractable scientific problems.

However, the union of scientific computing and machine learning is not without its challenges. It demands a robust understanding of both domains, ensuring that the integrity of scientific principles is maintained while employing ML techniques. To this end, researchers in SciML are continuously working on innovative algorithms and frameworks that can seamlessly integrate the strengths of both worlds. The goal is to create models that are not only highly accurate but also interpretable, trustworthy, and aligned with the underlying scientific principles. As SciML continues to evolve, its applications are anticipated to proliferate across various scientific sectors, heralding a new era of research and discovery.

Recent Developments

SciML has witnessed many innovations in recent years, dramatically reshaping the landscape of scientific research. One of the most notable advancements is the integration of neural ordinary differential equations (neural ODEs). This approach allows researchers to leverage neural networks within ordinary differential equations, enabling the simultaneous learning of dynamic system behavior from data while maintaining a structured model format. This blend aids in improving computational efficiency, reducing the amount of data needed for accurate predictions, and providing a more interpretable model structure that aligns with scientific principles. Neural ODEs have found applications in various domains, including fluid dynamics, pharmacokinetics, and even in some areas of finance.

Another example of SciML are physics-informed neural networks (PINNs) [23]. These networks incorporate known physical laws (like conservation laws) through a loss function, effectively bridging the gap between data-driven approaches and theoretical knowledge. By constraining neural network training with these laws, PINNs ensure that the resultant models are not just data-adaptive but also physically consistent. This integration has proven particularly valuable in scenarios with sparse or noisy data, where traditional machine learning models might overfit or produce non-physical results.

A groundbreaking current research direction in SciML is *operator learning*. Operator learning broadly refers to various machine learning strategies for learning maps between (formally infinite-dimensional) function spaces, such as those containing functions that parametrize PDEs, as well as the functions who solve them. Accurate operator approximations can be substituted for expensive physics simulations in "many-query" settings which required repeated PDE solutions for differing parameters. Examples of many query problems include inverse problems (both deterministic and Bayesian), optimization problems (e.g., design and control) among other tasks. A major recent breakthrough in operator learning are the so-called "neural operators", which use neural network approximations married with classical mathematical tools to exploit known structure of problems, while leveraging ML technologies to learn complex nonlinear representations from data and physics. For instance, Fourier Neural Operators [14] leverage the Fourier transform's efficiency to predict complex physical behaviors in various spatial dimensions. Another example is the DeepONet [15], which learns operators by treating them as mappings from functions to functions, utilizing two interacting neural networks — a branch network and a trunk network. The potential to represent and predict scientific phenomena with unprecedented accuracy is immensely expanded through these advanced neural architectures.

The 2023 BIRS Workshop on Scientific Machine Learning featured numerous talks on neural ODEs, PINNs, and neural operators, as well as numerous other emerging methodologies such as statistical inference methods and high performance computing considerations. The talks spanned algorithmic innovation, approximation theory and applications to complex physical systems.

Presentations

The following is a list of the presentations given at the workshop:

Monday, June 19, 2023

1. Panos Stinis, Pacific Northwest National Laboratory, Computational Mathematics Group: "Mutlifidelity Scientific Machine Learning". [10]

2. Romit Maulik, The Pennsylvania State University, Department of Information Science and Technology: "Multiscale Graph Neural Network Autoencoders for Interpretable Scientific Machine Learning" [2]

3. Scott Field, The University of Massachusetts, Dartmouth, Department of Mathematics: "Potential Applications of Scientific Machine Learning to the Binary Black Hole Problem" [12]

4. Bart van Bloemen Waanders, Sandia National Laboratories, Scientific Machine Learning group: “Learning control policies for high-fidelity models using hyper-differential sensitivities with respect to model discrepancy” [9]

5. Animashree Anandkumar, NVIDIA and California Institute of Technology, Computing and Mathematical Sciences: “Neural Operators for Accelerating Scientific Simulations ” [14, 22]

6. Michael Brennan, Massachusetts Institute of Technology, Computational Science & Engineering: “Exploiting Low-Rank Conditional Structure to Solve Bayesian Inverse Problems” [4]

Tuesday, June 20, 2023

7. Yunan Yang, Cornell University, Department of Mathematics: “Neural Inverse Operators for Solving PDE Inverse Problems” [19]

8. Aras Bacho, Ludwig Maximilian University of Munich, Department of Mathematics: “PoissonNet: Resolution-Agnostic 3D Shape Reconstruction using Fourier Neural Operators” [1]

9. Eric Cyr, Sandia National Laboratories, Computational Mathematics Group: “Exploiting time-domain parallelism to accelerate neural network training” [20]

10. Paolo Zunino, MOX - Modelling and Scientific Computing - Politecnico di Milano: “A Deep Learning approach to Reduced Order Modeling of parameter dependent Partial Differential Equations” [6]

11. Robert Scheichl, Heidelberg University, Department of Mathematics: “Structure-preserving learning of embedded closure models for fluid flows”.

12. N. Sukumar, The University of California, Davis, Department of Civil and Environmental Engineering: “Exact Imposition of Boundary Conditions in PINNs to Solve PDEs”

13. Mihai Nica, the University of Guelph, Department of Mathematics and Statistics: “A derivative-free method for solving elliptic partial differential equations with deep neural networks” [8]

14. Deepanshu Verma, Emory University:, Department of Mathematics “Advances and challenges in solving high-dimensional Hamilton-Jacobi-Bellman equations”

Wednesday, June 21, 2023

15. Jakob Zech, Heidelberg University, Department of Mathematics: “Nonparametric Distribution Learning via Neural ODEs” [18]

16. Nicholas Nelsen, The California Insitute of Technology, Division of Engineering and Applied Science: “Convergence Theorem for Vector-Valued Random Features” [13]

17. Margaret Trautner, The California Insitute of Technology, Department of Computing + Mathematical Sciences: “Learning Homogenized Constitutive Laws” [3]

18. Guang Lin, Purdue University, Departments of Mathematics, Statistics & School of Mechanical Engineering: “ Energy Dissipative Evolutionary Deep Operator Networks” [24]

19. Thomas O’Leary-Roseberry, The University of Texas at Austin, Oden Institute for Computational Engineering & Sciences: “Derivative-Informed Neural Operators for High-Dimensional Outer-Loop Problems” [21]

20. Jinwoo Go, Georgia Institute of Technology, School of Computational Science and Engineering: “Accelerating A-Optimal/D-Optimal Design of Experiments Using Neural Networks”

21. Dingcheng Luo, The University of Texas at Austin, Oden Institute for Computational Engineering & Sciences: “Efficient PDE-constrained optimization with derivative-informed neural operators” [16]

22. Bruno Despres, Jacques-Louis Lions Laboratory Sorbonne University: “Generating functions for polynomials with ReLU: application to training”

23. Marta D'Elia, Pasteur Labs: "GNN-based physics solver for time-independent PDEs" [7]

24. Shunyuan Mao, University of Victoria, Department of Physics and Astronomy: "PPDONet: Deep Operator Networks for Fast Prediction of Steady-State Solutions in Disk-Planet Systems" [17]

Thursday, June 22, 2023

25. Petros Koumoutsakos, Harvard University, Institute for Applied Computational Science: "AI/Scientific Computing: Alloys for Flow modeling and Control" [11]

26. Peng Chen, Georgia Institute of Technology, School of Computational Science and Engineering: " Projected variational inference for high-dimensional Bayesian inverse problems" [5]

27. Lu Lu, Yale University, Department of Statistics and Data Science: "Deep neural operators with reliable extrapolation for multiphysics, multiscale & multifidelity problems" [15, 25]

Scientific Progress Made

Participants of the 2023 BIRS Workshop on Scientific Machine Learning experienced a unique forum for knowledge exchange, collaborative problem solving, and exposure to cutting-edge theories and methods. The gathering served as a fertile ground for cross-pollinating ideas from various mathematical disciplines and applications while staying focused on developing new SciML methodologies. Senior researchers presented recent advancements, novel theories, and breakthrough methodologies, which helped to inspire younger researchers and offer fresh perspectives on both new and longstanding problems. Concurrently, the collaborative atmosphere of the workshop helped to catalyze the genesis of new research directions, as attendees often engaged in rigorous discussions that challenged existing paradigms and used the allotted time for unstructured interactions to brainstorm innovative solutions. Beyond the immediate dissemination of knowledge, we hope that the workshop has helped sow the seeds for future collaborations, publications, and even entirely new research domains, further solidifying BIRS's role as an essential catalyst in the relentless journey of mathematical discovery.

Hybrid format

The hybrid format was beneficial because it allowed us to significantly expand the scope of our workshop. Due to the hybrid format, we were able to bring in a number of very high-impact remote presenters that would otherwise not have been able to speak. Additionally, we were able to engage a number of remote participants, including researchers who had difficulties obtaining Canadian visas in time and early career grad students who were not able to attend.

The major benefit of the workshop, however, was the in-person interactions and open discussions that happened in response to and in addition to the technical talks. Part of what really made the workshop work as well was the scale of it: since there were fewer participants, people got to know each other better than they otherwise would have, which led to deeper engagement with the technical material and enlightening discussions about the future of the field. For this reason, the hybrid format was non-essential, and it would make sense to make the event only in person.

Participant Testimonials

We received the following public testimonial from Robert Scheichl of the Institute for Mathematics & Interdisciplinary Center of Scientific Computing, Heidelberg University:

> "The workshop provided an excellent venue for interaction on this exciting and booming new research area. I would like to congratulate and thank the organisers for putting together a very well-balanced

and high-profile line-up of talks. The interaction and engagement of participants at BIRS was great. It did showcase several new research avenues to me that will be highly influential in my future research; it did, however, also show the limitations of certain approaches. Participants were open for a non-competitive and unbiased discussion of competing potential approaches in the area of scientific machine learning. I particularly appreciated that a strong emphasis at this workshop was given to promising young and upcoming researchers, who brought a lot of enthusiasm, energy and willingness to engage and interact. The talks were well chosen and the things I took away will definitely influence my immediate hiring decisions. The concluding discussions on Friday on the wider discipline and on how this new field should be shaped was particularly useful and via an editorial piece we plan to write for a special journal issue should provide some lasting impact for the wider research field."

Outcomes of the Meeting

The workshop brought together an international and interdisciplinary group of researchers that would not otherwise have assembled to discuss the issues and challenges of the emerging field of scientific machine learning.

The countries represented were Canada, France, Germany, France, New Zealand, and the USA. The disciplines represented include computational and applied mathematics, information science, scientific machine learning, computational science & engineering, aerospace engineering, chemical engineering, civil engineering, mechanical engineering, physics, and astrophysics, spanning academia, labs, and industry.

In the workshop, key emerging areas of promise within SciML were identified, while honest critical debate led to a better understanding of the limitations of the field. This gave clarity to future directions of interest in the field. We hope that numerous collaborations and international research correspondences will arise due to this workshop. The organizers can attest personally that this is already a takeaway from this event.

The workshop will culminate in a special issue (SI) in the American Institute for Mathematical Sciences Foundations of Data Science (FoDS) journal. The SI will be guest-edited by the BIRS organizers. Additionally, a position paper summarizing the findings of some discussions at the workshop will accompany this issue.

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Chapter 20

Women in Operator Algebras III (23w5089)

June 25 – 30, 2023

Organizer(s): Astrid an Huef (Victoria University of Wellington), Therese Basa Landry (University of California, Santa Barbara), Sarah Plosker (Brandon University), Sarah Reznikoff (Kansas State University), Maria Grazia Viola (Lakehead University)

Overview

This report details the organization and scientific progress made at the third Women in Operator Algebras workshop held at the Banff International Research Station in Banff, Canada. WOA III was a 5-day workshop held from June 25 to June 30, 2023. Although predominantly an in-person event, 32% (12 of 38 participants) participated virtually.

To bolster the numbers and success of women in higher mathematics, various research fields have formed networks and held workshops for the purpose of initiating women-led and all-women research projects; among the first were "Women in Number Theory" and "Women in Topology". Women are particularly underrepresented in the field of operator algebras, with estimates indicating that less than 17% of researchers in that area are women ([7]). Moreover, as it is a small field with research nodes spread around the globe—including in Scandinavia, New Zealand, Australia, Germany, Italy, France, Japan, Poland, Canada, and scattered across the U.S.—geographic isolation can compound the marginalization of women in the field. For these reasons, the "Women in Operator Algebras" (WOA) network was formed several years ago. As the most recent effort in WOA's mission to broaden participation in the operator algebras community, WOA III provided a vital opportunity for women in operator algebras to

1. conduct cutting-edge collaborative research, and
2. build a network of women working in the field to drive future research collaborations.

Notably, these aims were accomplished while facilitating

1. horizontal integration amongst researchers from PhD-granting and primarily undergraduate institutions, and
2. vertical integration across career levels.

In particular, the format from the BIRS workshops "Women in Numbers," where the focus is on research in groups on current problems, provided the organizational groundwork for achieving these successes.

Leaders for WOA III group projects were solicited directly via emails to well-known senior and mid-career women researchers, as well as indirectly via bulk emails to lists for women in operator algebras and to major conferences in operator algebras. The workshop was advertised over social media (in particular, the Operator Algebras Facebook group).

Group leaders supplied a 1-2 page description of their proposed project. Some projects were led by leaders from past iterations of WOA, while a few were proposed by researchers who had never been involved with WOA. Notably, three fourths of the projects were proposed by past participants of Women in Operator Algebras workshops who had not previously led projects: Sara Azzali, Sarah Browne, Kristin Courtney, Marzieh Forough, Anna Duwenig, Sanaz Pooya, Lauren C. Ruth, Camila F. Sehnem, Dillian Yang. This marker of professional growth is another measure of the success and long-term impact of these workshops in increasing representation and visibility of women in operator algebras.

Once group leaders and projects were established, applications for participation in the workshop were sought through broad and repeated emails and social media posts. The application process included the descriptions of the projects and asked for a ranked list of the applicants' projects of interest, a brief statement describing how the applicant's expertise fitted with the projects and how they would benefit from participating, and a CV. Participants were selected according to the anticipated benefits to their careers, with consideration given to job security and advancement, existing collaborations, and isolation. Graduate students who had established thesis projects, and had the support of their advisors, were also considered. These selection criteria are similar to those used in previous WOA workshops.

Based on the responses, the organizers matched participants to projects according to their preferences and background. In most cases the participants received their preferred choice of project; however in some cases their second or third choice were used in order to keep group sizes to around 3-6 people, to optimize collaborations. One project was removed from the list of research options because of the lack of interest from potential participants, while another one was removed due to scheduling issues with the leaders. In the end, there were a total of eight research projects.

Group leaders were asked to provide background reading material and references for group members and encouraged to organize several virtual meetings in advance of the workshop. This gave the research groups opportunity to build a strong foundation for the week of the WOA III workshop.

The workshop had a total of 38 participants from 15 countries (Belgium, Canada, China, Czech Republic, England, France, Germany, India, Iran, Ireland, Italy, Korea, New Zealand, Norway and the USA). The majority of the workday was spent working in research groups.

A brief overview of each group project was presented on Monday morning. There was a formal talk by Arundhathi Krishnan (Markovianity and the Thompson Monoid F^+) on Tuesday afternoon. Thursday morning featured a discussion about questions related to projects as well as updates on projects. Thursday afternoon featured a formal talk by Preeti Luthra (C^*-envelopes and nuclearity of operator systems). Final presentations of research projects were given on Friday morning. Networking and informal discussions took place during the downtime throughout the week. In particular, there was a moderated discussion about workplace practices and issues on Wednesday evening after dinner. These opportunities enabled women to

1. have access to advice and feedback on career matters, and
2. forge connections between junior women and established researchers.

Overview of operator algebras, recent developments and open problems

The study of operator algebras was initiated in the late 1920's to formalize the algebra of observables in quantum mechanics; this branch of functional analysis now intersects with many fields, including algebra, dynamical systems, mathematical physics, ergodic theory, quantum information theory, and topology. The subject has seen many exciting developments over the last several decades. Among these are the work of Jones and Popa on the classification of finite factors; the solution of Tsirelson's problem in quantum information theory; and the recent advances in the classification program for unital, simple, separable, nuclear C^*-algebras, which was initially formulated by

Elliott in the early 1990's. Some of the fundamental open problems in the area are the classification of non-simple nuclear C^*-algebras, completion of the Toms-Winter conjecture, the classification of subfactors of certain indices, and the isomorphism question for free group factors.

Scientific progress made during the workshop: reports from the research groups

A rich dictionary of quantum analogues of classical subjects like directed graphs and harmonic analysis exists nowadays, as well as enriched operator algebraic perspectives of index theorems and elliptic operators on manifolds. The WOAIII projects proposed by the research leaders, who are well-known or rising experts in the field, deal with a wide variety of topics, reflecting the array of perspectives and research that falls under the umbrella of operator algebras. The research field not only gives a rigorous context for phenomena from quantum physics, but also for ideas from quantum information, as shown by extensive work on quantum channels and entanglement in recent years. For example, the project by Emanuela Sasso and Sachi Srivastava looks at decoherence of Gaussian Quantum Markov Semigroups, a phenomenon that may appear when there is interaction between the open quantum system and the environment. The concept is related to the ergodic behaviour of Quantum Markov Semigroups.

Non-commutative geometry is another major area of research for operator algebraists. Fundamental open problems in the area are the Baum-Connes conjecture, which suggests a link between the K-theory of the reduced C*-algebra of a group and the K-homology of the classifying space of proper actions of that group, and the coarse Baum-Connes conjecture. Two of the projects deal with the conjectures: the one by Sanaz Pooya and Hang Wang, and the one by Sara Azzali and Sarah Browne.

A few of the workshop's projects are related to the Elliott classification program and the rich structure theorems for C^*-algebras made possible by that framework. For example, dynamical systems yield an abundant supply of operator algebras, and some of the projects aim to advance our understanding of such structures and study the regularities properties of the associated C^*-algebras. The projects of Courtney-an Huef and Forough-Strung both focus on some aspects of dynamics and classification.

Questions about dynamics and equilibrium states can also be defined on C^*-algebras. Camila Sehnem's project studies such phenomena on noncommutative solenoids, which are C^*-algebras which can be identified with inductive limits of noncommutative tori, while Duwenig-Yang's project looks at non-amenable groups acting on a k-graph self-similarly and Cartan subalgebras of self-similar k-graph C^*-algebras.

Lastly the project led by Ruth sits at the intersection of operator algebras and finance. It is motivated by a result in options pricing called the Breeden-Litzenberger formula, and seeks to find an operator algebraic version of this result, by replacing the measure space with a von Neumann algebra, a common theme in operator algebras.

Actions of groups on C^*-algebras of twisted groupoids, and implications for group actions on classifiable C^*-algebras

Group members: Anshu (National Institute of Science Education and Research), Kristin Courtney (group co-leader; University of Münster), Magdalena Georgescu (Independent), Astrid an Huef (group co-leader; Victoria University of Wellington), and Maria Grazia Viola (Lakehead University)

Research synopsis and progress

The study of C^*-algebras is often described as the study of noncommutative topological spaces. Hence noncommutative generalizations of topological dynamics are a topic of intensive study in the field. These are studied via the crossed product C^*-algebra $A \rtimes H$ arising from an action of a locally compact Hausdorff group H on a C^*-algebra A (it is the C^*-algebraic version of a semi-direct product of groups). Of particular interest is determining, at the level of the dynamics, whether a given crossed product fits into the framework of the celebrated classification theorem (the culminating result of a major research program– see [37]). A useful tool in studying and modeling C^*-algebras is the theory of locally compact, Hausdorff and étale groupoids. The recent breakthrough in [27] (along with [35] and [13]) shows that all C^*-algebras which fit into the classification program admit twisted groupoid models, or equivalently, they possess a certain distinguished subalgebra called a Cartan subalgebra. From [5], a discrete group acting on a C*-algebra that admits a twisted groupoid model (including the so-called classifiable C*-algebras) and leaves the canonical Cartan subalgebra invariant, corresponds to an action on the underlying twisted groupoid. This gives rise to the natural question: can we study group actions on C*-algebras with twisted groupoid models via the corresponding actions on the groupoids? We made the first major strides towards this goal during WOA III at BIRS by proving the following theorem:

Theorem: Let $\mathcal{G}$ be a locally compact, Hausdorff, second-countable groupoid with a Haar system Let α be an action of a discrete group H on $\mathcal{G}$ satisfying an appropriate invariance condition on the Haar system. Then α induces an action $\tilde{\alpha}$ of H on $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$ such that $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}) \rtimes_{\tilde{\alpha},\mathrm{r}} H$ and $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G} \rtimes H)$ are canonically isomorphic.

We can show analogous results for

1. the full crossed product of the full groupoid C*-algebra $\mathrm{C}^*(\mathcal{G})$ by H, and
2. an action of H on a twist over $\mathcal{G}$.

We have also made promising progress in extending the above results to actions of locally compact groups. In future work, we plan to explore how properties of the action of H on $\mathcal{G}$ and properties of the semi-direct product $\mathcal{G} \rtimes H$ translate to those of the action H on $\mathrm{C}^*(\mathcal{G})$ and the crossed product $\mathrm{C}^*(\mathcal{G}) \rtimes_r H$ (and likewise in the full and twisted settings).

Self-similar k-graph algebras

Group members: Dawn Archey (University of Detroit Mercy), Anna Duwenig (group co-leader; KU Leuven), Shanshan Hua (University of Oxford), Kathryn McCormick (California State University, Long Beach), Rachael Norton (Macalester College), and Dilian Yang (group co-leader; University of Windsor)

Research synopsis and progress

A *self-similar k-graph* is a pair (G, Λ) consisting of a (discrete) group G and a k-graph Λ such that G acts on Λ in a 'self-similar' way: Loosely speaking, that means that Λ also 'acts' on G, and that the two 'actions' align in an appropriate sense. Self-similar k-graphs were first studied by Li-Yang [26], and they generalize Exel-Pardo's self-similar directed graphs [16].

The self-similar k-graph C*-algebra $\mathcal{O}_{G,\Lambda}$ associated to (G, Λ) is a C*-algebra universal for pairs (U, S) of representations, where U is a unitary representation of G on some C*-algebra $\mathcal{A}$ and S is a representation of the graph C*-algebra $\mathcal{O}_\Lambda$ of Λ on $\mathcal{A}$, such that U and S encode the self-similar action of (G, Λ). This class of C*-algebras encompasses many known important C*-algebras as special cases, such as k-graph C*-algebras introduced by Kumjian-Pask, Exel-Pardo algebras, Katsura algebras, and Nekrashevych algebras.

Li-Yang have studied self-similar k-graph C*-algebras in a series of works, including their properties [26], and under some conditions, their KMS states [24] and ideal structure [25].

During the week at BIRS, our different backgrounds enabled us to formulate and investigate various questions that arise for this new concept of self-similar k-graph algebras, such as the following.

Question: In [24], under some strong conditions, the authors obtain a canonical Cartan subalgebra for a self-similar k-graph. Is it possible to remove some of the restrictions?

Progress: To gain traction, we investigated a self-similar k-graph that is not locally faithful and found a likely candidate for a Cartan subalgebra in that particular case. We would like to generalize this.

Question: Can we find an example of a non-amenable group acting on a k-graph self-similarly?

Progress: Yes. In fact, we have found a whole family of examples.

Question: C*-algebra theory is replete with product-type constructions which turn given input data (such as a group action on a C*-algebra) into a new object (in this case, a crossed product C*-algebra), and then one studies the interaction between the two. What kind of product-type constructions make sense for a self-similar k-graph (G, Λ)?

Benefits of the WOA format: Months before the conference began, each participant was placed in a group in which they would be working on one of the proposed projects. Not only did this allow us to get to know each other via email beforehand, but it also allowed each member of our group to familiarize herself with the necessary mathematical background. This was particularly important since each group member has expertise in a different area. The assigned readings enabled everyone to feel more confident and contribute to the research during the week at BIRS.

Future plans: The group intends to have biweekly online meetings to make progress on the research project in their on-going collaboration. Furthermore, the group will meet at the *Fields Institute* in Toronto ON, Canada, during the *Thematic Program on Operator Algebras and Applications* in late 2023.

Equilibria on Toeplitz extensions of higher-rank noncommutative solenoids

Group members: Becky Armstrong (University of Münster), Mahya Ghandehari (University of Delaware), Larissa Kroell (University of Waterloo), and Camila F. Sehnem (group leader; University of Waterloo)

Project description Solenoids are inverse limits of tori. Latrèmoliére and Packer considered in [23] *noncommutative solenoids*. Following the equivalent characterisation of a noncommutative torus as a twisted group C^*-algebra of $\mathbb{Z}^2$, they defined a noncommutative solenoid to be a certain twisted group C^*-algebra, where the group involved is an abelian discrete group with a solenoid as its Pontryagin dual. They also described noncommutative solenoids as inductive limits of rotation algebras, that is, *noncommutative tori*.

Brownlowe, Hawkins and Sims [10] introduced a notion of Toeplitz extensions of the noncommutative solenoids of [23] as direct limits of certain building blocks, motivated by the description of a noncommutative solenoid as a direct limit of rotation algebras. Each of these building blocks is a Toeplitz-type extension of a noncommutative torus, obtained by replacing one of the two canonical unitary generators in the presentation of a noncommutative torus by an isometry. They studied KMS states and phase transitions of these Toeplitz noncommutative solenoids under natural dynamics. Following this construction, Afsar, an Huef, Raeburn and Sims [1] defined higher-rank versions of Toeplitz noncommutative solenoids using Toeplitz-type extensions of higher-dimensional noncommutative tori as building blocks (see [31, 33] for an account on higher-dimensional noncommutative tori), and studied their phase transition under a dynamics determined by a subgroup of the gauge action.

More recently, a higher-rank Toeplitz noncommutative torus was introduced in [2] using twisted semigroup C^*-algebras. The building blocks used in [1] are quotients of these Toeplitz noncommutative tori for particular choices of 2-cocycles. The main results on phase transitions of [2] generalize those of [1] on phase transitions for their Toeplitz extensions of higher dimensional noncommutative tori.

The purpose of this project is to define higher-rank Toeplitz noncommutative solenoids using twisted semigroup C^*-algebras, thus following [2] and the original approach proposed by Latrèmoliére and Packer in [23]. Once this is done we aim to investigate phase transitions on these Toeplitz noncommutive solenoids. This approach should allow us to generalize the main results of Afsar, an Huef, Raeburn and Sims [1].

Progress made during the week: We have greatly benefited from the workshop at BIRS as we had a very productive week, especially taking into account that this was the first time we all met in person. The group discussed the main concepts needed in the project and defined the class of semigroups and twisted semigroup C^*-algebras we aim to study. We found a concrete class of examples which fit into our approach and are not covered by previous work.

Future plans: We have allocated group members to certain tasks and we plan to meet as a group on Zoom for a week later this year. We plan to apply for funding to visit a research institution as a group to continue working on our project in person.

Explicit computations around the Baum–Connes conjecture

Group members: Sara Azzali (group co-leader; University of Bari), Sarah Browne (group co-leader; University of Kansas), Indira Chatterji (University of Côte d'Azur), Maria Paula Gomez Aparicio (Université Paris-Saclay), Therese Landry (UC Santa Barbara), and Hang Wang (East China Normal University)

The project "Explicit computations around the Baum–Connes conjecture" is a continuation of the project at WOA2. We had not met as an entire group for some time and we had a new group member so we spent some of the week working through definitions and results to be on the same page and be able to connect ideas as a team. Indira Chatterji and Hang Wang were both online, so we had scheduled times during the day to meet with them, given their time zone differences.

Activities during the week at BIRS:

1. Summarized the key aspects of goal intentions and directions for project.
2. Discussed and worked through Lafforgue's construction of the γ-element.

Specifically we worked through the definitions of hypotheses for the construction including:

- uniformly locally finiteness
- weakly δ-bolic property
- weakly δ-geodesic property
- strongly bolic property

and then we looked at examples and non-examples of these concepts. We also made connections with other references (Julg-Valette), some we had already explored as well as more recent work of (Brodski-Guentner-Higson [8] and Brodski-Guentner-Higson-Nishikawa [9])

3. We began working on the initial case for the project in order to progress to the main case we want to consider.
4. Therese gave a talk "Non-commutative analogues of geometric structures for twisted group C^*-algebras using length functions on groups" on her work and made connections.

Specifically Therese started speaking about her work with Carla Farsi, Nadia Larsen and Judith Packer, and spoke about the twisted structures and connections to some of the material we had spoken about during the week.

Our plan since BIRS is to meet on Zoom fortnightly and also distribute tasks among the group members as these arise. We also want to give Therese time to finish talking to us about her work with Farsi, Larsen, and Packer. We plan to apply for funding to be able to visit each other as an entire group or in subgroups as is possible given schedules. We hope to meet in-person within the next year as a group as well additionally with both Indira Chatterji and Hang Wang whom we spoke with online during our time here in Banff.

Some aspects of Gaussian Quantum Markov Semigroups

Group members: Priyanga Ganesan (University of California at San Diego), Arundhathi Krishnan (Munster Technological University), Yulia Kuznetsova (Université de Franche-Comté, Sarah Plosker (Brandon University), Emanuela Sasso (group co-leader; University of Genova), Sachi Srivastava (group co-leader; University of Delhi)

Our project is based on Gaussian Quantum Markov Semigroups (G-QMS). Recall that a QMS is called Gaussian when the underlying algebra is $\mathcal{B}(\Gamma(\mathbb{C}^d))$, i.e., the algebra of all bounded operators on the Fock space $\Gamma(\mathbb{C}^d)$, and the predual semigroup acting on trace class operators on $\Gamma(\mathbb{C}^d)$ preserves Gaussian states. Our goal is to investigate irreducibility and decoherence, phenomena that could appear when there is interaction between the open quantum system and the environment. In particular we have that the environment induces decoherence if the system can be decomposed in two parts: one that evolves as a closed quantum system and a second part which disappears in long time. Both the concepts of decoherence and irreducibility are related to the ergodic behaviour of a QMS. Our work starts from [12, 14, 18, 11] (decoherence for uniformly continuous QMSs) and [3, 4, 36] (G-QMSs)

During the week of the workshop, we started a lively and productive collaboration. Our plan was to study irreducibility and decoherence for G-QMSs. However, we immediately realized that [17] had already (recently) solved the characterization of irreducibility in $d-$dimension. So we approached the problem of decoherence and were able to achieve the following for a Gaussian QMS $(\mathcal{T}_t)$ with faithful normal invariant state ρ.

- Characterize the decoherence free algebra $N(\mathcal{T})$ completely when ρ is Gaussian and deduce that EID holds.
- Find a description of the set of fixed points $\mathcal{F}(\mathcal{T})$ of $(\mathcal{T}_t)$ in terms of Weyl operators and certain matrices that depend on the GKLS representation of the generator of the QMS.
- Express the fixed points set as
$$\mathcal{F}(\mathcal{T}) = \bigoplus p_i \mathcal{F}(\mathcal{T}) p_i,$$
where p_i are minimal projections on $\mathcal{F}(\mathcal{T})$ and $p_i\mathcal{F}(\mathcal{T})p_i$ is a type-I factor, and then define the set of reduced semigroups: $\mathcal{T}_t^i(p_i x p_i) := p_i \mathcal{T}_t(x) p_i$ on $p_i B(\mathfrak{h}) p_i$. This allows us to deduce important properties of $\mathcal{F}(\mathcal{T})$.

Our work thus far gives rise to several conjectures, involving behaviour of the minimal projections, faithful invariant states of the G-QMS and also $\mathcal{F}(\mathcal{T})$.

Plans for the near future: We plan to address our conjectures and obtain results about the structure of faithful invariant states for a G-QMS. Our long term goal is to investigate EID for G-QMSs. Finally by starting from [32] we want to explore how we can use some valid results for quantum channel for QMSs.

We will meet online to continue working on our project. We applied to the Fields Thematic Program on Operator Algebras and Applications and we have obtained funds for a research week in November in Toronto. We have also invited Veronica Umanità (University of Genova) to join this group project.

High dimensional expanders and the coarse Baum-Connes conjecture

Group members: Sherry Gong (Texas A&M University), Sanaz Pooya (group co-leader; Stockholm University), Hang Wang (group co-leader; East China Normal University), and Kun Wang (Texas A&M University)

The aim of our project is to construct new counterexamples of the coarse Baum-Connes conjecture for a metric space X with bounded geometry, via higher dimensional expanders. The coarse Baum-Connes conjecture states that the coarse assembly map
$$\mu : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(\mathrm{C}^*(X))$$
is an isomorphism. Let X be a countable family of higher dimensional expanders as in [28], with a uniform spectral gap at 0. Motivated by Higson's counterexamples to the coarse Baum-Connes conjecture [20], we would like to construct the associated higher Kazhdan projection p in the Roe algebra $\mathrm{C}^*(X)$. Then the hope is to establish functionals
$$\phi_1, \phi_2 : K_0(\mathrm{C}^*(X)) \to \frac{\prod \mathbb{R}}{\bigoplus \mathbb{R}}$$
so that $\phi_1(p) \neq \phi_2(p)$ while $\phi_1 = \phi_2$ over the image of the coarse assembly map μ.

During the focus week, we looked into relevant papers by Lubotzky [28], by Higson [20], by Higson-Lafforgue-Skandalis [21], by Willett-Yu [38], by Drutu-Nowak [15] and by Sawicki [34], as well as the survey paper by Gomez-Aparicio, Julg and Valette [19]. First, we observed that all existing counterexamples employ the idea of Higson [20]. Also, we noticed that most papers except for the one by Willett-Yu [38] construct the obstructing projections based on groups, rather than from expanders. Based on these findings, for the higher dimensional case, we need to find a suitable replacement to the notion of "large girth", so that the functionals τ_i for $i = 0, 1$ can be constructed. We also need to verify that the associated projection p is a ghost so that applying τ_0, one has $\tau_0(p) = 0$, or we need to find a characterization of p so that $\tau_0(p) \neq \tau_1(p)$.

At the end of the workshop, we made a plan to have regular online group meetings till the end of the year and a one-week gathering in 2024. Part of the group members will gather at some conferences during the summer of 2023. We expect to have a first draft of the article in the near future.

Nuclear dimension of crossed products by Hilbert bimodules over commutative C^*-algebras

Group members: Marzieh Forough (group co-leader; Czech Technical University in Prague), Zahra Hasanpour Yakhdani (University of Tehran), Ja A Jeong (Seoul National University), Preeti Luthra (University of Delhi),

Melody Molander (University of California San at Diego), Karen Strung (group co-leader; Institute of Mathematics, Czech Academy of Sciences)

This project is set out to study the nuclear dimension of crossed products by (full) Hilbert bimodules over commutative C*-algebras. This class of crossed products coincides with the class of C*-algebras of vector bundles twisted by homeomorphisms. The nuclear dimension is a noncomumtative version of covering dimension of topological spaces introduced by W. Winter and J. Zacharias (2010). This notion has played a key role in the classification program. Computing the nuclear dimension for C*-algebras, in particular, the ones which come from some sort of dynamical systems, has attracted a lot of attention. I. Hirshberg and J. Wu (2017) obtained an upper bound for the nuclear dimension of a crossed product of $C_0(X)$ by any automorphism where X is a locally compact Hausdorff space of finite covering dimension. Any such crossed product can be viewed as a C*-algebra of a trivial vector bundle twisted by a homeomorphism. The goal of this project is to extend the result of Hirshhberg and Wu to the case that the underlying vector bundle is not trivial. In particular, we aim to study the nuclear dimension of C*-algebras of a vector bundle twisted by a homeomorphism when the base space of the underlying vector bundle has finite covering dimension.

During the workshop at BIRS, we computed an upper bound for the nuclear dimension of the C*-algebra of a line bundle twisted by a periodic homeomorphism. Continuing working on this project after the workshop, we came up with an approach to deal with the general case by using our result for periodic homeomorphisms.

We are planning to continue our discussion via zoom and a shared overleaf. After some more progress, those members who are interested enough and collaborate actively will continue this project.

Towards an Operator-Algebraic Breeden–Litzenberger Formula

Group members: Sarah Reznikoff (Virginia Tech), Lauren C. Ruth (group leader; Mercy College), Lydia de Wolf (Siena College)

Background on Breeden–Litzenberger Formula

This project is motivated by a result in options pricing called the **Breeden–Litzenberger formula**. Let K be a random variable representing the strike price of an option. Let X be a random variable representing the stock value after one year. The payoff will be $\max\{X - K, 0\}$. Suppose you know the expected value of your payoff for every value k of the strike price. Surprisingly, it is then possible to determine the density function of the stock value, $f_X(x)$. Define

$$C(k) = E\left[\max\{X - k, 0\}\right].$$

Breeden and Litzenberger showed in [6] that

$$\frac{d^2}{dk^2}C(k) = f_X(k).$$

Their proof uses the calculus of finite differences, since they are working with discrete, real-world data. In the continuous setting, the proof begins as follows.

$$C(k) = E\left[\max\{X - k, 0\}\right] = \int_0^\infty \max\{x-k,0\} f_X(x)dx = \int_k^\infty (x-k) f_X(x)dx$$

To differentiate this with respect to k requires the formula for **differentiation under an integral with variable limits**:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} g(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial k} g(x,t)dx + g(b(t),t)\frac{db}{dt} - g(a(t),t)\frac{da}{dt} \tag{1}$$

Assuming finite expectation, we have

$$\begin{aligned}\frac{d}{dk}\lim_{b\to\infty}\int_k^b (x-k)f_X(x)dx &= \lim_{b\to\infty}\left(\frac{d}{dk}\int_k^b (x-k)f_X(x)dx\right)\\ &= \lim_{b\to\infty}\left(\int_k^b \frac{\partial}{\partial k}(x-k)f_X(x)dx + ((b)-k)\frac{d(b)}{dk} - ((k)-k)\frac{d(k)}{dk}\right)\\ &= -\int_k^\infty f_X(x)dx + 0 - 0 = -\left(1-\int_{-\infty}^k f_X(x)dx\right) = -(1-F_X(k)) = F_X(k)-1\end{aligned}$$

where F_X is the cumulative distribution function corresponding to the density function f_X. Differentiating once more, we obtain

$$\frac{d}{dk}\left(F_X(k)-1\right) = f_X(k)$$

thus the density function of X has been recovered as the second derivative of the expected value of max$\{X-k,0\}$ with respect to k.

Project Goals and Preparation

We seek to find the operator algebraic version of the Breeden–Litzenberger formula by replacing the measure space with a von Neumann algebra, a common theme in our field. In this non-commutative setting, the "expectation" from probability becomes the "trace" of an operator. For example, Ishan, Peterson and Ruth generalized the notion of measure equivalence to "von Neumann equivalence" in [22]. The experience of working on that paper prepared Ruth to undertake the proposed project with Reznikoff and de Wolf.

Our WOA III team is uniquely positioned to attack this problem, which sits at the intersection of operator algebras and finance: Ruth and Reznikoff have both taught courses in actuarial science (Ruth for Exam P, on probability, and Reznikoff for Exam FM, on financial mathematics); and Ruth and de Wolf have both passed actuarial exams.

Project Progress

The difficulty in the Breeden–Litzenberger generalization appears to lie in finding the right notion of "differentiation under the integral," formula (1) above. We thank Ben Hayes for the suggestion that what we are looking for might be closable derivations ([29], [30]). During WOA III, we reviewed pre-requisites for understanding the predual of a von Neumann algebra, and we concluded by reviewing pre-requisites for working with closable derivations. We look forward to applying what we have learned during WOA III towards our project in the coming months.

Impact on Participants' Careers

This fall, Reznikoff will begin serving as department chair at Virginia Tech, de Wolf will start a new position at Siena College, and Ruth will enter her fourth year on the tenure track at Mercy College. WOA III has provided us with a crucial opportunity to advance our research while managing heavy service responsibilities and teaching loads.

Outcomes of the Meeting

Underrepresentation of women is a particular concern in operator algebras, where it is particularly acute. Certain women will particularly benefit from the opportunity offered by the workshop: those with positions at small teaching-oriented colleges, those who have just finished their PhD and need multiple projects to move forward with a research career, and those who have postdoctoral or tenure-track positions in departments with no suitable collaborators.

Strengthening the research careers of women in the field will have broader impacts as well. As the participants return to their institutions, they will be better able to mentor and inspire women undergraduates and graduate students who may pursue higher degrees or careers in mathematics. In addition, having more accomplished and research-active women will improve the existing culture in operator algebras for the next generation.

All of the groups indicated that they will be continuing their collaboration virtually, with several groups meeting in the near future at the Fields Institute during the Thematic Program on Operator Algebras and Applications that is occurring in the Fall of 2023. Other groups will attempt to meet in-person at various upcoming conferences or plan their own research visits. These are good indications that group members see the benefits of the collaborations, and that the research may result in peer-reviewed journal articles in the long-run. Research groups have also been encouraged to seek further funding from other programs such as

1. MSRI's *Summer Research in Mathematics*,
2. Oberwolfach, and
3. the American Institute of Mathematics' *SQuaREs* program.

A follow-up survey will be administered to project leaders in six or eight months to determine projects' progresses, developments and career impacts.

The beneficial outcomes, both tangible and intangible, from the WOA workshops demonstrate the value of such an event in providing support and networking opportunities, particularly for women and others from under-represented gender groups.

The organizers led a discussion about a potential fourth iteration of WOA, and the response was overwhelmingly positive. Numerous participants put their names forward or nominated women who were not in attendance to help with the organizational aspects of a WOA IV, tentatively set for 2026.

Acknowledgement

We are grateful to BIRS for hosting our workshop. We thank the Clay Foundation, the Foundation Compositio Mathematica and the Fields Institute for Research in Mathematical Sciences for travel funds for selected participants. All the graduate students and early career researchers who requested it received travel support to attend the workshop.

Participants

an Huef, Astrid (Victoria University of Wellington)
Anshu, Anshu (National Institute of Science Education and Research)
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Armstrong, Becky (University of Münster)
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Courtney, Kristin (University of Southern Denmark)
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Duwenig, Anna (University of Victoria)
Forough, Marzieh (Czech Technical University in Prague)
Ganesan, Priyanga (University of California San Diego)
Georgescu, Magdalena (Independent)
Ghandehari, Mahya (University of Delaware)
Gomez-Aparicio, Maria Paula (University Paris Saclay)
Gong, Sherry (Texas A & M University)
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Hua, Shanshan (University of Oxford)
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Kroell, Larissa (University of Waterloo)
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Landry, Therese Basa (University of California, Santa Barbara)
Luthra, Preeti (University of Delhi)
McCormick, Kathryn (California State University Long Beach)
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Norton, Rachael (Macalester College)
Plosker, Sarah (Brandon University)
Pooya, Sanaz (Stockholm university)
Reznikoff, Sarah (Kansas State University)
Ruth, Lauren (Mercy College)
Sasso, Emanuela (Genova University)
Sehnem, Camila (University of Waterloo)
Srivastava, Sachi (University of Delhi)
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Chapter 21

New trends in fluids and collective dynamics (23w5002)

July 24 - 28, 2023

Organizer(s): Theodore Drivas (Stony Brooke University), Nancy Rodriguez (The University of Colorado, Boulder), Roman Shvydkoy (University of Illinois at Chicago), Eitan Tadmor (University of Maryland, College Park)

Overview of the subject

In the subject of partial differential equations (PDEs) which includes a broad range of models describing various physical phenomena, two distinct and seemingly unrelated branches may find themselves intricately connected. Such connections result in bursts of development as ideas are transferred between the subjects. An example of such merger was implementation of dispersive techniques into incompressible fluids in the 2000s. More recently, the geometric ideas of the Nash-Gromov immersion theory were brought to solve the long standing Onsager conjecture for energy conservation in the incompressible Euler system.

Currently, the subject of collective behavior, which spans studies of many biological, technological and social sysytems, is undergoing a similar development spurred by newly found links with fluid dynamics. The thread that runs through these fields stems from statistical mechanics of many particle systems – when the number of agents increases to infinity, the evolution equations describing collective behavior start to resemble equations of hydrodynamics.

One of the famous and notoriously challenging problems in collective studies is called *emergence*. Emergence is a paradigm stating that many naturally occurring global phenomena, such as swarming in biological systems or clustering in social networks, appear as a result of many locally communicating agents. Depending on the context and particular phenomenon in question, this paradigm can be formulated in precise mathematical terms. For example, occurrence of milling patterns in schools of fish governed by a 3Zone communication protocol is an emergent phenomenon.

A wide and recently popular class of alignment models such as Cucker-Smale and its variants has received a lot of attention in recent years. With the introduction of models with singular communication protocol new prospectives on collective behavior emerged with them new challenges in the analysis of fractional parabolic equations. Attempts to explain alignment through relaxation in kinetic equations brought adaptations of the hypocoercivity techniques from collisional models. Ideas coming from mixing and enhanced dissipation are now being implemented to chemotactic propulsion of micro-organisms. Methods of the regularity theory of fluid models such as modulus of continuity technique, generation of small scales, DiGiorgi-Nash-Moser iteration, are finding useful interpretations in the context of collective behavior. Models incorporating multiple scales of description and merging the classical hydrodynamical systems with systems of collective behavior have been very prominent in studies of immersed flocks where agents influence dynamics of the ambient fluid through a properly modeled stress. Applications to traffic control, droplet formation, cell migration, etc, are numerous.

Objectives planned and achieved

Such a rapid development of the new field necessitates active collaboration of specialists in various areas of PDEs. This makes our workshop timely and impactful. The main goal of this workshop was to provide a platform for an exchange of ideas between the subjects that can transformative to our understanding of collective phenomena and fluids.

This goal was achieved by bringing a diversified group of researchers from primarily two subjects of our focus – fluid dynamics and applied collective behavior. We gathered speakers involved in recent breakthroughs in regularity theory for fractional parabolic equations, aggregation, and chemotaxis models, as well as speakers more involved in implementation of analytical tools to specific models. The list of participants spans a broad range of carrier ages from young researchers to more senior and established ones.

Through careful planning and providing plenty of time for off-talk conversations we set the stage for active collaboration and exchange of knowledge between the participants, in person and virtual alike.

Format of the meeting

The meeting has been conducted in the hybrid format with most of the talks being delivered in survey / overview style accessible to participants in other fields and young researchers. We scheduled 4 one-hour talks on Monday, 6 talks on Tuesday, 2 talks on Wednesday, 6 talks on Thursday and 1 talk on Friday morning. Wednesday featured the traditional half-day schedule. Most talks have been recorded.

In total we hosted 9 in person talks and 7 virtual talks via Zoom. All talks, virtual and in person, have been broadcast to all the participants. Among the 16 speakers 6 were women of various seniority.

Abstracts and day-by-day activities

The first day featured four talks related to data assimilation, particle approximation methods and application to specific problems in collective droplet dynamics. These developments are important to understanding how and in what ways one can use the fundamental equations in real life measurements and how these measurements help calibrate the systems in question to improve their predictability. Mean-field limits and particle methods are crucial in justification and numerical implementation of some of the macroscopic models of fluid and collective phenomena. A surprising connection between non-local particle approximations and renormalization of fields in turbulence theory was discovered.

The schedule on this day was rather relaxed. We provided plenty of free discussion time.

Franziska Weber: *On Bayesian data assimilation for PDEs with ill-posed forward problems.*

> We consider Bayesian data assimilation for time-evolution PDEs, for which the underlying forward problem may be very unstable or ill-posed. We formulate assumptions on the forward solution operator of such PDEs under which stability of the posterior measure with respect to perturbations of the noisy measurements can be proved. We also provide quantitative estimates on the convergence of approximate Bayesian filtering distributions computed from numerical approximations. For the Navier-Stokes equations, our results imply uniform stability of the filtering problem even at arbitrarily

small viscosity, when the underlying forward problem may become ill-posed, as well as the compactness of numerical approximations in a suitable metric on time-parametrized probability measures. This is a joint work with Samuel Lanthaler and Siddhartha Mishra.

Hangjie Ji: *Coarsening of thin films with weak condensation.*

A lubrication model can be used to describe the dynamics of a weakly volatile viscous fluid layer on a hydrophobic substrate. Thin layers of the fluid are unstable to perturbations and break up into slowly evolving interacting droplets. In this talk, we will present a reduced-order dynamical system derived from the lubrication model based on the nearest-neighbour droplet interactions in the weak condensation limit. Dynamics for periodic arrays of identical drops and pairwise droplet interactions are investigated which provide insights to the coarsening dynamics of a large droplet system. Weak condensation is shown to be a singular perturbation, fundamentally changing the long-time coarsening dynamics for the droplets and the overall mass of the fluid in two additional regimes of long-time dynamics. This is joint work with Thomas Witelski.

Pierre-Emmanuel Jabin:*The mean-field limit of non-exchangeable integrate and fire systems*

We investigate the mean-field limit of large networks of interacting biological neurons. The neurons are represented by the so-called integrate and fire models that follow the membrane potential of each neurons and captures individual spikes. However we do not assume any structure on the graph of interactions but consider instead any connection weights between neurons that obey a generic mean-field scaling. We are able to extend the concept of extended graphons, introduced in Jabin-Poyato-Soler, by introducing a novel notion of discrete observables in the system. This is a joint work with D. Zhou.

Katy Craig: *Nonlocal particle approximations of constrained transport and diffusion, with applications to sampling.*

Given a desired target distribution and an initial guess of its samples, what is the best way to evolve the locations of the samples so that they accurately represent the desired distribution? A classical solution to this problem is to evolve the samples according to Langevin dynamics, a stochastic particle method for the Fokker-Planck equation. In today's talk, I will contrast this classical approach with two novel deterministic approaches based on nonlocal particle methods: (1) a nonlocal approximation of dynamic optimal transport, with state and control constraints, and (2) a nonlocal approximation of degenerate diffusion equations. I will present recent work analyzing the convergence properties of each method, alongside numerical examples illustrating their behavior in practice. This is based on joint works with Karthik Elamvazhuthi, Matt Haberland, Matt Jacobs, Harlin Lee,and Olga Turanova.

Second day was mode intense in terms of schedule of talks. We had 6 talks delivered altogether and active discussions after some of the talks. Most of the talks focused on regularity and long time behavior of mascroscopic hydrodynamic models of incompressible fluids which also play a central role for multi-scale modeling of self-organization of suspended active agents. The last talk featured a very recent groundbreaking development in regularity of solutions to the incompressible 3D Navier-Stokes system. This talk invoked active discussion afterwards.

Yao Yao: *Small scale formation for the 2D Boussinesq equation.*

In this talk, we consider the 2D incompressible Boussinesq equation without thermal diffusion, and aim to construct rigorous examples of small scale formations as time goes to infinity. In the viscous case, we construct examples of global-in-time smooth solutions where the H^1 norm of density grows to infinity algebraically in time. For the inviscid equation in the strip, we construct examples whose vorticity grows at least like t^3 and gradient of density grows at least like t^2 during the existence of a smooth solution. These growth results work for a broad class of initial data, where we only require certain symmetry and sign conditions. As an application, we also construct solutions to the 3D axisymmetric Euler equation whose velocity has infinite-in-time growth. This is a joint work with Alexander Kiselev and Jaemin Park.

Daniel Lear: *Time periodic solutions near shear flows for 2D Euler.*

In this talk we will consider the existence of time periodic solutions arbitrarily close to shear flows for the 2D incompressible Euler equations. In particular, we will present some results concerning the existence of such solutions near Couette, Taylor-Couette and Poiseuille flows. In the first part of the talk, we will introduce the problem and review some well-known results on this subject. In the second we will outline some of the ideas underlying the construction of our time periodic solutions. Joint work with Angel Castro.

Changhui Tan: *On the well-posedness of the Euler-alignment system.*

In this talk, I will provide an overview of recent advancements in the field of local and global well-posedness theories concerning the compressible Euler system with singular velocity alignment. This system, commonly referred to as the Euler-alignment system, serves as a mathematical model for studying the collective behavior of animal flocks. My discussion will primarily focus on two specific scenarios: (1) the establishment of global well-posedness for small rough initial data that reside within critical Besov spaces, and (2) the achievement of global well-posedness for large smooth initial data in suitable multi-dimensional settings.

Joonhyun La: *Singular coherent structures in 2D Euler equation and hydrodynamic limits toward them.*

In this talk, we will see that certain singular coherent structures in 2D Euler equation propagates. Also, we will see that these solutions can be approximated by solutions of Boltzmann equation. The talk is based on a joint work with Theodore Drivas and Tarek Elgindi, and a joint work with Chanwoo Kim.

Mikhail Perepelitsa: *Kinetic modeling of Myxobacteria motion with nematic alignment*

Motivated by motion of myxobacteria, we review several kinetic approaches for modeling motion of self-propelled, interacting rods. We will focus on collisional models of Boltzmann type and discuss the derivation of the governing equations, the range of their validity, and present some analytical and numerical results. We will show that collisional models have natural connection to classical mean-field models of nematic alignment.

Hussain Ibdah: *Bypassing Holder super-criticality barriers in viscous, incompressible fluids*

We will go over the main ideas used in showing that as long as a super-critical Holder semi norm of the classical solution to the incompressible Navier-Stokes system is under control, the solution remains smooth. The key idea is exploiting an enhanced regularity effect coming from the transport term at the level of a simple one-dimensional drift-diffusion equation, allowing us to break the criticality barrier. We then employ ideas introduced by Kiselev, Nazarov, Volberg, and Shterenberg to propagate this to abstract drift-diffusion equations, providing to our knowledge the very first reasonable extension of the celebrated parabolic regularity result of Nash to an equation that is not in divergence form. Such an approach coupled with subtle pressure estimates due to Silvestre also allows us to rigorously treat the incompressible Navier-Stokes as a perturbation of drift-diffusion, obtaining a super-critical regularity criterion.

Wednesday's schedule was half-day talks / half-day free time. We heard two excellent talks on recent results in regularity theory of systems governing chemotactic diffusion and macroscopic system of alignment dynamics.

Alexander Kiselev: *Suppression of chemotactic blow up by buoyancy*

Chemotactic blow up in the context of the Patlak-Keller-Segel equation is an extensively studied phenomenon. In recent years, it has been shown that the presence of fluid advection can arrest singularity formation given that the fluid flow possesses mixing or diffusion enhancing properties and its amplitude is sufficiently strong - an effect that is conjectured to hold for more general classes of nonlinear PDE. In this talk, I will discuss Patlak-Keller-Segel equation coupled with fluid flow obeying Darcy's law via buoyancy force. It turns out that in this case, the singularity formation is suppressed by arbitrarily weak coupling. The talk is based on joint work with Zhongtian Hu and Yao Yao.

Trevor Leslie: *Limiting configurations for solutions to the 1D Euler Alignment System*

> The Euler Alignment system is a hydrodynamic PDE version of the celebrated Cucker-Smale ODE's of collective behavior. Together with Changhui Tan (University of South Carolina), we developed a theory of weak solutions in 1D, which provide a uniquely determined way to evolve the dynamics after a blowup. Inspired by Brenier and Grenier's work on the pressureless Euler equations, we show that the dynamics of interest are captured by a nonlocal scalar balance law, the unique entropy solution of which we generate through a discretization involving the "sticky particle Cucker-Smale" system. In this talk, we will discuss the formation of clusters of mass in the Euler Alignment system, and we will describe how to predict these clusters using the flux from the associated scalar balance law.

On Thursday we had a very diverse program in terms of subjects discussed in the talks. The day featured a talk on implementation of machine learning to improving privacy in rapidly spreading applications of Artificial Intelligence. Alignment modeling, another take on chemotaxis-fluid systems, hydrodynamic limit of kinetic models, mass distribution of solution to variational problems, are the topics featured on this day. Interesting connections were found between these subjects. For example, the relative entropy method discussed by Aneta Wroblewska-Kaminska can potentially be implemented to handle more general system with friction/self-propulsion forces. The talk of Siming He presented a result on enhanced dissipation rates in a chemotaxis system that may be implemented to obtain optimal relaxation rate in a kinetic model of collective behavior based on Cucker-Smale communication protocol. Talk of Choi presented a very clever attempt to resolve the long standing problem of justifying kinetic Vlasov-Alignment model with singular communication.

Nicolas Garcia Trillos: *Adversarial machine learning, clustered federated learning, and how the analysis of particle dynamics can help implementing them.*

> Machine learning and its applications in AI have entered into a new stage in their development: while the use of AI algorithms is widespread and will continue expanding, it is imperative to ask how can we guarantee that as these algorithms penetrate into more domains of our lives they will also be sensitive to privacy concerns, make fair decisions, and be both reliable and robust to data corruption. Are we ready to certify when a given algorithm complies with specific requirements and behaves in the way it is intended to?
>
> In this talk, I will discuss adversarial machine learning in supervised learning and clustered federated learning, two examples of ML settings where model accuracy is not the sole criterion for training learning systems. I will present novel approaches for the training of models in these two settings that rely on the use of particle dynamics and their analysis. Our solution to the first problem is inspired by the literature of gradient flows in the space of probability measures under the Wasserstein-Fisher-Rao geometry, and our solution to the second problem is inspired by the literature of consensus-based optimization. With this talk I hope to convey the multiple opportunities for mathematicians to participate in the conversation about pressing societal questions in the development of AI.

Ruiwen Shu: *Interaction energy minimizers on bounded domains*

> I will discuss the behavior of interaction energy minimizers on bounded domains. When the interaction potential is more singular than Newtonian, the mass does not tend to concentrate on the boundary; when it is Newtonian or less singular, the mass necessarily concentrates on the boundary for purely repulsive potentials. We also draw a connection between bounded-domain minimizers and whole-space minimizers.

Aneta Wroblewska-Kaminska: *Relative entropy method for hydrodynamic models.*

> We show that weak solutions of degenerate compressible Navier-Stokes equations converge to the strong solutions of the pressureless Euler system with linear drag term, Newtonian repulsion and quadratic confinement. The proof is based on the relative entropy method using the artificial velocity formulation for the one-dimensional Navier-Stokes system. The result is based on the joint work with Jose A. Carrillo and Ewelina Zatorska.

Moreover we will shortly discuss how to obtain general nonlinear aggregation-diffusion models, including Keller-Segel type models with nonlinear diffusions, as relaxations from nonlocal compressible Euler type hydrodynamic systems via the relative entropy method. This result is based on the joint work with Jose A. Carrillo and Yinping Peng.

Siming He: *Enhanced dissipation and blow-up suppression in a chemotaxis-fluid system*

In this talk, we will present a coupled Patlak-Keller-Segel-Navier-Stokes (PKS-NS) system that models chemotaxis phenomena in the fluid. The system exhibits critical threshold phenomena. For example, if the total population of the cell density is less than 8π, then the solutions exist globally in time. Moreover, finite time blowup solutions exist if this population constraint is violated. We further show that globally regular solutions with arbitrary large cell populations exist. The primary blowup suppression mechanism is the shear flow mixing induced enhanced dissipation phenomena.

Jan Peszek: *Singular alignment dynamics*

I will present the latest results and ideas related to the micro- to meso- and macroscopic limit for singular alignment dynamics. This includes the heterogeneous gradient flows related to weakly singular alignment (joint with David Poyato, University of Granada) with matrix valued communication, and a monokineticity estimate for strongly singular alignment (joint with Michal Fabisiak, University of Warsaw). In particular, I will show that any weakly continuous solution to strongly singular Cucker-Smale kinetic equation is monokinetic. This information can be used to obtain (via direct micro- to macroscopic mean-field limit) existence of measure-valued solutions to the fractional Euler-alignment system in the whole space for general initial data admitting vacuum.

Young-Pil Choi: *ON THE EXISTENCE OF SOLUTIONS FOR THE VLASOV-ALIGNMENT MODEL WITH SINGULAR COMMUNICATION WEIGHTS*

In this talk, we discuss the existence theory for the Vlasov-alignment model with singular communication weights $\Phi(r) = r^{-\gamma}$ In the case $\gamma \in (0, d)$, we show the local-in-time existence of weak solutions, and the unique- ness is obtained for $\gamma \in (0, d-1]$. We also consider the hypersingular com- munication weight, where $\gamma \in (d, d+1/4)$, and establish the local-in-time well-posedness for the Vlasov-alignment model .

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The the last day, since many speakers had to leave early, we schedule only one morning talk via Zoom. Nonetheless, the talk was very informative as the model discussed is involved in many systems of collective behavior, and is attracted interest of many participants.

Ewelina Zatorska: *Analysis of traffic and collective behaviour models in 1D*

I will present our recent results on the 1-dimensional hydrodynamic models of traffic, lubrication and collective behavior in 1 dimension. I will discuss existence results, interesting two-velocity reformulations, singular limits (hard congestion) and long time behavior of solutions.

Conclusion

As organizers we feel like this workshop was a successful experiment, one of the first attempted, to bring together two disconnected groups of researchers from the fields of collective behavior and fluid dynamics. It also provided a platform for many young mathematicians to present their talks and connect with more senior participants. The workshop resulted in active collaboration and exchange of knowledge between the groups. New prospectives were found on different problems and new collaborations were initiated during the event.

This workshop will set an example for other similar meetings we will be planing in the near future.

Participants

Choi, Young-Pil (Yonsei University)
Craig, Katy (University of California, Santa Barbara)
Drivas, Theodore (Stony Brook University)
Garcia Trillos, Nicolas (University of Wisconsin-Madison)
He, Siming (University of South Carolina)
Ibdah, Hussain (University of Maryland)
Jabin, Pierre-Emmanuel (Pennsylvania State University)
Ji, Hangjie (North Carolina State University)
Kiselev, Alexander (Duke University)
La, Joonhyun (Imperial College London)
Lear, Daniel (University of Cantabria)
Lee, Yongki (Georgia Southern University)
Leslie, Trevor (Illinois Institute of Technology)
Perepelitsa, Mikhail (University of Houston)
Peszek, Jan (University of Warsaw)
Rodriguez, Nancy (University of Colorado Boulder)
Shu, Ruiwen (University of Georgia)
Shvydkoy, Roman (University of Illinois at Chicago)
Tadmor, Eitan (University of Maryland)
Tan, Changhui (University of South Carolina)
Weber, Franziska (Carnegie Mellon University)
Wróblewska-Kamińska, Aneta (Institute of Mathematics, Polish Academy of Sciences)
Yao, Yao (National University of Singapore)
Zatorska, Ewelina (Imperial College London)

Chapter 22

Curves: Algebraic, Tropical, and Logarithmic (23w5101)

August 6 - 11, 2023

Organizer(s): Yoav Len (University of St Andrews), Hannah Markwig (University of Tübingen), Dhruv Ranganathan (University of Cambridge)

Overview of the Field

The last decade has seen significant progress in the geometry of algebraic curves, their Jacobians, and their moduli. Highlights include long-sought progress on Brill–Noether theory for special curves [13, 18], the determination of the Kodaira dimension of M_{22} and M_{23} [7], a resolution of the maximal rank conjecture [14], a definitive understanding of universal double ramification cycles, and the development of the logarithmic Picard group. These achievements have come from researchers in different fields: logarithmic geometry, tropical geometry, and limit linear series, whose overarching theme revolves around degenerations of algebraic varieties.

Limit linear series are a collection of techniques developed by Eisenbud and Harris [6] to study the manner in which linear series degenerate as algebraic curves degenerate. A number of fundamental results in algebraic curve theory have been proved in this fashion, chief among them the celebrated Brill–Noether theorem. Limit linear series continue to be a central tool in the study of curves. However, despite the success of limit linear series in studying general curves, there is significant interest in studying special loci in the moduli space of curves. For example, such a theory would be crucial in constructing Brill–Noether varieties over the compact moduli space of stable curves, which in turn constrains the topology and geometry of M_g .

Tropical geometry encodes limits of linear series by discrete data structures on the dual graph of a degeneration and is well-suited to study maximal degenerations of curves. The theory is powerful, and has led to new results in Brill–Noether theory for special curves [17], Prym–Brill–Noether theory [5], and the determination of the Kodaira dimension of M_g [12]. The theory predicts a number of consequences pertaining to dimensions of Prym– Brill–Noether loci and to semistable reduction for Brill–Noether varieties, providing new directions for the classical theory. Its precise relation to limit linear series is still poorly understood.

Logarithmic geometry is a powerful set of techniques that deal with degenerate objects in algebraic geometry, and has a rich interaction with tropical geometry [1]. Following the recent development of the logarithmic Picard group, it is expected that both classical and tropical limit linear series are specializations of a single object in logarithmic geometry [16]. Additional evidence is provided by recent connections between logarithmic enumerative geometry and mirror symmetry [9], and the role of logarithmic geometry in the study of the double ramification cycle [10]. In a different vein, Brill–Noether constructions give rise to geometrically significant cycles on the moduli space of curves M_g, parameterizing curves that admit certain special linear series. A study of the extension of these constructions to the boundary of M_g is a natural and important open problem and is likely to shed additional light on the cohomology of the moduli space of curves. As evidence, we note that a special case of these constructions includes the double ramification cycles discussed above, which determine nearly all that is currently known about the tautological ring of $M_{g,n}$. These extension problems are typically well-served by logarithmic geometry methods, but it demands the development of a theory of logarithmic linear series.

These concrete questions, and others like it, provide fertile ground for cross-pollination of ideas, where experts from all three areas will have different perspectives. The tropical theory has been largely driven by applications to the geometry of curves, while logarithmic geometry has followed a ground-up conceptual view. By initiating a dialogue between these communities along with experts in the classical theory, we aimed to develop a shared understanding and appreciation of the recent breakthroughs and push for a new wave of developments in those areas.

Presentation Highlights

In order to give the participants a common ground for discussions, the first three talks gave an overarching view of the three main themes in the workshop: the classical, tropical, and logarithmic aspects of the geometry of curves. They were given by **Renzo Cavalieri**, **Nathan Pflueger**, and **Isabel Vogt**.

The introductory talks were pitched broadly, but included recent research breakthroughs as well. For example, Cavalieri explained how logarithmic intersection theory gave rise to a new and natural perspective on Hurwitz theory, as well as natural generalizations of it; Pflueger overviewed the basic techniques in the theory of tropical linear series, and also gave the first new construction of a family of Brill–Noether general tropical curves in a decade; Vogt introduced different classical perspectives on Brill–Noether theory, but presented a striking new proof of the Brill–Noether theorem based on stable map and deformation theoretic techniques, well-adapted to generalizations.

The first full-fledged research talk was given by **Diane Maclagan**, who spoke on the topic of toric and tropical vector bundles. The topic turned out to have a number of topics touched upon in talks and discussions later in the conference.

The Tuesday began with the talk of **Eric Larson** who discussed joint work with Isabel Vogt solving the famous interpolation problem for Brill–Noether curves. The solution brings to an end a long line of inquiry, which Larson and Vogt have both been involved in. The techniques involved degeneration and the study of vector bundles on singular curves, and was a strong fit for the themes of the conference.

In the second Tuesday talk was given by **Jonathan Wise**, who gave an introduction to his forthcoming work with Battistella–Carocci on moduli of linear series, beyond the compact type case, using logarithmic geometry. Again, vector bundles on degenerate objects (in this case, Artin fans of curves) played a crucial role. The talk would turn out to be closely related to the one given by Payne later in the conference.

Later on Tuesday **Navid Nabijou** spoke on the relationship between logarithmic and orbifold Gromov–Witten theory, reporting on his recent joint work with Battistella–Ranganathan. The work tied into Cavalieri's introductory lecture on the first day. The final full talk on Tuesday was given by Gavril Farkas, who reported on his recent work with Jensen–Payne, solving several problems concerning the Kodaira dimensions of moduli spaces, using tropical geometry. Farkas raised non-abelian Brill–Noether theory as an avenue for exploration through tropical geometry, where many fundamental open problems remain.

In addition to these, there were four talk by junior participants on Tuesday: **Shiyue Li, Siddarth Kannan, Patrick Kennedy-Hunt, and Alheydis Geiger**. These covered a range of topics, spanning tropical geometry, logarithmic geometry, matroids, and the moduli space of curves. It gave these early participants an opportunity to introduce themselves to the audience and served as a starting point for conversations later in the week.

The first talk on Wednesday was given by **Felix Röhrle** on the topology of tropical moduli spaces of p-cyclic covers. The line of inquiry is a natural and rich continuation of work of Chan–Galatius–Payne on the topology of spaces of tropical curves, in turn leading to several new results on the cohomology of the moduli space of curves.

Later on Wednesday, **Samouil Molcho** discussed logarithmic intersection theory in the context of the universal

Picard variety, and especially his joint work with Abreu and Pagani, who were both in attendance. Molcho's talk highlighted the new input of logarithmic geometry in the study of compactified Picard schemes. The latter has been a rich line of inquiry in the study of curves in the 20th century.

The final two talks on Wednesday concerned the arithmetic of curves and their Jacobians, and were given by **Padmavathi Srinivasan and Farbod Shokrieh**. Srinivasan's talk concerned recent progress in non-abelian Chabauty-Kim methods concerning rational points on curves, where the work of Litt–Katz on iterated p-adic integrals plays an interesting role. Shokrieh's talk concerned the theory of heighs on abelian varieties, and his work with R. de Jong using electrical network theory and tropical geometry to give a new explanation of the difference between different heights.

Wednesday also hosted a problem session, chaired by **Dan Abramovich**. Several interesting problems were put forward by experts, and these stimulated significant further discussion. A number of researchers put forward problem, ranging from concrete questions to inspiring speculations. For example, **Isabel Vogt** proposed interesting and fundamental questions about the classical curve theory, while **Dhruv Ranganathan** suggested a systematic study of the cohomology of the space of parameterized curves in projective space. Several smaller groups of researchers informally continued discussions after the session, late into the evening.

There were two talks each on Thursday and Friday. On Thursday, **Alex Abreu** discussed additional new results on the intersection theory on universal Picard varieties. Specifically, he reported on his recent work with Pagani, giving significant new insight into the structure of Brill–Noether classes on the universal Jacobian. The talk tied in with those of Cavalieri and Molcho earlier in the week. The second talk on Thursday was given by **Sam Payne**. He reported further aspects of his work with Farkas–Jensen, which Farkas had already spoken about. While Farkas's talk explained the context for the results, Payne's talk gave a detailed account of how calculations are actually done in their work, how tropical geometry enters the picture, and how the resulting tropical problems can be solved. The remainder of Thursday was taken up by the hike and evening discussions.

The final two talks were given by **Margarida Melo** and by **David Holmes**. Melo spoke about joint work with Molcho, Ulirsch, Viviani, and Wise on tropical universal Jacobians. This gave a complete account of the combinatorics lurking behind the universal logarithmic Jacobian, the compactified Jacobians, and the relationship between them. The talk nicely complemented those of Abreu and Molcho. Holmes discussed gluing maps for curves in tropical geometry, especially in the context surrounding logarithmic enhancements of CohFTs and punctured logarithmic curves due to Abramovich–Chen–Gross–Siebert.

Recent Developments and Open Problems

In the last five years, tropical geometry has matured significantly as a research area. The period from 2010–2015 saw the subject produce elegant new combinatorial proofs of results in classical geometry, including the Brill–Noether and Gieseker–Petri theorems [4, 11] and the enumerative geometry of plane curves [8]. However, recent contributions have gone far beyond what was previous known. These include developments in Hurwitz–Brill–Noether theory [13, 3], the determination of the Kodaira dimension of M_{22} and M_{23} [7], and advances in the cycle theory of Prym varieties [5]. Each of these developments have been anchored by theoretical advances from logarithmic geometry and Berkovich geometry. There has been remarkable contemporaneous progress in the classical side. For example, H. Larson developed a systematic study of degeneracy loci on $\mathbb{P}^1$-bundles [15] in order to address questions raised in Jensen and Ranganathan's work in Hurwitz–Brill–Noether theory, and E. Larson used Hilbert scheme and stable map techniques to resolve the maximal rank conjecture [14]. In another example, the search for a robust formalism for tropical Jacobians has led to the development of the logarithmic Picard group and related developments on the double ramification cycle [16, 10]

There are many overlapping themes and analogies among the breakthroughs on the two sides, and what appears to be missing is a bridge linking them together. For instance, there is no understanding of the relationship between the algebraic and tropical Riemann–Roch theorem [2], and the situation is similar for the tropical notion of the rank of a divisor. In light of recent developments in the tropical, logarithmic, and classical study of curves, it seems particularly timely to combine the expertise across these research communities.

Outcome of the Meeting and Scientific Progress Made

As mentioned earlier in this document, the conference brought together experts from three distinct communites, linked by their common interest in the theory of curves – logarithmic, classical, and tropical geometry. These communities each have expertise that promised to benefit the others, and we believe this promise was fulfilled. For example, the formalism and framework of logarithmic geometry is well-developed and gives a robust connection between tropical and algebraic geometry. At the same time, the expertise of tropical geometers in combinatorial aspects of the subject have helped clarify and direct developments in logarithmic geometry. Both these subjects have in turn, given new input to the classical theory of curves, as for example advanced in the talks of Farkas and Payne.

The lectures described in the previous section were well-received, and the introductory talks on the first day created a common language for the rest of the conference. The time between and after the lectures gave time for genuine discussions, and several researchers mentioned important conversations. There were also intangible benefits – community building and community linking, between these three groups. For example, strong connections appear to have been discovered between work of Battistella–Carocci–Wise and Jensen–Payne that will be explored at a future workshop, that may not have been discovered for a long time as the relevant papers are not yet public. Similarly, Kennedy-Hunt, one of the early career participants, had the chance to discuss the results of his PhD, on logarithmic moduli theory, with Jonathan Wise, who is one of world experts in this area.

The conference also included a mentorship activity. The participants were broken into groups of 4-5, with each group including a range in levels of seniority. The groups were encouraged to discuss some aspects of the profession. The groups were given a list of potential topics to discuss, such as what makes good research problems, how to deal with rejection, and how to transition between different phases of one's career. The organizers received positive feedback on the activity, with several participants noting the freedom to talk more freely in smaller groups than they might have in a larger activity.

Testimonials

Sam Payne said "I found the combination of different perspectives on algebraic curves represented at the conference especially useful. It was illuminating to see the commonalities between the logarithmic Picard theory that Jonathan Wise et al. are developing and the notions of tropical linear series that I have been considering with Farkas and Jensen. The most likely possibility for a new collaboration would be something with Dhruv, using tropical linear series and connections to log geometry to study the Castelnuovo Problem on the existence of smooth projective space curves of a given degree and genus."

Navid Nabijou said "The conference was one of the best I've been to. The quality of the talks was very high, with a great mix of topics. For me personally it was very stimulating, and I've come away with a couple of nascent collaborations. The mentoring activity worked extremely well. "

Alheydis Geiger said "I really enjoyed the conference! Here a few comments from me: The reflection groups were a really good idea, they provided an excellent opportunity to discuss questions within different career stages and get new perspectives and experience reports from others. The way you had set up the groups was also very good - to me it felt that you took pains to mingle the people, so that one would speak with people one probably hadn't approached with similar topics before. That was really great. The organization was good and smooth- only the "get-together part" on the first evening could maybe have happened in a more "mingle-friendly" environment, but I understand that the evening had been planned differently originally."

Siddarth Kannan said "As an early career mathematician, the BIRS meeting on curves provided a valuable opportunity to discuss a broad range of topics in algebraic geometry with potential collaborators at various career stages. These discussions led to two concrete research directions that I hope to explore further during my postdoctoral years: the weight filtration on the cohomology of moduli spaces of maps from genus one curves to projective space, and the topology of logarithmic Hilbert schemes. I was also able to present my past research on moduli spaces of weighted stable curves, and this led to several additional interesting conversations with workshop participants."

Participants

Abramovich, Dan (Brown University)
Abreu, Alex (Federal Fluminense University)
Álvarez González, Ernesto (Universidad Complutense de Madrid)
Baker, Matthew (Georgia Tech)
Bozlee, Sebastian (Tufts University)
Cavalieri, Renzo (Colorado State University)
Chan, Melody (Brown University)
Christ, Karl (Leibnitz University)
Coles, Desmond (University of Texas)
Farkas, Gavril (Humboldt Universität zu Berlin)
Geiger, Alheydis (Max Planck Institute for Mathematics in the Sciences)
Grushevsky, Samuel * (Stony Brook University)
Harada, Megumi (McMaster University)
Holmes, David (University of Leiden)
Ilten, Nathan (Simon Fraser University)
Jensen, Dave (University of Kentucky)
Kannan, Siddarth (UCLA)
Katz, Eric (The Ohio State University)
Kennedy-Hunt, Patrick (University of Cambridge)
Larson, Eric (Brown University)
Len, Yoav (University of St Andrews)
Levinson, Jake (Simon Fraser University)
Li, Shiyue (Institute for Advanced Study)
Li, Amy (UT Austin)
Maclagan, Diane (University of Warwick)
Markwig, Hannah (Eberhard Karls Universität Tübingen)
Melo, Margarida (Università Roma Tre)
Molcho, Samouil (ETH Zurich)
Nabijou, Navid (Queen Mary University of London)
Ortega, Angela * (Humboldt University)
Pacini, Marco (Universidade Federal Fluminense)
Pagani, Nicola Tito (University of Liverpool)
Payne, Sam (UT Austin)
Pflueger, Nathan (Amherst College)
Poiret, Thibault (University of St Andrews)
Ranganathan, Dhruv (University of Cambridge)
Ritter, Caelan (University of Washington)
Röhrle, Felix (Frankfurt University)
Satriano, Matthew (University of Waterloo)
Schwarz, Rosa (University of Leiden)
Shaw, Kris (University of Oslo)
Shokrieh, Farbod (University of Washington)
Smith, Gregory G. (Queen's University)
Srinivasan, Padmavathi (Boston University)
Ulirsch, Martin (Goethe University Frankfurt am Main)
Viviani, Filippo (Roma Tre University)
Vogt, Isabel (Brown University)
Wise, Jonathan (University of Colorado)
Zakharov, Dmitry (Central Michigan University)

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Chapter 23

New Directions in Applied Linear Algebra (23w5004)

August 27 - September 1, 2023

Organizer(s): John Pearson (University of Edinburgh), Jennifer Pestana (University of Strathclyde), David Silvester (University of Manchester), Valeria Simoncini (Università di Bologna)

Overview of the Field

Linear algebra is a fundamental component of pure mathematics. It also lies at the heart of many scientific, engineering, and industrial applications. Research and development in numerical linear algebra includes theoretical studies, algorithmic implementations on advanced computer architectures, and applications to various disciplines. The research landscape is constantly changing with an array of new theoretical problems continually arising as a result of the rapidly developing application areas of data science and uncertainty quantification.

Recent Developments and Open Problems

The distinctive aspect of the workshop was the focus on general theoretical problems, that is, on the development of rigorous numerical analysis. It is well established that bespoke numerical linear algebra techniques, including the design of direct and iterative methods, are a vital component of solvers for huge-scale problems arising in optimization constrained by deterministic physical models (partial differential equations). What is much less well established is the design of efficient linear algebra components in the contexts of machine learning (where the underlying model is associated with large data) and in nonlocal network dynamics (where deterministic physical models are defined on lattices), making optimal control more challenging.

Uncertainty quantification (UQ) has emerged as an important area of scientific computing within the last decade. This development is motivated by engineering practice. In a UQ setting one would like to model uncertainty in the input data (for example, material properties) in a mathematically rigorous way and then propagate the effect of the uncertainty in a quantity of interest (for example, the reliability of a component under loading). Efficient numerical linear algebra plays a crucial role in the design of efficient algorithms that accomplish this task (so-called *forward* UQ) and has led to significant interest and research developments in low-rank approximation and tensor decompositions.

Programme Structure

The hybrid workshop had 26 in-person participants with an additional 16 participants attending online. The in-person participants included 8 females and 7 early-career researchers. The list of in-person participants included experts from several EU countries (Belgium, Croatia, Germany, Italy, Ireland, Norway) as well as from the UK, USA, and Canada. All in-person participants were given the opportunity to give a talk. In addition to the technical program we organised three informal after-dinner social events and a hike/ramble on the Tuesday afternoon.

The programme included four discussion sessions. Two of these sessions were on Wednesday afternoon. The first of these was organised by Jon Cockayne and focused on the development of probabilistic linear solvers, where solution accuracy is guaranteed in a statistical setting. The second discussion session on Wednesday focused on EDIA aspects. This session also featured specific issues raised by early-career researchers. It was noted that the requirement to be mobile is clearly challenging for any postdoctoral researcher in a two-person relationship. The other two discussion sessions were held on Thursday afternoon. The first of these was organised by Michael Mahoney and focused on the development of library software for randomized linear algebra. The final discussion session returned to the issues raised by the early-career researchers. This was a productive session in the sense that strategies were identified by more senior participants that might alleviate issues raised by the early-career researchers on the previous day.

Programme Specifics

The general themes that provided a structure for the discussions were

- Optimization & Machine Learning
- UQ & Related Linear Algebra Challenges
- Low-Rank & Randomized Linear Algebra
- Applications to Physics & Network Science

All the talks that were given can be categorised into one of these 4 topics. We built in considerable flexibility, however, and several talks spanned multiple themes.

A key driver of state-of-the-art applied linear algebra techniques arises from the fast and robust solution of problems from high-dimensional optimization and machine learning. There has been much cutting-edge research on randomization for interior point methods [1, 2], preconditioning for double-saddle point systems [3], numerical linear algebra for four-dimensional variational data assimilation (4D-Var) [4], low-rank methods for data-driven optimization [5, 6], convergence of Anderson acceleration [7], stabilization for reduced-order models [8] with application to PDE-constrained optimization, interior point methods for optimal transport [9], preconditioners for such problems [10], and scalable preconditioned iterative solvers for PDE-constrained optimization [11]. The main focus of the workshop on Monday and Wednesday was on recent developments in these subject areas, and we received presentations about these topics. Abstracts for the talks as well as additional references are provided below.

Petros Drineas: Randomized linear algebra for interior point methods
Linear programming is a central problem in computer science and applied mathematics with numerous applications across a wide range of domains, including machine learning and data science. Interior point methods (IPMs) are a common approach to solving linear programs with strong theoretical guarantees and solid empirical performance. The time complexity of these methods is dominated by the cost of solving a linear system of equations at each iteration. In common applications of linear programming, particularly in data science and scientific computing,

the size of this linear system can become prohibitively large, requiring the use of iterative solvers which provide an approximate solution to the linear system. Approximately solving the linear system at each iteration of an IPM invalidates common analyses of IPMs and the theoretical guarantees they provide. In this talk we will discuss how randomized linear algebra can be used to design and analyze theoretically and practically efficient IPMs when using approximate linear solvers [1, 2].

Chen Greif: Numerical solution of double saddle-point systems
Double saddle-point systems are drawing increasing attention in the past few years, due to the importance of associated applications and the challenge in developing efficient numerical solvers. In this talk we describe some of their numerical properties and make some distinctions between this family and the more well-established class of classical 2-by-2 block-structured saddle-point systems. We derive eigenvalue bounds, expressed in terms of extremal eigenvalues and singular values of block sub-matrices. The analysis includes bounds for preconditioned matrices based on block diagonal preconditioners using Schur complements, and it is shown that in this case the eigenvalues are clustered within a few intervals bounded away from zero. Analysis for approximations of Schur complements is included [3].

Jemima Tabeart: Block alpha-circulant preconditioners for all-at-once diffusion-based correlation operators
I will present practical methods to apply block α-circulant preconditioners approximately to an all-at-once system from ocean data assimilation. We apply the preconditioner via a nested iterative method within an outer Chebyshev semi-iteration. We present theory on the rate of convergence and use this to design an adapted approach which ensures each sub-problem is solved to the same tolerance. A panel of numerical experiments reveal that using an appropriate choice of α our new approaches are competitive in terms of outer iterations and matrix-vector products compared to current choices of preconditioner.

Madeleine Udell: Low rank approximation for faster optimization
Low rank structure is pervasive in real-world datasets. This talk shows how to accelerate the solution of fundamental computational problems, including eigenvalue decomposition, linear system solves, composite convex optimization, and stochastic optimization (including deep learning), by exploiting this low rank structure. We present a simple method based on randomized numerical linear algebra for efficiently computing approximate top eigendecompositions, which can be used to replace large matrices (such as Hessians and constraint matrices) with low rank surrogates that are faster to apply and invert. The resulting solvers for linear systems (NystromPCG), composite convex optimization (NysADMM), and deep learning (SketchySGD) demonstrate strong theoretical and numerical support, outperforming state-of-the-art methods in terms of speed and robustness to hyperparameters [5, 6].

Hans De Sterck: Asymptotic convergence speed of windowed Anderson acceleration: an overview of results and open problems
Anderson acceleration with window size m is a nonlinear convergence acceleration mechanism for fixed-point iterative methods that is widely used in scientific computing, optimization and machine learning. For many applications $AA(m)$ dramatically improves the convergence speed, both when iteration counts are small, and asymptotically for large iteration counts. Nevertheless, there are still no known results yet that can bound or quantify the improvement in asymptotic convergence speed provided by windowed $AA(m)$. In this talk I will give an overview of what is known about the asymptotic convergence speed of windowed $AA(m)$ and highlight some recent results and open problems [7].

Howard Elman: A new method of model order reduction for parametrized constrained partial differential equations
Model order reduction techniques are effective for solving parametrized models involving PDEs, including models of incompressible flow, where the constraint is the incompressibility constraint, and in optimal control, where the constraints themselves are PDEs. However, reduced models may fail to be inf-sup stable. We present a new approach for generating reduced bases in this scenario, using a so-called stacked reduced basis, which avoids some of the difficulties associated with inf-sup stability. We show that this approach is effective although in some circumstances it also requires stabilization, which can be done using either classic methods of penalization or through Petrov-Galerkin methods. Computational tests are presented for models based on PDE-constrained optimization and incompressible flow. This is joint work with Kayla D. Davie (University of Maryland, College Park).

Jacek Gondzio: Applications of interior point methods: From sparse approximations to discrete optimal transport

A variety of problems in modern applications of optimization require a selection of a ‘sparse’ solution, a vector with preferably few nonzero entries. Such problems may originate from very different applications in computational statistics, signal or image processing or compressed sensing, finance, machine learning and discrete optimal transport, to mention just a few. Sparse approximation problems are often solved with dedicated and highly specialised first-order methods of optimization. In this talk I will argue that these problems may be very efficiently solved by interior point methods. The key to their success is a design of suitable preconditioners [9].

Enrico Facca: Linear algebra challenges in optimal transport problems
Optimal Transport (OT) is a branch of optimization theory that has prompted a number of successful applications in different fields in the last few years. However, the wide-spread application of OT ideas in science and engineering is still fragmentary, due to the lack of accurate and efficient numerical solvers. We will explore some recent advances and open issues in the construction of efficient preconditioners involved in the solution of PDE-based formulations of OT [10].

Tucker Hartland: A scalable interior-point method for PDE and bound-constraint optimization
We present a scalable method for large-scale elliptic PDE and bound constrained optimization. Such problems are a means to learn unknown aspects of PDE-based models and are particularly challenging as their optimality systems are coupled PDEs and complementarity conditions. We utilize an interior-point Newton method with a filter-line search strategy and solve the Newton linear systems with a block Gauss-Seidel preconditioned Krylov-subspace method. We show that, in certain problem regimes, the number of Krylov-subspace iterations is independent of discretization and ill-conditioning from the bound constraints. We conclude with scaling results on an example problem using the MFEM and hypre packages [11].

The main focus of the workshop on Tuesday was on uncertainty quantification and associated linear algebra challenges. Abstracts for the talks as well as additional references are provided below.

Elisabeth Ullmann: Uncertainty quantification with PDE-based models
In this talk I will outline recent trends in uncertainty quantification (UQ) with PDE-based models and connect the trends to linear algebra. The topics in UQ are: (1) advanced random field models such as hierarchical random fields, (2) dynamical systems with random coefficients and their sensitivity analysis, (3) reliability analysis and rare events. The associated linear algebra topics are: matrix factorization, matrix eigenvalue problems, numerical continuation, and dimension reduction [12].

Christopher Müller: Iterative solvers for stochastic Galerkin discretizations of Stokes flow with random data
We study the iterative solution of linear systems of equations arising from stochastic Galerkin finite element discretizations of symmetric saddle point problems with stochastic data. As preconditioners we consider block matrices with building blocks that have Kronecker product structure. We derive bounds for the eigenvalues of the preconditioned system matrix and show how the existence requirements of a Bramble-Pasciak-type conjugate gradient method can be met. Based on numerical experiments for Stokes flow with random viscosity, we compare a MINRES solver with block diagonal preconditioner to a conjugate gradient method with block triangular preconditioner [13].

Aretha Teckentrup: Smoothed circulant embedding and applications in multilevel Monte-Carlo methods
Parameters in mathematical models for physical processes are often impossible to determine fully or accurately, and are hence subject to uncertainty. In this talk, we employ the multilevel Monte Carlo (MLMC) method to compute expected values of quantities of interest related to partial differential equations with random coefficients. We make use of the circulant embedding method for sampling from the coefficient, and devise a smoothing technique to further improve the computational complexity of the MLMC estimator [14].

Giang Tran: Sparse random features and applications in time series data
Random feature methods have been successful in various machine learning tasks, are easy to compute, and come with theoretical accuracy bounds. They serve as an alternative approach to standard neural networks since they can represent similar function spaces without a costly training phase. However, for accuracy, random feature methods require more measurements than trainable parameters, limiting their use for data-scarce applications or problems in scientific machine learning. In this talk, we will introduce the sparse random feature expansion to obtain parsimonious random feature models. Specifically, we leverage ideas from compressive sensing to generate random feature expansions with theoretical guarantees even in the data-scarce setting. We also present a sparse random mode decomposition to extract intrinsic modes from challenging time-series data and propose a new method to predict the infectious using sparse random feature models and time-delay equations. Applications of our methods on

identifying important variables in high-dimensional settings as well as on decomposing music pieces, visualizing black-hole mergers, and epidemiologic forecasting will be addressed [15].

Jon Cockayne: BayesCG: A probabilistic linear solve
Probabilistic linear solvers are iterative solvers for linear systems that return probability measures. These measures are designed to capture uncertainty in the solution due to incomplete computation. In this talk we review the probabilistic linear solver BayesCG, and discuss ongoing research into applications of the solver. Specifically, we introduce a new application of the solver to the problem of multiple related linear systems, and discuss its role as a means of providing both warm starting and subspace recycling [16].

The availability of huge amounts of data, especially in data science, has created new great challenges for numerical linear algebra: the need for compact though flexible data representations and reduced (towards dimension-independent) computational costs. Migration from deterministic to probabilistic regime has led to the successful development of *randomized* linear algebra algorithms, which allow the approximate solution of problems with dramatic memory savings [17, 18]. These approaches are combined with strategic data allocations, such as low rank matrices, tensor representations, and structure-aware (hierarchical) formulations. Throughout the workshop many presentations have addressed these challenges, by focusing both on the theoretical foundations underlying randomized strategies, as well as on the impact of this methodology in various problems in data science and scientific computing. Abstracts for the talks as well as additional references are provided below.

Misha Kilmer: Structured matrix approximations via tensor decompositions
We provide a computational framework for approximating a class of structured matrices; here, the term structure is very general. Our matrix-to-tensor invertible mapping allows us to pose the matrix-approximation problem as a tensor-approximation problem. Mapping the tensor approximant back to matrix space, we obtain a structured matrix approximation that can be expressed as a sum of structured Kronecker products or written in block low-rank form. We illustrate the ability of our method to uncover block-low-rank format from applications such as system identification and show we can uncover sum of structured Kronecker products structure on several matrices from the SuiteSparse collection [19].

Martina Iannacito: An algebraic algorithm for blind source separation and tensor decomposition
The blind source separation problem is a classical signal processing problem. Mathematically it translates into the factorization of S, an observed signal matrix, into a mixing and a source signal matrix, i.e., $X = MS^T$. In [Domanov, 2016], the authors propose a criteria checking list to guarantee the decomposition generic uniqueness for a deterministic method class. From this and [De Lathauwer, 2006; Domanov, 2014] results, we design an algebraic algorithm to retrieve the unique factorization of X. Moreover, this algorithm finds application in the canonical polyadic tensor decomposition. The structure of the objects involved in the algorithm allows an efficient implementation. This is joint work with Ignat Domanov and Lieven De Lathauwer.

Zvonimir Bujanović: Randomized algorithms with rank-one random vectors
Randomized algorithms in numerical linear algebra typically generate random vectors from standard distributions, such as Gaussian. However, in certain applications it may be advantageous to run computations with vectors compatible with the underlying structure of the problem. In this talk we discuss algorithms that use rank-one vectors, i.e., Kronecker products of two random vectors, which may allow for faster operations with matrices represented as short sums of Kronecker products. Possible applications include trace estimation and large-scale eigenvalue computation; we provide theoretical and numerical evidence that the use of rank-one instead of unstructured random vectors still leads to good estimates [20].

Davide Palitta: Sketched and truncated polynomial Krylov subspace methods
Sketching can be seen as a random dimensionality reduction able to preserving the main features of the original problem with probabilistic confidence. Sketching is one of the most promising tools to boost numerical computations although its understanding is still limited. In this talk we present the main features of sketching and how this can be combined with Krylov methods for the solution of large-scale linear systems. However, thanks to the novel sketched Arnoldi relation we will illustrate, the results discussed in this talk can be extended to the numerical evaluation of matrix functions and the solution of matrix equations [21].

Lars Grasedyck: Challenges in Hierarchical Low Rank Tensor Approximation
The first part of the talk will provide a general introduction to low rank tensor formats for the data sparse representation of higher order tensors and multivariate functions (CP, Tucker, TT, HT). The second half of the talk is geared towards more recent results on multiparametric multigrid methods and challenges in approximation with regard to special distance measures (e.g. KL) and constraints on the set of admissible tensors (e.g. positivity) [22].

The focus of the workshop on Thursday afternoon and Friday morning was on applications to physics and network science. Abstracts for the talks as well as additional references are provided below.

David Silvester: Fast solution of incompressible flow problems with two-level pressure approximation
Reliable and efficient iterative solvers for models of steady incompressible flow emerged in the early 1990s. Strategies based on block preconditioning of the underlying matrix operators using (algebraic or geometric) multigrid components have proved to be the key to realising mesh independent convergence (and optimal complexity) without the need for tuning parameters, particularly in the context of classical mixed finite element approximation. The focus of this contribution is on efficient solver strategies in cases where (an inf-sup) stable Taylor-Hood mixed approximation is augmented by a piecewise constant pressure in order to guarantee local conservation of mass. The augmentation leads to over-specification of the pressure solution requiring a redesign of the established solver technology. This enrichment process causes over-specification of the pressure, which complicates the design and implementation of efficient solvers for the resulting linear systems. We first describe the impact of this choice of pressure space on the matrices involved. Next, we show how to recover effective solvers for Stokes problems, using a preconditioner based on the singular pressure mass matrix, and for Oseen systems arising from linearising the Navier-Stokes equations, by using a two-stage pressure convection-diffusion strategy [23]. This is joint work with Jennifer Pestana.

David Bindel: Linear algebra, invariant circles, and fusion plasmas
Stellarators are a class of non-axisymmetric toroidal magnetic confinement systems for fusion relevant plasmas. Confinement depends on nested magnetic flux surfaces that act as transport barriers. Flow along magnetic field lines has a Hamiltonian structure, and we generically expect such invariant surfaces to exist in stellarators by KAM theory. However, many devices exhibit not only nested flux surfaces, but also magnetic islands and chaotic regions. Physicists often explore this visually via Poincaré plots that show the structure of the monodromy map for magnetic field line flow around a device. However, to optimize devices we want fast and automatic methods to identify such structure. In this talk, we describe several such analysis methods, with an emphasis on new methods with roots in linear algebra, vector sequence extrapolation, and system identification. We illustrate our method with a variety of examples [24].

Igor Simunec: Computation of the von Neumann entropy of large matrices via trace estimators and rational Krylov method
We consider the problem of approximating the von Neumann entropy of a large and sparse symmetric positive semidefinite matrix A, defined as $\text{trace}(f(A))$ where $f(z) = -z\log(z)$, which is an important task in several fields, such as quantum information theory and network science. We consider the use of polynomial and rational Krylov subspace algorithms [25] within both randomized trace estimators [26] and probing methods based on graph colorings [27]. We develop error bounds and heuristics that are employed in the implementation of the algorithms. The performance of the methods is illustrated with several numerical experiments. The activities mentioned herein are performed in the framework of the project MUR_PNRR_PE_NQSTI_BELTRAM funded by the Italian Ministry for University and Research (MUR). A preprint is available on arXiv [28]. This is a joint work with Michele Benzi and Michele Rinelli.

Niall Madden: Linear solvers for singularly perturbed problems
We consider the solution of linear systems arising from *parameter robust* discretizations of singularly perturbed convection-diffusion problems, whose accuracy is proven to be independent of the perturbation parameter. There are many studies of such methods, but few of the robust solution of resulting linear systems. I'll discuss challenges for both direct and iterative solvers for such systems, and propose a multigrid-based preconditioning strategy tuned to the matrix structure induced by using layer-adapted meshes. Supporting theoretical and numerical results will be presented [29]. This is joint work with Anh Thài Nhan (SCU) and Scott MacLachlan (MUN).

Felix Kwok: Multigrid interpretation of a three-level Parareal algorithm
In this talk, we show how a recently introduced three-level variant of Parareal can be interpreted as a multigrid reduction in time (MGRIT) method for a particular choice of restriction/prolongation operators and relaxation scheme. Specifically, we show that when suitably initialized, the MGRIT V-cycle with F-relaxation and injection as the prolongation operator generates iterates that are identical to three-level Parareal. This equivalence allows one to analyze the convergence of three-level Parareal from two different perspectives, using either ODE or multigrid techniques [30]. This is joint work with Stephanie Friedhoff and Martin J. Gander.

Conclusion

The primary achievement of the workshop was that it enabled the goal of connecting expert researchers who would not interact otherwise. It also successfully enabled the informal interaction of early-career researchers with more senior experts: specifically by providing an opportunity for them to showcase their opinions and their research work in a non-intimidating atmosphere. This resulted in active collaboration and exchange of knowledge between the participants. New perspectives were found on a range of challenging problems and new collaborations were initiated. Follow-on workshops and activities are being planned and are likely to take place over the next two years.

Participants

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Chapter 24

Women in Geometry 3 (23w5062)

September 24 - 29, 2023

Organizer(s): Ghazal Geshnizjani (Perimeter Institute, University of Waterloo), Tracy Payne (Idaho State University), Raquel Perales (National Autonomous University of Mexico), Catherine Searle (Wichita State University)

Overview of the workshop

Women in Geometry 3 was a collaborative research workshop organized at Banff International Research Station (BIRS) that followed the model of the previous Women in Geometry 1 and 2 workshops. Specifically, the workshop hosted seven different collaborative research teams of five to seven women in the following areas: Geometric Flows, Riemannian Geometry, Mathematical General Relativity, Homogeneous Geometry, Symplectic Geometry, Spin Geometry and Special Geometric Structures, and Variational Problems in Geometry.

The WIG 3 workshop was timely and important because the overall representation of women in mathematics in general, and in geometry in particular, remains low. As of 2017-18, roughly 29% of all mathematics PhDs in the United States were awarded to women, while in the same year, women represented only 18% of the PhDs awarded in the United States in geometry/topology. Furthermore, at each successive academic career stage, the percentage of mathematicians who identify as female decreases. For example, as of 2017, the percentage of tenured math faculty members at PhD-granting institutions in the United States identifying as female was only 12%. (Statistics from the Mathematical and Statistical Sciences Annual Survey.) The WIG 3 workshop was designed to support, through research activity and mentorship, the retention and promotion of women at all stages of their mathematical careers.

The program structure was designed to maximize productive research exchanges during the five days of the workshop so that the discussions would lead to concrete outcomes such as published articles and future funding for ongoing research activity. The areas of geometry featured in the WIG 3 program were intentionally chosen because they are interrelated, thus ensuring strong potential for cross-collaboration. Furthermore, the program aimed to both strengthen the existing network of women mathematicians that had been established by the WIG 1 and WIG 2 workshops, and to extend the reach of these workshops to include more areas of geometry and more female geometers. The workshop had 42 participants, with 37 attending in person and 5 joining remotely. These women mathematicians came from different institutions, career stages, countries and continents, creating a vibrant and collaborative environment. A few months before the program, team leaders provided participants with a comprehensive overview of the research problems and essential background reading materials. Due to this advance preparation, it was possible to dedicate the majority of the workshop time to collaborative research. Seven plenary talks were scheduled over the first four mornings in the program, featuring one speaker from each group. These talks aimed to build a sense of community, and to foster discussion and collaboration within and between groups. Afternoons, with the exception of just one, were allocated to research collaboration. On the final day of the program, each team presented a 10-minute report to the full workshop, summarizing progress made and outlining future goals. This provided a comprehensive overview of the collective achievements and set the stage for continued collaboration. The program concluded with the remaining two hours devoted to research collaboration.

In addition to the scientific program, two insightful discussion sessions were conducted using different formats, enriching the overall workshop experience. The first event was a panel discussion titled "Collaboration in Mathematics," which took place on Tuesday evening. Both the moderators and the audience posed questions to a panel of 5 mathematicians, delving into the dynamics of collaboration, exploring effective collaboration models and offering guidance to junior researchers on initiating and maintaining healthy collaborations while avoiding common difficulties. The panel further addressed strategies to counteract power and gender dynamics within collaborative endeavors. The second discussion session on Thursday evening focused on community building. It began with breakout group discussions in which groups of 5 to 7 participants discussed specific topics in a more relaxed setting and shared experiences on topics such as how to build community, combat bias when starting new jobs, and mentor women in the early stages of their careers. The participants then all joined together to share highlights of earlier discussions, get further feedback from the larger group, and converse further.

Presentation Highlights

There were seven hour-long plenary talks scheduled on the first four mornings of the workshop: A senior member from each team gave a colloquium-style lecture related to their team's research area and the problem they were working on together. The talks were designed to be accessible to all WIG 3 participants. An important goal of the WIG workshops is for participants to not only come together to work with their own team, but also to learn more about the other participants' research areas and the important problems therein. WIG 3 aimed to establish strong working relationships within and between teams. These lectures reached beyond this to lay the groundwork for possible cross-team collaboration.

The plenary talks were as follows:

- Laura Starkston from UC Davis, representing the Symplectic Geometry group, spoke on *Symplectic and contact geometry: geometric properties, dynamics and invariants*.

- Meera Mainkar from Central Michigan University, representing the Homogeneous Geometry group, spoke on *Geometry of solvmanifolds*.

- Ruth Gregory from King's College London, representing the Mathematical General Relativity group, spoke on *Cosmic strings and spacetime singularities*.

- Ines Kath from Universität Greifswald (Germany), representing the Spin Geometry and Special Geometric Structures group, spoke on *Holonomy groups contained in the non-compact* G_2.

- Megan Kerr from Wellesley College, representing the Riemannian Geometry group, spoke on *Compact homogeneous Einstein manifolds*.

- Melanie Rupflin from Oxford University, representing the Variational Problems in Geometry group, spoke on *Quantitative estimates for geometric variational problems*.

- Julie Clutterbuck from Monash University, representing the Geometric Flows group, spoke on *The shape of solutions of elliptic problems in curved spaces*.

Scientific Progress Made: Reports from Research Groups

Report from Team 1 (Geometric Flows)

Group Members: Min Chen (McGill University), Julie Clutterbuck (Monash University), Yi Lai (Stanford University), Alina Stancu (Concordia University), and Valentina Wheeler (University of Wollongong), Mariel Sáez (P. Universidad Católica de Chile)
Research Synopsis and Progress: The most well-known curvature flow of hypersurfaces in Euclidean space is the flow by mean curvature (MCF). Without getting into technical details, mean curvature flow's equation is $X_t = -H\nu$ and describes an evolving hypersurface moving inward with a normal speed equal to its mean curvature. This flow has numerous features that have led to many applications.

Among the properties of the mean curvature flow, note that the flow admits a monotonicity formula proved by Huisken, first for compact hypersurfaces, then extended to noncompact hypersurfaces with certain decay at infinity. Huisken's monotonicity formula says that for any hypersurface Σ_t evolving by MCF the (n+1)-dimensional backward heat kernel ρ_{x_0,t_0}, centered at some point (x_0, t_0), decreases in time along Σ_t and stays constant on, and only on, the hypersurfaces that evolve self-similarly:

$$\int_{\Sigma_t} \rho_{x_0,t_0}(X,t)\, d\mu_t = -\int_{\Sigma_t} \left(H - \frac{X^\perp}{2t}\right)^2 \rho_{x_0,t_0}(X,t)\, d\mu_t.$$

Therefore, the monotonicity formula can be used to classify self-similar solutions of the mean curvature flow. In fact, more generally, as x_0 and t_0 are chosen as the time and position of a singularity of the evolving surface, the monotonicity formula is used to analyze the behavior of the surface as it evolves towards this singularity. Our project proposes to investigate the existence of monotonicity formulas for fourth order curvature flows, MCF being a second order parabolic partial differential equation for X or, roughly speaking, a heat equation. To start, we considered Willmore's flow which, without technical details, is $X_t = -\Delta_{\Sigma_t} H\nu$ and made preliminary calculations for non-compact hypersurfaces for which self-similar solutions exist (in the compact case there exist only stable solutions) with a bi-harmonic heat kernel as density. Further analysis is needed to conclude if the behavior of this convolution is monotone, possibly under a modified density. We intend to pursue this question during the coming months.

Report from Team 2 (Riemannian Geometry)

Group Members: Isabel Beach (University of Toronto), Erin Griffin (Northwestern University), Haydee Contreras Peruyero (UNAM Morelia), Megan Kerr (Wellesley College), Regina Rotman (University of Toronto), and Catherine Searle (Wichita State University)
Research Synopsis and Progress: The question of when a manifold M admits an Einstein metric, that is, a metric of constant Ricci curvature, is wide open. In dimensions greater than 4, there are no known obstructions, yet finding examples of Einstein metrics has historically been a piecemeal process. In the presence of symmetries, however, the search for examples has been more fruitful. Our group is working in the setting of compact homogeneous spaces, $M = G/H$, where $H \subset G$, and G and H are compact, connected Lie groups. A key fact is that Einstein metrics are critical points of the Hilbert action. Graev used this fact to prove that for such compact homogeneous spaces G/H, if the nerve $X_{G/H}$ is non-contractible, then G/H admits a G-invariant Einstein metric.

This result, an extension of an earlier result of Böhm, often provides quick data indicating that a space must admit an invariant Einstein metric. However, there are many compact homogeneous spaces that have a contractible nerve $X_{G/H}$, and yet admit a G-invariant Einstein metric. Our project is to address this gap. We consider the set of compact homogeneous spaces G/H for which the isotropy representation has 3 or more irreducible summands, contractible simplicial complex, and $\text{rank}(G) = \text{rank}(H)$, and denote this class of manifolds by $N_{0,\geq 3}$. We note that in the two-summand case, there is an algebraic marker, a discriminant, that tells us precisely when there are Einstein metrics. We expect that a better understanding of those spaces in $N_{0,\geq 3}$ which admit an Einstein metric will help us to identify an underlying algebraic marker for the existence of invariant Einstein metrics in this case.

At BIRS, we began work on the following problem.
Problem. Which of the manifolds in $N_{0,\geq 3}$ admit a G-invariant homogeneous Einstein metric? While at BIRS,

we worked to identify the full list of homogeneous spaces in $N_{0,\geq 3}$, for which there are no more than 4 isotropy summands, and we calculated the critical points of the scalar curvature functional for many examples in $N_{0,\geq 3}$. We have continued to have regular online meetings since BIRS to move our project forward. In the short term, we all met at the JMM in San Francisco in January 2023 to collaborate in person. We have also applied to several research centers in order to meet in person in the near future. We recently heard that our application to the IAS, Princeton Summer Collaborators program for summer 2024 was accepted.

Report from Team 3 (Mathematical General Relativity)

Group Members: Ghazal Geshnizjani ((Perimeter Institute and University of Waterloo), Melanie Graf (University of Hamburg), Ruth Gregory (King's College London), Sharmila Gunasekaran (Fields Institute), Christina Sormani (Lehman College, CUNYGC), and Ivonne Zavala (Swansea University)

Research Synopsis and Progress: Our team has been investigating the geometric properties of cosmic strings, gravitational anomalies resulting from symmetry-breaking processes in the early universe, such as phase transitions as the universe cools down. In the context of the simplest model of space-time, flat Minkowski space, which is a fusion of Euclidean space with time, endowed with a Lorentzian metric tensor, we consider a metric of the form:

$$-dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2.$$

The standard static straight cosmic string exhibits local flat Minkowski space characteristics. However, it possesses a conical singularity along the $z - t$ space-time plane, referred to as its world sheet, with a metric tensor given by:

$$-dt^2 + dz^2 + dr^2 + \alpha^2 r^2 d\theta^2,$$

where α is less than 1.

The understanding of other types of cosmic strings remains limited. Although they may adhere to the Nambu-Goto action, resolving the complete Einstein gravity around them is not straightforward due to the nonlinearities of the Einstein equations. Unlike in linearized Einstein gravity, defining curvature with a delta function value along the string is challenging. To grasp the geometry of these strings, understanding their conical singular geometry is crucial.

Our team is pursuing two approaches. First, we have developed a coordinate system for the cosmic string's neighborhood, showing the standard static cosmic string as the sole solution to the full Einstein equations on Minkowski spacetime with the removal of the z-axis. Next, we plan to explore more freedom by removing certain symmetries. Some cases we are considering are situations where we assume the metric tensor is the metric tensor of Minkowski space plus an extra $(1 - \alpha^2)d\theta^2$ term, where α is allowed to be a function depending on z.

Secondly, we have started to study initial data sets for cosmic strings using Einstein Constraint Equations. In the time symmetric setting, these are Riemannian manifolds with nonnegative scalar curvature or even the vacuum cases where the scalar curvature is 0. We can investigate various possible settings like $\mathbb{R}^3$ with a line removed or a circular ring. Again imposing symmetries along a line that has been removed, in the vacuum case, the only possibility satisfying certain limits is a cone crossed with a line. We are investigating the ring case using toroidal coordinates and imposing rotational symmetry along the ring. We will try similar ideas to those mentioned above for the spacetime setting in this simpler spacelike case. Once initial data sets are understood, it may be possible to solve the equations forward in time using numerical General Relativity or at least study the existence and properties without explicitly finding a solution.

Report from Team 4 (Homogeneous Geometry)

Group Members: Romina M. Arroyo (University of Cordoba, Argentina), Karen Butt (University of Chicago), Karla Garcíā (National Autonomous University of Mexico), Ruth Gornet (University of Texas–Arlington), Meera Mainkar (Central Michigan University), Tracy Payne (Idaho State University)

Research Synopsis and Progress: Homogeneous spaces with nonpositive sectional curvature are locally modelled on solvmanifolds. A solvmanifold is a simply connected solvable Lie group endowed with a left-invariant Riemannian metric. The general study of the geometry of solvmanifolds is quite difficult: the algebraic expressions for the curvature operator are complicated and messy, and solvable Lie groups do not have the nice structure that semisimple ones do. Research in this area tends to focus on studying nice families of solvmanifolds with special algebraic or geometric properties, or asking general qualitative questions.

Our group studies the sectional curvature and the eigenvalues of the curvature operator for nice families of negatively curved homogeneous spaces. We started meeting online before the workshop. During this time, we completed a literature review and typed up a set of preliminary calculations. On the first day of the workshop, we made a list of seven interrelated research questions about nonpositively curved solvmanifolds, and we decided to go through the list question by question, and feel out the difficulty of each problem, test out possible approaches to the problems, and discuss how the different strengths of group members were aligned with requirements of each problem. In some cases, discussion opened up new questions that we had to clarify by re-examining references and discussing.

As we discussed the potential problems, in parallel, we identified several classes of solvmanifolds of interest, which we formally defined, and we did preliminary geometric computations for these. Discussion during preliminary computations led us to decide on the right questions for us to work on. We removed four of the questions from consideration, keeping three interrelated questions to focus on.

We were able to completely answer our questions in the simplest class. In another class, which has more complicated geometry, we almost completed a full analysis of all of our questions. In doing these calculations, we discovered some general principles that we aim to frame as theorems as we progress.

On the last day, we reviewed our accomplishments from the week. We made plans for future collaboration and we decided on expectations for working together as a group going forward. After the workshop, we applied to AIM for an AIM Squares grant to meet in person at AIM to continue work on our project. We recently heard that our proposal was accepted.

Report from Team 5 (Symplectic Geometry)

Group Members: Aleksandra Marinković (Belgrade University), Jo Nelson (Rice University), Ana Rechtman (University of Grenoble), Laura Starkston (University of California–Davis), Shira Tanny (In- stitute for Advanced Study, Princeton), Luya Wang (Stanford University)

Research Synopsis and Progress: Symplectic and contact structures arose in the study of classical mechanical systems, leading to the development of global Floer theoretic invariants, which encode structural aspects of Hamiltonian and Reeb flows. Symplectic geometry also has key ties to subtle phenomena of smooth 4-manifolds. This has motivated the study of symplectic 4-manifolds and their contact 3-dimensional boundaries. A symplectic manifold is a smooth manifold which admits a closed nondegenerate differential 2-form, which measures signed area, forces the symplectic manifold to be even-dimensional, and whose n-fold wedge product is a volume form. A contact structure is a maximally nonintegrable hyperplane field, meaning it can be described as the kernel of a differential 1-form λ, which forces any contact manifold to be odd-dimensional and admit a volume form given by $\lambda \wedge (d\lambda)^n$. There is a close connection between symplectic and complex manifolds, as well as subtle interplay between the dynamics on the contact boundary of a symplectic manifold and the underlying topology of the latter. We studied the contact dynamics of 3-manifolds that are motivated by complex algebraic geometry and singularity theory. Complex curves and singular symplectic surfaces in symplectic 4-manifolds have standard symplectic neighborhoods, which under certain conditions have a convex or concave structure, inducing a contact form on the boundary 3-manifold. In some cases, these contact 3-manifolds are also links of surface singularities. We studied the dynamical properties of these contact manifolds and investigated connections to symplectic fillings and cappings. We worked on understanding the Floer theoretic invariant known as the order of algebraic torsion as defined using embedded contact homology, to obstruct the symplectic fillability of boundaries of certain concave plumbings of disk bundles over surfaces. Embedded contact homology is a three-dimensional gauge theory which is equivalent to monopole Seiberg-Witten Floer cohomology, but defined in terms of contact dynamics and holomorphic curves. Prior to our time at the Banff International Research Station, we had regular weekly online meetings, and had made significant progress on understanding the background on embedded contact homology and constructions of the Reeb vector field of boundaries of concave plumbings. During our time intensively working in person at BIRS, we were able to construct and sort out the details of a heuristic argument to compute the order of algebraic torsion in a model example. Through this model example, we developed key lemmas and arguments that we expect to need in more general examples. We are continuing to meet regularly and are writing up the details of the proof, which we are also working to generalize to all linear plumbings. In the linear plumbing case, we expect to recover results which could also be proven less directly using results of Lisca and McDuff, and possibly generate some unknown results. More generally, we will work with treelike plumbings, with the hope of establishing new classes of tight nonfillable contact manifolds, possibly with interesting values of algebraic torsion. We are currently investigating the additional tools we will need to handle these nonlinear plumbing examples. We are making plans to meet again in person this summer, for example, applying for the IAS Summer Collaborators program as well as utilizing our personal grants.

Report from Team 6 (Spin Geometry and Special Geometric Structures)

Group Members: Ilka Agricola (University of Marburg, Germany), Viviana Del Barco (State University of Campinas, Brazil), Jordan Hofmann (King's College, UK), Ines Kath (University of Greifswald, Germany), Marie-Amelie Lawn (Imperial College London, UK), Ana Cristina Ferreira (University of Minho)

Research Synopsis and Progress: The project of our team is part of the study of special geometries on pseudo-Riemannian manifolds. A special geometry exists if the holonomy group is a proper subgroup of the special orthogonal group. We studied pseudo-Riemannian manifolds of signature (4, 3) whose holonomy group is contained in the non-compact Lie group of type G2, which we denote by G^*2. The Lie algebras of the possible holonomy groups were classified by Fino and Kath [3, 4]. All corresponding simply-connected Lie groups can indeed be realized as the holonomy group of a metric [6, 7], but only some of them by a left-invariant metric. Since up to now there exist only sporadic examples of such left-invariant metrics, our aim is to start a systematic search for left-invariant, or more generally, homogeneous metrics with holonomy contained in G^*2. Two different approaches were tried in parallel:

Strategy I: In order to get some intuition for which Lie groups can occur, we wrote an appropriate Maple program which checks holonomy groups of left-invariant metrics that have a 3-dimensional socle (Type III holonomies). In this way we already found new examples of such metrics. Moreover, it led us to a conjecture how we can characterize the holonomy groups of Type III that can be realized by a left-invariant metric by the holonomy representation on the socle. We will continue this project and try to prove our conjecture.

Strategy II: Our second approach was to inspect all existing constructions for Riemannian holonomy groups inside $G2$ and to investigate which methods could be transferred to our particular pseudo-Riemannian case. It turned out that in the end, only cone constructions over 6-dimensional nearly pseudo-Kähler manifolds can be considered suitable, and hence promising [2, 5]. Of these, very little is understood; since we must exclude pseudo-Riemannian analogues of a Riemannian 'sibling' as they do not yield manifolds with holonomy representation of the type we are looking for, we are left with only one known candidate. This manifold is a solvable Lie group with a left invariant nearly pseudo-Kähler structure constructed by Alekseevsky, Kruglikov, and Winther in [1]. Although very interesting, this particular example has sectional curvature -1 and therefore it follows that the cone over it has vanishing holonomy. It was decided that it would be an interesting, but separate, project to try to construct further examples of nearly pseudo-Kähler manifolds.

Report from Team 7 (Variational Problems in Geometry)

Group Members: Theodora Bourni (University of Tennessee-Knoxville), Elena Mäder-Baumdicker (Darm- stadt University), Robin Neumayer (Carnegie Melon University), Jiewon Park (Korea Advanced Institute of Science and Technology), Raquel Perales (National Autonomous University of Mexico (UNAM) at Oaxaca), Melanie Rupflin (Oxford University)

Research Synopsis and Progress: We worked on the following problem: Asymptotic behaviour of free boundary area preserving curve shortening flow.

Curve shortening flow is the one-dimensional analogue of mean curvature flow, which is the gradient flow of the area. Area preserving curve shortening was introduced by Gage, who described it as the L^2-gradient flow of the length under an enclosed area constraint. With the term 'free boundary' we mean that the evolving curves have a boundary and we require that the evolving curves intersect a given support curve orthogonally at their boundary. In our setting, we consider a bounded convex domain $\Omega \subset \mathbb{R}^2$ and study area preserving curve shortening flow in the exterior domain $\mathbb{R}^2 \setminus \Omega$ with free boundary on $\partial\Omega$. This problem has been extensively studied by one of the members in our group, Elena Mäder-Baumdicker, who among other things, proved long time existence under natural geometric hypotheses. In particular, starting with an embedded convex curve that satisfies a certain natural length bound, the flow exists for all time and it subconverges to an arc of a circle.

Given a gradient flow whose solution exists for all times, a natural question that arises is whether one can expect that it converges to a unique limit, as $t \to \infty$. In general, solutions might exhibit winding behavior where the flow approaches a continuous set of critical points, and do so in a way that the total distance travelled is infinite, which could lead to different limits via different subsequences $t_j \to \infty$. In our project we investigate this and related questions concerning the structure of the set of critical points for the free boundary area preserving curve shortening flow.

During the time at BIRS we have already made progress on a model problem and have now started to have regular online meetings to take the next steps on our project. We have also discussed several possible opportunities to meet in person in the near future. We plan to manage this by meeting in smaller teams via personal grants or through "Research in Pairs" at Oberwolfach or BIRS. We have also discussed applying to the SWiM and/or SRiM program at SLM/MSRI and SQuaREs program at AIM, which will give us the opportunity to meet all together.

Outcomes of the Meeting

Each of the research teams made important progress on their research problems during the workshop. All teams have plans to continue their collaborations. Some teams are holding Zoom meetings regularly and in order to meet again are applying to research in teams programs, such as SWiM at MSRI; SQuaRE at AIM; Research in Pairs; and programs at the Max Planck Institute for Mathematics in Bonn, and the MFO, Oberwolfach Research Institute for Mathematics.

The feedback on the workshop that we have received from participants is quite positive and publications resulting directly from this workshop are expected.

Participants benefited on the individual level by building background knowledge on a new problem, by strengthening and broadening their research programs, and, in some cases, by being provided with a re-entry point after being sidetracked by any or all of family responsibilities, high service loads, or high teaching loads. By building teams that included women at all career stages, from advanced graduate students and recent PhDs to associate professors seeking to invigorate their research programs to senior researchers, the workshop formed mentoring and collaborative networks that will strengthen the careers of all participants. All attending gained an overview of seven exciting areas of current research in geometry, and all contributed to progress in their own area.

The community of women geometers was strengthened by the supportive research community, mentorship of women just beginning or at the middle of their research careers, and the new collaborative links forged between women geometers working within and between their respective areas of specialization at WIG3. It is important to note that the areas of geometry featured in the WIG3 program are strongly interrelated, so the potential for cross-area collaboration is high. The visibility of the community of women geometers was increased by highlighting the work of established female leaders in geometry, by bringing attention to the work of outstanding new women geometers, and, very simply, by having this many women together to do geometry research.

Finally, it bears mentioning that ten research articles resulted so far from the first two Women in Geometry workshops, the first held at the Banff International Research Station in Banff and the second at the Casa Matemática Oaxaca. Based on the preliminary information we have received from the participants of this third Women in Geometry workshop, we believe that at least as many, if not more, may be expected from WIG3.

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Chapter 25

Fluid Equations, A Paradigm for Complexity: Regularity vs Blow-up, Deterministic vs Stochastic (23w5040)

October 1 - 6, 2023

Organizer(s): Anna Mazzucato (Penn State University), Tai-Peng Tsai (University of British Columbia), Kazuo Yamazaki (University of Nebraska-Lincoln), Yao Yao (National University of Singapore)

Overview of the Field

Fluids are ubiquitous in nature; yet, our understanding remains far from complete. Mathematically rigorous investigation of fluid dynamics goes back to the pioneering work of Leray [13] in 1934. As hydrodynamics model in a spatial domain $D \subset \mathbb{R}^d$, a prominent system of equations is the Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla \pi = \nu \Delta u, \tag{25.0.1a}$$

$$\nabla \cdot u = 0, \tag{25.0.1b}$$

where $u : \mathbb{R}_+ \times D \mapsto \mathbb{R}^d$ represents the velocity vector field, $\pi : \mathbb{R}_+ \times D \mapsto \mathbb{R}$ the pressure scalar field, and $\nu \geq 0$ the viscosity coefficient. The case $\nu = 0$ reduces this model to the Euler equations, and there are many related equations such as the Boussinesq system, magnetohydrodynamics (MHD) system, Hall-magnetohydrodynamics (Hall-MHD) system, surface quasi-geostrophic (SQG) equations, incompressible porous media equations, and transport equation. Different types of variations include the compressible case, as well as the stochastic case. The Navier-Stokes equations has a wide array of real-world applications such as jet flow in aerodynamics, blood flow in medicine, as well as cash flow in finance, and there are decades of study and applications by physicists and engineers.

However, one of the most fundamental properties of any initial value problem of partial differential equations, namely the existence and uniqueness of a solution with bounded kinetic energy starting from an arbitrary smooth initial data with bounded kinetic energy, remains unknown (see [4]). To this day, the complexity of regularity versus blow-up and deterministic versus stochastic give researchers more questions than answers. Very recently, there have been many new developments in the research areas of blow-up, norm inflation, non-uniqueness via convex integration, andsingular stochastic case due to the very rough space-time white noise.

In the past decade, the research direction on PDEs in fluid dynamics is developing at an unprecedented rate, with many breakthrough results on singularity formation, non-uniqueness of weak solutions, etc. It is expected that the combined effort of this workshop participants from various fields with complementary strength will generate new ideas and innovative strategies that can allow us to continue to collaborate and obtain new results, as well as inspire young researchers within the workshop.

Recent Developments and Open Problems

There have been many remarkable developments in the areas of ill-posedness in terms of blow-up, norm inflation and non-existence. E.g., Elgindi [8] proved that there exists $\alpha > 0$ and a divergence-free odd initial data in $C^{1,\alpha}(\mathbb{R}^3)$ with certain growth condition such that the unique local odd solution to the Euler equations emanating from such initial data blows up in finite time. This proof is done by reformulating the problem into self-similar variables, and look for an asymptotically self-similar blow-up solution. It is an interesting question whether one can prove blow-up using other mechanisms without relying on the self-similar formulation. Recent progress in this direction has been discussed in the talk by Diego Cordoba.

On the other hand, convex integration is a technique with its roots from the work of Nash [15] in geometry that was adapted to the partial differential equations of fluid mechanics by De Lellis and Székelyhidi Jr. [7] to prove the non-uniqueness of weak solutions to the Euler equations in any dimension at the regularity level of $L^{\infty}(\mathbb{R} \times \mathbb{R}^d)$. Subsequently, it was extended by many authors and various conjectures that remained open for many decades were solved; the following are only some of such solved problems.

1. Physicist/Chemist Onsager [16] in 1949 conjectured the Hölder regularity of $\frac{1}{3}$ to be the critical threshold concerning the conservation of energy for the Euler equations:

   > "velocity field in such 'ideal' turbulence cannot obey any Lipschitz condition ... for any order n greater than 1/3."

2. Plasma physicist Taylor [19] in 1974 conjectured that the magnetic helicity is conserved in the infinite conductivity limit;

   > "[magnetic helicity] will be almost unchanged so long as departures from perfect conductivity are small."

3. Serrin [18] in 1963 conjectured

   > "whether a non-constant [weak] solution of the [3D Navier-Stokes equations] can ever come to rest in a finite time."

We refer to the surveys [2, 3] and references therein for more details. One of the major open problems is whether the Leray-Hopf weak solutions to the three-dimensional Navier-Stokes equations are unique or not; this is the context of Ladyzhenskaya's conjecture.

Finally, when partial differential equations are forced by random noise, they are called stochastic partial differential equations. Stochastic partial differential equations offer multiple advantages in applications over their deterministic counterpart. First, the stochastic forces are useful in understanding turbulence. Second, effects from external force, such as wind on ocean waves, are unpredictable. Third, for complex models consisting of many variables in real-world such as financial markets, an addition of the noise can turn it to become probabilistic and justify its simplification that is needed to make any mathematical analysis feasible. Finally, fluid in microscopic scales display collisions of molecules which are more accurately characterized as random and chaotic rather than a deterministic function of time and space.

It has been known that certain noise can regularize the solutions allowing one to deduce properties of the corresponding solutions that seem impossible in the deterministic case. A famous example is how a linear multiplicative noise can give a damping effect allowing one to prove probabilistic global well-posedness result starting from small initial data even in the inviscid case. Another famous example is that transport noise has allowed [9] to prove the global well-posedness of a transport equation with a given velocity field with regularity far below the well-known DiPerna-Lions criteria. Yet, using the technique of convex integration, Hofmanová, Zhu, and Zhu [11] were able to prove non-uniqueness even for the stochastic Navier-Stokes equations.

A related major open problem is the the Yang-Mills problem to prove that for any compact simple gauge group, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$ ([4]). In this regard, Parisi and Wu [17] proposed a stochastic quantization approach to prove the existence of the Yang-Mills measure as an invariant measure of the stochastic Yang-Mills equation forced by space-time white noise (STWN) ξ, where STWN ξ is a Gaussian field that is white in both time and space; i.e.,

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$$

and has been utilized in the physics literature very frequently for many decades. Much progress has been made very recently by the group of Chandra, Chevyrev, Hairer, and Shen [5] and there remains great interest to study such stochastic Yang-Mills equation.

Presentation Highlights

The conference was organized in a hybrid mode with 29 on-site participants and 26 more virtual participants who joined via Zoom. There were a total of 20 research talks: six on Monday, five on Tuesday, three on Wednesday, and six on Thursday. Each talk was 50 minutes long, leaving ample time for discussions. There were multiple questions and comments from virtual participants via Zoom as well. Roughly speaking, the presentations can be grouped into the following four topics, which we summarize below.

Singularity v.s. regularity in inviscid fluid equations

1. On Monday morning, Alexander Kiselev from Duke University gave a talk concerning regularity and singularity of vortex and SQG patches. One simple form of the surface quasi-geostrophic equations can be written as

$$\partial_t\theta + (u\cdot\nabla)\theta + \eta(-\Delta)^m\theta = 0, \tag{25.0.2a}$$
$$u = \mathcal{R}^{\perp}\theta, \tag{25.0.2b}$$

   where $\mathcal{R}$ represents the Riesz transform and $(-\Delta)^m$ is a fractional Laplacian with $m \in [0,1]$. He first gave an overall review on some recent progress on regularity properties of vortex and SQG patches. Then he presented an example of a vortex patch with continuous initial curvature that immediately becomes infinite but returns to C^2 class at all integer times only. One of the main reasons had to do with the Fourier symbol of a Hilbert transform. Kiselev also discussed another result of how the α-SQG patches interpolating between Euler and SQG cases are ill-posed in L^p for $p \neq 2$ or Hölder based spaces. The proofs involve derivation of a new system describing the patch evolution in terms of arc-length and curvature.

2. On Monday afternoon, Hongjie Dong from Brown University gave a talk on global well-posedness for the one-phase Muskat problem. He considered the free boundary problem for a 2D and 3D fluid filtered in porous media, which is known as the one-phase Muskat problem. He discussed the work by him and his collaborators that proved that if the initial free boundary is the graph of a periodic Lipschitz function, then there exists a unique global Lipschitz strong solution. The proof of the uniqueness relied on a new point-wise $C^{1,\alpha}$ estimate near the boundary for harmonic functions.

3. On Monday afternoon, Javier Gomez Serrano from Brown University talked about smooth imploding solutions for the three-dimensional (3D) compressible fluids. He presented results on singularity formation for the 3D isentropic compressible Euler and Navier-Stokes equations for ideal gases. These equations describe the motion of a compressible ideal gas, which is characterized by a parameter called the adiabatic constant. Finite time singularities for generic adiabatic constants were found in the recent breakthrough [14]

of Merle, Raphaël, Rodnianski, and Szeftel. Serrano discussed the result by him and his collaborators that allowed them to drop the genericity assumption and construct smooth self-similar profiles for all values of the adiabatic constant. Part of the proof is very delicate and required a computer-assisted analysis.

4. On Tuesday afternoon, In-Jee Jeong from Seoul National University gave a talk concerning well-posedness for generalized SQG equations with singular velocities. The work by him and his collaborators proved strong ill-posedness for the generalized SQG equations for smooth data, when roughly speaking the stream function is more singular than the advected scalar. The mechanism is due to degenerate dispersion near a quadratic shear flow. The case when the stream function is logarithmically singular is of particular interest ("Ohkitani model"); in this case, there is a well-posedness in a scale of Sobolev spaces with time-decreasing exponents, and this result has immediate applications to long-time dynamics of inviscid and viscous alpha-SQG models.

5. On Thursday morning, Diego Cordoba from Instituto de Ciencias Matematicas-CSIC gave a talk concerning blow-up of classical solutions for the incompressible Euler equations. He discussed recent results on the blow-up problem of classical solutions for the incompressible Euler equations with finite energy. Together with his collaborators, he constructed solutions that have instantaneous gap loss of Sobolev regularity in the plane and finite time singularities in the whole space. Previously, such blow-up results was usually done by reformulating the problem into self-similar setting and look for an asymptotic self-similar solution. The key novelty in their joint works is that the blow-up is not obtained through self-similar ansatz, but a different mechanism, where the interaction between localized solutions played a key role.

6. On Thursday morning, Christophe Lacave from Universite Grenoble Alpes gave a talk concerning point vortex for the lake equations. He presented the lake equations which can be considered as a generalization of the 3D axisymmetric Euler equations without swirl. This 2D model differs from the well-known 2D Euler equations due to an anelastic constraint in the div-curl problem. He explained how this new constraint implies a very different behavior of concentrated vortices: the point vortex moves under its own influence according to a binormal curvature law.

Ill-posedness, convex-integration and anomalous dissipation in fluid dynamics

1. On Monday afternoon, Jiahong Wu from the University of Notre Dame gave a talk on hyperbolic Navier-Stokes and hyperbolic MHD equations. The MHD system consists of the equations of

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi = \nu \Delta u + (b \cdot \nabla)b, \tag{25.0.3a}$$
$$\partial_t b + (u \cdot \nabla)b = \eta \Delta b + (b \cdot \nabla)u, \tag{25.0.3b}$$
$$\nabla \cdot u = 0, \tag{25.0.3c}$$

where $b : \mathbb{R}_+ \times D \mapsto \mathbb{R}^d$ represents the magnetic vector field and $\eta \geq 0$ the magnetic resistivity. The hyperbolic Navier-Stokes equations contain an extra double time-derivative term $\partial_{tt} u$ while the hyperbolic MHD system differs from the standard MHD system by a double time-derivative term in the magnetic field equation $\partial_{tt} b$. The appearance of these terms is not an artifact but reflects basic physics laws. Mathematically the global regularity problem on these hyperbolic equations is extremely difficult. In fact, even the L^2-norm of solutions to the two-dimensional (2D) equations are not known to be globally bounded in the general case. One of the results Wu discussed was the global existence of a wild solution to the 2D hyperbolic Navier-Stokes equations that doubles its initial kinetic energy size in one unit of time via convex integration technique.

2. On Tuesday morning, Gianluca Crippa from University of Basel gave a talk concerning anomalous dissipation in fluid dynamics. Kolmogorov's K41 theory of fully developed turbulence advances quantitative predictions on anomalous dissipation in incompressible fluids: although smooth solutions of the Euler equations conserve the energy, in a turbulent regime information is transferred to small scales and dissipation can happen even without the effect of viscosity, and it is rather due to the limited regularity of the solutions. In rigorous mathematical terms, however, very little is known. Crippa discussed his receont work in collaboration with Colombo and Sorella that consided passive-scalar advection equation where anomalous dissipation was predicted by the Obukhov-Corrsin theory of scalar turbulence, in similar vein to Onsager's conjecture

on the Euler equations. Crippa illustrated the main ideas behind their construction of a velocity field and a passive scalar exhibiting anomalous dissipation in the supercritical Obukhov-Corrsin regularity regime. He also described how the same techniques provide an example of lack of selection for passive-scalar advection under vanishing diffusivity, and an example of anomalous dissipation for the forced Euler equations in the supercritical Onsager regularity regime. The proofs partially used stochastic tools such as random characteristics.

3. On Tuesday morning, Mimi Dai from the University of Illinois at Chicago gave a talk on ill-posedness issues for fluid equations. She discussed some recent progresses in the effort to understand the classical problem by exploring ill-posedness behavior of solutions. The emphasis is on the construction of pathological solutions which either indicate non-uniqueness or develop finite time singularity. She described how the Euler equations, SQG equations, and even the electron-MHD system of the form

$$\partial_t b + \nabla \times ((\nabla \times b) \times b) = \eta \Delta b \tag{25.0.4}$$

can be considered as part of a large family of active scalars. In particular, she and her collaborators obtained a result on the forced SQG equations; specifically, one can find a force that is rough such that the non-uniqueness result can be obtained at a higher regularity level.

4. On Tuesday morning, Daniel Faraco from Universidad Autonoma de Madrid discussed the entropy solutions to macroscopic incompressible porous media (IPM) equations. The IPM equations over $\mathbb{T}^d$ may be written as

$$\partial_t \theta + (u \cdot \nabla)\theta + (-\Delta)^m \theta = 0, \tag{25.0.5a}$$
$$u = -(\nabla \pi + \gamma e_d), \tag{25.0.5b}$$
$$\nabla \cdot u = 0, \tag{25.0.5c}$$

where T is the liquid temperature, and u models Darcy's law. Faraco discussed how the convex integration has proven to be a successful technique for modeling instabilities in fluid dynamics (Kelvin-Helmholtz, Rayleigh-Taylor, or Saffman-Taylor instabilities) with a prime example being the unstable Muskat problem, which is the mathematical treatment of a two-phase, incompressible fluid evolving through porous media. Faraco gave a review of the existence theory and demonstrated how maximizing potential energy dissipation reconciles Otto's minimizing scheme on the Wasserstein space with the convex integration subsolution, leading to a unique equation for macroscopic evolution. Furthermore, he explained that such an equation admits entropy solutions.

5. On Tuesday afternoon, Sauli Lindberg from University of Helsinki gave a talk on the magnetic helicity and weak solutions of ideal MHD. In ideal MHD, smooth solutions conserve total energy, cross helicity and magnetic helicity. Nevertheless, in view of numerical evidence, ideal MHD should possess weak solutions that

   (a) arise at the ideal (inviscid, non-resistive) limit,

   (b) conserve magnetic helicity

   (c) but dissipate total energy and do not conserve cross helicity.

   Lindberg discussed a result directly on ideal MHD, in collaboration with Faraco and Székelyhidi Jr. that there exist bounded solutions of ideal MHD with prescribed total energy and cross helicity profiles and magnetic helicity. The proof used a new convex integration scheme on two-forms consistent with the conservation of magnetic helicity. Lindberg also discussed another result that proved a conjecture of Buckmaster and Vicol: $L^3_{t,x}$ is the L^p-threshold for magnetic helicity conservation in ideal MHD.

Dynamics and regularity of Navier-Stokes equations

1. On Monday morning, Ian Tice from Carnegie Mellon University gave a talk concerning stationary and slowly traveling solutions to the free boundary Navier-Stokes equations. The stationary problem for the

free boundary incompressible Navier-Stokes equations lies at the confluence of two distinct lines of inquiry in fluid mechanics. The first views the dynamic problem as an initial value problem. In this context, the stationary problem arises naturally as a special type of global-in-time solution with stationary sources of force and stress. One then expects solutions to the stationary problem to play an essential role in the study of long-time asymptotics or attractors for the dynamic problem. The second line of inquiry concerns the search for traveling wave solutions. In this context, a huge literature exists for the corresponding inviscid problem, but progress on the viscous problem was initiated much more recently in the work of Ian Tice and co-authors. For technical reasons, these results were only able to produce traveling solutions with nontrivial wave speed. Ian Tice discussed the well-posedness theory for the stationary problem and described how the solutions thus obtained lie along a one-parameter family of slowly traveling wave solutions.

2. On Monday morning, Giusy Mazzone from Queen's University gave a talk concerning fluid-solid interaction problems. Fluid-solid interaction problems are widely studied because of their connections with hemodynamics, geophysical and engineering applications. The differential equations governing this type of interactions feature a dissipative component (typically arising from the fluid, through the Navier-Stokes equations) and a conservative component (due to the solid counterpart, through either Euler equations of rigid body dynamics or Navier equations of elasticity). This dissipative-conservative interplay has a fundamental role in questions related to existence, uniqueness and stability of solutions to the governing equations. Giusy Mazzone discussed results concerning the existence and stability of solutions to equations characterized by the above-mentioned dissipative-conservative interplay, and described the dynamics of different mechanical systems featuring fluid-solid interactions.

3. On Wednesday morning, Alexis Vasseur from the University of Texas, Austin gave a talk concerning boundary vorticity estimate for the Navier-Stokes equation and control of layer separation in the inviscid limit. He discussed the results by him and his collaborators that provided a new boundary estimate on the vorticity for the incompressible Navier-Stokes equation endowed with no-slip boundary condition. The estimate is re-scalable through the inviscid limit and provides a control on the layer separation at the inviscid Kato double limit, which is consistent with the layer separation predictions via convex integration.

4. On Wednesday morning, Tobias Barker from University of Bath gave a talk concerning the dynamics of the 3D Navier-Stokes equations from initial data with zero third component. The study of the 3D Navier-Stokes equations under certain condition only on one of the three velocity field components has attracted much attention recently. In particular, it is known that as long as the third component of the velocity vector field satisfies a certain regularity criterion in a Sobolev scaling-invariant norm, then the solution remains smooth for all time. In 2017 Chemin, Zhang and Zhang posed the question of whether initial data with small third-component (with respect to a scale-invariant norm) implies global regularity of the associated 3D Navier-Stokes solution. Barker discussed results by him and his collaborators concerning the growth properties of solutions for certain initial data with zero third-component.

5. On Thursday morning, Helena Nussenzveig Lopes from Universidade Federal do Rio de Janeiro gave a talk on the conditions for energy balance in 2D incompressible ideal fluid flow. She described anomalous dissipation, flexibility versus rigidity, and focused on the side of rigidity in the 2D case. She pointed out that the Onsager scaling is not the last word on inviscid dissipation. There is a dynamical mechanism to avoid anomalous dissipation in 2D, whereas this is not the case in 3D. Namely, the class of 2D Euler physically realizable weak solution (which are the solutions that can be obtained in vanishing viscosity limit) conserve energy, therefore they are not attainable through convex integration/wild solutions.

6. On Thursday afternoon, Elizabeth Carlson from Caltech gave a talk on learning identifying properties of turbulent flows using analytical techniques in data-driven methods. She described recent developments in the research area of data assimilation that has seen much developments since the pioneering work of [1] by Azouani, Olson, and Titi. To explain how to learn the parameters of a chaotic system using partial observations, she used the Lorenz equations as an example of a chaotic system, and described the algorithm to dynamically learn its parameters from partial observations and its convergence proof. She also discussed a nonlinear-nudging modification of the Continuous Data Assimilation algorithm for the 2D incompressible Navier-Stokes equations, and demonstrated the numerical results.

Stochastic fluid equations

1. On Wednesday morning, Jonathan Mattingly from Duke University gave a talk concerning random splitting of fluid equations. He described some new models of randomly agitated stochastic dynamics in the context of systems with complex dynamics such as the 2D Euler and Navier-Stokes equations. The models introduced randomness onto the system through a random splitting scheme and can be viewed as a particular class of Random Interested Functions or Piecewise Deterministic Markov Processes. Mattingly went on to explain how the randomly split Galerkin approximations of the 2D Euler equations and other related dynamics can be shown to possess a unique invariant measure that is absolutely continuous with respect to the natural Liouville measure, despite the existence of other invariant measures corresponding to fix points of the PDEs. He also explained how one proves that the dynamics with respect to this measure have positive Lyapunov exponents almost surely. Finally, he discussed recent results that show that the system has a unique invariant measure even when damping is applied to part of the system.

2. On Thursday morning, Gautam Iyer from Carnegie Mellon University gave a talk concerning how mixing can accelerate the convergence of Langevin systems. A common method used to sample from a distribution with density proportional to $p = e^{-V/\kappa}$ is to run Monte Carlo simulations on an overdamped Langevin equation whose stationary distribution is also proportional to p. When the potential V is not convex and the temperature κ is small, this can take an exponentially large (i.e. of order $e^{C/\kappa}$) amount of time to generate good results. Iyer discussed about a method that introduces a "mixing drift" into this system, which allows us to rigorously prove convergence in polynomial time (i.e. a polynomial in $1/\kappa$).

3. On Thursday afternoon, Tommaso Rosati from University of Warwick gave a talk concerning a global-in-time solutions to perturbations of the 2D stochastic Navier-Stokes equations forced by STWN. When the STWN ξ is forced on an equation of the form $\partial_t - \Delta$ such as the Kardar-Parisi-Zhang equation, the spatial regularity of ξ is $C^\alpha(\mathbb{T}^d)$ for $\alpha < -\frac{d+2}{2}$ almost surely. Such roughness of the force transmits to the roughness of the solution making the product within the nonlinear term ill-defined according to Bony's paraproducts. The 2D Navier-Stokes equations forced by the STWN has similar difficulties; nevertheless, using Wick products and the explicit knowledge of invariant measure, Da Prato and Debussche [6] in 2002 were able to prove its global well-posedness for almost every initial data. Rosati discussed a new proof, which is a work together with his collaborator, of the global-in-time well-posedness for perturbations of the 2D Navier-Stokes equations driven by STWM.The proof relied on a dynamic high-low frequency decomposition, tools from paracontrolled calculus and an L^2 energy estimate for low frequencies. He described how their arguments require the solution to the linear equation to be a log-correlated field and that their arguments do not rely on (or have) explicit knowledge of the invariant measure: the perturbation is not restricted to the Cameron-Martin space of the noise. Their approach allows for anticipative and critical (L^2) initial data.

Scientific Progress Made and Outcome of the Meeting

There were various deep discussions by the participants through the workshop, after talks and during coffee breaks. For example, Rosati and Yamazaki discussed the contents of Rosati's talk in detail and obtained heuristic argument that one can extend [10] to the 3D case up to the exponent of $\frac{5}{4}$, as well as its consequent toward the stochastic Yang-Mills equation from [5]. There has also been some interactions between virtual participants and on-site participants. After some talks, there were some excellent questions raised by virtual participants.

Compared to the other fluid workshops held in the past few years, a key novelty of our workshop is the broad spectrum of people that have brought together with the workshop. We have paid special attention to select researchers that are working on different aspects of fluid equations (singularity v.s. regularity; deterministic v.s. stochastic; fixed-domain v.s. free-boundary; compressible v.s. incompressible; theoretical v.s. numerical...). We have also striven to select both leaders in the field as well as more junior, but up and coming, researchers, and established researchers that are working on areas of renewed interest, but have had less visibility in recent years. Many participants expressed to us that they enjoyed the broad topics covered in this workshop, and they especially appreciated meeting and discussing with other people who they rarely see in their own sub-areas.

Participants

Barker, Tobias (University of Bath)
Bowman, John (University of Alberta)
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Chae, Dongho (Chung-Ang University)
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Crippa, Gianluca (University of Basel)
Dai, Mimi (University of Illinois at Chicago)
Dong, Hongjie (Brown University)
Duguin, Mathis (École Polytechnique Fédérale de Lausanne)
Elgindi, Tarek (Duke University)
Enlow, Matthew (University of Nebraska-Lincoln)
Eyink, Gregory (JHU)
Faraco, Daniel (Universidad Autonoma de Madrid)
Farhat, Aseel (Florida State University)
Feng, Yuanyuan (East China Normal University)
Gie, Gung-Min (University of Louisville)
Gomez-Serrano, Javier (Brown University)
Hou, Thomas (Caltech)
Ignatova, Mihaela (Temple University)
Iyer, Gautam (Carnegie Mellon University)
Jeong, In-Jee (Seoul National University)
Jia, Hao (University of Minnesota)
Kelliher, James (University of California, Riverside)
Kiselev, Alexander (Duke University)
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Lacave, Christophe (Universite Grenoble Alpes)
Liang, Su (University of British Columbia)
Lindberg, Sauli (University of Helsinki)
Looi, Shi-Zhuo (Caltech)
Luo, Tianwen (Tsinghua University)
Mattingly, Jonathan (Duke University)
Mazzone, Giusy (Queen’s university)
Mazzucato, Anna (Penn State University)
Miura, Hideyuki (Tokyo Institute of Technology)
Nguyen, Quoc-Hung (Chinese Academy of Sciences)
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Chapter 26

Motives and Invariants: Theory and Applications to Algebraic Groups and Their Torsors (23w5146)

October 8 - 13, 2023

Organizer(s): Stefan Gille (University of Alberta), Detlev Hoffmann (TU Dortmund), Anne Quéguiner-Mathieu (Université Paris 13), Stephen Scully (University of Victoria)

Overview of the Field

The algebraic theory of quadratic forms was founded by Witt in 1937, when he introduced the Witt ring of a general field, thereby providing a rich new viewpoint on a topic that had been studied in number-theoretic contexts for many decades prior. After undergoing a major expansion throughout the second half of the 20th century, the subject reached a high point in the 1990s with Voevodsky's celebrated proof of the Milnor conjecture relating the graded Witt ring of a field to its mod-2 Galois cohomology and Milnor K-theory rings. This landmark result established a long-expected weak classification theorem for quadratic forms over fields in terms of (partially-defined) Galois cohomological invariants. At the same time, its proof, based on pioneering developments in motivic cohomology and motivic homotopy theory, has gone on to inspire a broad new algebraic-geometric approach to the subject. The central theme here is the study of algebraic cycles on (products of) quadrics and isotropic Grassmannians, which are homogeneous varieties for orthogonal groups and are naturally viewed in this context as objects of various motivic categories. The resulting theory has complemented existing approaches greatly, leading to the solution of old problems that seemed beyond the reach of classical methods, and enabling us to gain deeper insight into the classification of quadratic forms.

As has long been understood, the picture painted above finds its most natural generalization within the study of (linear) algebraic groups over general fields. Beyond the orthogonal groups defined by quadratic forms, it was shown by Weil in the 1960s that all reductive algebraic groups of classical type over a field can be described in terms of a larger class of algebraic objects, namely algebras with involution (or, equivalently, hermitian forms over involutive division algebras). From this perspective, the theory of quadratic forms appears as a key example, which despite its specificities, has been a constant source of inspiration for the wider study of algebraic groups and their torsors. This includes the study of groups of exceptional type, though more specialized tools and arguments are often needed here in the absence of a good general description of these groups in terms of concrete algebraic structures (still, many realizations of these groups also involve quadratic or hermitian forms in one way or another).

In this broader context, a cohomological invariant of an algebraic group G over a field F is understood as a natural transformation from the functor $H^1_{\acute{e}t}(-, G)$ of isomorphism classes of G-torsors on the category of extensions of F to a suitable abelian cohomological functor, typically taken to be Galois cohomology with coefficients in a torsion Galois module (although it is also natural to consider other cycle modules in the sense of Rost, as well as Witt groups of quadratic forms). Echoing the situation for quadratic forms (i.e., torsors for split orthogonal groups), the problem of determining the cohomological invariants of a given group G plays a central role in studying the classification of its torsors. The existence of non-trivial invariants already imposes lower bounds on the so-called essential dimension of G, a numerical invariant that informally measures the number of parameters needed to describe a generic G-torsor, and which has been the subject of much investigation in recent years. In the same spirit, the theory of cohomological invariants offers means by which to attack from the negative side the generalized Noether problem asking whether the classifying space of a connected algebraic group over an algebraically closed field is stably rational. The last decade has seen continued progress in these directions, not only for connected algebraic groups, but finite groups also. In the former case, a significant development came in work of Merkurjev that establishes an exact sequence describing the degree-3 (Galois) cohomological invariants of an arbitrary semisimple group, extending foundational work of Rost from the 1990s on the simply connected case. This goes some way towards establishing a complete understanding of the low-degree invariants of semisimple groups. While a general understanding of higher-degree cohomological invariants remains a very challenging goal, important progress has also been made here in special cases. For instance, Semenov exploited ideas in the proof of the Milnor conjecture to construct a (partially defined) degree-5 invariant for the split exceptional group E_8, answering in the process an open question of Serre on groups of type E_8 over the field of rational numbers.

Despite playing a central role in the general theory, the cohomological invariants discussed above are insufficient for many purposes. To this end, the motivic algebraic-geometric developments emerging from the proof of the Milnor conjecture have proved extremely fruitful, particularly in their application to index-reduction problems that feature prominently throughout the subject as a whole. A basic unifying question here is the following: Given projective homogeneous varieties X and Y under actions of semisimple algebraic groups, when does X admit a point over the function field of Y? In the case where X is a generalized Severi-Brauer variety, this was fully answered in the 1990s by Panin, Merkurjev, Wadsworth and others using tools from algebraic K-theory. Beyond this result, precise answers are only known in special cases. Over the past two decades, however, the advent of new tools with which to investigate rationality problems for algebraic cycles on projective homogeneous varieties has led to significant progress on cases where X is an isotropic Grassmannian attached to a quadratic or hermitian form (we note here the work of Karpenko, Merkurjev and Vishik, in particular). This progress has seen the introduction and systematic investigation of new discrete invariants of semisimple algebraic groups that capture important information of a motivic nature. For example, a significant advance made by Vishik on an old problem of Kaplansky concerning the possible values of the u-invariant of a field was founded on the introduction and study of the so-called elementary discrete invariant for special orthogonal groups. A certain component of this invariant, known as the J-invariant, has subsequently been adapted to the study of arbitrary semisimple groups (initially in work of Petrov, Semenov and Zainoulline), and has been applied with great effect to the study of exceptional groups in particular. While many aspects of the motivic approach have so far been somewhat isolated from the theory of cohomological invariants, other recent developments suggest a more unified picture. In particular, for a general algebraic group G, Smirnov and Vishik have proposed the investigation of refined characteristic-class invariants of G-torsors taking values in motivic cohomology groups of certain associated simplicial schemes. Implementing this for split orthogonal groups, they have constructed new invariants for quadratic forms – the subtle Stiefel-Whitney classes – that not only refine the classical Galois cohomological invariants (the ordinary Stiefel-Whitney classes), but also determine the J-invariant, among other things. This opens the door to a broader motivic-homotopic approach to the subject.

Finally, within the overall picture, the study of algebraic groups and their torsors over special fields of arithmetic or geometric interest has also occupied a position of central importance, and has served as a source of inspiration for the development of the general theory. Here, recent years have seen a great deal of progress, with a key driver on the arithmetic side being the general field patching techniques originally introduced by Harbater, Hartmann and Krashen, and further developed and applied in the work of these authors, Colliot-Thélène, Parimala, Suresh and others.

Recent Developments and Open Problems

- **Patching, local-global principles and field invariants.** Over the past 15 years, the development of field patching techniques originally introduced in the work of Harbater, Hartmann and Krashen has led to remarkable progress on the study of local-glocal principles for homogeneous spaces over function fields of arithmetic surfaces and other low-dimensional fields, as well as related problems concerning invariants of such fields defined in terms of torsors, e.g., the period-index problem for central simple algebras, and the u-invariant problem for quadratic forms. In a recent development, P. Gille and Parimala have exploited these methods to establish a general local-global principle for the existence of rational points on projective homogeneous varieties over semi-global fields, extending previously known results for generalized Severi-Brauer varieties and quadrics. On a more geometric side, refined tools from real and analytic geometry have been brought to bear. For instance, Benoist used methods from Hodge theory to determine the u-invariant of the function field of a real surface, and to give a positive answer to the period-index problem for function fields of real surfaces with no real points. Benoist has also applied étale cohomological techniques to obtain the best-known bounds on the Pythagoras numbers of Laurent series fields in several variables over real closed fields, thereby answering a long-standing question due to Choi, Dai, Lam and Reznick.

- **Cohomological invariants of algebraic groups.** Following Blinstein and Merkurjev's determination of the degree-2 cohomological invariants of reductive algebraic groups, Lourdeaux completed the picture in his PhD thesis by showing that the normalized degree-2 invariants of a smooth connected algebraic group G are in one-to-one correspondence with the extensions of G by the multiplicative group. A similar result for finite groups was also established by Bailey in her PhD thesis (with a more elementary and direct proof being given recently by S. Gille). A consequence of these results is the fact that these invariants arise from projective representations of the group. Building on the work of Merkurjev mentioned in §1, Baek completely determined the reductive indecomposable degree-3 invariants of all split semisimple groups of classical type. Despite recent progress, however, little is known in general about cohomological invariants of higher degree, the study of which remains a difficult and important challenge for the area.

- **Massey products in Galois cohomology.** The Milnor and Bloch-Kato conjectures, proved by Rost and Voevodsky, assert that the Galois cohomology ring of an absolute Galois group with coefficients in the integers modulo a prime p has generators in degree 1 and only one relation (the Steinberg relation) in degree 2, i.e., is a Koszul algebra. Closely related to these is the vanishing conjecture for Massey products, recently formulated by Mináč and Tân, which – if true – would give further constraints on the cohomology of absolute Galois groups. More specifically, this conjecture asserts that if the Massey product of $n \geq 3$ elements in the first Galois cohomology group of the absolute Galois group of a field F with coefficients in the integers modulo a prime p is defined, then this product (which is actually a set in the second Galois cohomology group) contains 0. This has been proven for number fields by Harpaz and Wittenberg, for $n = 3$ and arbitrary fields by Mináč and Tân as well as (independently) by Efrat and Matzri (after older results for $n = 3$ and number fields by Hopkins and Wickelgren). Recently, Merkurjev and Scavia have settled the case of four-fold Massey products at the prime 2 (after older results for $n = 4$ and number fields by Guillot, Mináč and Topaz). While the vanishing conjecture may on the surface seem tangential to the main themes of the workshop, the methods used to attack it, as well as its implications, are in fact intimately related to the theory of algebraic groups and their torsors, and the problem represents an important challenge towards establishing a better understanding of the structure of absolute Galois groups. The developments above are also in line with the recent work of De Clercq and Florence on so-called smooth profinite groups, which, as is now expected, may lead to more elementary proofs of the Milnor and Bloch-Kato conjectures.

- **Motives of projective homogenous varieties.** The focus of the research here is on understanding the possible decompositions of the motives of projective homogenous varieties under actions of semisimple algebraic groups (in various motivic categories) with a view towards applications to the classification of torsors over general fields. An important general role is played here by the aforementioned J-invariant, which governs the Chow-motivic decomposition of the variety of Borel subgroups into indecomposable motives (with coefficients in the integers modulo a prime). While this invariant was initially only defined for groups of inner type, Geldhauser and Zhykhovich have recently succeeded in extending the theory to certain groups of outer type, broadening the reach of its applicability. In other recent work of Geldhauser and Petrov, new constraints on the Chow-motivic decompositions of arbitrary projective homogeneous varieties for a given semisimple group G coming from the J-invariant have been established. This is based on a new uniform approach to the subject that exploits the Hopf algebra structure on the Chow ring of the split form of G. The results obtained have revealed new insights even in the extensively-studied case of quadrics. Another major theme here is the theory of upper indecomposable summands in the Chow motives of projective homogeneous varieties developed in the work of Karpenko. Recently, De Clercq and Quéguiner-Mathieu have revisited this theory by introducing and studying so-called Tate traces of Chow motives, i.e., maximal pure Tate summands. Using Karpenko's theory, they show that, with finite coefficients, the Chow motives of projective homogeneous varieties for semisimple groups of inner type are determined up to isomorphism by their Tate traces over all extensions of the base field. These investigations are closely related to the earlier work of De Clercq and Garibaldi on the notion of motivic equivalence for semisimple groups, and are also related to the recent work of Vishik on so-called isotropic motivic categories (the latter may provide a higher-viewpoint explanation on some of the isomorphism criteria obtained). Finally, replacing Chow groups with other oriented cohomology theories in the sense of Levine and Morel, such as Morava K-theories, provides further insight, and has been the subject of much recent work. In particular, Sechin and Semenov showed that the Morava K-theory motives of quadrics detect the vanishing of invariants of their underlying quadratic forms. As an application, they obtained strong new results on the torsion in the integral Chow groups of quadrics, something that remains poorly understood in general despite its significance for the theory of quadratic forms. These advances were, in part, made possible after the Rost nilpotence principle for Morava K-theory motives of projective homogeneous varieties was established by S. Gille and Vishik.

- **Quadratic forms in characteristic 2.** The last decade of the 20th century has seen rapid progress of the algebraic theory of quadratic forms in the wake of Voevodsky's proof of the Milnor conjecture. The methods that originated in Voevodsky's work have been further developed by Vishik, Karpenko and others (see also the previous paragraph on motives of projective homogeneous varieties) to solve many important problems related to isotropy questions, in particular isotropy and Witt indices of quadratic forms over function fields of quadrics. Much of that work was restricted to characteristic not 2, but over recent years these methods have been extended to characteristic 2. Crucial in this context was the the theory of Steenrod operations on mod-2 Chow groups of smooth varieties in characteristic 2 by Primozic that allowed Karpenko to complete the proof of the Hoffmann-Totaro conjecture on the first Witt indices of quadratic forms also in characteristic 2 and in that case also for singular forms. In fact, the theory of totally singular forms is of some independent interest as the methods to prove results on isotropy and Witt indices in the totally singular case are much more algebraic in nature. Such questions have been studied extensively by Scully. In particular, it was Scully who provided the proof of the Hoffmann-Totaro conjecture in the totally singular case.

Presentation Highlights and Scientific Progress Made

The programme consisted of 4 one-hour overview talks and 20 further scientific talks (mostly 50 minutes in length) on the latest developments in the field. All talks were given in person. We provide a brief summary of the presentations made, with quotes from speakers' abstracts being indicated by the use of italic letters.

We first report on a proposed talk by PAVEL SECHIN that unfortunately had to be cancelled due to visa problems. The intended topic of the talk was recent progress on *cohomological invariants of projective homogeneous varieties through Morava motives*, in particular a construction of an *injective functorial homomorphism from* $K^M_{n+1}(k)/2$ *to the group of invertible* $K(n)$*-motives over* k, which sends the class of a given nonsingular quadratic form q in the $(n+1)$st power of the fundamental ideal to an invertible summand of the $K(n)$-motive of the projective quadric associated to q. This is based on recent joint work of Sechin with Lavrenov.

Turning to the talks themselves, NIKITA KARPENKO gave an overview lecture on his recent work (partly in collaboration with Devyatov and Merkurjev) concerning the study of the Chow ring modulo torsion for generically twisted flag varieties of spin groups. As outlined in the talk, this work brings striking new applications to the study of degrees of partial splitting fields for nonsingular quadratic forms with trivial discriminant and Clifford invariant, enhancing an important 2005 paper of Totaro in which the torsion indices of the spin groups were completely determined.

In a related direction, CHARLES DE CLERCQ gave a talk *on the classification of direct summands of motives of projective homogeneous varieties, through the Tate motives they contain over field extensions*, a report on his recent work with Quéguiner-Mathieu that introduces and studies the notion of Tate traces for Chow motives. There were further talks on the Chow motives of projective homogeneous varieties by NIKITA GELDHAUSER and MAXIM ZHYKHOVICH, each presenting new results on the J-invariant, *a discrete invariant of semisimple algebraic groups which describes the motivic behaviour of the variety of Borel subgroups*. This invariant was *an important tool to solve several long-standing problems. For example, it plays an important role in the progress on the Kaplansky problem about possible values of the u-invariant of fields by Vishik.*

ALEXANDER VISHIK's talk embraced the more abstract setting of Voevodsky's motivic categories, and concerned the relation between the notions of numerical equivalence and the recently-introduced isotropic equivalence for Chow groups and Morava K-theories. Isotropic realizations *provide an algebro-geometric object with its local versions parametrized by various extensions of the base field, versions residing in the isotropic category whose complexity is similar to that of the topological category*. The talk culminated with a discussion of the speaker's recent proof that *over so called flexible fields, isotropic Chow groups coincide with numerical ones*.

Further talks related to motivic homotopy theory were given by BAPTISTE CALMÈS and OLIVIER HAUTION. The latter spoke on the so-called *concentration theorem for actions of linearly reductive groups on affine schemes* and a *consequence for equivariant stable motivic homotopy theory, asserting that, upon inverting appropriate elements, the equivariant cohomology of a scheme with a group action is "concentrated" on its fixed locus*. Calmès talk, based on joint work with Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus and Steimle, concerned *a new definition of Hermitian K-theory as a universal object in the context of quadratic functors and stable infinity-categories* and how it *enables us to simplify and generalize its classical properties, study the relationship between different objects of quadratic nature such as symmetric bilinear forms or quadratic ones, and completely remove or clarify the invertibility of* 2 *assumptions scattered in the theory until now*.

DIEGO IZQUIERDO gave an overview talk on relations between some Diophantine properties and cohomological properties of fields, such as the relation between the C_i property and cohomological dimension, or Serre's conjecture II, where the cohomological dimension controls the existence of rational points for certain homogeneous spaces. This talk was followed by a presentation of GIANCARLO LUCCHINI ARTECHE on his joint work with Izquierdo on some "higher versions" of Serre's conjecture II.

In a similar fashion, ALEXANDER MERKURJEV's overview lecture on Massey products in Galois cohomology was followed by the talk of FEDERICO SCAVIA on his recent joint work with Merkurjev on the vanishing conjecture for Massey products. As already mentioned in § 2, this topic is closely related to the (proven) Milnor and Bloch-Kato conjectures, which have been at the heart of MATHIEU FLORENCE's talk, where we heard about *a new approach to the norm residue isomorphism Theorem of Rost, Suslin and Voevodsky*, developed in particular with Charles De Clercq. More specifically, it was explained here that these famous conjectures follow from a lifting conjecture for Galois representations.

RAMAN PARIMALA spoke on her recent work with P. Gille that establishes a *Hasse principle for projective homogeneous spaces over semiglobal fields* (i.e., fields of transcendence degree one over a complete discretely valued field) using patching techniques. This extends previously known results on generalized Severi-Brauer varieties and quadrics, and proves for projective homogeneous spaces a conjecture of Colliot-Thélène, Parimala and Suresh initially made over function fields of p-adic curves. Parimala's talk was preceded by an overview talk by JEAN-LOUIS COLLIOT-THÉLÈNE on the degree-3 unramified cohomology of algebraic varieties over finite fields with torsion coefficients.

The theory of essential dimension for algebraic groups and related structures was addressed in the talks by DANNY OFEK and ZINOVY REICHSTEIN. Ofek reported on joint work with Reichstein in which valuation-theoretic techniques are exploited to establish a new *lower bound on the essential dimension of a Brauer class*, and a result on the *essential dimension of the Witt class of a Hermitian form, generalizing a theorem of Chernousov and Serre*. Reichstein's talk concerned recent work with Edens that establishes a new upper bound on the essential dimension of the finite symmetric group S_n over a field of odd characteristic for a certain infinite family of positive integers n (depending on the characteristic of the field). This bound runs contrary to the widely expected value of the essential dimension in characteristic 0. A fundamental concept in the study of essential dimension of algebraic groups is that of versal torsors. In his talk, URIYA FIRST spoke on recent work (partly joint with Florence and Rosengarten) that investigates the existence (or non-existence) of torsors over schemes satisfying various versality properties. Applications of these results to the symbol length problem for Azumaya algebras over semilocal rings (containing enough roots of unity) were also discussed. Continuing the theme of torsors over semilocal rings, THOMAS UNGER reported on his joint work with Astier that establishes *Pfister's local-global principle for hermitian forms over Azumaya algebras with involution over semilocal rings*, showing in particular that *the Witt group of nonsingular hermitian forms is 2-primary torsion*.

Talks on general structure of algebraic groups were given by PHILLIPE GILLE and SRIMATHY SRINIVASAN. The former spoke on joint work with Guralnick that studies *semi-continuity for the unipotent dimension of group schemes in view of application to finite groups*, and the latter on the classification of semisimple groups of classical type over a general base in terms of Azumaya algebras with involution.

ARTURO PIANZOLA reported on joint work with P. Gille that provides *a criterion for certain algebraic objects over Jacobson schemes to be forms of each other based on their behaviour at closed fibres*. An application to a question of Burban on loop algebras of simple finite-dimensional complex Lie algebras was also discussed.

Quadratic forms in characteristic 2 were addressed in two talks. ADAM CHAPMAN (based on joined work with Quéguiner-Mathieu) gave new constructions of so-called minimal quadratic forms for function fields of non-singular conics in characteristic 2. In characteristic not 2, the picture of such minimal forms is reasonably complete, but characteristic 2 poses significant problems because of the necessity to consider singular forms. DIKSHA MUKHIJA presented new results, joint with Ahmed Laghribi, on the (non)excellence of field extensions for quadratic forms in characteristic 2, including the fact that function fields of totally singular conics are generally not excellent (contrary to the case of non-singular conics considered in Chapman's talk). We also mention here CAMERON RUETHER's talk on triality, where he extended to the setting over schemes, the definition of the canonical quadratic pair of a Clifford algebra, recently defined by Dolphin and Quéguiner-Mathieu over fields of characteristic 2, and explored the consequences on triality.

Outcome of the Meeting

In this workshop, we brought together established specialists and young researchers working on topics related to the study of quadratic and hermitian forms, linear algebraic groups, homogeneous varieties and Galois cohomology. With a mix of expertise coming from the various sides of the overall picture outlined in § 1, we hoped for a synergistic impact, stimulating new developments and collaborative projects among different groups of researchers. The group of 40 in-person participants included 8 postdoctoral researchers, 1 PhD student and 1 undergraduate student.

During the workshop, we observed a lot of informal discussions between participants, some of which were reported to us after the conference. For instance, Jean-Louis Colliot-Thélène mentioned short discussions with six different people or groups of people, on various topics such as *an ongoing joint project on patching* and *explicit Brauer type counterexamples to the Hasse principle for biquadratic extensions as discussed in the literature*. Some discussions led to new results and new projects. The appendix by Alexander Merkurjev in Nikita Karpenko's recent preprint "Finite extensions partially splitting PGO-torsors" is an explicit outcome of the meeting. A joint project of Charles De Clercq, Nikita Karpenko, and Anne Quéguiner-Mathieu, which aims at extending the recent results of De Clercq and Quéguiner-Mathieu mentioned above to groups of outer type is work in progress, following a suggestion of several participants after De Clercq's talk.

This was the first major conference on the topics discussed in this report since the onset of the COVID-19 pandemic. As many of the participants communicated, the overall atmosphere was excellent. We had a chance to meet young colleagues, some for the first time, and several of them presented their work in talks. Zinovy Reichstein reported that *the two UBC students who attended the workshop were both happy to be invited. They enjoyed the workshop and learned a lot from it*. One of them has been following up some suggestions he got in response to his lecture, and the other has extended the result he talked about to the characteristic-2 case during the conference.

Participants

Alsaody, Seidon (Uppsala University)
Baek, Sanghoon (KAIST)
Barry, Demba (Université des Sciences des Techniques et des Technologies de Bamako)

Bayer-Fluckiger, Eva (Ecole Polytechnique Federale de Lausanne (EPFL))
Bingöl, Fatma Kader (Galatasaray University)
Blanks, Tamar (Fordham University)
Calmes, Baptiste (Université d'Artois)
Chapman, Adam (Tel-Aviv-Yaffo Academic College)
Chernousov, Vladimir (University of Alberta)
Colliot-Thélène, Jean-Louis (Université Paris-Saclay)
De Clercq, Charles (Université Sorbonne Paris Nord)
Edens, Oakley (University of British Columbia)
First, Uriya (University of Haifa)
Florence, Matthieu (Sorbonne Université)
Garibaldi, Skip (Center for Communications Research La Jolla)
Garrel, Nicolas (Université de Tours)
Geldhauser, Nikita (LMU Munich)
Gille, Stefan (University of Alberta)
Gille, Philippe (Université Lyon 1)
Gosavi, Saurabh (Technion-Israel Institute of Technology)
Grimm, David (Universidad de Santiago de Chile)
Hartmann, Julia (University of Pennsylvania)
Haution, Olivier (University of Milano-Bicocca)
Hoffmann, Detlev (TU Dortmund)
Izquierdo, Diego (Ecole Polytechnique)
Karpenko, Nikita (University of Alberta)
Krashen, Daniel (University of Pennsylvania)
Laghribi, Ahmed (Artois University)
Lee, Ting-Yu (National Taiwan University)
Lorenz, Nico (Ruhr-Universität Bochum)
Lucchini Arteche, Giancarlo (Universidad de Chile)
Mackall, Eoin (University of California, San Diego)
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Parimala, Raman (Emory University)
Pianzola, Arturo (University of Alberta)
Quéguiner-Mathieu, Anne (Université Sorbonne Paris Nord)
Reichstein, Zinovy (University of British Columbia)
Ruether, Cameron (Memorial University of Newfoundland)
Scavia, Federico (University of California Los Angeles)
Scully, Stephen (University of Victoria)
Sechin, Pavel (Universität Regensburg)
Sobiech, Marco (TU Dortmund)
Srinivasan, Srimathy (Tata Institute of Fundamental Research)
Suresh, Venapally (Emory university)
Unger, Thomas (University College Dublin)
Vishik, Alexander (University of Nottingham)
Zaninelli, Marco (University of Antwerp)
Zaynullin, Kirill (University of Ottawa)
Zemkova, Kristyna (University of Alberta)

Zhykhovich, Maksim (Ludwig Maximilian University of Munich)

Chapter 27

The Mathematics and Physics of Moiré Superlattices (23w5058)

October 22 - 27, 2023

Organizer(s): Mitchell Luskin (University of Minnesota), Svetlana Jitomirskyaya (UC Berkeley), Eunah Kim (Cornell University), Lin Lin (University of California, Berkeley), Allan MacDonald (University of Texas)

Overview of the Field

Moiré materials are formed by overlaying two dimensional crystals that have slightly different lattice constants, or that have a small relative rotation, thus creating larger moiré patterns like the beat effect. The advantage of moiré materials over real atomic materials is that their effective lattice constants are typically on the tens of nm length scale. At this length scale, the number of electrons per period of the pattern can be adjusted experimentally over a range of 1 to 10 simply by adjusting electrical gate voltages. The electron density is typically the most critical control parameter for any type of electronic phenomena. When the number of electrons per atom in a material changes by an amount of order 1, it changes its chemistry, i.e., it changes its quantum physics even at a high energy scale. Achieving an understanding of how smooth changes at the highest energy scales control transitions between quantum ground states with distinct properties, for example between superconductors and insulators, is the essential challenge of quantum materials physics. In moiré materials, quantum properties of interest are routinely studied experimentally over the full relevant range of electron density without introducing disorder from chemical dopants. The list of interesting known states of matter that have already been realized in moiré materials includes superconductors, Mott insulators, Chern insulators, charge-transfer insulators, ferromagnetic metals, and charge density-wave states. It is abundantly clear that many other states of matter are waiting in the wings.

Motivated by theoretical work that pointed to new opportunities, tremendous progress has been made over the past several years in using moiré superlattices formed from two-dimensional materials as a laboratory for the study of quantum materials. This development has opened up an exciting new scientific opening, one in which mathematics and theoretical physics have a very large role to play by identifying moiré systems that are likely to exhibit new or poorly understood electronic phenomena, by inventing mathematical methods that enable testable predictions of physical properties, and by unravelling the meaning of experimental observations.

This BIRS workshop combined expertise from mathematics and physics to push the frontiers of this field. The presentations and discussions identified new moiré superlattice systems of interest, derived and solved models that accurately describe their electronic properties, and invented new concepts to explain experimental surprises.

All steps above require active interactions between physicists and mathematicians. This field thus presents an excellent opportunity to foster cross-fertilization between different branches of mathematics as the building and simulation of the reduced models, as well as the analysis and physical interpretation of the results require expertise from partial differential equations, operator theory, dynamical systems, geometry, topology, probability theory, optimization, and numerical linear algebra.

An exciting feature of the study of moiré theory is the recent and continuing development of experimental methods to probe the phenomena being investigated by theorists contemporaneously. This workshop included the active participation of several of the pioneering experimental moiré physicists.

Recent Developments and Open Problems

The nm length scale periodicity of moiré materials makes it possible for the first time to fabricate devices in which the number of magnetic flux quanta per period exceeds 1. This development opens new opportunities to study the math and physics embedded in Hofstadter-type electronic structure models that account for incommensurability between Aharonov-Bohm phase variations and matter periodicity. The emergence of moiré materials thus highlights new opportunities to make math progress by looking to nature, and to make physics progress by learning from math.

The natural incommensurability of twisted lattices has received little rigorous attention. At the same time, recently developed rigorous methods have unexplored potential to identify novel phenomena and inform our search for new phases of matter. Our workshop sought to motivate significant progress in both directions by bringing together leading mathematicians and physicists to collaborate on this search.

Closely related Hamiltonians can be generated simply by fabricating incommensurate moiré of moiré systems with two or more distinct twist angles or lattice constant differences, such as the twisted trilayer systems recently investigated by several workshop participants and the workshop organizer.

Moiré materials also present an exciting platform for designing topological phases with an unprecedented level of control. It has already been established that the alignment of magic-angle graphene with a boron nitride substrate creates a topological Chern insulator in which broken sublattice symmetry leads to a Dirac mass and therefore to non-zero Chern numbers in moiré bands. Several theoretical and experimental presentations described recent developments and opportunities for the development of mathematical foundations.

Moiré systems such as tBLG present significant and perhaps long-lasting challenges from a computational perspective. First, bilayer moiré systems at magic angles have a very large effective unit cell (called the moiré unit cell) consisting of approximately 10^4 atoms. The moiré lattice consists of many such unit cells, and therefore the physics of interest only occurs on a mesoscopic scale and the system size is well beyond 10^5 atoms. Bistritzer and MacDonald developed an emergent low-energy model in 2011 that is the basis for most current theoretical and computational investigations of twisted bilayer graphene.

Even more challenging is direct simulation at the mean-field (Hartree-Fock or DFT) level due to the cubic scaling with respect to the number of atoms. Because the physics of interest is an energy scale that is on the order of meVs around the flat energy bands, compared to the atomic energy scale that is at least 10^3 times larger, direct numerical simulation must be very accurate, or the behavior at the scale of the moiré unit cell may not exceed numerical inaccuracy tolerances. Hence, reduced order modeling (ROM) is essential for studying moiré physics. Reduced order modeling is also important in order to study correlated electronic structures beyond the mean-field level, where current processes for constructing effective many-body models from first principles are highly empirical, and standard correlated solvers can often only treat systems with up to 100 orbitals. Such a reduction often involves identifying relevant localized orbitals in the moiré unit cell. However, recent works indicate that the localization step is not simple either, due to topology obstructions. This gives rise to the challenge of finding a mathematically principled way to faithfully and systematically derive the reduced model for interacting electrons of moiré systems. Even when the model is identified, the accurate solution of the reduced model at the interacting level is also very expensive. Therefore, it is challenging to solve the interacting model at the proper level of quantum many-body physics that balances the efficiency and accuracy.

These examples demonstrate that accurate modeling and simulation of moiré systems is a very challenging and interdisciplinary field. The standard methods of electronic structure theory fail. All steps above require active interactions between mathematicians and physicists. This field is also an excellent opportunity to foster cross-fertilization between different branches of mathematics as the building and simulation of the reduced models, as well as the analysis and physical interpretation of the results require expertise from partial differential equations, operator theory, dynamical systems, geometry, topology, probability theory, optimization, and numerical linear algebra.

Presentation Highlights

Allan MacDonald: Some math and physics moiré material challenges

In this talk I will highlight three interesting and important challenges facing moiré materials researchers: i) How can we build a better models of moiré materials ? ii) Can we open new ground in the math and physics of quasi-periodic Hamiltonians? iii) Can we develop a predictive understanding of fractional Chern insulator states in moiré materials? For i) twisted bilayer graphene seems to be the most challenging case. Current models do not fully account for the non-local nature of the interactions between flat band electrons and the negative energy sea, and this may be necessary if we are ever to achieve a quantitative understanding of the competition between superconducting, Fermi liquid, and magnetic states. For ii), the moiré material platform allows creation of flexibly tunable quasiperiodic two-dimensional Hamiltonians. The physical properties of these systems will be influenced by interactions as well as by single-particle physics. There is an opportunity, I believe, to pose new mathematical questions. One important goal is to learn how to calculate and how to measure the quantum numbers that characterize spectral gaps. For iii), we are in early days. The discovery of fractional Chern insulator states in moiré materials is a breakthrough event in condensed matter physics. Can we use the tunability of moiré materials to stabilize fractionalized states that support non-Abelian quasiparticles?

Kin Fai Mak: Magnetism in doped moiré Mott insulators

Moiré materials provide a highly controllable platform to explore the strong electronic correlation phenomena. Specifically, Mott insulators with a lattice of local magnetic moments have been observed in semiconductor moiré bilayers in the flat band limit. In this talk, I will discuss experiments studying magnetism in doped moiré Mott insulators. One observation involves the spin polarons—bound states of a doped hole and a spin flip—in a hole-doped triangular lattice Mott insulator. The second involves the emergence of ferromagnetism at the onset of a Kondo breakdown transition in a moiré Kondo lattice (a lattice of local moments exchange-coupled to conduction electrons).

Lin Lin: Exact ground state of interacting electrons in magic angle graphene

One of the most surprising theoretical discoveries of magic angle twisted bilayer graphene is that in the chiral limit, certain single Slater determinants can be the ground state of the flat-band interacting Hamiltonian. This provides an explanation of the correlated insulating phase at integer filling. We investigate the symmetry attributes of the interacting Hamiltonian and the resulting ground states. This allows us to study systems beyond TBG, including TBG-like systems with 4 flat bands per valley, and equal twist-angle trilayer graphene systems (joint work with Simon Becker and Kevin Stubbs).

Fabian Faulstich: Interacting models for twisted bilayer graphene: Towards a quantum chemistry approach

We present a numerical study of an interacting Bistritzer-MacDonald (IBM) model of TBG using a suite of methods in quantum chemistry, including Hartree-Fock, coupled cluster singles, doubles (CCSD), and perturbative triples (CCSD(T)), as well as a quantum chemistry formulation of the density matrix renormalization group method

(DMRG). At integer filling, all numerical methods agree in terms of energy and $C_{2z}T$ symmetry breaking. Additionally, as part of our benchmarking, we explore the impact of different schemes for removing "double-counting" in the IBM model. Our results at integer filling suggest that cross-validation of different IBM models may be needed for future studies of the TBG system. After benchmarking our approach at integer filling, we perform a systematic study of the IBM model near integer filling. In this regime, we find that the ground state can be in a metallic and $C_{2z}T$ symmetry breaking phase. The ground state appears to have low entropy, and therefore can be relatively well approximated by a single Slater determinant. Furthermore, we observe many low entropy states with energies very close to the ground state energy in the near integer filling regime.

Shiwei Zhang: Treating electron interactions in moiré systems

The interplay between strong interactions and the unique environment created by moiré superlattices is believed to be a key for many of the most exciting phenomena seen in these two-dimensional materials. It is then important to treat both one- and two-body effects in a balanced and accurate way. This presents outstanding theoretical, algorithmic, and computational challenges. I will describe our preliminary work on developing and applying quantum Monte Carlo methods to treat moiré systems. This includes a study of the interacting Bistritzer-MacDonald model beyond specialized fillings which are sign-problem-free, calculations in a continuum model of two-dimensional electron gas in moiré potentials, and the parametrization of an exchange-correlation functional for density-functional theory calculations in two-dimensional materials.

Leni Bascones: Heavy quasiparticles and cascades without symmetry breaking in twisted bilayer graphene

Twisted bilayer graphene (TBG) exhibits a plethora of electronic phases. Among the variety of correlated states, the cascades in the spectroscopic properties and in the compressibility happen in a much larger energy [1,2,3], twist angle and temperature range than other effects, pointing to a hierarchy of phenomena. In this work [4], we show that the spectral weight reorganization associated to the formation of local moments and heavy quasiparticles, and not a symmetry breaking process, is responsible for the cascade phenomena. Among the phenomena reproduced in this framework are the cascade flow of spectral weight, the oscillations of the remote band energies and the asymmetric jumps of the inverse compressibility. Due to the fragile topology of TBG, we predict a strong momentum differentiation in the incoherent spectral weight. In the talk, I will also address other possible measurements which may help distinguishing the phenomenology of the cascades discussed here from proposals involving symmetry breaking.

[1] Wong et al, Nature 582, 198 (2020) [2] Zondiner et al, Nature 582, 203 (2020) [3] Polski et al, arXiv:2205.05225 [4] A. Datta, M.J. Calderón, A. Camjayi, E. Bascones, Nature Comms. 14, 5036 (2023)

Eslam Khalaf: Spin polarons in topological ferromagnets: application to graphene moire heterostructures

Understanding the phase diagram of twisted bilayer graphene and related moiré systems is a central theoretical challenge. While the ground states at integer fillings have been shown in many cases to be simple flavor ferromagnets, the charge excitations above such states can be non-trivial due to band topology. Conventional approaches to understand such excitations as real space topological textures fail to account for the distinct momentum space features of Chern bands and obscures their comparison to single particle excitations. Here, we present a general fully momentum space formulation for the problem of charge excitations in Chern bands. In the limit of (normal-ordered) contact interactions in an ideal flat band, we construct exact analytical wavefunctions for the lowest energy excitation with charge ±e and spin n+1/2 , a spin polaron. Away from this ideal limit, we show that these analytical wavefunctions are excellent variational states describing a bound state of an electron/hole with n spin flips. We show that the ansatz can be cast in the form of an antisymmetrized electron-hole geminal power and develop a diagrammatic approach to evaluate the expectation values of operators in such states, allowing us to study relatively large number of spin flips and large system sizes. We apply our formalism to study charge excitations in twisted bilayer graphene and find that (i) in the chiral limit, multispin flip polarons are the lowest energy charge excitations at charge neutrality and at non-zero integer fillings when doping towards neutrality. In the realistic limit, we find that the multispin flip states are the lowest charged excitations at $\nu = \pm(1-\varepsilon)$ for any strong coupling state and at $\nu = \pm(2-\varepsilon)$ for the time-reversal intervalley coherent state (TIVC) but not the Krammers intervalley coherent state (KIVC). We discuss the experimental implications of these results for low strain devices.

Eunah Kim: Fractionalization in 1/3 filled twisted bilayer graphene

The best-established example of fractionalization starts from the partially filling flat kinetic energy dispersion, namely the fractional quantum Hall effect. We will show that twisted bilayer graphene systems present new platforms for arriving at fractionalization: fractional correlated insulating state. Various 1/3 filled twisted bilayer graphene are expected to form Chern number=0 incompressible states driven by the extended fidget spinner-shaped Wannier orbitals. The geometric frustration due to the orbital shapes leads to charge 1/3 excitations with restricted mobility. The restricted mobility limited to the sub-dimension gives fracton-like character to the charge 1/3 excitations. I will discuss theoretical predictions on how to detect such fractional correlated insulators and preliminary experimental results.

Simon Becker: The mathematics of the chiral limit – What we know and what we don't know!

I will report on 3-4 years of mathematical analysis on the chiral limit of the BMH model and report on our understanding for bilayer and multilayer systems, and what open questions remain. My understanding of the subject has been shaped by collaborations with Mark Embree, Tristan Humbert, Ryan Kim, Izak Oltman, Martin Vogel, Jens Wittsten, Mengxuan Yang, Xiaowen Zhu, and Maciej Zworski.

Patrick Ledwith: Vortexable Chern bands and fractional Chern insulators in moiré graphene and transition metal dichalcogenides

Fractional Chern insulators realize the remarkable physics of the fractional quantum Hall effect (FQHE) in crystalline systems with Chern bands. The lowest Landau level (LLL) is known to host the FQHE, but not all Chern bands are suitable for realizing fractional Chern insulators (FCI). Previous approaches to stabilizing FCIs focused on mimicking the LLL through momentum space criteria. Here instead we take a real-space perspective by introducing the notion of vortexability. Vortexable Chern bands admit a fixed operator that introduces vortices into any band wavefunction while keeping the state entirely within the same band. Vortexable bands admit trial wavefunctions for FCI states, akin to Laughlin states. In the absence of dispersion and for sufficiently short-ranged interactions, these FCI states are the ground state – independent of the distribution of Berry curvature. Vortexable Chern bands emerge naturally in chiral twisted graphene, and fractional Chern insulators were subsequently observed experimentally. Recently, zero-field fractional Chern insulators, and potentially a zero-field composite Fermi liquid, were also observed in the nearly-vortexable twisted $MoTe_2$. New and exciting nearly-vortexable platforms are also appearing, including periodically strained graphene and helically twisted graphene.

Senthil Todadri: Quantum anomalous Hall physics in moiré bilayers: proximate phases and phase transitions

Quantum Hall phases are the most exotic experimentally established quantum phases of matter. Recently they have been discovered at zero external magnetic field in two dimensional moiré materials. I will describe recent work on their proximate phases and associated phase transitions that is motivated by the high tunability of these moire systems. These phase transitions (and some of the proximate phases) are exotic as well, and realize novel 'beyond Landau' criticality that have been explored theoretically for many years. I will show that these moiré platforms provide a great experimental opportunity to study these unconventional phase transitions and related unconventional phases, thereby opening a new direction for research in quantum matter.

Raquel Queiroz: Stability of chiral Landau levels and its implications for twisted heterostructures

Perfectly flat bands in moiré materials have intimate connections to Landau levels of Dirac fermions. In this talk, we will investigate how the robustness of zeroth Landau level to chiral disorder relates to the stability of moiré flat bands against certain types of disorder. In light of moiré TMDs, we will also discuss the implications for massive Dirac fermions where the chiral symmetry is explicitly broken.

Jie Shan: Fractional Chern insulators and electric-field-induced topological phase transitions in moiré $MoTe_2$

The recent discovery of fractional Chern insulators (FCIs), which can exhibit the fractional quantum anomalous Hall effects, has attracted much scientific interest. I will discuss thermodynamic studies on the FCIs and the non-topological states in twisted bilayer $MoTe_2$. I will particularly focus on the nature of the electric-field-induced transitions between the FCIs and the non-topological states. I will also compare our thermodynamic studies with recent transport studies and discuss its implications.

Francisco Guinea: Superconductivity in graphene stacks

Superconductivity has been observed in a number of twisted and untwisted graphene multilayers. The dependence of the superconducting properties on the geometry of the graphene stack will be discussed. The possibility of novel phenomena due to non trivial order parameters will also be highlighted.

Mikito Koshino: Topological moiré trilayers

In addition to the extensive study of twisted moiré bilayers in the past decade, the scope of investigation has extended to encompass multilayer systems including three or more layers. Particular attention has recently been directed toward twisted trilayer systems which consists of three layers arranged in a specific rotational configuration. A twisted trilayer is characterized by two twist angles between adjacent layers, offering a vast parameter space that remains largely unexplored. In the first part of my talk, we will present systematic theoretical studies on the lattice relaxation and the electronic structures in general twisted trilayer graphenes [1]. We show that the relaxed lattice structure forms a patchwork of moiré-of-moiré domains, where a moiré pattern given by layer 1 and 2 and another pattern given by layer 2 and 3 become locally commensurate. The electronic band calculation reveals a wide energy window featuring sparsely distributed highly one-dimensional electron bands. These one-dimensional states exhibit a sharp localization at the boundaries between supermoiré domains, and they are identified as a topological boundary state between distinct Chern insulators. In the latter part of our discussion, we will explore the electronic structure of hBN/graphene/hBN trilayer system with arbitrary twist angles. We find that the electronic spectrum displays fractal minigaps akin to the Hofstadter butterfly. We demonstrate that each of minigaps is uniquely labeled by six topological numbers associated with the quasicrystalline Brillouin zones, and these numbers can be expressed as second Chern numbers through a formal connection with the quantum Hall effect in four-dimensional space [2,3].

[1] N. Nakatsuji, T. Kawakami, and M. Koshino, arXiv:2305.13155; Phys. Rev. X, in press. [2] M. Koshino, H. Oka, Phys. Rev. Research 4, 013028 (2022) [3] H. Oka and M. Koshino, Phys. Rev. B 104, 035306 (2021)

Jennifer Cano: Topological flat bands in bilayer graphene with a superlattice potential

We propose an externally imposed superlattice potential as a platform for engineering topological phases, which has both advantages and disadvantages compared to a moiré superlattice. We show that a superlattice potential applied to Bernal-stacked bilayer graphene can generate flat Chern bands, similar to those in twisted bilayer graphene, whose bandwidth can be as small as a few meV. Further, the flat band has a favorable band geometry for realizing a fractional Chern insulator at partial filling. The superlattice potential offers flexibility in both lattice size and geometry, making it a promising alternative to achieve designer flat bands without a moiré heterostructure.

Oskar Vafek: Interacting narrow bands of twisted bilayer graphene in magnetic field

In the first part of the talk I will discuss magneto-transport experiments on twisted bilayer graphene at 1.32 degree twist angle, i.e., away from the magic value. Despite the absence of correlated states at B=0, the theoretical explanation of these experiments provides insight into the origin of the Landau level degeneracy near the charge neutrality point and the role of heterostrain [1]. Equipped with this understanding, I will present a comprehensive Hatree-Fock study of interacting electrons in finite magnetic field while varying the electron density, twist angle and heterostrain. Within a panoply of correlated Chern phases emerging at a range of twist angles, I will present a unified description for the ubiquitous sequence of states with the Chern number t for $(s,t) = \pm(0,4), \pm(1,3), \pm(2,2)$ and $\pm(3,1)$. Correlated Chern insulators at unconventional sequences with $s + t \neq \pm 4$ are also found, as well as with fractional s. I will discuss their nature [2].

[1] Xiaoyu Wang et al. PNAS2023 Vol. 120 No. 34 e2307151120 [2] Xiaoyu Wang and O.Vafek arXiv.2310.xxxx

Dumitru Calugaru: Heavy-fermion physics in twisted bilayer graphene

Twisted bilayer graphene (TBG) displays two seemingly contradictory characteristics: (1) quantum-dot-like behavior in STM suggesting electron localization; (2) transport experiments indicating an itinerant nature. Both aspects can be naturally captured by a topological heavy-fermion model where topological conduction electron bands interact with local moments. We study the local-moment physics and the Kondo effect within this model. We reveal that at integer fillings ($\nu = -2, -1, 0, 1, 2$), the RKKY interactions favor ferromagnetic states satisfying a U(4) Hund's rule. Conversely, at non-integer fillings, the Kondo effect becomes significant, resulting in a Kondo resonance in the spectral function. Through our heavy-fermion model, we also explore the transport properties of TBG. We identify two primary types of carriers: incoherent f electrons and coherent c electrons. The coherent c electrons dominate the transport properties and give rise to a fully negative Seebeck coefficient at positive fillings. We also show that our model can also reproduce various aspects of the physics of TBG, including a natural explanation of the IKS state and its wavevector, as well as the reason for the existence of stronger correlated states for positive integer fillings, despite the bare band structure being more dispersive on that side.

Eric Cances: Semiclassical analysis of moiré Hamiltonians

The method introduced in [1] allows one to construct an approximate Kohn-Sham Hamiltonian for (incommensurate) twisted bilayer graphene. In the first part of the talk, I will show how an effective moiré-scale continuum model, similar though not identical to the Bistritzer-MacDonald model, can be derived from this Hamiltonian by simple variational approximation [2]. In the second part of the talk, I will show that methods from semiclassical analysis can be used to study the density-of-states of this Hamiltonian in the limit of small twist angles [3].

[1] G. Tritsaris, S. Shirodkar, E. Kaxiras, P. Cazeaux, M. Luskin, P. Plechac, and E. Cancès, Perturbation theory for weakly coupled two–dimensional layers, J. Mater. Res. 31 (2016) 959–966 [2] E. Cancès, L. Garrigue and D. Gontier, Simple derivation of moiré-scale continuous models for twisted bilayer graphene, Phys. Rev. B 107 (2023) 155403. [3] E. Cancès and L. Meng, Semiclassical analysis of two-scale electronic Hamiltonians for twisted bilayer graphene, in preparation.

Guillaume Bal: Robust asymmetric interface transport in topological insulators

The surprising robustness to perturbation of the asymmetric transport observed along interfaces separating distinct insulating bulks has a topological origin. This talk reviews recent classifications of partial differential operators modeling such systems. A classification by means of domain walls provides a topological invariant whose calculation as an explicit integral is straightforward. A general bulk-interface correspondence then proves that the invariant also describes the quantized aforementioned asymmetric transport. The theory is illustrated on several examples of applications and in particular gated twisted bilayer graphene models.

Dionisios Margetis: Chirality and edge plasmons in the twisted bilayer graphene

In this talk, I will discuss recent progress in understanding implications of the electrical conductivity tensor, coming from the Kubo formalism, for the twisted bilayer graphene (TBG) and similar heterostructures. The use of a spatially homogeneous and isotropic tensor conductivity has led us to the analytical derivation of a dispersion relation for non-retarded edge plasmons. This relation explicitly depends on the chiral response of the system. I will describe a correspondence of the chiral optical plasmon in the TBG to the magnetoplasmon in the single-layer graphene, by introducing an effective magnetic field. If time permits, I will also discuss related extensions of the theory to the twisted trilayer and quadrilayer graphene systems. In the analysis, the long-range electrostatic interaction is retained via application of the Wiener-Hopf method of factorization to systems of integral equations for scalar fields.

Daniel Massatt: Observables of an incommensurate bilayer linear Schrödinger equation

We formulate a plane-wave basis representation of an incommensurate bilayer linear Schrödinger equation. We use the representation to find algorithms for a number of fundamental electronic observables such as the local density of states of spatial configurations, total density of states, and the local density of states in momentum space, which is a parallel object to electronic band structure in the absence of periodicity. We further prove the equivalence of the density of states in the plane-wave formulation to that of the density of states of the real space Schrödinger equation through a properly averaged thermodynamic limit. The methodology relies on tracking the plane-wave scattering between incommensurate potentials and using these 'hopping' parameters to construct a matrix describing the coupling of all interacting plane-waves, which we find to be indexed by a four-dimensional lattice. The algorithm relies on truncation of the matrix via an energy truncation and hopping distance truncation with rigorous convergence rates.

Alexander Watson: Moiré materials from atomic to moiré scales

I will review progress towards realistic yet manageable models of the electronic properties of twisted bilayer graphene and other moiré materials. First, I will present a general approach to modeling atomic relaxation in moire materials using interatomic potentials. Then, I will discuss a general framework for deriving corrections to effective continuum models such as the Bistritzer-MacDonald model.

Wencai Liu: Algebraic geometry, complex analysis and combinatorics in spectral theory of periodic graph operators

In this talk, we will discuss the significant role that the algebraic properties of Bloch and Fermi varieties play in the study of periodic graph operators. I will begin by highlighting recent discoveries about these properties, especially the irreducibility. Then, I will show how we can use these findings, together with techniques from complex analysis and combinatorics, to study spectral and inverse spectral problems arising from periodic graph operators.

Xiaowen Zhu: Cantor spectrum of a 1D moiré model

It is known that the magnetic Schrödinger operator on a 2D lattice with irrational flux has Cantor spectrum, which illustrates the well-known picture of "Hofstadter butterfly". In this talk, I will introduce the proof of Cantor spectrum for another model - a 1D moiré model. In particular, the moiré pattern plays a key role in the exhibition of Cantor spectrum in a relatively robust way that is intrinsically different from magnetic fields. This implies further potential of understanding other moiré-pattern models. The talk is based on a joint work with Simon Baker and Svetlana Jitomirskaya.

Daniele Guerci: Interplay of moiré patterns: TBG on aligned hBN and helical trilayer graphene

In my presentation, I will explore two illustrative examples where the interplay of two moiré patterns to the formation of large moiré domains where commensuration is restored: TBG on aligned hBN and helical trilayer graphene. The latter gives rise to a topological Chern mosaic dominated by regions of ABA/BAB stacking forming a periodic triangular pattern on the moiré of moiré scale. I will provide a detailed exploration of the origins of the electronic bands in the chiral limit. Exact results will reveal the existence of a Chern 2 band with unique properties that cannot be reduced to a single lowest Landau level [1-3]. Notably, our predictions are consistent with recent experimental findings [4], underscoring the significance of these moiré patterns in uncovering novel topological phenomena.

[1] D. Guerci, Y. Mao, C. Mora, arXiv:2305.03702 (2023) [2] D. Guerci, Y. Mao, C. Mora, arXiv:2308.02638 (2023) [3] Y. Mao, D. Guerci, C. Mora, PRB 107, 125423 (2023) [Editors' Suggestion] [4] L. Xia, P. Jarillo-Herrero et al., arXiv:2310.12204 (2023)

Daniel Bennett: Twisted bilayer graphene revisited: where is the "magic"?

The moiré pattern observed experimentally in twisted bilayer graphene (tBLG) clearly shows the formation of different types of domains. These domains can be explained by the atomic relaxation, both in-plane and out-of-plane, using continuum elasticity theory and the Generalized Stacking Fault Energy (GSFE) concept. Moreover, the atomic relaxation significantly affects the electronic states, leading to a pair of flat bands at the charge neutrality point which are separated by band gaps from the rest. These features appear for a small range of twist angles, that we call the "magic range", around the twist angle of 1 degree. We discuss how all these aspects of the system are crucial for understanding the origin of correlated states and superconductivity in tBLG. We also present a minimal model that can capture these features with 2 flat and 2 auxiliary bands and explore the implications of the model for correlated electron behavior in the context of the Hubbard model.

Scientific Progress Made

Active discussions occurred during the presentations, breaks, meals, and along hiking trails on Wednesday afternoon. Informal sessions were also held the hour before dinner, For example, an informal session was held on Wednesday on interacting Bistritzer-MacDonald models and on Thursday on configuration-based relaxation models for twisted trilayer graphene and other multi-layer heterostructures.

Outcome of the Meeting

The moiré mathematics community has recently begun to investigate interacting models for twisted bilayer graphene and other 2D heterostructures. The opportunity for the mathematical community to have detailed discussion with the theoretical physicists who developed these interacting models will enable further collaboration between mathematicians and physicists on the investigation of correlated phases in moiré systems and the development of rigorous mathematical foundations.

Participants

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Chen, Huajie (Beijing Normal University)
Clark, Drake (University of Minnesota)
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Fu, Liang (Massachusetts Institute of Technology)
Guerci, Daniele (Flatiron Institute)
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Zhu, Ziyan (Stanford University)

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Chapter 28

Astrostatistics in Canada and Beyond (23w5094)

Octover 30 - Nov 3, 2023

Organizer(s): Pauline Barmby (Western University), Gwen Eadie (University of Toronto), Gregory Sivakoff (University of Alberta), David Stenning (Simon Fraser University)

Overview of the Field

Astrostatistics is the development and application of statistical methods to measurements of the Universe beyond the Earth's atmosphere. As one of the oldest quantitative sciences, astronomy was once linked closely with statistics—for example, problems in Newtonian celestial mechanics were the driver for the development of the least-squares method [7]. In the last century, this link weakened as the field shifted from solely collecting astronomical observations to considering their astrophysical intepretation. In the past few decades, astrophysics and statistics have begun to re-approach one another [3, 4, 6].

Astrophysics has gone from being a data-poor field to one with an embarrassment of data riches, with complex, high-dimensional datasets that require sophisticated methods for statistical inference. Astronomy runs the gamut in terms of data type, including data products such as images, spatial data, time series data, and categorical data. Most of these data are collected through observational studies without the ability for repeated sampling or controlled experiments. Thus, characteristic challenges of astrophysics data include but are not limited to heteroscedastic measurement errors, truncated samples, incomplete data, and non-repeatability [8]. At the same time, issues in astrophysics models include parameter degeneracy, uncertainty quantification, over-simplification, and over-fitting [10]. The statistical community is becoming more aware of the interesting problems and opportunities provided by astrophysics, and the astronomy community is recognising the need for increased connections and collaborations with many different fields in statistics.

Recent Developments and Open Problems

The Canadian statistical and astrophysical communities have not worked together extensively in the past, but times are changing, with several joint faculty positions recently being filled and with the potential for interdisciplinary

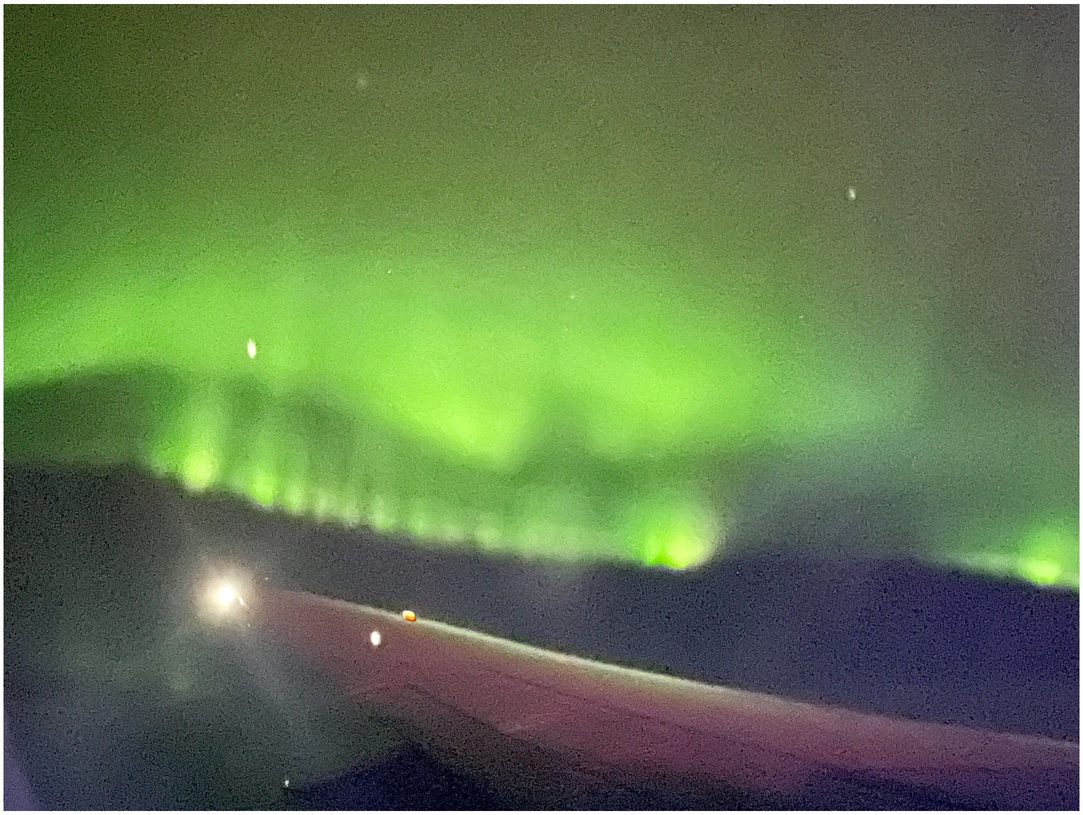

Figure 28.1: Airborne aurora borealis from ”Astrostatistics in Canada and Beyond” return trip. Credit: D. Li (PhD student, University of Toronto)

collaboration through grant programs of the Canadian Statistical Sciences Institute (CANSSI; e.g. the CANSSI Collaborative Research Team Projects). The explosive interest in big data is driving a need for the development of new statistical tools and approaches, and astrophysics datasets provide an ideal avenue for exploration and testing. Canadians are involved in many new projects that generate large quantities of data, including the Canadian Hydrogen Intensity Mapping Experiment (CHIME; [2]), the Legacy Survey of Space and Time (LSST; [9]) with the Vera C. Rubin Observatory, and the SKA Observatory [11] and its precursors. The massive datasets produced by these world-class facilities require statistical expertise to be most efficiently exploited for physical understanding.

The novelty of this workshop was both in the imminent arrival of massive datasets from new facilities and on the emphasis on what astrophysicists and statisticians can offer to one another as research partners. Unlike other recent efforts, the workshop focused not on a particular astrophysical application but on providing a broad introduction to the possibilities for collaboration, particularly between the Canadian communities. We provided structured space and time for groups of participants who do not normally meet to identify links between astronomy and statistics and start new collaborations and projects. Remote participation worked well for participants who were unable to travel for various reasons; one travel highlight was an amazing view of the aurora borealis from a participant’s return flight to Toronto (Figure 28.1)!

Presentation Highlights

The first day of the meeting included four review talks, on the history of astrostatistics, big datasets in astronomy, statistical learning methods for big data, and astrostatistics with ’small data.’ All meeting participants introduced themselves with ’lightning talks’: 60-second talks using a single presentation slide. This format was new to the statistician participants, and enjoyed by all. The day finished with a panel discussion about funding opportunities for interdisciplinary projects, such as in astrostatistics.

All participants who wanted to present their research were able to do so: early-career researchers gave 20-minute talks on the mornings of day 2 and 4, and more senior researchers gave 10-minute talks on the morning of day 3. Most talks were in person, but several participants appreciated the opportunity to present and attend remotely; the Zoom hybrid meeting technology worked very well for presentations. The ECR talks in particular were all very well done and well-received by both astronomers and statisticians. There was a wide range of presentation topics, from fast radio bursts [5] to stratified learning [1], solar flares [12] and copulas [13], with slides available at `bit.ly/birs-astro-talks`. Two presentations later led to an impromptu self-directed discussion on copulas in astronomy. Notably, two ECR presentations sparked productive afternoon discussions between astronomers and statisticians about statistical methods for fast radio bursts and galaxies. The morning of day 5 featured summary talks from two of the organizers ("Statistics" Summary Talk from an "Astronomy" Point-of-View and vice versa) and a workshop-wide discussion of next steps (see Section 28).

Scientific Progress Made

The afternoons of days 2 and 4 were spent in self-organized discussions: from a list of topics, participants voted on the ones they were most interested in, and broke into smaller groups for discussions. Some sessions were repeated so that participants could discuss multiple topics. Discussion topics included:

- Simulation-based and likelihood free inference
- MCMC / Bayesian
- Noisy / biased / incomplete data
- Electromagnetic transients
- Data handling
- Astronomical surveys
- Probabilistic catalogues
- Astrostatistics resources

For most of the discussions summary notes were taken and shared with all participants via Google Docs. These notes included links to relevant projects, publications, and tutorials as well as records of the discussion. The shared meals and excursions also led to several small group impromptu discussions about individual students' work, which were very productive and led to initiation of new collaborations. Most remote participants indicated in advance that they would likely not participate in these discussions; we were able to arrange Zoom participation for those who were interested.

Outcome of the Meeting

Participants were polled for their feedback during the closing session using Mentimeter. They reported becoming more familiar with the terminology of either statistics or astronomy (Figure 28.2) and appreciated the chance to meet researchers from other fields and learn from them (Figure 28.3).

The 'next steps' discussion on the workshop's last day led to a host of ideas for further developing astrostatistics in Canada and beyond. Joining existing professional groups, such as the International Astrostatistics Association (IAA), the Astrostatistics Interest Group (AIG) of the American Statistical Association, or the Working Group on Astroinformatics and Astrostatistics (WGAA) of the American Astronomical Society, is one way to keep in touch with the community. The idea of creating a Canadian group such as a chapter of the Statistical Society of Canada, or a mailing list of interested researchers, was also discussed. The workshop Slack channel workspace is another

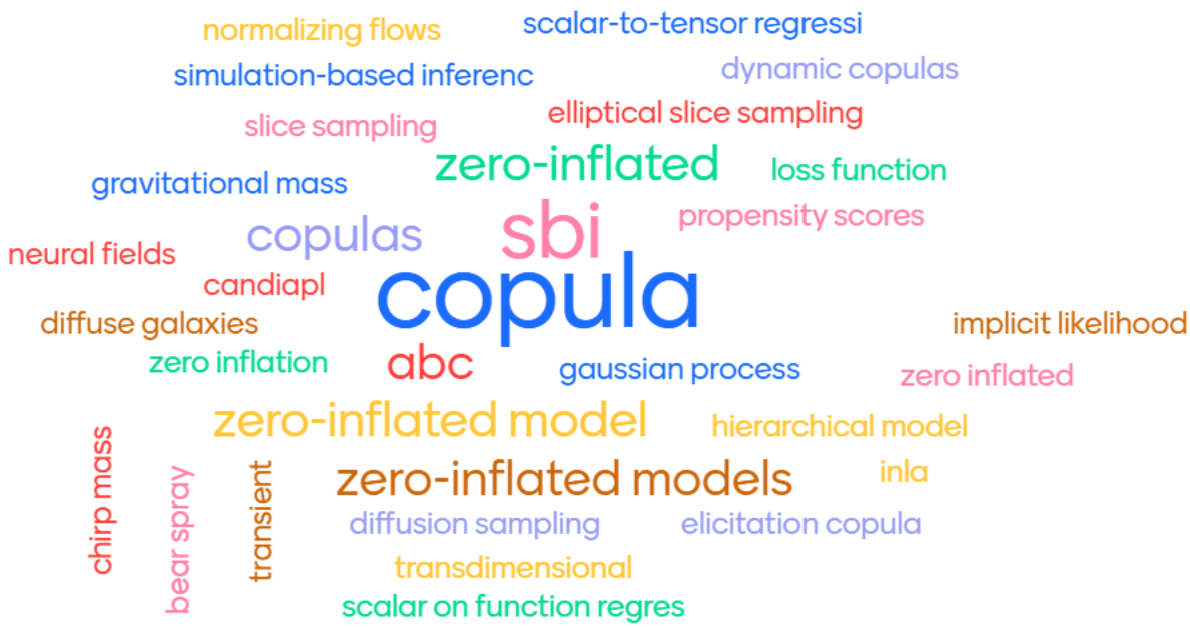

Figure 28.2: Terminology and lingo participants learned at the workshop

possible forum, although like many conference Slacks it has seen little use after the meeting. Maintaining a more permanent web presence for the Canadian astrostatistics community, including a list of funding opportunities, would be desirable: one possibility is to partner with one of the existing groups listed above.

Potential follow-on activities include a future BIRS workshop, perhaps focused on explainability and expanded to include researchers in machine learning and computer science, or taking advantage of the American Astronomical Society's "Meeting in a Meeting" structure through WGAA. Existing and future astrostatistics training opportunities were also discussed. Workshops already exist in Italy, Spain, Chile, USA (Penn State), Greece and Germany; adding a workshop to a future Canadian Astronomical Society meeting, or pursuing funding for a wider interdisciplinary training program through NSERC's CREATE program, are other possibilities. Co-supervision of graduate students is another, smaller-scale way to pursue interdisciplinary training. Writing community papers (also called white papers) that promote astrostatistics for the development of human capital, guides to documentation, or updates to the dictionaries maintained by the International CHASC Astro-Statistics Collaboration (`https://hea-www.harvard.edu/astrostat/` are further examples of possible future activities for interested participants.

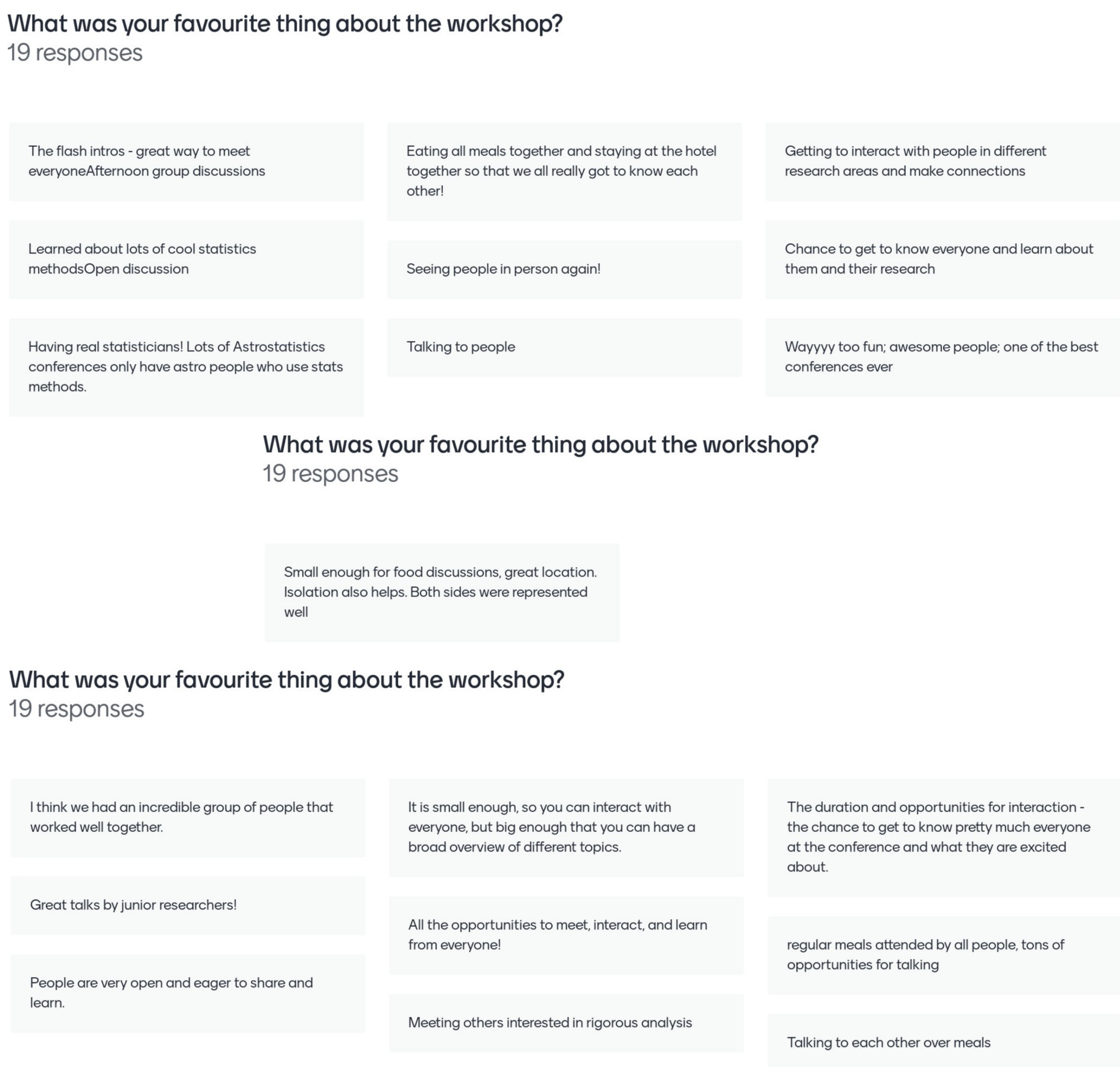

Figure 28.3: Participant feedback gathered with the Mentimeter platform during summary talk

Participants

Algeri, Sara (University of Minnesota)
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Barmby, Pauline (Western University)
Berek, Sam (University of Toronto)
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Cook, Amanda (University of Toronto)
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Chapter 29

Infinite-dimensional Geometry and Fluids: 23w5020 (23w5020)

November 5 -10, 2023

Organizer(s): Martin Bauer (Florida State University), Boris Khesin (University of Toronto), Klas Modin (Chalmers University of Technology / University of Gothenburg), Stephen Preston (Brooklyn College/CUNY Graduate Center), Cornelia Vizman (West University of Timişoara)

Overview of the Field

The overarching theme of the workshop was the use of differential geometric tools to model and analyze the governing equations of fluid dynamics. The idea of extending the terminology of manifolds to infinite dimensions can be traced back to the very roots of Riemannian geometry: in his Habilitationsschrift, Bernhard Riemann [31] discussed the possibility of extending his concepts from finite dimensions to the infinite-dimensional settings. Since then infinite-dimensional Riemannian geometry has matured into an active research area, where the interest in the field has been driven by a diverse set of applications in areas such as mathematical physics, hydrodynamics, computer vision, data science and medical imaging. Working in the infinite-dimensional setting comes with a variety of surprises and difficulties, as many results from finite-dimensional Riemannian geometry cease to exist. For example the well-known theorem of Hopf-Rinow [21], which connects metric completeness, geodesic completeness and existence of minimizers, has been shown to be wrong in a series of papers by Atkin [3, 4]. Understanding these purely infinite-dimensional phenomena has led to an active area of research, which comprised one of the fundamental topics of the proposed workshop.

Geometry of fluids.

The main motivation for this workshop is related to the fundamental role of infinite-dimensional Riemannian geometry in the governing equations of fluid mechanics. This approach, also called the Arnold approach to fluid dynamics, dates back to Arnold's seminal paper from 1966 [1], in which he recasts Euler's equations

$$\partial_t v + \nabla_v v = -\nabla p$$

for the motion of an inviscid incompressible fluid as the geodesic equation for the energy Riemannian metric on an infinite-dimensional Lie group, namely for the right-invariant L^2-metric on the group of volume-preserving diffeomorphisms. The geodesic property of the Euler equation is best seen in Lagrangian coordinates: for an incompressible flow $(t,x) \mapsto g(t,x)$ of fluid particles defined by its velocity field $v = \partial_t g \circ g^{-1}$, the Euler equation is equivalent to

$$\partial^2_{tt} g(t,x) = -(\nabla p)(t, g(t,x)).$$

The latter form represents a geodesic g on the group of volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$ regarded as a submanifold in $L^2(M, \mathbb{R}^n)$ (since the acceleration of the flow is opposite to the pressure gradient, hence always L^2-orthogonal to divergence-free fields). The motion of an ideal fluid filling an arbitrary Riemannian manifold M is given by the same Euler equation, where $\nabla_v v$ stands now for the Riemannian covariant derivative of the field v in the direction of itself, and the geometric picture continues to hold.

The framework proposed by Arnold was much more general and applicable to geodesic flows on arbitrary (finite- or infinite-dimensional) Lie groups with respect to a suitable one-sided invariant Riemannian metric, and the resulting equations are referred to as Euler-Arnold equations. Over the past decades many other conservative dynamical systems in mathematical physics have been interpreted as Euler-Arnold equations, i.e., they have been found to describe geodesic flows on appropriate Lie groups. Examples include the Kirchhoff equation for a body in a fluid, the Hopf (or inviscid Burgers) equation, magnetohydrodynamics equations, Constantin-Lax-Majda-type equations, and the Korteweg-de Vries and Camassa–Holm equations. See Arnold-Khesin [2] for a survey.

As acknowledged by Arnold himself, his paper concentrated mostly on the geometrical ideas and not on the analytical difficulties that are inherent when infinite-dimensional manifolds are involved. In 1970, Ebin and Marsden [16] reconsidered this breakthrough geometric approach from the analytical point of view, looking at the Fréchet Lie group of smooth diffeomorphisms as an inverse limit of Hilbert manifolds, following some ideas of Omori [29]. The remarkable observation is that, in this framework, the Euler equation (a PDE) can be recast as an ODE (the geodesic equation) on these Hilbert manifolds. Furthermore, following their approach, if one can prove local existence and uniqueness of the geodesics (ODE), then the Euler equation is well-posed as well, and one is able to recover the well-posedness in the smooth (Fréchet) category using a no-loss-no-gain of regularity argument. In contrast to these results it was established by Himonas and Misiołek [19] that continuity of the data-to-solution map in Eulerian coordinates (i.e., of the Cauchy problem for the PDE) is the best one should expect. Similarly to Arnold's geometric picture, their analytic approach extends to many of the above mentioned PDEs; see for example Misiołek-Preston [26].

These well-posedness results are, however, only one example of analytic results that one can obtain from studying the geometric picture. In 2006 Ebin, Misiołek, and Preston [17] studied the Fredholm properties of the exponential map and proved that for two-dimensional fluids the Riemannian exponential operator is Fredholm. Furthermore, this exponential map turns out to be Fredholm quasiruled, i.e., it has a rigid global structure, as was established by Shnirelman [34] by using paradifferential calculus. These investigations also allowed them to establish a Morse Index theorem for 2D fluids and partial results for the group diameters and the Hopf–Rinow theorem. For 3D fluids it was shown by Misiołek and Preston [26] that the Fredholm property of the exponential map fails; it is however regained if one considers a slightly stronger metric of class H^s for $s > 0$. As a final example we would like to mention a Morse–Littauer theorem for 2D fluids, which was proved by Misiołek [24]: in any H^s neighbourhood of a conjugate point in $\mathrm{Diff}^s_\mu(M)$ for a two-dimensional M there always exist fluid configurations that can be reached from the initial configuration by two distinct geodesics in the same amount of time. These studies of the exponential maps shed new light on other properties of the hydrodynamical Euler equation and are again applicable to many other equations of interest in mathematical physics.

Fluids and optimal transport.

Otto's geometric picture [30] for the optimal transport problem directly links Arnold's approach for fluid dynamics and the field of optimal mass transport. Namely, given a Riemannian manifold M, equip the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M with a "flat" Riemannian L^2-metric

$$\langle \partial_t g, \partial_t g \rangle_g := \int_M (v, v)\, g_* \mu\,.$$

for $g(t) \in \mathrm{Diff}(M)$, a vector field $v = \partial_t g \circ g^{-1}$, a Riemannian metric $(\cdot,\cdot)$, and a volume density μ on M. While this metric is not right invariant on the whole group $\mathrm{Diff}(M)$ of all diffeomorphisms, it is right invariant on the subgroup of volume-preserving ones, and it gives rise to the energy right-invariant L^2-metric there, thus recovering Arnold's geometric interpretation of the Euler equation.

To relate it to optimal mass transport, one regards the volume form μ as a reference density on M of unit total mass, and considers the projection $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ of diffeomorphisms onto the space of (normalized) smooth densities on M. The diffeomorphism group $\mathrm{Diff}(M)$ is fibered over $\mathrm{Dens}(M)$ by means of this projection π as follows: the fiber over a volume form $\tilde{\mu}$ consists of all diffeomorphisms g that push μ to $\tilde{\mu}$, $g_*\mu = \tilde{\mu}$.

Consider now the optimal mass transport problem: *find a map* $g : M \to M$ *that pushes the measure* μ *forward to another measure* $\tilde{\mu}$ *of the same total volume and attains the minimum of the* L^2*-cost functional* $\int_M \mathrm{dist}^2(x, g(x))\mu$ *among all such maps.* We call the resulting distance on the space of densities the Kantorovich-Wasserstein distance. Brenier [10] established that the mass transport problem admits a unique solution for Borel maps and densities (defined up to measure-zero sets), called the optimal map $\tilde{g}$. Furthermore, a one-parameter family of maps $g(t)$ joining $g(0) = \mathrm{id}$ with the map $g(1) = \tilde{g}$, such that $g(t)$ pushes μ to $\mu(t) := g(t)_*\mu$ in an optimal way for every t, defines a geodesic $\mu(t)$ between μ and $\tilde{\mu}$ in the space of densities with respect to the Kantorovich-Wasserstein distance.

In the smooth setting, by results of Benamou-Brenier [8], the Kantorovich-Wasserstein distance is generated by a Riemannian metric on the space of smooth densities. According to Otto's seminal work, the bundle map $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ is a Riemannian submersion from the diffeomorphism group with the L^2-metric onto the density space with the Kantorovich-Wasserstein Riemannian metric. To summarize, one obtains a beautiful fiber bundle picture of the diffeomorphism group $\mathrm{Diff}(M)$ with a Riemannian metric, where the base $\mathrm{Dens}(M)$ (the horizontal geometry) is the main setting of optimal transport problems, while the (vertical) fiber $\mathrm{Diff}_\mu(M)$ carries the Euler equation of an ideal fluid, according to Arnold's interpretation.

Recent Developments and Open Problems

There are many long-standing open questions in hydrodynamics – turbulence, global well-posedness, and stability to mention a few. Infinite-dimensional geometry offers new ways to address many of these questions, already proven to be vital in the field of optimal transport, closely related to hydrodynamics. The overarching objective of the workshop was to drive these infinite-dimensional geometric techniques further, as well as exploring new, clever ways of using existing techniques. The specific objectives were:

- To extend exploring the underlying infinite-dimensional Riemannian and symplectic structures of fluid type equations from Euler's incompressible setting (as Arnold, Ebin and Marsden did) to include many other settings: compressible fluids, the nonlinear Schrödinger equation, geophysical and shallow water equations, Dirac equations, or equations of vortex sheets are such examples. This exploration has become possible only due to recent development of techniques such as infinite-dimensional Newton's equations, the study of Riemannian geometry of infinite-dimensional Lie groups and groupoids, as well as dual pairs in infinite-dimensional symplectic geometry.

- To establish even stronger links between optimal mass transport and hydrodynamics. This includes, but is not limited to, constructing proper analytical settings for infinite-dimensional reductions and submersions following the geometric approaches by Otto, Brenier, and Shnirelman.

- In the converse direction, to explore the subtleties of geometric analysis for infinite-dimensional manifolds and Lie groups of diffeomorphisms, as well as their quotients and embedding theorems, for which fluid-type equations are a source of peculiar properties and various counterexamples.

- To continue exploring the fruitful connections between information geometry and hydrodynamics, with a particular focus on infinite-dimensional Kähler geometry of the Madelung transform and possible applications to hydrodynamics – a quantum mechanics analogy related to bouncing droplets, pilot waves, and other effects.

- To share ideas and results about model equations for 3D fluid mechanics which have some common geometric features (including finite-dimensional approximations of the Lie algebra, one- and two-dimensional hyperbolic PDEs such as SQG and the De Gregorio equation, and geodesic equations in different Riemannian metrics) in order to further understand the fundamental principles underlying the major questions in the field.

These newly gained insights will not only have an impact on geometric, topological and analytic fluid dynamics, but will also be a new source for other applications of infinite-dimensional geometry, to such domains as mathematical shape analysis and geometric data science.

Presentation Highlights

Monday

The theme of Monday's talks was about topological and symplectic structures in Hamiltonian systems. Talks included explorations on metriplectic structures, gradient flows on the space of densities, and the difference between Hamiltonian vector fields and general divergence-free fields. We also had three "flash" talks from graduate students discussing their research in progress, as well as two tutorial sessions aimed at graduate students on both Euler-Arnold equations and optimal transport.

- Philip Morrison (University of Texas at Austin)

  On an inclusive curvature-like framework for describing dissipation: metriplectic 4-bracket dynamics

  Philip Morrison spoke about mathematical tools that generalize Poisson brackets which are used to give Hamiltonian descriptions of dynamical systems. Because such systems must be conservative, they cannot describe dissipative phenomena. Instead the dynamics can be described by a "metriplectic" bracket with four arguments rather than two. This generalization allows one to handle entropy production, and Prof. Morrison spoke about both finite-dimensional and infinite-dimensional examples, as in his recent paper with Updike [28].

- François-Xavier Vialard (Université Gustave Eiffel)

  On the global convergence of the Wasserstein gradient flow of Coulomb discrepancies.

  François-Xavier Vialard spoke about the gradient flow under the Wasserstein metric in optimal transport, and the issue of convergence of solutions to a minimizer. In some situations there is no local minimum, while in others one can prove linear convergence toward a solution and global regularity of the minimizer, as shown in the recent preprint of Boufadéne-Vialard [9].

- Albert Chern (University of California San Diego)

  Dynamics of fluid's cohomology

  Albert Chern spoke about an often-neglected issue in fluid mechanics: the effect of nontrivial cohomology on the equations of fluid mechanics. On a simply-connected domain, any divergence-free vector field can be written as the skew gradient of a stream function, but in nontrivial topology there may be extra harmonic terms. Prof. Chern demonstrated the effects of these harmonic terms, writing explicitly the equations for the coupling and the conservation laws that are generated, in addition to showing numerics of some interesting special solutions under nontrivial topology. The presentation was based on the recent paper [37].

- Ioana Ciuclea (West University of Timişoara)

  Flash talk: Coadjoint orbits of weighted nonlinear flags via dual pairs

  Ioana Ciuclea, a graduate student working with Cornelia Vizman, gave a short talk about her work on the group of diffeomorphisms that preserve a weighted nonlinear flag. In particular she spoke about aspects such as symplectic reduction, coadjoint orbits, and dual pairs. This was a report on her joint work with Prof. Vizman and Stefan Haller.

- Daniil Glukhovskiy (Stony Brook University)

  Flash talk: Pensive billiard system in vortex motion

  Daniil Glukhovskiy, a graduate student working with Theodore Drivas, spoke on his work on point vortices on surfaces as described in his paper [15]. On a plane a pair of opposite-circulation vortices will move in a straight line, and the question is what happens on more general domains, in particular domains in the plane with boundaries. This talk described some interesting results on the limit where two vortices approach each other: the dipole travels in a straight line until it reaches the boundary, at which point it splits into its two monopole components which move along the boundary with constant speeds in opposite directions.

- Sadashige Ishida (Institute of Science and Technology Austria)

  Flash talk: Exploration of an implicit representation for space curves

  Graduate student Sadashige Ishida gave the third flash talk, describing a new method of representing space curves in terms of level sets of two Clebsch variables rather than via explicit parameterization. This method avoids some issues related to singular points of curves, and allows one to consider a Marsden-Weinstein symplectic structure for such generalized curves.

- Cornelia Vizman (West University of Timişoara)

  Tutorial: Geometry of Euler-Arnold equations

  Workshop co-organizer Cornelia Vizman gave an introduction to the theory of Euler-Arnold equations. These arise as the equation in the Lie algebra that describes geodesics on a diffeomorphism group under a right-invariant metric, and typically they become partial differential equations that describe conservative continuum mechanical systems. Prof. Vizman explained in a talk directed to graduate students how to derive these equations, along with the general conservation law along a coadjoint orbit, and what this implies for reducing the degrees of freedom in general.

- Klas Modin (Chalmers University of Technology/University of Gothenburg)

  Tutorial: Wasserstein-Otto geometry

  Workshop co-organizer Klas Modin gave the second tutorial of the workshop, explaining aspects of optimal transport from the geometric point of view. In general the Wasserstein distance can be viewed, as expounded by Otto [30], as the quotient of a space of diffeomorphisms, where diffeomorphisms are related to each other if they generate the same density under pullback. The standard Riemannian metric on the space of all maps, where geodesics in the space of maps involve each particle moving along its own pointwise geodesic, generates the Wasserstein distance on the space of densities, and this quotient map is a Riemannian submersion. This fact has a variety of useful consequences, as expounded by Prof. Modin.

Tuesday

The main theme of Tuesday's session was the geometry of diffeomorphism groups, particularly those with right-invariant metrics, where the geodesic equation leads to an Euler-Arnold equation. Examples include the Euler equations for ideal fluids, the surface quasigeostrophic equation, the template matching equation in shape analysis, and the Hunter-Saxton equation. We also included three more flash talks from graduate students on similar topics.

- Gerard Misiołek (University of Notre Dame)

  On continuity properties of solution maps of the SQG family

  Gerard Misiołek gave an online talk in which he discussed the β-equation, a family of equations which interpolate between the surface quasiogeostrophic (SQG) equation and the 2D Euler equation (essentially involving the vorticity being given as some power of the Laplacian of a stream function). The SQG equation is considered a good two-dimensional model of the 3D Euler equation. In general all of these equations have similar properties, and for example Prof. Misiołek described how to extend an ill-posedness proof for the usual Euler equations to the more general family: the solution operator that takes an initial condition u_0 to a solution $u(t)$ is continuous, but cannot even be differentiable, much less smooth, expounding upon a paper with Truong Vu [25].

- Stephen Preston (Brooklyn College/CUNY Graduate Center)

  Liouville comparison theory for blowup of Euler-Arnold equations

  Stephen Preston described a new result [7] on using comparison theory to prove breakdown of smooth solutions to an Euler-Arnold equation arising from a Sobolev H^k metric on the diffeomorphism group of $\mathbf{R}^n$. In the Lagrangian point of view, this equation is an ODE, and the presentation described how comparison with the Liouville equation in a Banach space yields a general technique for proving breakdown, since the Liouville equation is an ODE in a Banach space with a simple explicit solution.

- Theodore Drivas (Stony Brook University)

  Irreversible features of the 2D Euler equations

  Theodore Drivas discussed some results from his paper [14] on the long-time behavior of two-dimensional ideal fluids. Although one expects ideal fluids to be theoretically reversible due to nondissipation, there are various ways in which a fluid becomes more complicated over time as particles twist around each other, in measurable ways. This paradoxical behavior is of primary importance in understanding the long-time behavior of fluids.

- Peter Michor (University of Vienna)

  Regularity and Completeness of half Lie groups

  Peter Michor discussed his recent paper [6] on the concept of "half Lie groups," a generalization of Lie groups which allows for the composition and inversion maps to not necessarily be smooth. This generalization is essential in practice, since for example in diffeomorphism groups of finite smoothness (C^k or H^s), the right translations will be smooth while the left translations will only be continuous, and these groups are the configuration spaces of fluids. Prof. Michor described a set of axioms for such spaces in general, and some results that hold in this level of generality.

- Alexander Shnirelman (Concordia University)

  Geometric structures on the group of volume preserving diffeomorphisms

  Alexander Shnirelman gave an online talk about the global topological properties of the group of volume-preserving diffeomorphisms. In two dimensions this group has a Fredholm exponential map [17], which implies that it behaves in many ways like a finite-dimensional manifold. Furthermore Prof. Shnirelman showed that it has the structure of a quasiruled map [34], which means that in some sense it can be approximated well by affine maps. He also discussed the connection between the intrinsic geometry of the space of volume-preserving maps and the extrinsic geometry (when viewed as a subspace of the space of all diffeomorphisms under the non-invariant metric). This is related to properties of weak solutions of the Euler equations, which can make sense in the L^2 closure of the volume-preserving diffeomorphisms despite not being elements of the group of smooth diffeomorphisms.

- Anton Izosimov (University of Arizona)

  Geometry of generalized fluid flows

  Anton Izosimov discussed a geometrization of generalized fluid flows, which are needed to solve the two-point minimization problem in the group of volume-preserving diffeomorphisms. Since minimizing geodesics may not exist between two volume-preserving diffeomorphisms, Brenier and Shnirelman introduced the concept of generalized flows, where fluid particles may split into clouds and later come back together. Prof. Izosimov described his paper with Boris Khesin [22] wherein a geometric structure is introduced on such flows: since they do not form a group, one must represent them as a groupoid. Some of the group-theoretic results of Arnold extend to this context, while others need to be modified, as described in this talk.

- Patrick Heslin (National University of Ireland)

  Geometry of the generalized SQG equations

  Patrick Heslin described the results he and his coauthors obtained in the recent paper [5] on the family of generalized SQG equations, as discussed by Gerard Misiołek earlier in the day. In particular he focused on the geometric aspects, including Fredholmness of the exponential map (which works for all parameters except in the actual SQG case). As the parameter in the family approaches the critical value corresponding to the SQG equation, the conjugate points along a geodesic can be seen to cluster until they collapse in the critical case, via explicit formulas.

- Levin Maier (University of Heidelberg)

  Flash talk: On Mañé's critical value for the Hunter-Saxton system

  In the first flash talk of the day, graduate student Levin Maier discussed "magnetic geodesic flows," an extension of geodesics that can be used to describe motion of charged particles in a magnetic field. Applying this to the Hunter-Saxton equation, a well-known and particularly simple infinite-dimensional Euler-Arnold equation, he described the geometry and the solution operator in this situation, showing in particular how to solve the two-point boundary problem.

- Luke Volk (University of Toronto)

  Flash talk: Simple Unbalanced Optimal Transport

  The second flash talk of the day featured graduate student Luke Volk describing his work with workshop co-organizers Boris Khesin and Klas Modin [23] on unbalanced optimal transport, which entails finding minimizing geodesics between two densities under a Wasserstein-type metric, but allowing for the second endpoint to vary (with a penalty for being far away from the desired endpoint). Some of the properties of this model can be best understood via finite-dimensional models.

- Archishman Saha (University of Ottawa)

  Flash talk: Symmetry and Reduction for Stochastic Differential Equations

  Graduate student Archishman Saha gave the final flash talk of the workshop, describing stochastic versions of Euler-Arnold equations. Because the geodesic equation is second-order on the diffeomorphism group, it is necessary to understand second-order stochastic differential equations (SDE) on manifolds in order to make sense of this. Mr. Saha described aspects of reduction and reconstruction in the second-order SDE models.

Wednesday

Wednesday's session was focused on Hamiltonian aspects of the equations of 3D fluids, in particular the relationship with contact geometry and its uses in determining the topological structure of steady fluid flows, along with new Hamiltonian principles that can be used to derive equations of fluids with viscosity and heat.

- Eva Miranda (Universitat Politècnica de Catalunya)

  Navigating Uncharted Waters: Bridging Geometry and Fluid Dynamics

  Eva Miranda gave a far-reaching survey of the relationship between contact geometry and fluid dynamics. Beltrami fields (eigenfields of the curl operator) form a special class of steady solutions of the 3D Euler equation, and all such fields are the Reeb field of some contact form under a certain Riemannian metric. Hence the vast topological information we have about contact geometry can tell us about properties of steady fluids. Prof. Miranda surveyed a wide variety of results that she and her collaborators have obtained using this correspondence, some of which is summarized in her paper [11].

- Daniel Peralta-Salas (Instituto de Ciencias Matemáticas - Madrid)

  Obstructions to topological relaxation for generic magnetic fields

  Daniel Peralta-Salas spoke about a theorem he proved with Alberto Enciso [18] relating to magnetohydrostatic (MHS) equilibria, i.e., steady solutions of the equations of an ideal fluid with a coupled magnetic field. They specifically analyzed generic obstructions for a divergence-free vector field to be topologically equivalent to some MHS equilibrium, and showed that for any axisymmetric toroidal domain there is a locally generic set of divergence-free vector fields that are not topologically equivalent to any MHS equilibrium.

- François Gay-Balmaz (Ecole Normale Supérieure de Paris)

  Geometry and Numerics of Navier-Stokes-Fourier Fluids

  François Gay-Balmaz discussed a variational geometric setting for nonequilibrium thermodynamics, which extends the Hamilton principle and the geometric formulation of classical mechanics to include irreversible phenomena. This allows one to study the same geometric techniques used in ideal fluids for those that include irreversible processes such as viscosity and heat conduction. In addition he presented new work on discretizing this setting to yield structure preserving and thermodynamically consistent finite element schemes, particularly in the context of the Navier-Stokes-Fourier equations.

Thursday

In Thursday's session, we focused on analytical aspects of the Euler equations. In particular this includes the long-time behavior, such as the breakdown of smooth solutions of the Euler equations, one of the most famous open problems in the field. Long-time behavior of solutions also includes the counterintuitive phenomenon of inviscid solutions settling down to quasi-periodic solutions in a low-dimensional space (without viscosity, one would not expect such convergence), and several presenters discussed both special finite-dimensional families of solutions as well as quasi-periodic solutions in general.

- Javier Gómez-Serrano (Brown University)

  Self-Similar Blow up Profiles for Fluids via Physics-Informed Neural Networks

  Javier Gómez-Serrano described an exciting new field of research [36] aimed at finding smooth self-similar solutions for different equations in fluid dynamics. The approach is numerical and computational, using interval-based arithmetic in order to make the results rigorous. The primary innovation is in using physics-informed neural networks to find the solutions, and he showed that the framework is both robust and readily adaptable to several situations.

- Jiajie Chen (Courant Institute)

  Sharp functional inequalities related to singularity formation in incompressible fluids

  Jiajie Chen described his recent work with Thomas Hou [12] on the breakdown of both the Boussinesq equation and the 3D Euler equation, the latter of which has been a famous open problem, and the solution of which is of deep importance in fluid mechanics. In addition to the use of approximately self-similar smooth solutions which are shown to exist through numerics (and related to the work in the previous talk), sharp analytical inequalities in function spaces are crucial in the proof for estimating the nonlocal terms. He particularly explained those inequalities that are originally based in optimal transport and geometry.

- Milo Viviani (Scuola Normale Superiore Pisa)

  On the infinite-dimensional limit of steady states for the Euler–Zeitlin equations

  Milo Viviani discussed his work with Klas Modin [27] on the analysis of the long-time behavior of 2D fluids on a sphere. The Zeitlin model is a particularly successful finite-dimensional approximation of the infinite-dimensional geometry which uses the special structure of the sphere to obtain close analogues of the fluid structure in a Lie group of large but finite dimension. In particular the numerical simulations demonstrate the same behavior as those seen for 2D fluids: evolution towards simple quasi-periodic solutions, which likely represent the weak-* limits in Shnirelman's general theory.

- Gigliola Staffilani (Massachusetts Institute of Technology)

  Energy transfer for solutions to the nonlinear Schrödinger equation

  Gigliola Staffilani discussed a 2D version of the cubic defocusing nonlinear Schrödinger equation on the periodic (torus) domain. Energy transfer on different scales can be shown, but the dynamics of solutions differs in the case of the rational torus and the irrational torus.

- Jia Shi (Massachusetts Institute of Technology)

  On the analyticity of the Muskat equation

  Jia Shi described two results [32, 33] on the Muskat equation, which describes the interface of two liquids in a porous medium. The first result is that if a solution to the Muskat problem is sufficiently smooth (with the same viscosity and different densities), then it must be analytic except at a point where the fluids turn over. Her second result is analyticity also at the turnover points under some additional conditions.

- Francisco Javier Torres de Lizaur (University of Seville)

  Finite dimensional invariant manifolds of the Euler equation and their dynamics

  Francisco Javier Torres de Lizaur described special solutions of the Euler equation of ideal fluids from his recent paper [13]. He showed that there are invariant manifolds of nontrivial, nonsteady solutions (generalizing simple cases such as steady flows where the invariant manifold is a point, or slightly more complicated cases such as the harmonic fields discussed earlier in the week by Albert Chern). If the Euler equation is on a manifold of sufficiently high dimension and the Riemannian metric can be specified arbitrarily, then the Euler equation on these invariant manifolds is general enough to capture all possible dynamical systems behavior.

- Peter Topalov (Northeastern University)

  Spatially quasi-periodic solutions of the Euler equation

  Peter Topalov described a framework for studying quasi-periodic maps and diffeomorphisms on $\mathbf{R}^n$. One application is proving that the Euler equation is locally well-posed in the space of quasi-periodic vector fields, which implies that the equation preserves the spatial quasi-periodicity of the initial data. In addition he showed some results on the analytic dependence of solutions on both time and the initial data, as detailed in his recent paper [35].

Friday

On the final day of the workshop, our speakers discussed some topics related to fluids that did not quite fit in earlier sessions.

- Sonja Hohloch (University of Antwerp)

  Hypersemitoric systems: Recent developments and advances

  Sonja Hohloch gave an overview of hypersemitoric systems based on her paper [20]. These are integrable Hamiltonian systems with two degrees of freedom on 4-dimensional compact symplectic manifolds, possibly with mild degeneracies, where one of the integrals gives rise to an effective Hamiltonian circle action. She gave an update on recent developments like important new examples, links with Hamiltonian S^1-actions, bifurcation theory, symplectic features like (non)displaceability of fibers, and steps towards a symplectic classification.

- Tsuyoshi Yoneda (Hitotsubashi University)

  Mathematical structure of perfect predictive reservoir computing for autoregressive type of time series data

  Tsuyoshi Yoneda discussed a particular form of machine learning called reservoir computing (RC), used for building prediction models for time-series data since it has low training cost, high speed, and high computational power. In particular he focused on some hidden structure in RC machine learning which leads to perfect prediction for autoregressive data as in his paper [38]. He argued that this structure is expected to be similar to the relationship between Lie algebras and Lie groups, which is used to derive the Euler-Arnold equations.

Scientific Progress and Outcomes of the Meeting

The workshop involved many high-quality presentations and discussion sessions, with enthusiastic participation of the audience. Moreover, particular highlights of the workshop were the tutorial sessions on the geometry of Euler-Arnold equations and the Wasserstein-Otto geometry for graduate students, as well as young mathematicians' flash talks. The small discussion rooms provided by BIRS were used extensively by groups of participants to brainstorm and work on their projects. We expect that many future papers and research programs have their beginnings rooted in this BIRS workshop. Participants also raised many interesting questions during and after the presentations that suggested new directions of research and different approaches to established topics, some of which were mentioned in this report. The workshop fostered interactions between several seemingly remote domains of hydrodynamics: geometric, numerical, and analytical. We anticipate that quite a few of these novel interactions and research questions will be influential in the ongoing development of these areas of infinite-dimensional geometry and fluid dynamics. Judging from the above observations and the informal feedback received by the organizers, the workshop was very successful at making new connections and fostering progress in these active research areas.

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Chapter 30

The Canadian Network for Modelling Infectious Diseases: Progress and Next Steps (23w5151)

November 12 -17 2023

Organizer(s): David J. D. Earn (McMaster University), Caroline Colijn (Simon Fraser University), Irena Papst (Public Health Agency of Canada)

Background & Motivation

Infectious disease modelling that aims to contribute to public health decision-making has a long history, going back to Daniel Bernoulli's work on smallpox control in the 18th century [1]. In the early 20th century, Ronald Ross developed a mathematical model to help identify effective malaria control strategies [2], and Kermack and McKendrick developed the foundations of modern mathematical epidemiology [3]. While theoretical work on disease modelling continued [4, 5, 6], attention from decision-makers and politicians was scarce until the 2001 Foot and Mouth Disease epidemic in the UK, and the SARS epidemic in 2003. Over the last 20 years, the perception of mathematical modelling as a valuable tool in the public health policy process has become more and more common, especially in the context of influenza pandemic preparedness, which set the stage for immediate, serious engagement with modellers when the SARS-CoV-2 pandemic exploded in early 2020.

From the start of the pandemic, many disease modellers around the world found themselves in constant demand from public health agencies and policy-makers. Recognizing the importance of this development, NSERC invested \$10M with the aim of creating networks of disease modellers, public health professionals, and policy-makers. A quarter of the investment (\$2.5M) was allocated to our CANMOD network (the remaining 75% was allocated to four other networks).

CANMOD aims to increase Canada's capacity for data-driven emerging infectious disease modelling (EIDM) to directly support short, medium, and long-term public health decisions. Our network comprises collaborative teams of modellers, statisticians, epidemiologists, public health decision-makers, and those implementing and delivering interventions. The questions we are tackling are grounded in public health needs and generated in partnership between research investigators and knowledge users – public health leaders, health administrators and policy-makers. This collaborative research supports data collection, curation and access, with the hope that it will increase the speed with which critical information is made available.

The COVID-19 pandemic in Canada has made clear the urgent and immediate need for modelling of local context to inform decisions that are often implemented in local jurisdictions, and across diverse epidemiologic and health system contexts. Effective public health benefits from the support of engaged modellers who understand the local data, local epidemiological, socio-cultural, and health-system contexts, and who are passionate about collaborating on public health research problems. Our 44 co-applicants and dozens of collaborators have been engaged in this kind of work since the beginning of the COVID-19 pandemic, and are ideally placed to ensure that the collaborations we continue to build are effective. Enthusiasm from public health institutions was clear from the immediate high level of engagement in early 2020, the regular use of our research results in decision-making, and from numerous and enthusiastic letters we have received from a wide range of public health institutions across Canada, spanning municipal, regional, provincial and national jurisdictions. Some of the high-priority scientific questions that have emerged through this collaborative research were discussed at the workshop are summarized by this report.

As researchers have moved from the daily challenges of decision-making during the COVID-19 pandemic to working on longer-term policy questions, CANMOD continues to build and coordinate national capacity in infectious disease modelling at the forefront of public health. This capacity will position public health across Canada for better control of any infectious disease, and will build better preparedness and resilience in case of future pandemics. We are providing extensive experiential training opportunities for postdoctoral fellows (PDFs), graduate and undergraduate students at the intersection of infectious disease modelling, public health policy and decision making, and we are committed to increasing equity, diversity, and inclusion in the next generation of infectious disease modellers. Our trainees are well-placed for quantitatively-oriented careers in academia, industry and the public sector, both in Canada and abroad.

CANMOD's multi-disciplinary and multi-sectoral network is addressing all infectious disease challenges with principles of equality, diversity and inclusion through its recruitment, training, and research, and through events like the November 2023 BIRS meeting that this report summarizes.

Structure of the Workshop

This five-day hybrid workshop featured several special sessions, 27 short (30 minute) talks from in-person participants, and one short talk from an online participant.

Special sessions included an introduction to the Historic Disease Data Portal (Chapter 30), a discussion about challenges in infectious disease surveillance across Canada (Chapter 30), and a workshop on the *macpan2* modelling software (Chapter 30).

Short talks were grouped thematically in the schedule and spanned many domains, including behavioural modelling, infectious disease surveillance, pathogen evolution and genetics, vaccination and within-host immunity, policy-making in response to infectious disease outbreaks, as well as more general mathematical and statistical methods to support epidemiological modelling. Chapter 30 features summaries for several talks presented at the workshop.

All talks were streamed live to online participants. Talk recordings and slides can be found online.

Presentation Highlights

Special Sessions

Historic Disease Data Portal

Steven C Walker

In addition to funding research, training, and networking opportunities for building applied infectious disease modelling capacity in Canada, CANMOD also funded a project to provide straightforward and convenient access to historical and publicly available incidence, mortality, and population data. This project led to three long-term Canadian datasets.

Although there is a great wealth of historical data on infectious diseases in Canada that is or could be technically available to the public, it tends to be locked up in inconvenient formats like handwritten documents and internal databases at Statistics Canada. With this project, we are building a Canadian data archive that will provide straightforward and convenient access to historical public Canadian infectious disease data.

We systematically contacted data stewards across Canada to access the disparate source documents that contain Canada's public historical infectious disease data, in an effort to be comprehensive. We entered the information provided by the source documents into spreadsheets such that they can be compared with the original sources, and have produced automated pipelines for converting the digitized spreadsheets into convenient csv files with metadata. We worked with Statistics Canada to obtain mortality data that they have not made public before, collaborating with their analysts to balance strict anonymity requirements with our intention to make these data publicly-accessible. We systematically digitized official documents on Canadian populations for normalizing disease incidence and mortality data. All-in-all, we now have a systematic and comprehensive data archive ($\sim$ 2 million records) of the following communicable disease incidence, mortality, and population data:

- Notifiable Communicable Disease Incidence (CDI) (1924-2000)
  - 1924-1980 (weekly), 1980-1990 (monthly), 1990-2000 (quarterly)
  - Broken down by province/territory
  - Broken down by disease
  - Some diseases broken down by age and sex (before 1956)
- Mortality (1950-2010)
  - Weekly
  - Broken down by province/territory
  - Broken down by 12 cause groups selected by Statistics Canada
  - Extends the public data portal back from 2010 to 1950
- Population (1881-present)
  - Population estimates every ten years (1881-1921), every year (1921-present)
  - Broken down by sex
  - Broken down by age
  - Broken down by province/territory

To enhance accessibility of this archive we developed the following tools, which will be made public once we have a pre-print describing this work:

- All data available on GitHub with open data pipelines
- Web-based and R-based APIs for programmatically accessing and searching the data on GitHub
- Dashboard for searching, downloading, and combining data

We provided meeting participants with sample API commands for accessing the archive.

We identified and adopted best practices for archiving research data. We developed a controlled data dictionary and CSV format used by all datasets in the archive, making it easier to combine data from different historical sources. We used the widely-adopted DataCite standard for metadata on research datasets, which allows our archive to be integrated in any number of pre-existing data portals. Ultimately, we plan to contribute our archive to a long-term storage and data access service for researchers (likely the Federated Research Data Repository).

This project has the potential to contribute to Canadian pandemic preparedness by providing long-term data on a diversity of infectious diseases that have significantly impacted the health of Canadians. The convenient access to these data provided by this project will help epidemiologists to learn from past public health challenges.

Surveillance Discussion

The effective collection, analysis, and application of diverse infectious disease surveillance data streams are paramount to understanding and managing infectious disease threats. Recently, the WHO released guidelines for ethical public health surveillance, arguing that societies have an obligation to design and deliver effective public health surveillance. We explore infectious disease surveillance from the point of view of researchers, primarily infectious disease modellers, working at the interface between research and policy during the COVID-19 pandemic.

We held a roundtable discussion on surveillance. Our discussion moved through infectious disease surveillance (and related) data types, their collection, and the ways in which they inform surveillance questions.

There is a wide range of data types that play important roles in understanding and managing infectious diseases; here we focus primarily on respiratory infectious disease and issues that arose during the COVID-19 pandemic. Many of these apply to influenza, RSV and other respiratory illnesses, and some (like the "First 100") are relevant mainly to a potential new emerging infectious disease.

Data streams include:

- **Lab-Based Data**: These data comprise the results of testing, which indicate whether an individual is infected, for a specific virus or infectious agent. There is typically some stratification by age, perhaps sex, location.
- **Case-Level Data**: This refers to detailed information on each case, such as line lists and contact tracing records.
- **First 100 cases**: especially in the early stages of a new emerging infectious disease, this can provide essential characterization of the course of infection, exposure, pace of transmission, severity, clinical needs
- **Genomics**: Genomic analysis (viral sequencing) helps track the mutation and spread of pathogens. Interpretation can be challenging, due to lack of linkage with epidemiological, clinical, demographic or immunity data, and due to sampling that is a mixture of travel-related cases, random sampling (but only within the schema of the testing system), and priority sequencing (for example of outbreaks)
- **Outcomes**: hospitalization, acute care needs, by age (aggregate; individual outcomes would be in individual-level data)
- Tools like the **WHO Ordinal Scale**, which measure the severity of cases: however, there are challenges in standardizing these scales across different regions, hospitals, or countries, particularly when data is incomplete or inconsistent. If hospitals are overwhelmed, data on these scales will be impacted
- **Mortality Data**: this can lack timeliness, consistency across jurisdictions and completeness
- **Immunity**: serology, vaccination levels
- **Denominator/population-level data**

Additional non-health data that are relevant for interpretation and modelling of infectious disease surveillance data:

- **Policy data**:
- **Behavioural data**: this is a large area, not typically considered part of routine respiratory surveillance data. It comprises mobile phone (mobility) data, contact data derived from surveys, and other information about behaviour relevant to infectious disease transmission (use of NPIs, response to illness). This could also include information about test-seeking and health-care-seeking behaviour.
- **Travel and Movement Data**: Data on travel, both international and interprovincial
- **Demographic changes over time**

Our discussion continued, to explore the epidemiological pyramid, which connects some of these layers of data together, conditional on others. For example, the relationship between detected cases and infections depends on test seeking, testing policy, immunity including vaccination, and the intrinsic severity of the virus variant, to name a few.

Several participants emphasized the importance of data linkage, and of context. The integration of genomic data with epidemiological, clinical, vaccination, and demographic data is especially important. Without these connections, the full potential of genomic insights remains largely untapped. For instance, determining whether a new variant is transmitting among vaccinated individuals requires a synthesis of genomic, epidemiological and vaccination data. Determining severity requires information about clinical outcomes, in individuals with and without the new variant.

One of the main focal points for our discussion was to solicit modellers' input on how infectious disease surveillance data could be made more useful to modellers, as throughout the pandemic, modellers were asked to support policy-makers in questions about COVID-19 scenarios, forecasts and healthcare impacts.

The utility of data is intrinsically linked to the specific question it aims to answer. For prediction purposes, it's useful to clarify what is being predicted. For example, predicting the dynamics of infections requires different data compared to predicting the impact on healthcare resources. The following aspects were identified as important ways that surveillance systems could take these needs into account.

Timeliness: The value of surveillance data is heavily dependent on its currency. Real-time or near-real-time data acquisition enables public health officials to respond swiftly to emerging threats, adjust strategies based on current trends, and predict future outbreaks with greater accuracy. Delayed data can lead to missed opportunities in containing and mitigating outbreaks.

Testing policy: Knowing who is being tested and recognizing any shifts in this demographic is important. Changes in testing patterns can significantly impact the interpretation of surveillance data. For instance, if testing becomes more widespread or targeted at specific groups, this shift needs to be factored into the analysis to avoid misinterpretation of disease trends.

Stratification (lab data): Stratifying lab data by variables such as age, sex, and immune status can help unpack nuance and changes in testing, and help build a more nuanced understanding of a disease's impact and spread. Stratification may be helpful in understanding changes in testing policy or test-seeking behaviour. Including vaccination status (including the recency of vaccination and time since the last dose) adds another layer of depth, enabling improved understanding of immunity in the population and the effectiveness of vaccines against current strains.

Linkage: Linking across datasets enhances the richness and depth of analysis and adds value. For instance, connecting laboratory data with clinical outcomes, vaccination records, and demographic information provides a comprehensive picture of the disease's impact, spread, and evolution (and see above for the benefits of genomic data with linkage).

Consistency: In addition to the above, it is important for data to be consistently reported across local jurisdictions, hospitals, and laboratories, and as such, the creation of standards can help move the needle forward when thinking about surveillance data for infectious disease modeling.

`macpan` Modelling Software

Steven C Walker, Irena Papst

`McMasterPandemic` is an `R` package for compartmental modelling that was developed to provide forecasts and insights to public health agencies throughout the COVID-19 pandemic. Forecasts created with `McMasterPandemic` were prepared for the Public Health Agency of Canada, the Ontario COVID-19 Science Table, the World Health Organization, and Public Health Ontario. Much was learned about developing general purpose compartmental modelling software during these experiences, but the pressure to deliver regular forecasts to these organizations made it difficult to focus on the software itself. With the support of CANMOD, the `macpan` project was launched to re-imagine `McMasterPandemic` , building it from the ground up to address lessons learned while responding to a global public health emergency.

The special session on `macpan` started with a presentation by Steve Walker introducing the project. Steve traced the history of `McMasterPandemic` and `macpan`, using it to argue that impactful modelling requires many interdisciplinary steps along the path from epidemiological research teams to operational decision-makers. Researchers must quickly tailor a model to an emerging public-health concern, validate and calibrate it to data, work with decision-makers to define model outputs useful for stakeholders, configure models to generate those outputs, and package up those insights in an appropriate format for stakeholders. `macpan` targets bottlenecks along this path that can be solved with thoughtful software engineering. The goal is to ease the software development burden on modellers, especially when they are working on an urgent public health response, so that they can devote their time and energy to the modelling itself.

After discussing the project's history and motivation, the presentation transitioned to exploring `macpan`'s modular model building, a key feature meant to address a commonly-encountered bottleneck in modelling. New public health concerns often demand new modules to be added to existing models. For example, as vaccines against COVID-19 were deployed, models needed to be modified to include vaccination. Even beyond responding to a public health emergency, a common paradigm in modelling (and in writing code) is to start simply and add complexity incrementally, testing outputs at every step of the way. Experience shows that it can be surprisingly difficult to add new modules to a modelling pipeline if your existing toolkit is not designed for modular model building. Steve briefly reviewed existing approaches to modular compartmental modelling based on mathematical tools from graph theory and category theory. Steve described how modules in `macpan` can be represented by tables (like tables in a database), and that widely-understood table manipulation tools (like join and group-by) can be used to combine modules without the need for advanced mathematical concepts.

After the presentation, participants were invited to a hands-on session to explore `macpan`, led by Steve Walker, Irena Papst, and Ben Bolker. There were roughly 20 participants in the session, representing a wide range of career phases, from graduate students to tenured faculty. We started the session by helping participants download and install the software on their computers. There were several installation hiccups that we were able to troubleshoot on the fly. These issues gave us valuable insight into potential difficulties deploying this tool more widely, and have inspired further work on the software.

We then invited participants to work through a getting started vignette to enable them to further familiarise themselves with the software's model specification grammar, which enables modular model building. The vignette walks users through specifying a very simple epidemiological model, and then introduces software features that make it easy to add additional structure to models, such as modules for multiple infection types (*e.g.,* asymptomatic, symptomatic), multiple locations (often referred to as "metapopulation" models), stratification by vaccination status, and more. The vignette specifically works through the example of specifying a two-strain model while demonstrating `macpan` functions key to easily specifying "structured" models.

After participants worked through the vignette, some worked on specifying other models in `macpan`, as a way to test their understanding of the model specification grammar and to experiment with other features of the package. Two participants worked together to try to specify a Lotka-Volterra predator-prey model, and their attempts revealed interesting points of friction in the software that have directly inspired further development. These attempts also spurred the addition of Lotka-Volterra models to `macpan`'s model library. Three participants independently provided the same feedback about how calibrating models to data is a bigger bottleneck than modular model building, which has inspired us to make existing calibration tools more accessible via the `macpan` interface.

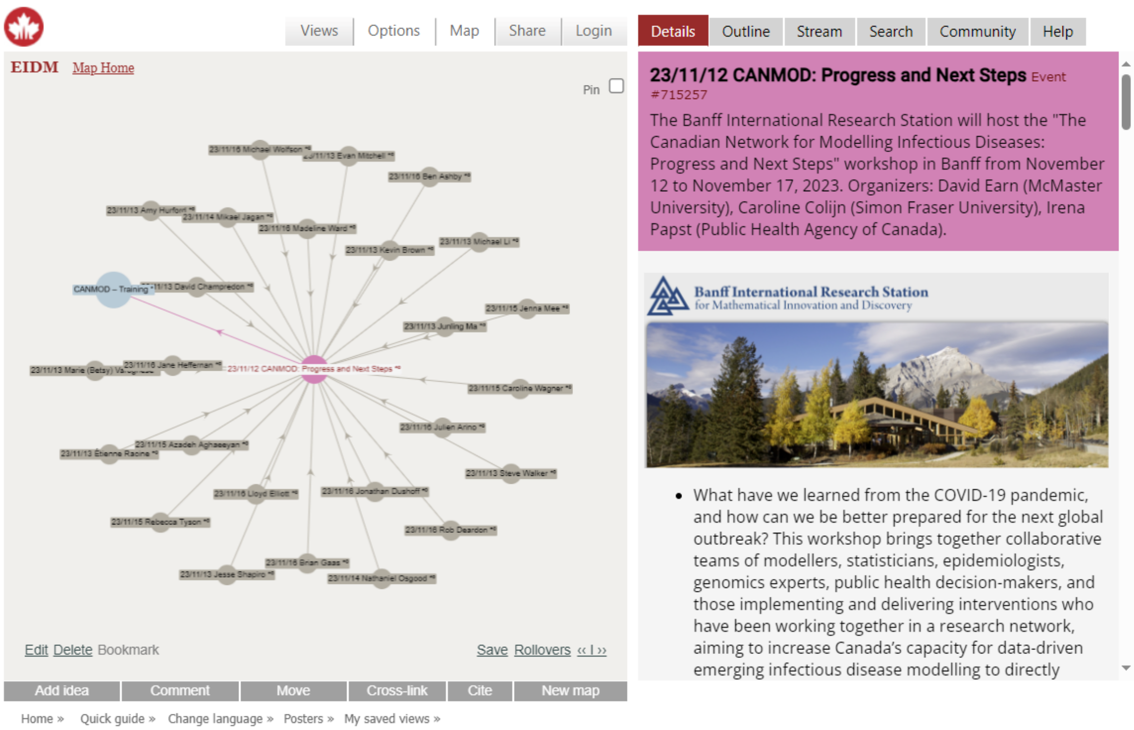
This session was the first `macpan` training ever run, and overall, we received a lot of valuable feedback from participants on it. We continue to use this feedback to both improve guides for the software, as well as the software itself.

Short Talks

The CANMOD/EIDM Knowledge Graph

David Price, DebateGraph

This opening talk of the workshop explored the content, structure, and rationale of the EIDM dynamic knowledge graph (`https://eidm-mmie.net`), which is being developed by CANMOD to support, document, and interconnect the work conducted across the five EIDM networks. Workshop participants were guided through the exploration and use of the graph as a repository of knowledge and as a resource for search-based discovery. As illustrated in the figure below, the graph identifies and interweaves multiple aspects of the EIDM initiative, including: the participants, their organizational affiliations and collaborations, research interests, publications, goals, datasets, software, and training materials. Interactive visualisations make it simple to traverse the graph (which already contains thousands of nodes and edges), focusing on the immediate connections around the individual elements and zooming out to see the wider patterns and connections emerging as the network continues to grow.



Association between Delayed Nursing Home Outbreak Identification and SARS-CoV-2 Infection and Mortality in Ontario, Canada

Kevin Brown, Public Health Ontario

Delayed outbreak identification is likely an important driver of respiratory infection transmission in nursing homes. Most studies examining outbreak identification have been descriptive and there are no measures of delayed outbreak identification in nursing homes. We conducted a longitudinal cohort study of SARS-CoV-2 outbreaks from 623 nursing homes in Ontario, Canada in the March 1, 2020 to November 14, 2020 period prior to the rollout of COVID-19 vaccination. Our exposure was the timeliness of outbreak identification, defined as late (≥ 3 resident-days of infection pressure) versus early (≤ 2 resident-days of infection pressure) on the date of outbreak identification. Residents were considered to contribute infection pressure from 2 days prior to onset to 8 days afterwards while non-residents (including staff and visitors) were not considered to contribute infection pressure. Our outcomes were 30-day secondary infections and mortality, defined as the proportion of at risk residents with a laboratory-confirmed SARS-CoV-2 infection with onset within 30-days of the outbreak identification date, and mortality among these residents.

We identified 632 SARS-CoV-2 outbreaks across 623 Ontario nursing homes during the study period. Of these, 230 (34.3%) outbreaks were identified late. Outbreaks identified late had higher secondary infections (10.3%, 4,437/43,078) and mortality (3.2%, 1374/43,078) compared to outbreaks identified early (infections: 2,015/61,061, $p < 0.001$, mortality: 0.9%, 579/61,061, $p < 0.001$). After adjustment for 12 nursing home risk factors, the incidence of secondary infections in outbreaks identified late was 2.90-fold larger than that of outbreaks identified early (OR=2.90, 95%CI: 2.04, 4.13). Each 1-person-day increase in infection pressure at the time of outbreak identification was associated with an 1.10-fold increase in the secondary infections (OR=1.10, 95%CI: 1.08, 1.12) and a 1.07-fold increase in secondary mortality (adjusted OR=1.07, 95%CI: 1.06, 1.09). In the nursing home setting, SARS-CoV-2 outbreaks identified late evolved to be much larger than outbreaks identified early. The timeliness of outbreak identification can be used to predict the trajectory of an outbreak and plan for increased staffing demands, infection control measures, and antiviral administration, with the goal of mitigating harms to residents.

The need to evaluate existing data resources and knowledge gaps to support future needs for respiratory disease surveillance and modelling

Michael Li, Public Health Agency of Canada

Infectious disease surveillance and health data sharing have always been topics of discussion, even before the SARS-CoV-2 pandemic. The SARS-CoV-2 pandemic amplified the value of these discussions, providing researchers and government scientists with a small glimpse of the data possibilities—such as the high frequency of time series data reporting positive cases, testing, sequencing, hospitalization, and death. However, what we had for SARS-CoV-2 is still far from the ideal data structure (e.g., linkable data, health status, etc.) needed to learn more about the questions of interest. Before we can make further progress, the capacity diminishes due to low-frequency/quality reporting.

Lack of surveillance and data sharing are often viewed as the same problem; however, it is important to recognize that they are separate issues. Our focus should be on learning from data, not just on the data itself. The key is not sharing data per se, but fostering effective collaborations to obtain information from data—referred to as "data-info or data-knowledge sharing"

This talk proposes a pandemic and peacetime preparedness vision called the "PREP" vision, which stands for Profiling, Reflection, Exploring alternative options, and Proof of concept. Profiling identifies what different people want to know and the bottlenecks of knowledge gaps. Reflection evaluates existing resources used to seek answers to understand what needs improvement. Exploring alternative options goes beyond the current status quo to see what can be done to improve access and, eventually, enhance the resources. Lastly, a proof of concept aims to validate whether the ideas are worth implementing using the model world.

The Decision Uncertainty Toolkit

Megan Wiggens, Marie Betsy Varughese, Ellen Rafferty, Jeff Round, Sasha van Katwyk, Erin Kirwin
Presented by: Marie Betsy Varughese, Institute of Health Economics

Infectious disease (ID) models played an important role in decision making during the COVID-19 pandemic. While ID modelling has methods to address structural and parameter uncertainty, communicating decision uncertainty is another important interface between ID modelers and decision-makers. This talk presented the Decision Uncertainty Toolkit aimed to address and develop methods for communicating uncertainty to decision makers where there are multiple policy options. The toolkit includes visualizations, risk measures, descriptions, and interpretations. As this work is on-going, we included an opportunity for further involvement through a planned workshop in 2024 to try out the tool and provide additional feedback on the codes, visualizations, and descriptions.

Academic collaborations to improve wastewater-based modelling at the Pubic Health Agency of Canada

David Champredon, Public Health Agency of Canada

Since the COVID-19 pandemic, the surveillance of respiratory viruses in municipal wastewater has emerged as a valuable new data source for modellers. However, there are still many unknowns regarding the various causes that can impact the viral concentration in wastewater during the journey of viruses in the sewer system. Understanding the processes that can affect viral concentration in wastewater is critical for epidemiological surveillance at the Public Health Agency of Canada, and it involves many scientific fields, not all represented within the Agency. In this talk, I highlighted several projects done in collaboration with academic groups that brought their expertise to help better understand how various processes can impact viral concentration in wastewater. I also presented the different ways academic groups can collaborate with PHAC.

Infections, hospitalizations, and deaths prevented by COVID-19 vaccines in Canada

Evan Mitchell, McMaster University

How many infections, hospitalizations, and deaths did COVID-19 vaccines prevent during the pandemic in Canada? This talk presented research aimed at answering this question. A compartmental model was fit to daily infection report and hospitalization occupancy data for Ontario from the start of the pandemic through the end of the Delta wave in December 2021. We use this model to simulate counterfactual scenarios where vaccines were not present, vaccine introduction was delayed 60 days, or vaccines were 25% less effective. Results from these simulations show that the predicted numbers of infections, hospitalizations, and deaths would be orders of magnitude larger than they actually were. To follow this up, we consider the effects of introducing a hypothetical stay-at-home order in an attempt to control these counterfactual scenarios. Our main finding from these explorations is that we would have a much easier time controlling the situation in the case of less effective vaccines than in the other two cases, suggesting that it is important to release a vaccine as early as possible during a pandemic even if that vaccine is less effective than it might be otherwise. We are currently in the process of extending these results to five other provinces: Alberta, British Columbia, Manitoba, Québec, and Saskatchewan.

Within-host diversity of SARS-CoV-2 across animal host species

Jesse Shapiro, McGill University

Viral transmission across different host species makes eradication very challenging and also opens new evolutionary trajectories for the virus. Since the beginning of the ongoing COVID-19 pandemic, SARS-CoV-2 has been transmitted from humans to several different animal species, and novel variants of concern could plausibly evolve in a non-human animal. Previously, using available whole genome consensus sequences of SARS-CoV-2 from four commonly sampled animals (mink, deer, cat, and dog) we inferred similar numbers of transmission events from humans to each animal species but a relatively high number of transmission events from mink back to humans (Naderi et al., 2023). In a genome-wide association study (GWAS), we identified 26 single-nucleotide variants (SNVs) that tend to occur in deer, more than for any other animal, suggesting a high rate of viral adaptation to deer. Here we show that deer harbor more intra-host SNVs (iSNVs) than other animals, providing a larger pool of genetic diversity for natural selection to act upon. Deer contain more distinct viral lineages than other animals, indicating possible co-infections, but this effect is unlikely to explain the overall higher diversity within deer. Compared to other animals, iSNV frequencies in deer are skewed toward higher frequencies, which is unexpected after a recent population bottleneck or population expansion and therefore suggests that deer are sampled relatively late in the course of infection. Combined with extensive deer-to-deer transmission, the high levels of within-deer viral diversity help explain the apparent rapid adaptation of SARS-CoV-2 to deer.

Estimating phenomenological epidemic models with mixed effects

Mikael Jagan, McMaster University

When dealing with emerging or historical epidemics, modelers must contend with uncertainty about the disease of interest. Sparse knowledge about the pathogen, the natural history, and primary modes of transmission impedes selection of appropriate mechanistic models and complicates interpretation of estimated model parameters. In this situation, much can still be learned from simple, phenomenological models that capture salient features of available disease incidence data without making strong assumptions about disease characteristics or mechanisms of spread.

In this talk, I motivated the use of generalized logistic models to estimate the initial rate of exponential growth of an epidemic, a quantity that, in an outbreak context, informs how fast public health interventions must be deployed in order to meaningfully curtail spread and reduce burden on health systems. I introduced statistical software (R package **epigrowthfit**) that implements our methods for both estimating growth rates and investigating variation in growth rates between waves and across jurisdictions. Discussion with workshop participants after the talk centered on the theoretical distinction between mechanistic and phenomenological epidemic models.

Opinion dynamics and disease: One wave or many?

Rebecca Tyson, UBC Okanagan

Opinion dynamics, that is, changes of opinions/behaviours in a population arising from interactions between individuals in the population, can have a strong effect on disease dynamics. In most modelling efforts however, such behaviours are considered to be fixed within a given subpopulation (divided by, e.g., age or socioeconomic class), or altered by top-down public policies. In this talk we present a suite of models coupling disease and opinion dynamics, and show how the interaction between these two processes can have a profound effect on the disease dynamics, creating, e.g., multiple epidemic waves, changes in peak size, and changes in final size. While there is a long history of modelling disease dynamics, the field of opinion dynamics modelling is still fairly new. We call for more research on how best to model opinion dynamics, particularly within the context of new diseases.

Revealing the unseen: What portion of the Americans relied on others' satisfaction when deciding to take the COVID-19 vaccination

Azadeh Aghaeeyan, Brock University

Efficient coverage for newly-developed vaccines requires knowing which groups of individuals will accept the vaccine immediately and which will take longer to accept or never accept. In this study, we assumed that, within the context of COVID-19 vaccination, non-vaccine refuser Americans behaved as either success-based learners, making decisions based on others' satisfaction, or as myopic rationalists, attending to their own immediate perceived benefit. We used COVID-19 vaccination data to fit a mechanistic model capturing the distinct effects of the two types on the vaccination progress. We estimated that about half of Americans behaved as myopic rationalists with a high variation across the states. The proportion was correlated with the vaccination coverage, proportion of votes in favor of Democrats in 2020 presidential election, and education score. The findings reveal the impact of the proportions of the decision-makers on the vaccination speed and, consequently, overall vaccination coverage.

Multi-Pathogen Agent-Based Models for Disease Surveillance and Mitigation

Caroline Wagner, McGill University

Understanding the dynamics of emerging infections and the efficacy of detection technologies in the context of the endemic circulation of other pathogens is a critical aspect of effective public health responses against infectious diseases. The use of compartmental models to simulate the effectiveness of different detection technologies is complicated by the importance of heterogeneity in numerous aspects of these systems, including underlying patterns of technology distribution within a population and variable in-host immune responses. Compartmental models also present challenges when modeling large numbers of co-circulating pathogens with specific disease characteristics and immunological rules for pathogen-pathogen interactions.

In light of this, we presented Pathosim, a multi-pathogen agent-based model (ABM) that builds on the open-source COVID-19 model Covasim. Like Covasim, Pathosim allows for flexible population and transmission network structures, and can simulate individual in-host viral kinetics, the implementation of pharmaceutical interventions, and testing and quarantine procedures. In addition, Pathosim allows for the flexible characterization of any pathogen of interest along with the specification of immunological rules for pathogen-pathogen interactions (*i.e.,* cross-immunity and altered disease course during co-infection). We demonstrated the utility of Pathosim in terms of simulating and modeling various detection and surveillance systems including protocols for serosurveillance and early-detection systems based on sequencing data.

Assessing the Impact of Non-Pharmaceutical Interventions on COVID Prevalence Using A Predator-Prey Lotka-Volterra Model Approach

Lisa Kanary, Public Health Agency of Canada

Understanding the dynamics of disease transmission and effective interventions is crucial due to a virus' rapid spread. Non-Pharmaceutical Interventions (NPIs) play a vital role in mitigating the spread of a virus through a population, especially when pharmaceutical interventions are limited or ineffective. Implementing NPIs (such as social distancing, face masks, hand hygiene, travel restrictions, and quarantine measures) has been essential in controlling the pandemic and reducing the burden on healthcare systems.

This study aims to investigate the relationship between NPIs and COVID prevalence. By incorporating NPIs into a modeling framework, this assessment will help determine the effectiveness of NPIs in reducing the transmission and overall prevalence of COVID-19, and effectively, disease in general.

To investigate the relationship between COVID prevalence and NPIs, we employ a predator-prey Lotka-Volterra model. The predator-prey model offers a theoretical framework that allows for the examination of the impact of NPIs on COVID transmission dynamics and provides insights into the effectiveness of these interventions in controlling disease prevalence. The insights provided by these mathematical models can inform decision-making processes for policy-makers, public health officials, and researchers, and can guide the development of targeted interventions, helping to control the spread of the virus and mitigate its impact on public health and society. Several methods for fitting model coefficients will be explored in this exercise.

Better modeling through chemistry: quantifying COVID vaccine hesitancy

Brian Gaas, Government of Yukon

Much of the literature on vaccine hesitancy focuses on whether an individual receives a vaccine. However, the rate of vaccination—the number of people getting vaccinated in a given amount of time—is equally important. This work presents a conceptual framework for understanding and predicting vaccine adoption rates, following the transition state theory of chemistry.

The vaccine uptake framework hypothesizes people will only get vaccinated if their personal Vaccine Motivation exceeds a population-averaged Vaccine Hesitancy. Within the framework, Vaccine Motivation and Vaccine Hesitancy are functionally equivalent to temperature and activation energy, respectively, within the Arrhenius equation. The proportion of unvaccinated individuals getting vaccinated per unit time is related to the negative exponential of the Vaccine Hesitancy Ratio, defined as Vaccine Motivation divided by Vaccine Hesitancy. Neither Vaccine Motivation nor Vaccine Hesitancy are observable, but the Vaccine Hesitancy Ratio for a given time period can be estimated as the negative log-odds of vaccination status (individuals who changed vaccination status versus individuals who did not change status within that period). Logistic regression can be used to test whether the Vaccine Hesitancy Ratio varies over time, since it has the same log-odds form.

Dose 1 uptake rates from the Yukon (Canada) for COVID-19 were analyzed using the vaccine uptake framework. The population could be clustered into four groups of people based on how the Vaccine Hesitancy Ratio changed over time: low, medium, and high Vaccine Hesitancy Ratios, and one group who never got vaccinated. Further work could include applying clustering algorithms to better differentiate groups, identifying predictors that classify individuals into each of the four groups, and applying the vaccine uptake framework to forecast future dose uptake or uptake rates of different vaccines.

Antigenic evolution of SARS-CoV-2 in immunocompromised hosts

Ben Ashby, Simon Fraser University

Prolonged infections of immunocompromised individuals have been proposed as a crucial source of new variants of SARS-CoV-2 during the COVID-19 pandemic. Longitudinal sampling of SARS-CoV-2 from immunocompromised hosts reveals evidence of accelerated adaptation relative to the wider population. In principle, sustained within-host antigenic evolution in immunocompromised hosts could allow novel immune escape variants to emerge more rapidly, but little is known about how and when immunocompromised hosts play a critical role in pathogen evolution. This talk discussed how a relatively simple mathematical model can provide powerful insights into the effects of immunocompromised hosts on the emergence of immune escape variants. Specifically, when the pathogen does not have to cross a fitness valley for immune escape to occur, a small number of immunocompromised hosts have no qualitative effect on antigenic evolution at the population-level. But if a fitness valley exists, then persistent infections of immunocompromised individuals allow mutations to accumulate so that the fitness valley can be traversed, thereby facilitating large jumps in antigenic space at the population-level. Our results suggest that better genomic surveillance of infected immunocompromised individuals and better global health equality, including improving access to vaccines and treatments for individuals who are immunocompromised (especially in lower- and middle-income countries), may help to prevent the emergence of future immune escape variants of SARS-CoV-2.

Toward Support for Epidemic Preparedness via Digital Twin Data + "Think Big" Proposal

Michael Wolfson, University of Ottawa

An essential component for pandemic preparedness is adequate data supporting ongoing analytical capacity. Since support for specialized pandemic-oriented data collection (+ analytical capacity) tends to wane between acute events, data for pandemic preparedness should, as much as possible, be designed to be useful during quiescent periods. One such kind of data is a realistic but synthetic "digital twin" that closely resembles detailed census data but is non-identifiable, hence non-confidential. Such digital twin data would provide individual-level details of Canada's population by small area geography and a range of socio-economic and infectious disease-relevant characteristics, substantially reflecting real-world heterogeneities. In turn, these data could provide a richly textured basis for more sophisticated infectious disease modeling, especially with regard to contact patterns more readily incorporated into agent-based modeling as compared to the more usual compartment models.

Such a digital twin database could be constructed by starting first with published census cross-tabs at the census tract level using simulated annealing, and then synthetically matching (with replacement) individual and household level records from census public use files, thereby substantially preserving important kinds of correlations. In addition, data from other key microdata files like the Canadian Community Health Survey (CCHS), the Labour Force Survey (LFS), and the time use results from the General Social Survey could also be synthetically matched. As these data sources are either already in the public domain, or versions could be so constructed, the resulting digital twin would also be non-confidential, hence completely open data.

Constructing this digital twin database and regularly updating it would incur considerable costs. However, it would also have a much broader range of uses than only to support epidemic modeling. With a broad range of users, it would be more easily sustained over time as a new and important addition to Canada's statistical system.

In order to keep the digital twin data current, "nowcasting" using a public domain version of Statistics Canada's DEMOSIM model could be used, along with other regular monthly data sets including the LFS and CCHS.

With the advent of an epidemic and the imposition of public health measures such as lockdowns, behaviors would change. These could be tracked via appropriately anonymized yet still geographically detailed real-time cell phone mobility data.

In sum, a well-conceived digital twin database could provide both a substantially improved real-time basis for infectious disease modeling and a substantial addition to Canada's statistical system that would have a broad range of other uses, thereby assuring its longer term sustainability.

Based on some of the preceding discussion in the conference, this presentation also offered the suggestion of "thinking big" – developing a brief for senior officials and governments outlining the idea of a digital twin along with other key improvements in linkable and coherent data flows such as infections, hospitalizations, vaccinations, and genotyping of infections. While this kind of initiative is not something fundable using the current Canadian and provincial granting council structures, there is a window of opportunity with recent reports on data from the Public Health Agency of Canada and Health Infoway, and the 2023 federal budget proposing over half a billion dollars for health data. There is also the recent Report of the Advisory Panel on the Federal Research Support System which could well open up the right kind of funding opportunities.

Conceptual models of immunity

Jonathan Dushoff, McMaster University

A better understanding of patterns and processes underlying cross-immunity is needed to better understand future dynamics and burden of important diseases. Simple dynamical models of cross-immunity can often be classified as either history-based (classifying individuals by history of infections), or status-based (classifying individuals by what strains they are currently effectively immune to). There is a very strong analogy between these classifications and those of "leaky" and "polarized" vaccine models.

This talk discussed the importance of immune-boosting—that is, enhanced immune protection following an unsuccessful infection challenge—and demonstrated that a model with boosting provides a practical way to bridge between the dynamics of these two paradigms, while arguing that this mechanism is highly biologically relevant. Differences in such conceptual assumptions can lead both to different estimates of vaccine effectiveness, and different predictions about future dynamics.

Modelling Immunity

Jane Heffernan, York University

'Immunity' is both a population- and an individual-level characteristic. We have developed individual- and population-level models of immunity using in-host modelling of vaccination, immune system and infection dynamics, and epidemiological models of disease spread and public health programming. In this talk, we discussed the multi-level models of immunity and their integration, which can be used to inform public health decision-makers. We focused on COVID-19 in Ontario.

Testimonials

"The CANMOD BIRS workshop was a fabulous experience, both overall and in its specifics. Direct shuttles between Calgary and Banff, lodging and included meals at the site of the conference, on-hand IT staff for presentations, giant mountains to look at...I'm not sure what else one might want for logistics. I imagine conference attendees everywhere were happy to see the people behind the Zoom and Teams screens for the first time in a few years, and this was no exception. It is perhaps the goal of every conference to provide a venue where people can learn what their colleagues are up to, and collaborate on future projects. The CANMOD workshop succeed admirably in this, and I am currently following up with a few different groups on work that was presented. My two complaints are me not bringing climbing shoes, and eating too much at the buffet dinners... both which are perfectly good problems to have."

Brian Gaas, Government of Yukon

"The CANMOD BIRS workshop was easily one of the best I have ever attended. All of the talks were excellent and the focus of the workshop on infectious disease modelling meant all of them were of interest to everyone. Communal meal times were also a great way to get to interact with other attendees and the small workshop size meant that, by the end of the four days, I was able to have a conversation with everyone at least once. I thoroughly enjoyed my time in Banff and left the workshop feeling reinvigorated about working on infectious disease research projects."

Evan Mitchell, McMaster University

"As a infectious disease epidemiologist working as a scientist at Public Health Ontario, I know the acute need for strong connections between the academic mathematical modeling community, public health, and government decision makers. At the BIRS-CANMOD meeting in November 2023, I made great connections that will support future public health responses to infectious disease threats, by supporting such connections. In addition to excellent applied presentations, we had beneficial discussions about how to strengthen public health surveillance data for respiratory infections, that could help support more robust forecasting of respiratory infections in the future."

Kevin Brown, Public Health Ontario

"Attending the conference was an invaluable experience that provided me with a unique platform for fostering meaningful research connections and collaboration. The opportunity to meet people in person was instrumental in developing new innovative ideas and projects geared towards enhancing future pandemic preparedness. The insights gained not only deepened my understanding of the research our CANMOD team is engaged in but also fueled a collective passion for addressing critical challenges.

This conference served as a catalyst for interdisciplinary discussions, sparking the exchange of ideas that have the potential to shape the future of pandemic response strategies. The immersive environment facilitated networking, enabling me to connect with professionals who share a commitment to advancing our shared goals and learn from the diverse expertise of others. The experience has underscored the importance of such gatherings, emphasizing the need for more opportunities like this in the future."

Lisa Kanary, Public Health Agency of Canada

Conclusion

A common experience since the start of the COVID-19 pandemic has been that we often meet people first virtually, and potentially never in-person. A large proportion of CANMOD's co-applicants had never met in-person before the BIRS workshop, and the opportunity to do so—for those of us who were able to attend in Banff—was greatly appreciated. For both in-person and virtual participants, the presentations and discussions were extremely valuable, and the connections made during the meeting will influence research and public health policy-making in the coming years. These types of events—which generate enthusiasm among participants for continued interactions between academics, government scientists, and policy-makers—are critical to the vision for EIDM modelling in Canada described in a white paper [**?**] drafted by several of the attendees together with members of other EIDM networks at an earlier BIRS meeting in January 2023. We are certain that all the attendees at the CANMOD BIRS workshop in November 2023 are looking forward to continuing to contribute to the realization of that vision!

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Chapter 31

Harmonic Analysis and Convexity (23w5053)

November 19 - 24, 2023

Organizer(s): Alexander Koldobsky (University of Missouri), Dmitry Ryabogin (Kent State University), Kateryna Tatarko (University of Waterloo), Vladyslav Yaskin (University of Alberta), Artem Zvavitch (Kent State University)

Overview of the Field

Convexity is a very old topic which can be traced at very least to Archimedes. These days the area is especially active due to its numerous applications to the linear programming, tomography, medicine, information theory, to name a few. For the last twenty five years Harmonic Analysis has been the major tool for solving the most challenging open problems in Convex Geometry. Since convexity is a very natural notion and an excellent choice for graduate students and postdocs, the workshop brought together a number of top and junior researchers with the aim of discussing most recent developments in the areas.

The topics of the workshop included harmonic analysis in $\mathbb{R}^n$ and on the sphere, spherical operators and special classes of bodies, geometric inequalities, discrete and differential geometry, topology, probability and random matrices.

Presentation Highlights

We start our highlights with the results of *Hermann König*. His talk was about *non-central sections of the ℓ_1^n-ball and the regular simplex*. Let $n \in \mathbb{N}$, $n \geq 2$, let S^{n-1} be a unit sphere in $\mathbb{R}^n$ and let K be a convex body in $\mathbb{R}^n$. Denote by $A(a,t) = A_K(a,t)$ its t-section function,

$$A(a,t) = \mathrm{vol}_{n-1}(K \cap (a^\perp + ta)), \qquad t \in \mathbb{R}, \quad a \in S^{n-1}$$

where $a^\perp$ is the hyperplane passing through the origin and orthogonal to a unit vector $a \in S^{n-1}$. Given K and $t \in \mathbb{R}$, how to find $a_{max}, a_{min} \in S^{n-1}$ so that

$$A(a_{min},t) \leq A(a,t) \leq A(a_{max},t) \qquad \forall a \in S^{n-1} \ ?$$

For $k \in \mathbb{N} \cap [1,n]$ let $a^{(k)} = \frac{1}{\sqrt{k}}(1,\dots,1,0,\dots,0)$, where the coordinate 1 is taken k times. Then, for the unit cube $K = \frac{1}{2}B_\infty^n = [-\frac{1}{2},\frac{1}{2}]^n$ and $t=0$ one has the classical results that

$$1 = A(a^{(1)},0) \le A(a,0) \le A(a^{(2)},0) = \sqrt{2}.$$

also, if $p>0$ and $K = B_p^n = \{x=(x_1,\dots,x_n)\in\mathbb{R}^n : \sum_{j=1}^n |x_j|^p \le 1\}$, then it is known that

$$A(a^{(n)},0) \le A(a,0) \le A(a^{(1)},0), \qquad 0<p<2.$$

What about non-central sections of the bodies, i.e., what if $t \neq 0$? If $K = \frac{1}{2}B_\infty^n$, $n \ge 3$, and $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$, then $A(a,t) \le A(a^{(n)},t)$ $\forall a \in S^{n-1}$, and the same result holds for $n \ge 5$ and $t \in (\frac{\sqrt{n-2}}{2}, \frac{\sqrt{n-1}}{2}]$. However, for $K = B_1^n$, $t \in (\frac{1}{\sqrt{2}},1]$, one has

$$A(a,t) \le A(a^{(1)},t) = \frac{2^{n-1}}{(n-1)!}(1-t)^{n-1} \qquad \forall a \in S^{n-1}.$$

Also, for $n \ge 4$ and $t \in (\frac{1}{\sqrt{3}},1]$, for $n=3$ and $t \in (\sqrt{2}-1-\sqrt{5-\frac{7}{\sqrt{2}}},1]$, and for $n=2$, $t\in(\frac{3}{4},1]$, it is shown that the same result holds. Moreover, the explicit formula for the t-sections of $K = B_1^n$, $1 > a_1 > a_2 > \cdots > a_n > 0$, is given,

$$A(a,t) = \frac{2^{n-1}}{(n-1)!}\sum_{j=1}^n \frac{a_j^{n-2}(a_j-t)_+^{n-1}}{\prod_{k=1,k\neq j}^n (a_j^2-a_k^2)}.$$

If

$$\Delta^n = \{x \in \mathbb{R}_+^{n+1} : \sum_{j=1}^{n+1} x_j = 1\}$$

is the n-dimensional simplex of side-length$\sqrt{2}$, then its centroid is $c = \frac{1}{n+1}(1,\dots,1)$. Assuming that the hyperplane $x\cdot a = 0$ passes through c, one has

$$A(a,0) \le A(\bar{a},0) = \frac{\sqrt{n+1}}{(n-1)!}\frac{1}{\sqrt{2}} \qquad \forall a \in S^{n-1} : \quad \sum_{j=1}^{n+1} a_j = 0,$$

where $\bar{a} = \frac{1}{\sqrt{2}}(1,-1,0,\dots,0)$. Several other results that allow to determine the maximal and minimal sections of the simplex were given for some $t \neq 0$.

Carsten Schütt presented results related to measuring *the Banach-Mazur distance* $d(l_r^{n^2}, l_p^n \otimes_\varepsilon l_q^n)$ *between the spaces* $l_r^{n^2}$ *and* $l_p^n \otimes_\varepsilon l_q^n$ in the cases $r=1,2$ and $1 \le p,q \le \infty$. Here

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\| : T : X \to Y\},$$

where T is an isomorphism between the Banach spaces X and Y, the norm of the vector (the matrix) $A_{n\times n}$ in $l_p^n \otimes_\varepsilon l_q^n$ is defined as

$$\|A\|_{l_p^n\otimes_\varepsilon l_q^n} = \|A\|_{L(l_{p^*}^n, l_q^n)} = \sup_{\|x\|_{p^*}=1,\, \|y\|_q=1} \sum_{i,j=1}^n A_{ij}x_iy_j,$$

and p^* is defined via $\frac{1}{p^*}+\frac{1}{p}=1$. It is shown that

$$d(l_2^{n^2}, l_p^n \otimes_\varepsilon l_q^n) = \begin{cases} n^{\frac{1}{p}+\frac{1}{q}-1}, & \frac{3}{2} \le \frac{1}{p}+\frac{1}{q} \\ \sqrt{n}, & 1 \le p,q \le 2 \quad \text{and} \quad \frac{3}{2} \ge \frac{1}{p}+\frac{1}{q} \\ n^{\frac{1}{p^*}}, & 1 \le q \le 2 \le p \le q^* \\ n^{\frac{1}{q}}, & 1 \le q \le q^* \le p \\ n^{\frac{1}{q^*}}, & 2 \le q \le p, \end{cases}$$

and

$$d(l_1^{n^2}, l_p^n \otimes_\varepsilon l_q^n) = \begin{cases} n^{\frac{5}{2}-\frac{1}{p}-\frac{1}{q}}, & \frac{3}{2} \leq \frac{1}{p} + \frac{1}{q} \\ n, \quad 2 \leq p, q. \end{cases}$$

Tomasz Tkocz gave a talk titled *Hardwired... to Szarek and Ball* about his joint results with Alexandros Eskenazis and Piotr Nayar. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of i.i.d random variables which are $Unif(\{-1, 1\})$, and let $\{\xi_k\}_{k=1}^\infty$ be a sequence of i.i.d random variables which are $Unif(S^{n-1})$. In 1976 Szarek proved that

$$\mathbb{E}\Big|\sum_{j=1}^n a_j \varepsilon_j\Big| \geq \mathbb{E}\Big|\frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}\Big|.$$

On the other hand, in 1986 K. Ball showed that

$$\mathbb{E}\Big|\sum_{j=1}^n a_j \xi_j\Big|^{-1} \leq \mathbb{E}\Big|\frac{\xi_1 + \xi_2}{\sqrt{2}}\Big|^{-1};$$

both results hold for all $a = (a_1, \ldots, a_n) \in S^{n-1}$ and all $n \geq 1$. Such inequalities were also known only for some other distributions such as $Unif(S^d)$, $Unif(B_2^d)$, GMs and the marginals of l_p-balls. The main result is that the above estimates hold for essentially all distributions sufficiently close to ε/ξ, provided $\|a\|_\infty \leq \frac{1}{\sqrt{2}}$.

Dylan Langharst delivered a lecture *On the measures satisfying a monotonicity of the surface area with respect to Minkowski sum* on his joint results with M. Fradelizi, M. Madiman and A. Zvavitch. One of the theorems of Fradelizi, Madiman and Zvavitch says that

$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A + B + C) \geq \mathrm{vol}_n(A + B) + \mathrm{vol}_n(A + C).$$

Does a similar result hold for other measures? On a similar note, recently, G. Saracco and G. Stefani proved that if μ on $\mathbb{R}^n$ is absolutely continuous with respect to the Lebesgue measure, and for any two convex bodies one has $\mu^+(\partial K) \leq \partial^+(\partial L)$, then μ must be a constant multiple of the Lebesgue measure. Here

$$\mu^+(\partial K) = \lim_{\varepsilon \to 0} \frac{\mu(K + \varepsilon B_2^n) - \mu(K)}{\varepsilon}.$$

The main result is that if μ is absolutely continuous with respect to the Lebesgue measure such that for any convex bodies K and L one has $\mu^+(\partial(K + L)) \leq \partial^+(\partial K)$, then μ is a constant multiple of the Lebesgue measure.

Bartłomiej Zawalski solved a problem of Louis Montejano by showing that *the star-convex bodies with rotationally invariant sections* are the bodies of revolution. One of the most famous questions in Banach space theory belongs to Banach himself and asks the following. Let B^n be a Banach space of finite dimension n and let $k \in \mathbb{N}$ be such that $1 < k < n$. If all the k-dimensional subspaces of B^n are isometrically isomorphic to each other, is B^n a Hilbert space? In the 30's of the last century H. Auerbach, S. Mazur and S. Ulam solved the case $n = 3$. At the end of 1960's M. Gromov settled it for odd n. More recently, J. Bracho and L. Montejano obtained several results on a complex version of Problem 1, S. Ivanov, D. Mamaev and A. Nordskova solved the case $n = 4$, and G. Bor, L. Hernández-Lamoneda, V. Jiménez de Santiago and L. Montejano solved the question $n = 4k + 2 \geq 6$, $n \neq 134$. One of the key elements of the proof of the last authors was to show that the hyperplane sections of the unit ball of B^n must be the body of revolution, which prompted the authors to ask the following. Let $K \subset \mathbb{R}^n$, $n \geq 4$, be a convex body containing the origin in its interior. If every hyperplane section of K passing through this point is a body of affine revolution, is K necessarily a body of affine revolution? The main result of Zawalski gives an affirmative answer to this question, provided the boundary of K is C^3 and K is origin-symmetric.

Maud Szusterman presented her results on *Vector balancing and lattice coverings: inequalities via the Gaussian measure*. Let U, V be origin-symmetric convex bodies in $\mathbb{R}^n$ and let

$$\beta(U, V) = \inf\{\beta > 0 : \forall u_1, u_2, \ldots, u_n \in U, \exists \varepsilon \in \{\pm 1\}^n : \sum_{i=1}^n \varepsilon_i u_i \in \beta V\} =$$

$$= \max_{u_1, \ldots, u_n \in U} \min_{\varepsilon \in \{\pm 1\}^n} \|\sum_{i=1}^n \varepsilon_i u_i\|_V.$$

It is known that $\beta(B_2^n, B_2^n) = \sqrt{n}$, $\beta(B_\infty^n, B_2^n) = c_n n$ (with $c_n = 1$ if Hadamard matrices exist in $\mathbb{R}^n$), $\beta(B_1^n, B_\infty^n) \leq 2$, $\beta(B_\infty^n, B_\infty^n) \leq c\sqrt{n}$. Komlos conjecture asks if $\beta(B_2^n, B_\infty^n) = O_n(1)$. The best result related to this conjecture belongs to Banaszczyk, who proved that $\beta(B_2^n, B_\infty^n) \leq 5\sqrt{2}\sqrt{\log n}$. It is also known $\beta(B_2^n, V) \leq 5$ if $\gamma_n(V) \geq \frac{1}{2}$, where γ_n is a Gaussian measure on $\mathbb{R}^n$. Let

$$\mu(L, V) = \inf\{t > 0 : \ L + tV = \mathbb{R}^n\}$$

be the covering radius of the lattice L with respect to V (if $L = \mathbb{Z}^n$, $V = B_2^n$, then $\mu(\mathbb{Z}^n, B_2^n)) = \frac{\sqrt{n}}{2}$) and let

$$\lambda_k(L, V) = \inf\{t > 0 : \ \dim(\text{span})(tV \cap L) \geq k\}.$$

If

$$\alpha(U, V) = \sup_L \frac{\mu(L, V)}{\lambda_n(L, V)} = \sup_{L, \lambda_n(L,V)=1} \mu(L, V),$$

then $\alpha(B_2^n, V) \leq \frac{1}{\psi^{-1}(\frac{1}{2})}$ for any convex body V satisfying $\gamma_n(V) \geq \frac{1}{2}$. Here

$$\psi(x) = 2\Phi(x) - 1, \qquad \Phi(x) = \gamma_1((-\infty, x]).$$

It is also shown that for any convex body $V \subset \mathbb{R}^n$ one has $\alpha(B_2^n, V) \leq \frac{1}{\psi^{-1}(\gamma_n(V))}$.

Oscar Adrian Ortega Moreno spoke about *The complex plank problem, revisited.* Ball's complex plank theorem states that if $v_1, \ldots, v_n$ are unit vectors in $\mathbb{C}^d$, and $t_1, \ldots, t_n$, non-negative numbers satisfying $\sum_{k=1}^{n} t_k^2 = 1$, then there exists a unit vector v in $\mathbb{C}^d$ for which $|\langle v_k, v\rangle| \geq t_k$ for every k. Oscar presented an elegant version of Ball's original proof. He started with the case when all t_k's are $\frac{1}{\sqrt{n}}$. In this case, he shows that if $v_1, \ldots, v_n$ are unit vectors in $\mathbb{C}^d$ and u maximizes $\prod_{k=1}^{n} |\langle v_k, u\rangle|$ among unit vectors, then $|\langle v_k, u\rangle| \geq \frac{1}{\sqrt{n}}$ for every k. For general positive t_k's satisfying $\sum_{k=1}^{n} t_k^2 = 1$ and unit vectors $v_1, \ldots, v_n$ in $\mathbb{C}^d$ one proves that the vector u maximizes $\prod_{k=1}^{n} |\langle v_k, u\rangle|^{t_k^2}$ among unit vectors, then $u = \sum_{k=1}^{n} \frac{t_k^2}{\langle v_k, u\rangle} v_k$. Then the proof is reduced to the previous case, provided one analyzes the function $p(z) = \prod_{k=1}^{n} \left|\frac{\langle v_k, zv_1 + u\rangle}{\langle v_k, u\rangle}\right|^{t_k^2}$ applying the maximum principle (the function $\log|p(z)|$ is subharmonic).

Mark Rudelson delivered a talk *Approximately Hadamard matrices and random frames* about his joint results with his student *Xiaoyu Dong*. Let A be an $N \times n$ matrix with i.i.d. symmetric non-degenerate entries and let A_I be its submatrix such that its columns belong to a subset $I \subset \{1, 2, \ldots, N\}$. Then there exist constants c, C, α, β depending on the distribution of entries of A with the following property: if $N \geq e^{Cn}$, then there exists $L \geq e^{cn}$ such that the probability of the existence of disjoint subsets $I_1, \ldots I_L$ of $\{1, \ldots, N\}$ with $|I_j| = n$ and $\frac{s_{max}(A_{I_j})}{s_{min}(A_{I_j})} < \alpha \ \forall j \in \{1, \ldots, L\}$, is greater or equal to $1 - e^{-e^{\beta n}}$. Here $s_j(A) = \sqrt{\lambda_j(AA^T)}$, λ_j are the eigenvalues of AA^T, and s_{max}, s_{min} are the maximal and minimal singular values among s_j's. It is also shown that there exist constants $0 < c < C$ such that for any $n \in \mathbb{N}$ one can find an $n \times n$ matrix V with ± 1 entries satisfying $c\sqrt{n} \leq s_{min}(V) \leq s_{max}(V) \leq C\sqrt{n}$. Finally, it is proved that finding a sub-matrix with a bounded condition number requires an exponential number of columns for matrices with sub-gaussian entries.

Alexander Litvak brought up a discussion about volume ratio of convex bodies and spoke on *volume ratio between projections of convex bodies*. The volume ratio of two convex bodies $K, L \subset \mathbb{R}^n$ containing the origin in their interior is defined as $vr(K, L) = \inf(\frac{|K|}{|TK|})^{\frac{1}{n}}$, where the infimum is taken over all affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$ and $|K|$ stands for the n-dimensional volume of K. What is the maximal possible volume ratio? Giannopoulus and Hartzoulaki proved that $vr(K, L) \leq c\sqrt{n}\log n$, while Khrabrov showed that there are bodies K and L such that $vr(K, L) \geq c\sqrt{\frac{n}{\log\log n}}$. One sees that there is a log gap between these results. Another well-known measurement of the distance between bodies is the Banach-Mazur distance $d(K, L)$ mentioned in the first talk,

$$d(K, L) = \inf\{\lambda > 0 : \ K - x \subset T(L - y) \subset \lambda(K - x), \quad x, y \in \mathbb{R}^n\}$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear. If K and L are origin-symmetric, i.e., $K = -K$ and $L = -L$, then Gluskin proved that $d(K, L) \geq cn$, while Rudelson showed that $d(K, L) \leq cn^{\frac{4}{3}} \log^9 n$. It is open if Rudelson's result can be improved to $n \log^\alpha n$ for some $\alpha > 0$. Another Rudelson's result states that for origin-symmetric convex bodies K and L one has $\delta_k(K, L) \leq C \max\{\frac{k^2}{n}, \sqrt{k} \log n\}$, where $\delta_k(K, L) = \inf\{d(PK, QL) : rank(P) = rank(Q) = k\}$, $1 \leq k \leq n$, and the infimum runs over all projections P, Q of rank k. Here the threshold is $k = n^{\frac{2}{3}}$ (up to logs). The main result is that for any K and $k \geq n$ there exists $L = -L$ such that

$$\delta'_k(K, L) = \inf\{vr(PK, QL) : rank(P) = rank(Q) = k\} \geq \frac{ck}{\sqrt{n \log n}}.$$

Moreover, if $k \geq n^{\frac{2}{3}}$, then $\delta'_k(K, L) \leq \frac{ck}{\sqrt{n}}$ and this result (sharp for $K = B_1^n$) holds for any L.

Piotr Nayar presented several joint results with J. Melbourne and C. Roberto on *Minimum entropy of a log-concave random variable for fixed variance*. Let $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$, $h(X) = h(f) = -\int f \log f$, $h_\alpha(X) = h_\alpha(f) = \frac{1}{1-\alpha} log\Big(\int f^\alpha\Big)$, where $\alpha \in (0, +\infty) \setminus \{1\}$. It is proved that for $\alpha \in (\frac{1}{3}, +\infty) \setminus \{1\}$, $h_\alpha(f)$ is maximal under fixed variance for $f(x) = c_0(1 + (1-\alpha)(c_1 x)^2)_+^{\frac{1}{\alpha-1}}$. For $\alpha = 1$, $h(X) \leq \frac{1}{2} \log Var(X) + \frac{1}{2} \log(2\pi e)$. M. Bialobrzeski, M. Madiman, P. Nayar and M. Fradelizi showed that for a symmetric log-concave random variable and α^* solving $\frac{1}{1-\alpha} \log \alpha = \frac{1}{2} \log 6$ one has

$$h_\alpha(X) \geq \begin{cases} \frac{1}{2} \log Var(X) + \frac{1}{2} \log 12, & \alpha \leq \alpha^* \\ \frac{1}{2} \log Var(X) + \frac{1}{2} \log 2 + \frac{\log \alpha}{\alpha - 1}, & \alpha \geq \alpha^*. \end{cases}$$

J. Melbourne, P. Nayar and C. RobertoIt proved that for a log-concave real random variable one has $h(X) \geq \frac{1}{2} \log Var(X) + 1$ with equality for $f(x) = e^{-x} 1_{[0, +\infty)}$, and for $\alpha \geq 1$, $h_\alpha(X) \geq \frac{1}{2} \log Var(X) + \frac{\log \alpha}{\alpha - 1}$. The case $\alpha < 1$ is open.

Eli Putterman talked about *Small-ball probabilities for mean widths of random polytopes* and presented his joint work with J. Haddad, D. Langharst, M. Roysdon, and D. Ye. Given a convex body $K \subset \mathbb{R}^n$, define the l-th higher order projection body $\Pi^l K \subset \mathbb{R}^{nl}$ via its support function $h_{\Pi^l K}(\theta) = \max_{\{x \in \Pi^l K\}} x \cdot \theta$, $\theta = (\theta_1, \ldots, \theta_l) \in \mathbb{R}^{nl}$, as

$$h_{\Pi^l K}(\theta_1, \ldots, \theta_l) = \int \max_{i=1,\ldots,n} u \cdot \theta_i \, dS_K(u),$$

where S_K is the surface area measure of K. The above expression is equal to $n\text{vol}_n(C_\theta, K[n-1])$, the multiple of the mixed volume of the simplex $C_\theta = \text{conv}(0, \theta_1, \ldots, \theta_l)$ and $n-1$ copies of K. One shows that for $\Pi^{0,l} K = (\Pi^l K)^*$, where K^* stands for the polar of K, one has

$$\text{vol}_n(K)^{(n-1)l} \text{vol}_{nl}(\Pi^{0,l} K) \leq \text{vol}_n(B_2^n)^{(n-1)l} \text{vol}_{nl}(\Pi^{0,l} B_2^n).$$

Eli poses the following problem. Let $(\theta_1, \ldots, \theta_l)$ be i.i.d. random vectors uniformly distributed on S^{nl-1} and let $W_\theta = W(C_\theta)$ be the mean width of C_θ. Compute $\mathbb{E}_\theta(W_\theta^{-nl})$. So far, it has been evaluated up to a constant, $\mathbb{E}_\theta(W_\theta^{-nl})^{\frac{1}{nl}} \approx \max(\sqrt{\frac{nl}{\log l}}, \sqrt{l})$, the transition between the quantities occurs when $l \sim c^n$.

Orli Herscovici presented her joint results with Galyna Livshyts, Liran Rotem and Alexander Volberg on *the stability and the equality cases in the Gaussian B-inequality*. They considered the B-inequality of D. Cordero-Erausquin, M. Fradelizi, and B. Maurey, staying that $\gamma(\sqrt{ab}K) \geq \sqrt{\gamma(aK)\gamma(bK)}$ for all $a, b > 0$ and any convex set $K = -K$, i.e., the function $t \to \gamma(e^t K)$ is log concave with respect to t. Here Γ is the Gaussian measure on $\mathbb{R}^n$. The stability result asserts that if $0 \leq a < b < \infty$, $K = -K$, convex and satisfying $\gamma(\sqrt{ab}K) \geq \sqrt{\gamma(aK)\gamma(bK)}(1 + \varepsilon)$ for some $\varepsilon > 0$, then for the inradius $r(K)$ (the largest r such that $rB_2^n \subset K$) one has $r(K) \geq \frac{1}{b} \sqrt{\log \frac{c \log(\frac{b}{a})^2}{n^2 \varepsilon}}$ and $r(K) \leq \frac{c\sqrt{n}}{a} \varepsilon^{\frac{1}{n+1}} (\log \frac{b}{a})^{-\frac{2}{n+1}}$. It is shown that the lower bound is sharp, while the sharpness of the upper bound is open.

Michael Roysdon discussed his joint results with Alexander Koldobsky and Artem Zvavitch related to *Comparison problems for Radon Transforms*. Inspired by the Busemann-Petty problem in Convex Geometry, they examined similar tomography questions concerning estimates of the L_p-norms of even continuous functions given information about their Radon-type transforms. In particular, they studied comparison problems for the spherical and classical Radon transforms by introducing families of functions which extended the class of intersection bodies of star bodies due to Lutwak. Michael also discussed comparison problems for the $(n-k)$-dimensional Radon and spherical Radon transforms. One of the results is that for an even infinitely smooth positive function g on S^{n-1} and $p > 1$ there exists an infinitely smooth f on S^{n-1} such that $\|f\|_{L^p(S^{n-1})} \geq \|g\|_{L^p(S^{n-1})}$, provided $g^{p-1}(\theta)\frac{1}{r}$ is not positive definite in $\mathbb{R}^n$ (here one puts $x = r\theta \in \mathbb{R}^n$). He explained why the L^p-analogue of the implication

$$\int\limits_{\theta^\perp + t\theta} f \leq \int\limits_{\theta^\perp + t\theta} g \quad \forall t \in \mathbb{R}, \theta \in S^{n-1} \quad \Longrightarrow \quad \|f\|_{L^1(S^{n-1})} \geq \|g\|_{L^1(S^{n-1})}$$

(with the L^1-norm replaced by the L^p-norm) does not hold. He also gave a technical sufficient condition on the function g being a "section function of f" (a generalization of the notion of a convex body being the intersection body of a convex body) in terms of the Fourier transform of distributions.

Wen Rui Sun and his advisor *Beatrice-Helen Vritsiou* talked about *Illumination Conjecture for Convex Bodies with many Symmetries*. Suppose one wanted to illuminate a solid object with convex shape, that is, illuminate its surface, by placing a number of light sources around it. What is the smallest number of light sources one would need? This seemingly innocent question has actually turned into a longstanding conjecture in Convex and Discrete Geometry, called the Illumination Conjecture. The conjecture states that for an n-dimensional object, one should need less than $2n$ light sources, except if the object "looks like" a cube (which then needs $2n$). The result is known on the plane and for symmetric bodies in $\mathbb{R}^3$. Tikhomirov proved the result for 1-symmetric bodies (that is, convex bodies with the symmetries of the cube) in all sufficiently large dimensions. The main results show that the illumination conjecture is now verified (along with its equality cases) for 1-symmetric convex bodies in all dimensions and some cases of 1-unconditional convex bodies as well (that is, convex bodies with the symmetries of a rectangular box). In particular, let $B \subset \mathbb{R}^n$ be a 1-unconditional convex body with the boundary ∂B, and let

$$m_B = \max\{k \in \{1, \ldots, n\} : e_1 + e_2 + \cdots + e_k \text{ or some permutation of it } \in \partial B\}.$$

If $m_B = n - 1$ or $m_B = n - 2$ and B contains all the permutations of $e_1 + e_2 + \cdots + e_{n-2}$, then B is not an image of the cube and number of the light sources does not exceed $2^n - 2$.

Grigoris Paouris presented his joint results with Kavita Ramanan on a *probabilistic approach to the geometry of p-Schatten balls*. Let $K \subset \mathbb{R}^n$ be a convex body and let X be an anisotropic random vector on $\tilde{K} = \frac{K}{\mathrm{vol}_n(K)^{\frac{1}{n}}}$, i.e., the correlation matrix of X is a multiple of identity. The vector X is called sub-Gaussian if $\langle X, \theta \rangle$ satisfies

$$\Big(\mathbb{E}(|\langle X, \theta \rangle|^q\Big)^{\frac{1}{q}} \leq k\sqrt{q}\Big(\mathbb{E}(\langle X, \theta \rangle|^2\Big)^{\frac{1}{2}} \quad \forall q \geq 2, \, \forall \theta \in S^{n-1},$$

and it is called super-Gaussian if

$$\Big(\mathbb{E}(|\langle X, \theta \rangle|^q\Big)^{\frac{1}{q}} \geq \frac{\sqrt{q}}{k}\Big(\mathbb{E}(\langle X, \theta \rangle|^2\Big)^{\frac{1}{2}} \quad \forall q \leq n, \, \forall \theta \in S^{n-1},$$

For example, $\tilde{B_p^n}$, $p \geq 2$ is sub-Gaussian and $\tilde{B_p^n}$, $p \in (1, 2)$ is super-Gaussian. Let $M_{n \times l}$ be a space of matrices, $l \geq n \geq 2$ and let $\langle A, B \rangle_F = \mathrm{tr}(A^T B)$ be the Frobenius scalar product of $A, B \in M_{n,l}$. One says that A belongs to the Schatten class $S_p^{n,l}$ if $\|A\|_{S_p^{n,l}} = \|\Sigma_A\|_p < \infty$, where Σ_A is the finite set of singular numbers of A. If $B(S_p^{n,l})$ is a unit ball in $S_p^{n,l}$ and a random matrix W_p is uniformly distributed in $B(S_p^{n,l})$, one can compute sharp upper and lower bounds for the moments of marginals of this random matrix. If a matrix Γ has singular values $\gamma_1, \ldots, \gamma_n$, then

$$\Big(\mathbb{E}|\langle W_p, \Gamma \rangle_F|^q\Big)^{\frac{1}{q}} \simeq \begin{cases} \sqrt{q}\|\Gamma\|_F \\ q^{\frac{1}{p}} l^{\frac{1}{2} - \frac{1}{p}} \Big(\sum\limits_{i=1}^{[\frac{q}{l}]} |\gamma_i^*|^{p'} \Big)^{\frac{1}{p'}} + \Big(\sum\limits_{[\frac{q}{l}]+1}^{n} |\gamma_i^*|^2 \Big)^{\frac{1}{2}} \qquad l \leq q \leq nl \\ \sqrt{l} n^{\frac{1}{p}} \|\gamma\|_{p'}, \qquad q \geq nl \end{cases}$$

Here $\|\gamma\|_{p'}$ stands for the $l_{p'}^n$-norm of the vector $(\gamma_1, \ldots, \gamma_n)$, $\frac{1}{p'} + \frac{1}{p} = 1$, and $\gamma_1^* \geq \ldots, \geq \gamma_n^*$ are the decreasing rearrangements of γ_i's. As a corollary one obtains, for example, that W_p is sub-Gaussian for $p \geq 2$, W_p is super-Gaussian, if $1 < p < 2$.

Galyna Livshyts delivered a lecture *Gaussian principle frequency and convexity* on her joint results with A. Colesanti, E. Francini and P. Salani. Let K be a convex domain. Its principal frequency $\kappa(K)$ is defined as

$$\kappa(K) = \inf_{u \in W^{1,2}(K),\, u|_{\partial K}=0} \frac{\int\limits_K |\nabla u|^2 dx}{\int\limits_K u^2 dx}.$$

Equivalently, $\kappa(K)$ is the smallest positive number such that there exists a non-zero $u \in W^{1,2}(K)$ satisfying $\Delta u = -\kappa(K)u$ on K, $u|_{\partial K} = 0$. G. Faber and E. Krahn proved that $\kappa(K) \leq \kappa(RB_2^n)$, provided $\mathrm{vol}_n(K) = \mathrm{vol}_n(RB_2^n)$. H. Brascamp and E. Lieb showed that $\kappa(tK_0 + (1-t)K_1)^{-\frac{1}{2}} \leq t\kappa(K_0)^{-\frac{1}{2}} + (1-t)\kappa(K_1)^{-\frac{1}{2}}$, provided K_0, K_1 are convex bodies $\mathbb{R}^n$ and $t \in [0,1]$. What are the analogues of these results for $\kappa_\mu(K)$ instead of $\kappa(K)$,

$$\kappa_\mu(K) = \inf_{u \in W^{1,2}(K),\, u|_{\partial K}=0} \frac{\int\limits_K |\nabla u|^2 d\mu}{\int\limits_K u^2 d\mu},$$

where $d\mu(x) = e^{-v(x)}dc$, v is convex on $\mathbb{R}^n$? One can show that $\kappa_\mu(K)$ is the smallest positive number κ_μ such that there exists a non-zero $u \in W^{1,2}(K)$ satisfying $\Delta u - \langle \nabla u, \nabla v \rangle = -\kappa(K)u$ on K, $u|_{\partial K} = 0$. The main result is the following. Let γ be the Gaussian measure on $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a convex set. If κ_γ is the smallest non-trivial u such that $\Delta u - \langle \nabla u, \nabla v \rangle = -\kappa_\gamma(K)u$ on K, $u|_{\partial K} = 0$, then u is log-concave. It is shown also that $\kappa_\gamma(tK_0 + (1-t)K_1) \leq t\kappa_\gamma(K_0) + (1-t)\kappa(K_1)$.

Julián Haddad presented several results related to *Fiber symmetrization and the Rogers-Brascamp-Lieb-Luttinger inequality*. Given a positive concave function f, $f(x) = \int\limits_0^\infty \chi_{\{f \geq t\}}(x)dt$, its Steiner symmetrization $f^{(v)}$ in the direction v is performed on the level sets and is defined as $f^{(v)}(x) = \int\limits_0^\infty \chi_{S_v\{f \geq t\}}(x)dt$. In 2016 G. Paouris and P. Pivovarov proved that if positive $F_i : \mathbb{R}^{nd} \to \mathbb{R}$ are concave, $1 \leq i \leq k_2$, $f_1, \dots, f_{k_1}$ be non-negative integrable functions on $\mathbb{R}^n$, $a_j^{(i)}$ are real numbers, $j = 1, \dots, d$, $i = 1, \dots, k_1$, then

$$\int\limits_{\mathbb{R}^n} \cdots \int\limits_{\mathbb{R}^n} \prod_{i=1}^{k_2} F_i(x_1, \dots, x_d) \prod_{i=1}^{k_1} f_i\Big(\sum_{j=1}^{d} a_j^{(i)} x_j\Big) d\mu(x_1) \dots d\mu(x_d) \leq$$

$$\int\limits_{\mathbb{R}^n} \cdots \int\limits_{\mathbb{R}^n} \prod_{i=1}^{k_2} F_i(x_1, \dots, x_d) \prod_{i=1}^{k_1} f_i^{(v)}\Big(\sum_{j=1}^{d} a_j^{(i)} x_j\Big) d\mu(x_1) \dots d\mu(x_d),$$

provided $\mu \geq 0$ is an absolutely continuous measure on $\mathbb{R}^n$ (with respect to the Lebesgue measure) which is rotationally invariant and with density having convex level sets. Julián explained that inspired by Rogers-Brascamp-Lieb-Luttinger inequality and the above result, he obtained a matrix analogue of several known inequalities. In particular, if $M_{d \times m}(\mathbb{R})$ stands for the space of matrices, $L_i \in M_{d \times m}(\mathbb{R})$, $f_i : M_{n \times m}(\mathbb{R}) \to \mathbb{R}$, $i = 1, \dots, k$, then

$$\int\limits_{M_{n \times d}(\mathbb{R})} \prod_{i=1}^{k} f_i(xL_i)dx \leq \int\limits_{M_{n \times d}(\mathbb{R})} \prod_{i=1}^{k} f_i^{(v)}(xL_i)dx,$$

where the "matrix Steiner symmetrization" is properly defined on f_i.

Elisabeth Werner gave a talk *Approximation of convex bodies in Hausdorff distance by random polytopes* about her joint work with J. Prochno, C. Schuett and M. Sonnleitner. While there is extensive literature on approximation, deterministic as well as random, of general convex bodies in the symmetric difference metric, or other metrics coming from intrinsic volumes, very little is known for corresponding random results in the Hausdorff distance. For a polygon Q in the plane, the convex hull of n points chosen at random on the boundary of Q gives a random polygon Q_n. They determine the exact limiting behavior of the expected Hausdorff distance between Q and a random polygon Q_n as the number n of points chosen on the boundary of Q goes to infinity. More precisely, if the boundary ∂K of a convex body K is smooth, then

$$\lim_{n\to\infty} n^{\frac{2}{n-1}} \int\limits_{\partial K} \cdots \int\limits_{\partial K} \delta_\Delta(K, K_n) d\mathbb{P}(X_1) \ldots d\mathbb{P}(X_n) = c_d \int\limits_{\partial K} k(x)^{\frac{1}{d+1}} d\mu_K(x).$$

Here μ_K is the affine surface area measure and δ_Δ is the symmetric difference between K and K_n. A similar result is obtained in the case when K is a simple polytope (every vertex meets d facets). In particular, it is shown that the asymptotic behavior of $\mathbb{E}\delta_\Delta(K, K_n)$ (the integral expression in the left-hand side of the above equality) is $n^{-\frac{2}{d+1}}$ in the smooth case and it is $n^{-\frac{d}{d+1}}$ in the polytopal case.

Andrii Arman gave a talk *On covering problems related to Borsuk's conjecture* where he spoke about his recent results with A. Bondarenko and A. Prymak. Borsuk's number $b(n)$ is the smallest integer such that any set of diameter 1 in n-dimensional Euclidean space can be covered by $b(n)$ sets of a smaller diameter. K. Borsuk proved that $b(1) = 2$, $b(2) = 3$. In 1993 J. Kahn and G. Kalai showed that $b(n) \geq 1.2^{\sqrt{n}}$ for n large enough. Later, A. Raigorodskii improved it to $b(n) \geq 1.2255^{\sqrt{n}}$ for n large enough. It is unknown what the smallest n is for which $b(n) > n + 1$. Exponential upper bounds on $b(n)$ were first obtained by O. Schramm (1988) and later by J. Bourgain and J. Lindenstrauss (1989), while a lower bound (exponential in $n^{\frac{1}{2}}$) was obtained by J. Kahn and G. Kalai (1993). To obtain an upper bound on $b(n)$, C. Rogers showed that $b(n) \leq (\sqrt{2} + o(1))^n$, M. Lassak proved that $b(n) \leq 2^{n-1} + 1$, while O. Schramm (1988) and J. Bourgain and J. Lindenstrauss (1989) provided exponential upper bound $b(n) \leq (\sqrt{\frac{3}{2}} + o(1))^n$, where O. Schramm considered the case of bodies of constant width and J. Bourgain and J. Lindenstrauss were covering the body by Euclidean balls of smaller diameter. Let $g(K)$ be the smallest number of balls of diameter $< d$ and let $g(n) = \sup\{g(K) : K \subset \mathbb{R}^n, \text{diam}(K) = 1\}$. It is known that $g(n) \geq 1.003^n$ and $g(n) \leq (\sqrt{2} + o(1))^n$. Let $I(K)$ be the minimal number of smaller homothetic translates of a convex body K needed to cover K. It is known that any set of diameter d can be covered by a set of constant width d. Hence, $h(n)“=”b(n)$ for constant width and $b(n) \leq h(n)$. Also, $h(n) \geq b(n) \geq 1.2255^{\sqrt{n}}$. Since O. Schramm provided an exponential upper bound on the illumination number of n-dimensional bodies of constant width, G. Kalai (2015) asked for a corresponding lower bound, namely if there exists an n-dimensional convex body of constant width with the illumination number exponential in n, i.e., if there exists $C > 1$ such that $h(n) > C^n$? The result is that $h(n) \geq \frac{c_1}{\sqrt{n}}(\frac{1}{\cos\frac{\pi}{14}})^n$.

Deping Ye presented his (joint with N. Li and B. Zhu) results on *the dual Minkowski problem for unbounded sets*. Let C be a fixed pointed closed convex cone in $\mathbb{R}^n$. A closed convex set $A^* \subset C$ is called C-close if $A = C \setminus A^*$ has positive finite volume. The set A is called C-coconvex. The set A^* is called C-full if the C-coconvex set A is bounded and non-empty. Let also $\Omega_{C^o} = S^{n-1} \cap \text{int}(C^o)$, where C^o is the dual cone of C, i.e., $C^o = \{x \in \mathbb{R}^n : x \cdot y \leq 0 \, \forall y \in C\}$. Schneider posed (and established the existence and uniqueness of the solution to) the Minkowski problem for C-coconvex sets: given a finite Borel measure μ on Ω_{C^o}, does there exist a C-coconvex set A such that $\mu = \bar{S}_{n-1}(A, \cdot)$? Let $\tilde{C}_q(K, \cdot)$ be the q-th dual curvature measure, introduced by Y. Huang, E. Lutwak, D. Yang and G. Zhang in 2016. Let $0 \notin E \subsetneq C$ be a non-empty set and let $\text{conv}(E, C)$ be the closed convex hull of E about C, i.e., $\bigcap\{\tilde{E} : \tilde{E}$ is a C-close set such that $E \subset \tilde{E}\}$. One calls $E \subsetneq C$ a C-compatible set if $0 \notin E = \text{conv}(E, C)$. The analogue of Schneider's result is proved for the q-th dual curvature measure of (C, q)-compatible sets (C-compatible with finite q-th dual volume): given a positive Borel measure μ, finite on Ω_{C^o} and satisfying $\text{supp}\mu \subset w \subsetneq C^o$, there exists a C-full set A such that $\tilde{C}(A, \cdot) = \mu$ for $0 \neq q \in \mathbb{R}$. Also, for any $q > 0$ there exists a (C, q)-close set $\mathbb{A}$ such that $\tilde{C}(\mathbb{A}, \cdot) = \mu$.

Sergii Myroshnychenko, gave a lecture *Information-theoretic extensions of Kneser-Poulsen conjecture* on his joint results with G. Aishwarya, I. Alam, D. Li and O. Zatarain-Vera. Using methods of rearrangement and majorization, they affirmatively answer the following information-theoretic question that is directly related to the famous Kneser-Poulsen conjecture: suppose Alice wants to communicate with Bob using a collection of points K in space. However, the night is foggy, so Bob receives the random point $x + W$ when Alice sends x, where W is uniformly distributed on the unit ball. Does communication suffer if the points in K are brought pairwise closer together? More precisely, let $\alpha > 0$, $\alpha \neq 1$, and let $h_\alpha(X) = \frac{1}{1-\alpha} \int\limits_{\mathbb{R}^d} f^\alpha dx$, where X is an $\mathbb{R}^d$-valued random variable with distribution having density f with respect to the Lebesgue measure (for $\alpha = 0, 1, \infty$, the corresponding expressions are obtained by passing to the limit). Does one have $h_\alpha(Tx + W) \leq h_\alpha(X + W)$, where $T : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction, W is a random variable with symmetry. It is shown that for any radially symmetric random variable with convex level sets and any contraction, the above inequality holds. The natural behavior of some intrinsic volumes of convex bodies under contractions are also described.

Petros Valettas gave a talk *Probabilistic Padé Problems* about his joint results with S. Dostoglou. It has been observed, by Froissart (1969), that zeros and poles of high order Padé approximants of random perturbations of a deterministic Taylor series tend to form unstable pairs. These pairs appear at loci characteristic of the random part in the coefficients of the Taylor series. While this phenomenon has only been confirmed experimentally, it has been suggested, and indeed broadly used, as a noise detection tool. In his talk Petros explained how one can combine methods from high-dimensional probability and logarithmic potential theory to rigorously establish and quantify this phenomenon for the "pure noise" case, when the coefficients come from some distribution with anti-concentration properties. Recall that, given a power series $F(z) = \sum_{n=0}^{\infty} a_n z^n$, the polynomials $P_m(z) = \sum_{k=0}^{m} p_k z^k$ and $Q_n(z) = \sum_{k=0}^{n} q_k z^k$ give $[m,n]$-Padé approximation of F, provided $F(z) - \frac{P_m(z)}{Q_n(z)} = O(|z|^{m+n+1})$ as $q_0 = 1$. In a probabilistic setup, given a random vector $\xi = (\xi_0, \xi_1, \ldots, \xi_N)$ in $\mathbb{R}^{N+1}$, for $m + n \leq N$, one applies $[m,n]$-Padé approximation $\frac{P_m}{Q_n}$ to the random signal $f_\xi(x) = \sum_{k=0}^{N} \xi_k z^k$, where P_m and Q_n are as above. One of the results reads as follows. Let $\xi = (\xi_0, \ldots, \xi_N)$ be a random vector in $\mathbb{R}^N$ which satisfies, $\mathbb{E}|\xi_k| \leq K < \infty$ $\forall k$, $\sup_{\{v\in\mathbb{R}\}} \mathbb{P}(|\xi_k - v| < \varepsilon) \leq \kappa\varepsilon$, $\varepsilon > 0$. Then, given $\varepsilon, \delta \in (0,1)$ and $m \geq C\varepsilon^{-4} n \log(e\kappa K \frac{n}{\delta})$, for every $N \geq m + n$ and for any random vector ξ on $\mathbb{R}^{N+1}$ satisfying the above conditions, the numerator P_m of the $[m,n]$-Padé approximant for $f_\xi(x)k$ satisfies $d_{BL}(\nu_{P_m}, \mu) < \varepsilon$ with probability greater than $1 - \delta$. Here d_{BL} stands for the bounded Lipschitz metric.

Paul Simanjuntak discussed his joint results with R. Adamczak and P. Pivovarov concerning *Central Limit Theorem for Volume of Sections of* B_p^n. They established Central Limit Theorem (CLT) for the volumes of intersections of B_p^n, $0 < p < 2$, with uniform random subspaces of fixed co-dimension d as n tends to infinity. The result is obtained using volume representation as sum of Gaussian mixtures: for $p \in (0,2)$. There exist constants $a_{p,d}$, $b_{p,d}$ and $\Sigma_{p,d}^2$ such that

$$\sqrt{n}\Big(\frac{\mathrm{vol}(B_p^n \cap H_n)}{\mathrm{vol}(B_p^{n-d})} - a_{p,d} - \frac{1}{n} b_{p,d}\Big) \xrightarrow{d} N(0, \Sigma_{p,d}^2) \quad \text{as} \quad n \to \infty.$$

As a corollary the higher order approximations for expected volumes are also obtained, refining previous results by Koldobsky and Lifshits and approximation obtained from the Eldan-Klartag version of CLT for convex bodies.

Chase Reuter gave a lecture *The Euclidean ball is locally the only fixed point for the p-centroid body operators*. Characterizing the Euclidean space among all normed spaces is one of the aims of the ten problems formulated in 1956 by Busemann and Petty. These problems lead to the study of certain integral operators on convex bodies, such as the intersection body operator for the first Busemann-Petty problem. In the class of convex bodies, obtaining global statements about the fixed points of such operators is difficult. The local study of these problems appears to be a more approachable initial step, which has yielded local solutions to problems 5 and 8 by M. Alfonseca, F. Nazarov, D. Ryabogin and V. Yaskin. Chase applied similar techniques to study the fixed points up to dilation of the p-centroid body operator in a neighborhood of the Euclidean ball. Given an origin-symmetric convex body K, this body can be identified by its radial function $\rho_K(\theta) = \max\{r > 0 : r\theta \in K\}$ or its support function $h_K(\theta) = \max_{\{x\in K\}} x \cdot \theta$, where $\theta \in S^{n-1}$. For $p \geq 1$, the p-centroid body $\Gamma_p K$ is defined for all $\theta \in S^{n-1}$ as

$$h_{\Gamma_p K}(\theta) = \frac{1}{\mathrm{vol}_n(K)} \int_K |x \cdot \theta|^p dx = \frac{1}{\mathrm{vol}_n(K)} \int_{S^{n-1}} |\sigma \cdot \theta|^p \rho_K^{n+p}(\sigma) d\sigma.$$

When $p = 1$, the boundary of $\Gamma_p K$ can be described physically: If K has density $\frac{1}{2}$ and were allowed to float in a particular orientation, then the boundary of Γ_p is the locus of the centers of masses of the submerged portions for all orientations. In the integral definition, passing to polar coordinates yields $h_{\Gamma_p K} = c_K \mathcal{C}_p \rho_K^{n+p}$, where c_K is some constant depending on the body and $\mathcal{C}_p$ is the p-cosine transform (the integral over S^{n-1} in the right-hand side of the previous integral equality). Since the eigenspaces of the cosine transform are spanned by the spherical harmonics, one uses techniques from harmonic analysis to show that if K is close to the Euclidean ball and $K = c\Gamma_p K$ for some real number c, then K is the Euclidean ball up to a linear transformation.

Katarzyna Wyczesany gave a final talk of the conference titled *A Blaschke-Santalo type inequality for dual polarity*, where she presented her joint results with S. Artstein-Avidan and S. Sadovski. Let T be an order reversing quasi involution acting on all subsets $\mathcal{P}(X)$ of the given set X, i.e., $K \subset TTK$ and $L \subseteq K$ yields $TL \supseteq TL$. Assume also that $C = \{K \subset X : \exists L \subset X : K = TL\}$, then $T|_C$ is a duality (order reversing involution). If $c : X \times X \to (-\infty, \infty]$ is such that $c(x, y) = c(y, x)$, and the c-dual of $K \subset X$ is defined as $K^c = \{y \in X : \forall x \in K, c(x, y) > 0\}$, it is shown that for T as above there exists $c : X \times X \to \{\pm 1\}$ such that for every $K \subset X$ one has $TK = K^c$. The characterization of the order reversing quasi involutions is given and the new Blaschke-Santalo type inequality is proved. Let K be essentially symmetric (for some $e \in S^{n-1}$, one has $x + te \in K$, $x \in e^{\perp}$, yields $-x + te \in K$) and let $T : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be an order reversing involution. If $TK = \{x \in \mathbb{R}^n : \forall y \in K, x \cdot y \geq 1\}$, then $\gamma_n(K)\gamma_n(TK) \leq \gamma_n(K_0)^2$ holds. Here $K_0 = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^+ : |x|^2 + 1 \leq t^2\}$ and γ_n is the Gaussian measure on $\mathbb{R}^n$.

Outcome of the Meeting

The meeting was very successful, we brought together mathematicians from many countries and many research areas, such as convex geometry, discrete geometry, probability, functional and harmonic analysis. Besides the leading scientists, we also had 8 graduate students and 8 postdocs participating in-person in the workshop. Female participation was above 27%. The friendly atmosphere created during the workshop helped many participants not only to identify the promising ways to attack the old problems but also to get acquainted with many open new ones.

Participants

Alfonseca-Cubero, Maria de los Angeles (North Dakota State University)
Arman, Andrii (University of Manitoba)
Artstein, Shiri (Tel-Aviv University)
Boroczky, Karoly (Alfréd Rényi Institute of Mathematics)
Chasapis, Giorgos (University of Crete)
Chasioti, Effrosyni Maria (Case Western Reserve University)
Dann, Susanna (Universidad de los Andes)
Eskenazis, Alexandros (CNRS, Sorbonne Université)
Florentin, Dan (Bar-Ilan University)
Haddad, Julián (Universidad de Sevilla)
Hernandez Cifre, Maria A. (University of Murcia)
Herscovici, Orli (St. John's University)
Hosle, Johannes (MIT)
Jimenez, Carlos Hugo (University of Sevilla)
Koenig, Hermann (Universitaet Kiel)
Koldobsky, Alexander (University of Missouri)
Langharst, Dylan (Kent State University)
Letwin, Brayden (University of Alberta)
Li, Dongbin (University of Alberta)
Litvak, Alexander (University of Alberta)
Livshyts, Galyna (Georgia Institute of Technology)
Manui, Auttawich (Kent State University)
Milman, Emanuel (Technion)
Mui, Stephanie (New York University)
Myroshnychenko, Sergii (Lakehead University)
Nayar, Piotr (University of Warsaw)
Ortega Moreno, Oscar Adrian (Technische Universität Wien)
Paouris, Grigoris (Texas A & M University)
Pivovarov, Peter (University of Missouri)
Polavarapu, Achintya (University of Alberta)

Putterman, Eli (Tel Aviv University)
Reuter, Chase (NDSU)
Rotem, Liran (Technion)
Roysdon, Michael (Brown University)
Rudelson, Mark (University of Michigan, Ann Arbor)
Ryabogin, Dmitry (Kent State University)
Schütt, Carsten (Christian-Albrechts-Universitaet)
Shaw, Vincent (Kent State University)
Simanjuntak, Paul (Texas A & M University)
Slomka, Boaz (The Open University of Israel)
Stancu, Alina (Concordia University)
Sun, Wen Rui (University of Alberta)
Szczepanski, Tomasz (University of Alberta)
Szusterman, Maud (Tel Aviv University)
Tatarko, Kateryna (University of Waterloo)
Tikhomirov, Konstantin (Carnegie Mellon University)
Tkocz, Tomasz (Carnegie Mellon University)
Valettas, Petros (University of Missouri)
Vritsiou, Beatrice-Helen (University of Alberta)
Werner, Elisabeth (Case Western Reserve University)
Wyczesany, Katarzyna (Carnegie Mellon University)
Xing, Sudan (University of Arkansas at Little Rock.)
Yaskin, Vladyslav (University of Alberta)
Ye, Deping (Memorial University)
Zawalski, Bartłomiej (Polish Academy of Sciences)
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Zhang, Gaoyong (New York University)

Two-day Workshop Reports

Chapter 32

Stochastic Modelling of Big Data in Finance, Insurance and Energy Markets (23w2004)

May 19 - 21, 2023

Organizer(s): Anatoliy Swishchuk (University of Calgary, AB, Canada), Rudi Zagst (TUM, Munich, Germany), Wenjun Jiang (University of Calgary,AB, Canada)

Overview of the Field

Since the 2008 global financial crisis, insurance and energy markets have become closely linked with financial markets, with energy prices exhibiting more financial characteristics, e.g., similar volatile behaviour, creating a need for cooperation in such area as data science, and to communicate with highly trained professionals who work with financial, insurance and energy market data, and who can model those data. Big data and data analytics have recently become an essential part of the financial, insurance and energy markets services industry. The 2-Day workshop "Stochastic Modelling of Big Data In Finance, Insurance and Energy markets" has brought together academics, industry people and graduate students to share their ideas, to share their research experience, and to learn more about big data and to know how to model those data. This is the first data science workshop that was devoted to stochastic modelling of big data in finance, insurance, and energy markets. The topics of the workshop included stochastic and statistical modelling of big data in insurance, finance, and energy markets.

Recent Developments and Open Problems

Systemic risk, or instabilities, occur in many complex systems: In ecology (diversity of species), in climate change, in material behavior (phase transitions), in finance (financial crisis), etc. In finance, e.g., two types of trading in equities are widely practiced today: High-frequency (limit-order and market) trading and statistical arbitrage or market neutral (generalized) pairs trading. These types of trading account for well over two thirds the volume traded today. It is not yet clear how to quantify the systemic risk, or the market instabilities generated by these types of trading. Main problem: how to deal with big data arising in electronic markets for algorithmic and high-frequency (milliseconds) trading that contain two types of orders, limit orders and market orders. More than half of the markets in today's highly competitive and relentlessly fast-paced financial world now use a limit order book (LOB) mechanism to facilitate trade. Same problems arise in insurance and energy markets associated with big data. The workshop was intended to bring together professionals from finance, insurance and energy markets areas to discuss these open problems and to find the ways for solving them.

Presentation Highlights

From the **insurance** side the talks were presented by Wenjun Jiang, Ye Wang and Rudi Zagst. Wenjun Jiang's was devoted to revisiting the optimal reinsurance design under adverse selection and Value-at-Risk. Thus, the design of optimal reinsurance contracts from a monopolistic reinsurer's perspective in the presence of information asymmetry was revisited. The reinsurer aims to maximize its expected profit and the insurers select the contracts based on their Value-at-Risk (VaR) preferences. It was shown that the optimal contract menu is also the first-best solution. Some numerical examples are presented to show more implications of our results [3, 4]. Ye Wang's talk considered optimal reinsurance with Vajda condition and Range-Value-at-Risk It was studied an optimal reinsurance problem where the insurer's risk-adjusted liability gets minimized. To consider both robustness and tail risk, the insurer is assumed to apply Range-Value-at-Risk (RVaR) to evaluate its risk. The closed-form solution to the problem, which includes the results in Chi and Weng (2013) as special cases, was derived. Some comparative studies and sensitivity analysis were also carried out through numerical examples [6]. Rudi Zagst's talk was devoted to Optimal investment strategies for pension funds with regulation-conform dynamic pension payment management in the absence of guarantees. It was considered the post-retirement phase optimization problem for a specific pension product in Germany that comes without guarantees. The continuous-time optimization problem is defined consisting of two specialties: first, we have a product-specific pension adjustment mechanism based on a certain capital coverage ratio which stipulates compulsory pension adjustments if the pension fund is underfunded or significantly overfunded; second, due to the retiree's fear of and aversion against pension reductions, it was introduced a total wealth distribution to an investment portfolio and a buffer portfolio to lower the probability of future potential pension shortenings. A numerical case study on optimization and simulation completed the work with highlighting the benefits of the proposed model [1, 2].

The talks from the **finance** side were presented by Alex Badescu, Dominik de Witte, Luca Lalor, Jinniao Qiu, Ana Karen Contreras, Matthias Scherer and Yang Yang. Alex Badescu's talk studied the impact of long memory on modelling asset returns and pricing options in discrete time. A general pricing framework based on affine multi-component volatility models that admit $ARCH(+\infty)$ representations, which not only nests a plethora of option pricing models from the literature, but also allows for the introduction of novel fractionally integrated processes for valuation purposes was proposed. An extensive empirical analysis which includes single and joint calibrations of a variety of short and long memory models to historical returns and $S\&P500$ options was carried out [11, 12]. Dominik de Witte's talk had shown how to practically obtain the optimal portfolio withrespect to expected power utility maximization if the price processes are drivenby a geometric Levy process. Based on a result by Nutz (2012), who showedthat the optimal portfolio weights are constant in this setting even if we includebudget constraints, we make use of the Stochastic Gradient Descent Algorithmto arrive at the optimal solution even for a high-dimensional asset universe. The main subject of Luca Lalor's talk was to introduce an algorithmic and High-Frequency Trading model where the price process is of the jump-diffusion type. Previous research modelled the jumps through a diffusion approximation, while here the jumps are modelled directly. Preliminary results, using an Implicit-Explicit Finite Difference Scheme, for an Optimal Acquisition algorithmic trading problem have been presented. Here the jump part of the Jump-Diffusion price process was a function of a Poisson process. The future models will account for the non-Markovian property seen in LOB data [7]. Jinniao Qiu talked about the rough volatility. Rough volatility is a new paradigm in finance. He discussed about the option pricing problems for rough volatility models. As the framework is non-Markovian, the value function for a European option is not deterministic; rather, it is random and satisfies a backward stochastic partial differential equation (BSPDE) or so-called stochastic Black-Scholes equation. The well-posedness of such kinds of BSPDEs will be discussed. These BSPDEs are also used to approximate American option prices. Moreover, a deep learning-based method was investigated for the numerical approximations to such BSPDEs and associated non-Markovian pricing problems.

Examples were presented for both European and American options. This talk was based on joint work with Christian Bayer and Yao Yao [13]. In Ana Karen Contreras' talk, review solutions for optimal control problems such as optimal acquisition, optimal liquidation, and market making, which are considered primary purposes in trading activity, were presented. Diffusion processes model the price. The optimal solution using the Semi-Markov process versus the general compound Hawkes process in the limit order market was computed. Comparison the optimal solutions expressed in terms of parameters describing the arrival rates and the mid-price and find a more general solution that more authentically describes the observed price process in HFT markets. In addition, the comparison was explicit with real-world data from LOBster data [8]. Matthias Scherer's talk considered a vector of bankruptcy times with Marshall-Olkin multivariate exponential distribution implies a simple, yet reasonable, continuous-time dynamic model for dependent credit-risky assets with an appealing trade-off between tractability and realism. Within this framework the maximization of expected power utility of terminal wealth requires the maximization of a concave function on a polygon, a numerical problem whose complexity grows exponentially in the number of considered assets. It was demonstrated how to solve this seemingly impractical numerical problem reliably and efficiently in order to prepare the model for practical use cases. Joint work with J-F. Mai and A. Blagoeva [1, 2, 21]. Yang Yang's talk investigated some stochastic path-dependent volatility (SPDV) models as well as their performance in stock and energy markets. First the SPDV models was shown to capture the rich spot-volatility dynamics in various financial markets. Then it was moved on to a specific kind of SPDV model and discuss its applications in the natural gas market. Both calibration and pricing problems were discussed. The talk is based on joint work with Jinniao Qiu and Antony Ware [9].

Talks from **energy markets** were presented by Tylar Jia, Ruediger Kiesel, Deniz Sezer and Zuming Sun. Tylar Jia's talk presented a methodology to incorporate large-scale atmospheric information into short-term wind speed forecast in Alberta using two publicly accessible datasets. The first dataset was used for atmospheric clustering by applying the k-means algorithm and the hidden Markov model on atmospheric variables related to wind speed. The second dataset was used to test the proposed time series regime-switching models and mixture models that integrate the clustering results to predict 6-hour ahead wind speed at 23 weather stations in Alberta, Canada [5]. Ruediger Kiesel's talk was devoted to managing climate risk. By now it is widely agreed that climate change poses a substantial risk to financial markets and institutions. He discussed risk management strategies in this context and advocate the use of a pre-commitment approach. Utilizing our general framework, the speaker turned to several specific examples relating to the measurements of risks for insurance and financial companies. The presenter also developed a method to assess the credibility of net-zero commitments, which may be applied to control the carbon budget in loan and investment portfolios. The talk relied on joint work with Gerhard Stahl (HDI), Andrej Bajic (Deloitte FSI-Audit-Garage), Alexander Blasberg and Kateryna Chekriy (both University Duisburg-Essen [16, 17]). Deniz Sezer's talk, was devoted to an overview of various methodologies used by my research team for meso- and sub-meso scale problems related to wind power. Meso-scale research focuses on probabilistic modeling of wind power produced at wind farms distributed across a large geographical region such as a province to make inferences about the aggregate wind power and applications to power system planning and management. Sub-meso-scale modeling involves resolving finer features of the wind flow that can affect wind production in nearby farms. Alberta specific wind data was considered in this talk [5]. Zuming Sun's talk was devoted to the Polynomial Process and Polynomial Regression Model for French Electricity Prices. Empirical experience reveals that electricity prices can be affected by the dynamics of residual demand, power generation capacity of each commodity and spot prices of each generation technology. A model involving a polynomial map of polynomial processes, a so called PMPP model, for electricity prices was presented. The work was focused on the French electricity market. Joint talk with Tony Ware and Thomas Deschatre (EDF Lab, Paris-Saclay, France [10]). Statistical properties of big data were discussed and presented in Florian Brueck, Aleksey Min and Ralf Werner's talks. Florian Brueck investigated the influence of parameter estimation on the asymptotic distribution of the two-sample test based on the Maximum-Mean-Discrepancy proposed by Gretton et al and show that its asymptotic distribution is influenced by parameter estimation. To circumvent the problem of determining covariances of an infinite sum of Chi-squared distributed random variables and the parameter estimators, we propose a new two-sample test based on Maximum-Mean-Discrepancy. Moreover, it was deduced a new test for model comparison based on Maximum-Mean-Discrepancy, which is also robust under parameter estimation [21]. Aleksey Min's talk was devoted to Stationary vine copula models for multivariate time series. A time series consists of multiple observations indexed by time. Classical time series models allow for only linear dependence between variables and time points. More recently, several nonlinear time series models based on copulas were proposed. Vine copulas are graphical models for the dependence and can conveniently capture cross-sectional and temporal dependence of multivariate time series. It was also proposed computationally efficient methods for estimation, simulation, prediction, and uncertainty quantification. The theoretical results allow for misspecified models. The talk is based on joint work with Daniel Krüger and Thomas Nagler [18]. Ralf Werner's talk was devoted to asymptotic non-parametric confidence intervals for the uniform estimator in nested simulations. In the European Union, life insurers face severe challenges when computing the Solvency Capital Requirement (SCR) based on internal models. Mathematically speaking, one has to compute the 99.5% quantile (VaR) of the 1-year loss distribution of basic own funds (BoFs). Therefore, it was suggested a novel non-parametric asymptotic confidence interval for the quantile estimator which significantly improves upon the only existing one (which, albeit, is not specialized to the quantile estimation case). Besides a few illustrating numerical examples and comparison to the existing method, it was also briefly discussed the main ideas for the corresponding proofs if time allows. The last part of the talk was devoted to a discussion of the implications concerning the application of modern machine learning methods. This was a joint work with Maximilian Klein [19].

There exists a need for highly trained professionals such as data specialists who can work with financial and energy markets data and who can model those data. In addition, big data and data analytics have become an essential part of the financial services industry. Thus, the workshop has brought together academics, industry people and graduate students to share their ideas, to share their research experience and to learn more about big data and their modelling. This workshop has also aim to train young talents and included mentorship component in the program. There were 4 people from industry, who shared their industry experience with 9 graduate students, presented at the workshop: Scott Dalton (Ovintiv), Clifford Kitchen (NBC), Kevin Malenfant (Auspicecapital) and Nima Safaian (Cenovus). It was performed during the **industry panel** of the workshop.

The social program included two walks to Tunnel Mountain and along the Bow River.

Scientific Progress Made

The workshop have brought together academics, industry people and graduate students to share their ideas, to share their research experience and to learn more about big data and their modelling.

This workshop also accomplished its aim to train young talents and included mentorship component and industry panel in the program.

Outcome of the Meeting

Several research projects and working papers were created based on the workshop's topics. They were associated with stochastic modelling of insurance, finance and energy markets real data. Below are just two out of many testimonies from the workshop's participants:

"Dear professor, I wanted to thank you for giving me the opportunity to join the BIRS workshop. It was insightful and inspiring. The workshop was very well organized and it was a very good learning experience for me. Thank you for giving me the opportunity to participate."

"Thank you so much for your invitation to this wonderful workshop! The workshop gave me some new ideas for my research and a lot of motivation!"

Participants

Badescu, Alexandru (University of Calgary)
Brueck, Florian (Technical University of Munich)
Dalton, Scott (Ovintiv)
de Witte, Dominik (Technical University of Munich)
Jia, Tianxia (University of Calgary)
Jiang, Wenjun (University of Calgary)
Kiesel, Ruediger (University Duisburg-Essen)

Lalor, Luca (University of Calgary)
Malenfant, Kevin (Auspice Capital Advisors)
Min, Aleksey (Technical University of Munich)
Nawodh, Sudeesha (University of Calgary)
Qiu, Jinniao (University of Calgary)
Roldan Contreras, Ana (University of Calgary)
Safaian, Nima (Cenovus)
Scherer, Matthias (Technical University of Munich)
Sezer, Deniz (University of Calgary)
Sun, Zuming (University of Calgary)
Swishchuk, Anatoliy (University of Calgary)
Vithana, Shanukie (University of Calgary)
Wang, Ye (University of Calgary)
Ware, Tony (University of Calgary)
Werner, Ralf (University of Augsburg)
Yang, Yang (University of Calgary)
Zagst, Rudi (Technical University of Munich)

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[7] Luca Lalor: A Numerical Solutions to an Algorithmic and HFT Problems with a Jump-Diffusion Price Processes. Lunch at the Lab finance seminar talk, Apr 4, 2022.

[8] Ana Karen Roldan Contreras: Stochastic Optimal Control Problems in Limit Order Books for Semi-Markov Process vs General Compound Hawkes Process. Lunch at the Lab finance seminar talk, Jan 24, 2023.

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Chapter 33

Alberta-Montana Combinatorics and Algorithms Days (23w2008)

June 23 - 25 2023

Organizer(s): Hadi Kharaghani (University of Lethbridge), Ryan Hayward (University of Alberta), Mark Kayll (University of Montana), Robert Woodrow (University of Calgary)

Overview

The Banff International Research Station (BIRS) hosted the second *Alberta-Montana Combinatorics and Algorithms Days* on June 23–25, 2023. The essential purpose of the event was to bring together, for a second time, faculty and students from post-secondary institutions in Alberta and Montana. The participating universities were: University of Lethbridge, University of Calgary, University of Alberta, and University of Montana.

Combinatorics at its heart is the study of discrete structures. This pithy description, however, belies the inherent complexity of these objects and their deep connections both with each other and with other branches of mathematics such as algebra and number theory. Moreover, combinatorics is seeing ever proliferating applications in fields as diverse as cryptography and algorithm analysis in game theory and elsewhere. All of these exciting and cutting-edge topics and more were touched upon at this year's workshop at BIRS as outlined in the abstracts reproduced below.

Demographics (briefly)

The workshop attracted participants from a broader geographical region than originally envisioned; to wit, there were speakers from Calgary, Edmonton, Lethbridge, Missoula, Vancouver, Burnaby, Hartford, and Bhopal in India. Likewise, the "vertical" representation went even deeper than the organizers had originally hoped. Of the participants, there were six undergraduate students (one of whom spoke), three graduate students (two of whom spoke), one postdoctoral fellow (who spoke), one MITAC summer research student from Bhopal in India, and a freshman at the University of Waterloo. Thus, the total of fifteen talks was rounded out by eleven lectures given by junior and senior faculty.

Presentation highlights

Here follows a précis of each of the talks in their original chronological order.

Davoud Abdi (University of Calgary) opened the workshop by presenting on equimorphic objects, that is, objects which are at once embeddable into each other. In the case the objects are infinite, one does not necessarily obtain isomorhpic objects. Speaking on conjectures specific to this phenomenon, Davoud touched upon those of Thommassé and Bonato-Tardif. Thomassé conjectured that a countable relation has either one or many siblings, up to isomorphism. The Bonato-Tardif conjecture is related to this and states that a locally finite tree has one or infinitely many siblings. The talk concluded with counterexamples to these two conjectures.

Following the opening address, Martin Mueller (University of Alberta) spoke on combinatorial games with sum structure, namely, games that are an aggregation of independent subgames. By considering only boolean outcomes;"Did player A win or loose?", using elementary means, he showed how one can improve drastically upon the typical computation of a canonical representative, a known computational bottleneck. A case study for the game of 1-dimensional linear Clobber was presented.

Because of Xinyue Chen's absence, Ryan Hayward (University of Alberta) presented his talk on his behalf. Ryan spoke on the game of Hex, a two-player zero-sum game. Evidently, the first player to move is always guaranteed to win if only they can play a "perfect game." He went on to explain the use of the Benze Hex Solver with the DFPN algorithm in finding the starting positions available for a player to have a "perfect game." In particular, he gave valuable insights into how proof-number search and how depth-first-proof-number search algorithms work and can be used to solve such problems.

Continuing the topic on games, Invited Speaker Svenja Huntemann (Concordia University, Edmonton) presented on the family of so-called placement games. Such games take place on a finite graph where two players take turns placing tokens, without moving or replacing them. Specifically, Svenja outlined the differences between placement games such as Col and Domineering to games with strong placement such as Snort, NoGo, and Hex. She gave a detailed survey about essential combinatorial game theory terminology, and the idea of disjunctive sums for placement games. Svenja went on to introduce game values as a metric for quantifying how much of an advantage the winning player holds and temperature to determine the urgency of making a particular move. She later spoke about enumerating all possible positions of a game to develop the complexity of a complete analysis. The talk concluded by giving an overview of open research areas within this problem context.

Mahya Jamishidian (University of Alberta) gave a review of the related clustering problems Minimum Sum of Radii (MSR), Minimum Sum of Diameters (MSD), and Minimum Sum of Squared Radii (MSSR). In improving upon what was previously known, Mahya presented her findings on developing a 3.389-approximation for MSR and a 6.546-approximation for MSD, improving over respective 3.504 and 7.008. Furthermore, a 11.078-approximation algorithm for MSSR with a similar approach to the MSR and MSD cases known as well. The techniques employed include bi-point solutions, LP, Lagrangian Relaxation, followed by rounding technique and binary search.

Invited Speaker Stephanie Van Willigenburg (University of British Columbia) spoke on the Stanley-Stembridge (3+1) free conjecture and introduced the chromatic symmetric functions, the (3+1)-free conjecture, new cases and tools for resolving it. She continued on an answer to a question of Stanley of whether the (3+1)-free conjecture can be widened, see [3].

Zac Friggstad (University of Alberta) spoke on the difficult subject of the chromatic theory of graphs. Focusing on the minimum sum colouring problem (MSC), Zac recapitulated the facts that linear time algorithms can recognize 2-colourable bipartite graphs and optimally colour cubic graphs. In general, however, the problem is known to be NP-hard. By specializing to the case of chordal graphs, Zac uses linear-programming techniques for minimum latency problems obtaining a 1.796-approximation algorithm for the MSC problem in chordal graphs.

Caleb Van't Land (University of Lethbridge) spoke about the Butson Hadamard matrices and a Bush-type Butson Hadamard matrices. He went over what it means for two Hadamard matrices to be unbiased, and demonstrated a construction for a maximal family of unbiased Bush-type Butson Hadamard matrices, see [5].

Invited Speaker Dave Morris (University of Lethbridge) recapped the conjecture that cycles are the only finite regular graphs that have unique hamiltonian cycle and an eventual proof in 2014 that the conjecture is true in the special case where the graph is vertex-transitive. Dave then spoke on the possible generalization of the vertex-transitive case to infinite graphs.

Van Magnan (University of Montana) presented three directions of generalizations of the Erdős-Ko-Rado theorem and generalized diversity for families of sets. He then talked about the flower base as a set of special transversals which inherits the family's key properties. Every member of the family contains a member of the flower base. Throughout the remainder of his talk, Van went through ways that can work with the flower base to solve small design problems.

Amarpreet Rattan (Simon Frazer University) spoke about how parking functions, trees, and factorizations are connected to one another. He then continued on the statistics of these mathematical objects, and how it can be generalized to k-parking functions, see [4].

Invited Speaker Cory Palmer (University of Montana) began his survey of extremal co-degree problems for hypergraphs by introducing three parameters including Turán number. After a detailed history of the three parameters in hypergraphs he focused on recent developments on the positive co-degree problem, see [1].

Anna Halfpap (University of Montana) gave a detailed review of the general properties of the positive co-degree Turán number. She explained why these properties are desirable, and then talked further about the hypergraph removal lemma towards establishing supersaturation.

For the penultimate address, Invited Speaker Rei Safavi-Naini (University of Calgary) gave an in-depth synopsis of the history of cryptography and fostered an appreciation for the methodology and computational underpinnings behind this technology. Rei spoke about challenges to modern cryptography algorithms such as posed by the realization of quantum computers.

The concluding talk was given by Daniel Johnston (Trinity College, CT) who defined and spoke to the saturation of graphs, and then demonstrated what it means for a graph to be rainbow saturated. Similar to the rainbow Turán number introduced by Keevash, Mubayi, Sudakov, and Verstraëte, Daniel went on to introduce the rainbow saturation number of graphs, see [2].

Meeting outcomes

Nestled in Banff, the heart of the Canadian Rockies, BIRS speakers gathered under one roof from: University of Calgary, University of Alberta, University of Montana, University of Lethbridge, Trinity College, University of British Columbia, and Simon Fraser University. Undergraduates, Masters, and PhD students all felt welcome and were encouraged to participate in the student-friendly environment. Participants of this BIRS workshop found themselves to be an equal part of the group which shared a joint curiosity for combinatorics and algorithms with underlying mathematical structures. The organizers facilitated the development of new connections between people from different walks of life that, hopefully, will lead to fruitful collaborations in the near future.

A last minute cancellation due to illness and the timely action by the Program Coordinator provided an opportunity for a MITAC summer intern from India a chance to participate at the workshop. In his words "The BIRS workshop was one of a kind experience for me. I couldn't think of any place more beautiful and engaging than BIRS. The talks were well organized, perfectly delivered and were very engaging. I learnt many new concepts. And above all, everyone was very friendly and helpful".

Participants

Abdi, Davoud (University of Calgary)
Friggstad, Zachary (University of Alberta)
Halfpap, Anna (University of Montana)
Hasham, Kamillah (University of Alberta)
Hayward, Ryan (University of Alberta)
Holzmann, Wolfgang (University of Lethbridge)
Huntemann, Svenja (Concordia University)
Jamshidian, Mahya (University of Alberta)
Johnston, Daniel (Trinity College)
Kayll, Mark (University of Montana)
Kharaghani, Hadi (University of Lethbridge)
Laflamme, Claude * (University of Calgary)
Magnan, Van (University of Montana)
Morris, Dave (University of Lethbridge)
Morris, Joy (University of Lethbridge)
Mousavi, Ramin (University of Alberta)
Müller, Martin (University of Alberta)
Palmer, Cory (University of Montana)
Rattan, Amarpreet (Simon Fraser University)
Safavi-Naini, Rei (University of Calgary)
Schultz, Luke (University of Alberta)
Shamsundar, Khobragade Yash (Indian Institute of Science Education and Research Bhopal)
Van Willigenburg, Stephanie (UBC)
Van't Land, Caleb (University of Lethbridge)

Woodrow, Robert (The University of Calgary)
Zaitsev, Vlad (University of Lethbridge)

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Chapter 34

Open-Source Tools to Enable Geophysical Data Processing and Inversion (23w2014)

July 28 - 30, 2023

Organizer(s): Santiago Soler (University of British Columbia) Lindsey Heagy (University of British Columbia) Craig Miller (GNS Science) Leonardo Uieda (University of Liverpool)

Overview

One of the main goals of geosciences is to obtain information about the structure of Earth's subsurface. Geophysics relies on measurements of physical quantities such as seismic waves, and gravitational and electromagnetic fields to characterize the inner structure of the Earth through its physical properties. The process through which geoscientists build this understanding of Earth's interior can be summarized as: (i) acquiring the data, (ii) processing it, and (iii) producing a model of the subsurface, through inversion, that is consistent with the observed data. This workflow requires mathematical and computational methods that include numerical simulations of partial differential equations, solving large-scale optimization problems, and methods for assimilating a variety of data types across a range of scales. Software tools that provide computational solutions to these problems are a major resource for geoscientists. They will surely play a fundamental role in enabling the research needed to address upcoming geoscientific problems related to energy transitions and climate change remediation.

In the last decade, there has been a transformative move within the scientific community towards reliance on and contribution to open-source software. Within the field of geophysics, the Fatiando a Terra [12, 13, 10, 5, 4] and SimPEG [2, 6] projects were early pioneers that have driven the development of Python-based open-source tools for data processing, numerical simulations, inversion, and visualization. These two projects share a common goal of providing open-source Python tools to solve geophysical problems, building foundational frameworks that simplify the implementation of state-of-the-art methodologies, and facilitating their usage in applied and basic research. Their ongoing and future development benefits not only their own communities but also the broader geoscientific community.

During this 2-day BIRS Workshop members of the communities of users, developers and contributors of Fatiando a Terra and SimPEG were invited to gather and discuss the future development roadmaps for both projects, strengthen the collaboration networks between the two projects, identify issues and points of improvement, and build solutions for them.

A total of 17 people were able to travel to Banff for the Workshop, representing nine different nationalities; while 18 people were able to join virtually for most of the sessions.

Invited talks

Some participants were invited to give brief talks during the time span of the Workshop, covering topics from the history of the two projects up to how researchers and professionals use these and other open-source tools for carrying out their research and work in industry and government agencies.

Some personal reflections on a decade of SimPEG

Lindsey Heagy, one of the founders of SimPEG, gave her own personal reflections on how the project got started by a small group of PhD students, motivated by a desire to work together and an interest in creating shared tools that implement all the pieces necessary to simulate and invert data for a range of geophysical applications. She also shared some insights on what it takes to create a movement, build a community and inspire other people to join.

The Fatiando a Terra Project

During this talk, Leonardo Uieda introduced the Fatiando a Terra project to all participants, providing details about its origins in Brazil, created by a group of PhD students, how it evolved in time along with a growing open-source geoscientific ecosystem, and set future goals for the project. One of these goals was to nurture the collaboration with SimPEG by jointly developing and maintaining tools that can be used by the two projects.

Complete Open-source Software Processing Workflow: From Field Data to Publication

Craig Miller showcased the workflow he and his coworkers developed in GNS Science (New Zealand) for processing, inverting, and interpreting gravity and magnetic data using open-source tools like gsolve, Fatiando a Terra and SimPEG. Craig also listed the advantages of relying on open-source software for generating this type of workflows: the value of the communities around open-source projects, the level of customization of these tools, the transparency of their algorithms, the time-saving aspect of not having to build tools from scratch, and the capacity to combine these different tools in the same framework.

Using open-source tools to bridge the gap between Geology and Geophysics

During her talk, Andrea Balza Morales presented her advances on her latest research projects, which involve methodologies for bringing together open-source tools for geology and geophysics, to produce geologically plausible models that also fit the geophysical data. Her work is oriented towards geothermal energy applications, with the goal of improving the efficiency and safety of these types of operations. In these projects, she used GemPy [3], an open-source package for geological implicit modelling, to create geologic models based on the known geology of the study area, and physical properties obtained through drilling. She then used pyGIMLi [9], an open-source Python package for geophysical modelling and inversion, to carry out deterministic inversions that adjust the position of some pilot points, that ultimately modify the initial geological model to adjust to the observed geophysical data. She also made extensive use of SimPEG for running forward simulations, and Fatiando a Terra for processing and interpolation of gravity data.

Building 3D quasi-geology models and predicting mineral resources using joint inversion and open-source code

Jiajia Sun presented the results of one of his latest projects. Using SimPEG to run a set of L_p norm joint inversions of gravity and magnetic data, he was able to generate quasi-geology models by applying a probabilistic geologic differentiation technique. From the entire set of recovered models by different L_p norm joint inversions, the probabilistic geologic differentiation classifies each cell of the 3D mesh as belonging to one of a few different rock units, based on its recovered physical property values. The probabilistic differentiation assesses the uncertainty of the classification by considering a whole set of recovered models. Finally, he showed the results of applying this technique to a real dataset over North Decorah, Iowa, US, showing its usefulness in localizing units of interest for mineral prospectivity.

Results from a survey on open-source software in the industry

At the end of the Workshop, Sean Walker joined virtually to present the results of a survey he ran to collect information about the usage of open-source tools in the industry. The survey questions brought insight into which open-source tools professionals in the industry are currently using, how relevant are they in their workflows, what value they see in them, and why they aren't using them in certain applications. The survey results showed that members of industry see a lot of value in open-source tools. Most of the survey participants rely on them for their daily tasks. The barriers to extending their usage are related to the pre-existence of workflows that make use of proprietary software, the lack of programming proficiency of their teams, and the lack of graphical user interfaces (GUIs) for some of these tools.

Group Discussions

In-person participants were split into groups of four people each, while virtual participants formed their own group. During the Discussions Sessions, each group was provided with a set of guiding questions to foster the discussion around three main topics:

- improving the communication within our communities,
- future developments of our projects, and
- increasing the diversity and sustainability of the projects.

Each group was encouraged to capture all outputs of these discussions and report them back to the other participants after each Discussion Session was finished. Based on the ideas developed during these sessions, the Workshop Organizers compiled a selection of actionable tasks. On the last day of the Workshop, participants chose one of those actionable items to start working on them, forming new sets of groups. In-person and virtual participants were encouraged to work and collaborate together within the groups.

In the following sections, we'll summarize the main discussion points for each topic, and list the progress made during the Workshop to develop solutions and take action after the identified issues and new ideas.

How can we improve the communication within our communities?

Participants identified avenues for improvement in the various communication channels of the two communities and came up with different ways to solve them. These strategies include improving the documentation of the software libraries, creating more user-friendly resources like comprehensive tutorials and examples, and exploring various outreach methods like newsletters, YouTube videos, and workshops. These initiatives aim to foster better communication, streamline collaboration, and ensure valuable information reaches community members, effectively preventing information fragmentation.

Amongst the other ideas that were discussed were:

- creating pages for frequently asked questions (FAQs) in the documentation of the software packages;
- establishing cross-links between the websites of the two projects pointing to relevant information;
- expanding the "Getting Started" guides;
- creating a comprehensive guide for new developers;
- scheduling periodic joint meetings between the communities of Fatiando a Terra and SimPEG;
- establishing office-hours for helping users and contributors to solve issues and ask questions;
- collecting all scientific articles that make use of the two projects and upload a list of them to the projects' websites;
- and extending the references to literature in the functions and classes of the packages, especially to provide details about the mathematical and physical backgrounds of their implementations.

What future research can be enabled by SimPEG and Fatiando?

Participants discussed the types of software tools that might be needed to perform current and further research, and which features are missing in the current packages of both projects.

One of the main discussion points was expanding the capabilities of Fatiando tools to process potential fields data from raw observations, such as drift and base station corrections, tide correction, and data levelling amongst others. Participants also discussed:

- optimizing terrain corrections using rectangular prisms for high-resolution elevation models;
- implementing geometry inversions for undulating layers like Mohorovicic discontinuity [11], the basement of sedimentary basins, and bathymetry under ice sheets;
- and FFT-based inversions like the Parker-Oldenburg method [8, 7].

Regarding SimPEG, participants proposed: improving and extending the magnetotellurics inversions; improve the ability to interoperate with other open-source packages that implement other forward models (e.g. ground-penetrating radar, empymod [14]); enabling the use of p-norms for the data misfit terms; and further developing advanced inversion methods that include geological and petrophysical knowledge (e.g. PGI [1]).

Other ideas were discussed, like speeding up the gravity and magnetic simulations in SimPEG using Choclo (one of the Fatiando a Terra libraries); developing functionalities for procedural model generation; exploring stochastic inversions; and smoothing the interoperability between the two projects.

How can we increase the diversity of our communities?

All groups started the discussion by acknowledging that there are many different types of diversity: cultural, social, racial, gender, sexual orientation, disability, by technical proficiency, amongst others. Considering this diversity spectrum, participants discussed strategies to increase the involvement of people from underrepresented groups, why this is a major goal for our communities, and proposed to write a Diversity Statement to establish the communities' scope and commitment to increase the diversity within them.

Participants came up with actionable tasks for lowering barriers that some underrepresented groups find while approaching these two projects. Amongst them we can find: translating tutorials and essential documentation pages to other languages to lower the language barrier for non-English speakers; implementing mechanisms of inclusion and accessibility that other organizations usually make use of, like accessible colormaps, alt-text for figures, and more accessible documentation websites; optimize the software tools so they can be used on low-end computers, allowing users without access to high-performance computational resources to use them.

Groups also explored the opportunity of reaching existing programs and organizations geared to underrepresented groups and undeveloped regions of the world for collaborations, like Geoscience Without Borders, SEG Women Networking Committee, GeoLatinas, Earth Science Women's Network, Women in Mining, the Equator project, sub-committees in professional associations (e.g. American Geophysical Union, European Geophysical Union). They also proposed extending our participation in other open-source software communities, like the Software Sustainability Institute, pyOpenSci, and Open Science Labs.

They explored the possibility of creating mentorship programs to train new developers and contributors from underrepresented groups, pairing mentee and mentors that have common backgrounds; hosting local workshops and events in the native language of the participants; organizing workshops, meet-ups and sessions at major geoscientific conferences to engage with a broader audience.

How to make these projects more sustainable?

All groups agreed that in order to promote long-term sustainability for the two projects, two key aspects must be addressed: (i) identifying avenues to increase representation from a wider variety of industry and academic institutions, and (ii) new generation of developers and contributors should be trained. Aligned with the diversity discussions, people from underrepresented groups should be encouraged to make contributions to both projects and trained to become developers and maintainers. One actionable task that came up during these discussions is to improve the guides to contribute in the documentation pages of the two projects: adding more clear and updated instructions on how to set up a development environment; providing details and examples on code style and design; giving insight on how to write and execute tests; build the documentation; and ultimately, requesting contributions to be included in the main branch of the source code.

Outcome of the Meeting

The discussion that participants held during the workshop helped to identify points of improvement for the future of the SimPEG and Fatiando a Terra projects. The main topics were around future developments for the software packages within the two projects; how to promote more mutual collaboration and interoperability between the two of them; and how to lower barriers to increase diversity within our communities and train a new generation of developers and maintainers of these open-source tools. Since the participants were mostly graduate students and early-career researchers, most of the discussions were about how can we start contributing back and make our communities more open to people from underrepresented groups.

From all the discussions, some actionable items were selected and participants formed new groups to start working on them. Among them we can find: the creation of a comprehensive list of scientific publications that make use of SimPEG and Fatiando a Terra, as a way to promote the work of authors that rely on these tools and to provide an overview of the impact these two projects have in the geoscientific community; improving the guidelines for contributing to the two projects, by extending the documentation pages that help developers to setup the development environment and contribute with new code; move SimPEG tutorials to their own website built using JupyterBooks.

The discussions also led to some more long-term goals for the two projects, like developing a new joint tutorial that covers processing some geophysical data up to running an inversion using both Fatiando a SimPEG; promoting the collaboration between the two projects, for example by speeding up SimPEG's potential fields simulations with Fatiando's tools; scheduling joint periodic virtual meetings between the two projects to promote more interactions between the two communities; define joint and better roadmaps for the two projects; and keep improving the guides for new developers to join.

This workshop proved to be an excellent opportunity for the communities of Fatiando a Terra and SimPEG to get together and create new bonds that will trigger future scientific and technical collaborations. It provided graduate students and early-career researchers the room to share their perspectives and show their motivations and expectations for the future of these two projects. Covering from new developments that will help their current and future research and industry applications, to considering ways to build stronger and more diverse communities with better communication channels, their discussions were fruitful and enlightening.

Participants

Antunes, Vinicius (University of Western Australia)
Balza, Andrea (RWTH Aachen University)
Capriotti, Joseph (University of British Columbia)
Chakraborti, Prithwijit (University College Dublin)
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Cowan, Devin (University of British Columbia)
Ferreira de Souza Junior, Gelson (Universidade de São Paulo)
Fournier, Dominique (Mira Geoscience)
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Kang, Seogi (Stanford University)
Kuttai, Johnathan (Dias Geophysical)
Le Mével, Hélène (Carnegie Institution for Science)
Li, Lu (The University of Western Australia)
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MacLennan, Kris (University of North Dakota)
Martens, Kalen (SJ Geophysics)
Melo, Aline (University College Dublin)
Miller, Craig (GNS Science)
Nikolaidis, Prodromos (n/a)
Nzikou Mamboukou, Michel (University of Western Australia)
Odiegwu, Chukwukelu (Nnamdi Azikiwe University Awka Nigeria)
Peacock, Jared (US Geological Survey)
Pesce Lopez, Agustina (Fatiando a Terra)
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Soler, Santiago (University of British Columbia)
Sun, Jiajia (University of Houston)
Tankersley, Matt (Victoria University of Wellington)
Toqeer, Muhammad (Quaid-I-Azam University)
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Weis, John (Zonge)
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Chapter 35

2023 Math Attack Summer Camp for Girls (23w2013)

August 18 - 20, 2023

Organizer(s): Lauren DeDieu (University of Calgary),
Sean Graves (University of Alberta)

Description

The 2023 Math Attack Summer Camp for Girls was an 8-day overnight camp that was held at the University of Calgary and the Banff International Research Station (BIRS) from Sunday, August 13th - Sunday, August 20th.

The camp brought 21 grades 6 - 10 students who identify as girls together to engage in fun mathematical activities and build connections. Students stayed in the university residence for the first five nights of the camp and stayed at the Banff Centre for the last two nights.

The camp aimed to encourage girls to pursue their passion for mathematics and make connections with peers who shared similar interests. Throughout the week, students engaged in mathematical sessions that explored topics such as graph theory, topology, data science, statistics, and actuarial science. They investigated how x-ray machines work using tomography techniques, explored the mathematics behind blockchains, and competed in a Crypto Hunt and math-based Escape Room. These sessions exposed students to 14 female role models, including recent high school graduates, undergraduate math students, graduate math students, mathematics faculty, and mathematicians in industry.

During the camp, there was also plenty of time for friendship building and physical activity. Evening activities included sports, swimming, board games, and karaoke.

There was no registration fee for the camp and all meals and accommodations were provided.

Schedule

Time	Sunday August 13th	Monday August 14th	Tuesday August 15th	Wednesday August 16th	Thursday August 17th	Friday August 18th	Saturday August 19th	Sunday August 20th
8 – 9am		Breakfast	Breakfast	Breakfast	Breakfast	Breakfast	Breakfast	Checkout + Breakfast
9 – 10:15am		Cryptology (Lauren DeDieu)	Sharing Secrets (Tahera Fahimi)	Unleashing the Power of Computational Thinking: Empowering Girls in a Digital World (Mary Grant)	Callysto Hackathon	Finish Packing (9 – 9:45am) Bus to Banff (9:45am)	Mathematical Communication (Lauren DeDieu)	Cryptology (Lauren DeDieu)
10:15 – 10:30am		Break	Break	Break	Break		Break	Break
10:30 – 11:45am		Crossing Bridges (Ryan Hamilton)	A Quick Look at Blockchains (Sepideh Avizheh)	An Introduction to Tomography (Tracey Balehowsky)	Callysto Hackathon		Mathematical Card Tricks (Lauren DeDieu)	Feedback + Closing Ceremony
11:45am – 1pm		Lunch	Lunch	Lunch	Lunch	Explore Banff (*participants purchase their own lunch*)	Group Photo + Lunch	Lunch
1 – 2:15pm		Topology (Rachel Hardeman Morrill)	Walk the Line (Madeline Ward)	The Crypto Hunt (Taylor Markham)	Callysto Hackathon	Hike (Tunnel Mountain)	Escape Room (Dami Wi)	Bus to Calgary (1pm)
2:15 – 2:30pm		Break	Break	Break	Break			
2:30 – 3:45pm		Finite State Automata (Ryan Morrill)	Walk the Line (Madeline Ward)	The Crypto Hunt (Taylor Markham)	Callysto Hackathon			Departure (International House)
3:45 – 4pm		Break	Feedback	Feedback	Feedback		Feedback	
4 – 5:30pm		Actuarial Science (Ella Charpentier) Feedback (5:15 – 5:30pm)	Free Time	Free Time	Free Time	Check-in/ Free Time	Free Time	
5:30 – 6:30pm		Dinner	Dinner	Dinner	Dinner	Dinner	Dinner	
6:30 – 9pm	Arrival (6:30 – 7:30pm, International House) Ice Breaker Activities (7:30 – 9pm, MS 431)	Walk	Board Games	Sports	Movie	Origami (Dami Wi)	Walk (Bow Falls Trail)	

BIRS Highlights

During the BIRS portion of the camp, the focus was on helping students develop their mathematical logic and communication skills. Sessions began on Friday night with a session led by Dami Wi where students explored the mathematics behind origami. The focus of Saturday morning was mathematical communication. Since many K-12 math classes do no emphasize communication, this concept was new to many students. We discussed the importance of communicating results precisely, using correct notation and prose to help the reader navigate. Students then broke into teams and went to the breakout rooms in the basement of BIRS to solve mathematical logic problems and write their solutions as elegantly as possible; students then ranked the other groups' solutions based on the quality of communication and Dr. Lauren DeDieu ranked them as well and provided feedback. At the end of the session, a winner was announced. Afterwards, students continued to develop their mathematical proof-writing skills by learning mathematical card tricks and working to explain why the tricks work.

On Saturday afternoon, students engaged in a 3-hour-pirate-themed mathematical escape room designed by Dami Wi. Stations were set up at various outdoor locations at the Banff Centre (e.g., Shaw Amphitheater, outside of the TransCanada Pipelines Pavilion). At each station, students completed a mathematical puzzle then moved to the next station. The goal was to figure out the route to *Treasure Island*.

On Sunday morning, students engaged in a modern cryptology session led by Dr. Lauren DeDieu, where they learned about the RSA public-key cryptosystem and the number theory needed to understand why it works (e.g., multiplicative inverses, Euler's Theorem). The day wrapped up with a Closing Ceremony where students were awarded certificates and prizes.

Over the weekend, students also took some time to explore the town of Banff. On Friday afternoon we hiked up Tunnel Mountain, and on Saturday evening we went for a walk along Bow Falls Trail.

Outcomes of the Meeting

This camp helped inspire our female participants to pursue their passion for mathematics by making connections with female role models and peers who share similar interests. This is reflected in the following quotes from our participants:

- It was encouraging for me to meet more girls that excel in STEM. We really bonded over the course of this camp by working on challenges together and talking about similar interests.

- I really enjoyed attending this camp. I learned a lot, and my appreciation for mathematics grew beyond measure. Leaders were so amazing, impactful, inspiring and insightful. I wish I never had to leave, I had such a blast!

- I am extremely grateful for the opportunity to attend this camp. I was able to connect with people who I wouldn't have met otherwise as they came from all over alberta and we attend different schools. I was able to make friends who I intend to keep in contact with even after the camp is over. After I complete high school, I would like to attend the University of Calgary, and this program allowed me to meet professors that I may meet again in a class a couple years down the road. Additionally, this has given me insight into what's in store for me in classes like this.

- Before I came to camp, I thought math was anything to do with numbers, if there were numbers it was math, if there weren't numbers, it wasn't math. But I realized that math is everywhere, even the smallest thing in the whole universe is math. Before I didn't know if I would be interested in math, but know I know even if I don't like one small thing about math, there are so many more things I can do in the math field.

- This camp reinforced the idea that mathematics is such a field of various areas, and ties in with so many different applications, pretty much anywhere within our world. We use mathematics to understand, describe, develop and sharing knowledge about our world, and it connects everyone no matter race, gender, religion, or sexuality, math is inclusive!

- Prior to this camp, I thought that math was only about numbers and performing complicated calculations. However, I learned that excelling in mathematics also requires good communication skills thanks to Dr. DeDieu. I also rarely connected math with the medical field, but I learned about how CT scans require geometry through our tomography session with Tracey Balehowsky.

- This camp has shown me that math extends beyond the classroom, and plays a vital role in many of my hobbies. From robotics to math puzzles, this camp has done a fantastic job of balancing the realistic applications of mathematics, with the recreational side that makes people fall in love with math to begin with.

- This camp provided an environment that was inclusive and open. All the chaperones and speakers were brilliant and encouraging, which definitely did inspire me to explore my own passions for STEM opportunities.

- Originally, I had no idea about the numerous branches under mathematics that we studied in this camp. Throughout this week, I was able to explore so many interesting topics that I would never have the chance to otherwise.

- At this camp, I met many likeminded peers with whom I have developed strong connections and relationships with, and will stay in touch with for many years to come. I will leave this camp better aware of the different areas of mathematics and possible applications. I believe that a good balance of these outcomes were met, as we learnt so much, and were encouraged to get excited about mathematics in a lighthearted real-world way.

- The people I met at this camp are just as passionate about math and I have formed many connections with my peers over the course of the week. Additionally, being able to enjoy math and excelling and studies without feeling embarrassed or ashamed was really nice.

Additional Information

Additional photos and information about the camp can be found in the *Final Report* that is available here: https://science.ucalgary.ca/math statistics/engagement/educational-outreach/math-attack.

Participants

Cao, April (Student)
DeDieu, Lauren (University of Calgary)
Ding, Alice (Student)
Falasinnu, Tomi (Student)
Hua, Jasmine (K-12 Student)
Jiang, Ava (Student)
Josan, Saachee (Student)
Liu, Cassandra (Student)
Mao, Catherine (Student)
Mutti, Amryn (Student)
Okunlola, Michelle (Student)
Plumb, Abby (Student)
Qiu, Sian (Student)
Rong, Cathy (Student)
Tian, Keren (Student)
Venkateshwaran, Vaishnavi (Student)
Wang, Heather (Chaperone)
Wi, Dami (University of Alberta)
Wu, Amy (Student)
Xie, Janice (Student)

Xu, Heather (Student)
Yu, Jeannie (Student)
Zhang, Sophia (Student)
Zhang, Lillian (Student)
Zheng, Fogil (Chaperone)

Chapter 36

Emerging Mathematical Challenges in Synthetic Biological Network Design (23w2007)

August 25 - 27, 2023

Organizer(s): Enoch Yeung (University of California, Santa Barbara), Andras Gyorgy (New York University Abu Dhabi), Mustafa Khammash (ETH Zurich), Ophelia Venturelli (University of Wisconsin-Madison)

Thirty one researchers met in hybrid format at the Banff International Research Station to discuss emerging mathematical challenges in designing synthetic gene networks. The workshop was organized around the following themes: (i) distributed & multi-cellular biological control; (ii) from modularity to robustness; (iii) biological context & control; and (iv) quantitative design & discovery.

Overview of the Field

Synthetic biology seeks to understand and control living processes by building them. Nearly twenty years after its inception, the field now faces emerging challenges as it builds increasingly complex genetic circuits and networks. These challenges can be categorized roughly into five areas: (i) scalability, (ii) robustness, (iii) modularity, (iv) context, and (v) data-driven approaches.

First, genetic circuits rely on biochemical energy borrowed from living cells. As a result, seemingly disconnected network components influence each other due to the limited availability of shared cellular resources [5, 11]. Regarding the second challenge, the cellular environment is replete with uncertainty due to the lack of precise knowledge of endogenous and engineered molecular components and their interactions, in addition to external perturbations and stochastic noise. As a result, models of cellular function are necessarily uncertain, and designing circuits that function reliably in the face of this uncertainty requires circuit architectures that are robust by design [12]. The third major challenge that synthetic biologists often face centers on modularity. The ability to accurately predict the behavior of a complex system from that of the composing modules has been instrumental to the development of engineering systems. Unfortunately, despite the fact that biological networks are rich of frequently repeated motifs [1], suggesting a modular organization, our current ability of predicting the emergent behavior of a network from that of the composing modules remains limited. Fourth, while synthetic biology bears many similarities to traditional engineering disciplines, designing synthetic gene circuits is often time consuming due to their context-dependent behavior, frequently leading to unexpected and perplexing phenomena [4]. Thus, the construction of even simple systems typically relies on massive DNA libraries that need to be iteratively refined, involving high-throughput screening and testing in a lengthy and expensive process [13, 3]. Finally, synthetic biological networks are consistently represented through experimental data, but translating data to generalizable biophysical insight and parameters remains a challenge [8]. While multiple data streams have become commonplace in characterizing engineered living systems, data-driven methods for extracting and embedding insight from these into an integrated model for learning representation are lacking.

Recent Developments and Open Problems

Recent years have seen tremendous progress in synthetic biology ranging from the development and implementation of insulation devices to multiple controllers facilitating both the modular and rational design as well as the robust functioning of genetic circuits [6, 15, 2, 10, 7, 9]. These results have been made possible by combining mathematical tools from a variety of fields with experimental techniques.

For instance, in addition to distributing complex genetic functions across multiple heterogeneously programmed cells [14], mathematical tools aided by experimental approaches were also developed to capture the previously unmodeled interactions among circuit components to overcome the challenge of scalability [17]. Motivated by the success of this approach, there is an urgent need for developing new mathematical (and experimental) approaches for engineering synthetic microbiomes and consortia [16]. While control theory from engineering offers some clues on the structural requirements of robust designs [12], the unique nature of molecular systems necessitates the development of new mathematical methods and accompanying experimental tools for characterizing and engineering biological circuits that are insensitive to the highly uncertain environment peculiar to the living cell. Additionally, it has been proposed that biological networks may have a modular organization similar to that of engineered systems, and that core processes, or motifs, have been conserved through the course of evolution and across different contexts [1]. In addition to having profound consequences from an evolutionary perspective, this view implies that biology can be understood, just like engineering, in a modular fashion. Unfortunately, genetic circuits often display context-dependent behaviors, undermining the modular design of complext systems [5, 11]. Therefore, it is imperative to properly model and abstract biological interconnections and the emergent behavior of interconnected, low-dimensional, nonlinear systems upon their composition. While the iterative re-design and tuning of synthetic systems can prove successful for modules of modest complexity [13, 3], it rapidly becomes infeasible with increasing circuit size. Context-dependence thus poses a critical limitation in synthetic biology by undermining the rational design of large-scale systems [4]. It is becoming clear that new mathematical and experimental approaches are urgently needed to facilitate a deeper understanding of context-dependence and to devise practical approaches for overcoming it. Finally, despite the overabundance of biological data (e.g., transcriptomics, proteomics, flow cytometry, time-lapse single cell microscopy, photospectometry), a generalized mathematical theory integrating experimental measurements with biological system representation is lacking for the development of data-driven approaches for engineering complex synthetic biological systems [8].

Presentation Highlights

The focus of this workshop was the coordination, unification, and cross-fertilization of myriad mathematical methods in designing synthetic biological gene networks. The informal yet focused environment provided by the BIRS station was the perfect place to have critical conversations about technology gaps and how to increase scalability, robustness, and define the next generation of challenge problems in biological design. This has been highlighted by some of the testimnonials: *This BIRS workshop was very stimulating and interesting: I could hear about new results, meet friends as well as new contacts (whom I would have probably never met at the conferences that I typically attend) and learn a lot, especially thanks to the interdisciplinary nature of the workshop, and have new research ideas (Giulia Giordano)*; *The BIRS workshop on "Emerging Mathematical Challenges in Synthetic Biological Network Design" provided a unique venue for like-minded synthetic biologists to discuss their most recent* *research results in excellent mathematical detail. Unlike other meetings in this field, presenters could provide complete derivations and explanations of their results without fear of losing their audience (Howard Salis).* The hybrid format allowed us to have not only detailed discussion in person, but also accommodate the schedule of those who could not attend personally.

Scientific Progress Made

To address the major challenges highlighted above, progress has been made in four thematic areas: (i) distributed and multi-cellular biological control; (ii) modularity and robustness; (iii) biological context and control; and (iv) quantitative design and discovery.

First, considering distributed and multi-cellular biological control, we learned about the possibility of predicting and controlling gene transfer in microbial communities (Lingchong You) and their heterogeneity due to stochastic promoter switching (Yili Qian), as well as how to implement robust dynamical biosystems with layered controls (Chelsea Hu) and the principles of microbial community efficiency, robustness, and controllability (Ophelia Venturelli). These experimental talks were complemented by theoretical results ranging from reaction order analysis to reveal global polyhedral constraints on the behavior of biomolecular reaction systems (John Marken) to Bayesian optimization of microbiomes using tailored machine learning (Jaron Thompson).

Second, participants highlighted the role that modularity plays to facilitate achieving robustness. For instance, we learned not only about the structural and topology-independent stability of biological systems (Giulia Giordano) but also how to characterize and implement maximally robust genetic control circuits (Mustafa Khammash). In addition to quantifying the impact of resource limitations (Eduardo Sontag), a general framework was presented that quantifies the contribution of a biological feedback control mechanism to adaptation (Mariana Gómez-Schiavon), as well as concrete application examples of biological control for the automatic detection of gut infection (Michaelle Mayalu) and for the governing of bottom-up engineered bacteria consortia by synthetic gene circuits (Xiao Wang).

Third, it was illustrated not only how supercoiling can couple the behavior of unrelated genes (Enoch Yeung), but also how inducible plasmid copy number control can facilitate the characterization of these surprising interactions (Andras Gyorgy) with the help of the proper dynamical framework for characterizing gene expression dynamics (Terence Hwa). Furthermore, cell-free transcription-translation systems were highlighted as an efficient platform to unentangle the complicated relationship of genetic circuits and their context (Vincent Noireaux) and ribocomputing as a promising tool for implementing large-scale cellular computations (Jongmin Kim).

Fourth, illustrating the role of quantitative design and discovery, talks centered around the mutually beneficial fusion of computational approaches and experimental techniques. Topics ranged from the rational design of artificial phosphorylation signaling networks in human cells (Caleb Bashor) to engineering bacteria for mastering tic-tac-toe through accelerated adaptation (Alfonso Jaramillo) and to the control of complex biological systems interfaced with bioelectronics, showcasing the current frontier of experimental approaches. These were complemented by cutting-edge computational techniques, including leveraging Fisher information to harvest fluctuation information while rejecting measurement noise (Brian Munsky), verification- and model-guided gene circuit design and prototyping (Lucia Marucci and Chris Myers), as well as the dynamic modelling of mixed microbial populations (Brian Ingalls).

Outcome of the Meeting

This workshop had four outcomes. First, presenting the latest advances from expert synthetic biologists to illustrate and highlight emerging challenges in biological scale-up, modular design, context robustness, data-driven representation. Second, allowing theorists and engineers to share their recent advances in mathematical modeling and data-driven design that can aid in addressing these challenges. Third, incorporating the input from the biologists to spur development and organization of new mathematical strategies or approaches, integrating ideas from control theory, systems theory, nonlinear analysis, functional analysis, behavioral systems theory, and pure mathematics to address the fundamental differences in these emerging challenges. Fourth, providing a venue for theoreticians to forge new collaborations with leading practitioners in systems and synthetic biology.

Invited speakers were selected to ensure accessibility for a mixed audience of experimentalists and theorists. Focused discussions on challenges of interest emphasized the need to solicit multiple perspectives from practitioners and theorists. Key junior speakers were identified who serve as emerging thought leaders in synthetic biology and biological control, specifically those who represented underrepresented minorities in the STEM fields and our field at large. We also placed an emphasis on inviting more female engineers and scientists, which historically have been underrepresented in the field of synthetic biology and biological control. Additionally, the international roster of speakers represented great geographical diversity from North and Central America, Europe, Asia, and the MENA region.

Participants

Bashor, Caleb (Rice University)
Bennett, Matthew (Rice University)
Briat, Corentin (–)
Cheung, Kevin Pak Lun (University of Wisconsin Madison)
Clark, Harris (UCSB)
Giordano, Giulia (University of Trento, Italy)
Gomez, Marcella (University of California at Santa Cruz)
Gómez-Schiavon, Mariana (UNAM)
Guy-Bart, Stan (Imperial College London)
Gyorgy, Andras (New York University Abu Dhabi)
Hu, Chelsea (Texas A & M University)
Hwa, Terence (UC San Diego)
Ingalls, Brian (University of Waterloo)
Jaramillo, Alfonso (CSIC)
Jaron, Thompson (University of Wisconsin, Madison)
John, Marken (Caltech)
Johnson, Charles (BCCL UC Santa Barbara)
Khammash, Mustafa (ETH Zurich)
Kim, Jongmin (Pohang University of Science and Technology)
Marucci, Lucia (University of Bristol)
Mayalu, Michaelle (Stanford)
Munsky, Brian (Colorado State University)
Myers, Chris (University of Colorado Boulder)
Noireaux, Vincent (University of Minnesota)
Qian, Yili (University of Wisconsin-Madison)
Salis, Howard (Pennsylvania State University)
Sontag, Eduardo (Northeastern University)
Venturelli, Ophelia (University of Wisconsin-Madison)
Wang, Xiao (Arizona State University)
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Chapter 37

Mathematical Challenges in Adaptation of Quantum Chemistry to Quantum Computers (23w2015)

September 1 - 3, 2023

Organizer(s): Sergey Gusarov (National Research Council Canada), James Brown (Good Chemistry Company), Prashant Nair (UBC), Valera Veryazov (Lund University)

Introduction

The ”Mathematical Challenges in Adaptation of Quantum Chemistry to Quantum Computers” workshop, held in Banff, Alberta, from September 1 to September 3, 2023, was a significant event that brought together experts from various fields to discuss the intersection of quantum chemistry, mathematics, and quantum computing. The workshop aimed to address the current mathematical and computational challenges in quantum chemistry and explore how modern hardware architectures, particularly quantum computers, can revolutionize this field. The presentations jointly covered a wide range of topics, from quantum computing fundamentals to practical applications in quantum chemistry and related fields, offering attendees a comprehensive view of the mathematical and computational challenges and opportunities at the intersection of quantum chemistry and quantum computing.

Recent Developments and Open Problems

The workshop was more than just a series of presentations and discussions; it was an affirmation of the exponential advancements and possibilities that lie at the intersection of quantum computing and quantum chemistry. The event stood out as a testament to the dedication and passion of the global scientific community, striving to bridge gaps and foster collaborations in these rapidly evolving fields. In a world where technology is advancing at an unprecedented pace, such gatherings become crucial platforms for cross-disciplinary dialogue, bridging theoretical frameworks with practical applications. They represent not just the exchange of knowledge but the synthesis of new ideas that have the potential to drive forward both scientific understanding and technological innovation.

Quantum computing, with its potential to process and analyze vast amounts of data in ways previously thought impossible, promises to be the key that unlocks many of quantum chemistry's most intricate puzzles. From simulating complex molecular interactions to accelerating the discovery of new materials and pharmaceuticals, the potential applications of this synergy are vast and varied. However, with these advancements come challenges that require not just technical prowess but also visionary thinking and robust collaboration. The workshop highlighted the necessity of an interdisciplinary approach, where experts from diverse backgrounds—be it physics, chemistry, mathematics, or computer science—come together to share insights and craft solutions that no single discipline could achieve on its own.

Moreover, the workshop echoed a broader sentiment prevalent in the scientific community: the importance of constant learning and adaptability. As the lines between disciplines blur and new challenges emerge, the need to stay updated, reevaluate traditional methodologies, and be open to novel approaches becomes more critical than ever.

Key Takeaways:

1. Quantum Advantage: The workshop reiterated the concept of a "quantum advantage" in quantum chemistry. Quantum computers have the potential to perform calculations exponentially faster than classical computers in specific areas of quantum chemistry, offering the promise of groundbreaking advancements in simulating molecular structures and chemical reactions. Multiple presentations highlighted recent accomplishments, encompassing novel methodologies, advancements in software development, and the utilization of quantum machine learning for tackling the Schrödinger equation.
2. Optimization Challenges: Quantum chemistry often involves complex optimization problems. While quantum computers hold the promise of solving these problems efficiently, there is a continued need for the development of efficient quantum optimization algorithms. The discussions revolved around the significance of quantum optimization and explored emerging directions in its application to quantum chemistry.
3. Interdisciplinary Collaboration: A recurring theme throughout the workshop was the critical importance of interdisciplinary collaboration. Success in adapting quantum chemistry to quantum computers hinges on cooperation between quantum physicists, chemists, mathematicians, and computer scientists. These multidisciplinary teams are at the forefront of developing and testing quantum algorithms and applications.
4. Real-World Applications: The integration of quantum chemistry with quantum computers has far-reaching implications across various industries, including pharmaceuticals, materials science, and environmental science. Quantum computing can accelerate drug discovery, aid in designing new materials, and enhance our understanding of complex chemical processes.

Scientific Progress Made

The "Mathematical Challenges in Adaptation of Quantum Chemistry to Quantum Computers" workshop was a resounding success, emphasizing the transformative potential of quantum computing in the field of quantum chemistry. It provided a forum for in-depth discussions, knowledge sharing, and the exploration of cutting-edge research in this exciting interdisciplinary domain. This workshop underscored the critical role that collaboration and innovation play in advancing the boundaries of science and technology. The organizers, presenters, and participants contributed significantly to making this event a valuable platform for the exchange of ideas and the pursuit of groundbreaking solutions. As we look to the future, it is evident that the integration of quantum chemistry and quantum computing will continue to push the boundaries of scientific discovery, leading to new frontiers in understanding the quantum world and addressing complex computational challenges.

Outcome of the Meeting

In conclusion, the "Mathematical Challenges in Adaptation of Quantum Chemistry to Quantum Computers" workshop was not just an event, but a milestone in the ongoing journey of scientific exploration and discovery. As attendees returned to their respective institutions, they carried with them not just the knowledge acquired but also a renewed sense of purpose and commitment to the collective goal of pushing the boundaries of what's possible in quantum chemistry and quantum computing. The ripple effects of this gathering will undoubtedly be felt in research papers, collaborations, and breakthroughs in the years to come, further solidifying the workshop's importance and impact on the global scientific landscape. The workshop holds great significance for Canadian research, as it serves as a focal point for advancing the country's contributions to the field of quantum chemistry and quantum computing. Also, it serves as a catalyst for enhancing both the quality and reach of Canadian research by promoting fruitful collaborations between Canadian scientists and leading experts from around the world. This collaborative spirit is fundamental in driving advancements in quantum chemistry and quantum computing on a global scale.

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Focused Research Group Reports

Research in Teams Reports

Chapter 38

Spectral Synthesis and Weak Amenability of Uniform Algebras (23rit009)

April 16 - 30, 2023

Organizer(s): Alexander J. Izzo (Bowling Green State University, Mathematics and Statistics), Joel F. Feinstein (University of Nottingham, School of Mathematical Sciences)

Overview of the Research Project

The goal of the research project we pursued while at BIRS is to solve two distinct, but related, longstanding problems concerning the relationship between uniform algebras and two important Banach algebra properties, spectral synthesis and weak amenability, with connections to abstract harmonic analysis.

A uniform algebra on a compact Hausdorff space X is an algebra of continuous complex-valued functions that contains the constants, separates the points of X, and is uniformly closed in the algebra $C(X)$ of all continuous complex-valued functions on X. On every compact Hausdorff space there is the trivial example of a uniform algebra, namely $C(X)$ itself. A typical example of a nontrivial uniform algebra is the disc algebra which consists of the continuous complex-valued functions on the closed unit disc that are holomorphic on the open unit disc. The uniform algebras form a class of Banach algebras that is important both in the field of Banach algebras and in complex analysis. Uniform algebras also have applications to operator theory. The particular properties we are investigating, spectral synthesis and weak amenability, are of fundamental importance in abstract harmonic analysis.

As the example of the disc algebra suggests, there is a close connection between uniform algebras and analyticity, and there were once a number of conjectures that asserted the existence of various forms of analyticity associated with every nontrivial uniform algebra. These conjectures were disproved in the 1960s. However, in the current century, we and our coauthors have shown that under a variety of additional natural hypotheses, these conjectures are in fact true, and we have also constructed additional counterexamples further elucidating the extent to which the conjectures fail in general [1–6, 10–13, 15–22]. We believe that the time is ripe for developing a much better understanding of the connection between analyticity and uniform algebras and Banach algebras more generally.

One form of analyticity that was once conjectured to be present in every nontrivial uniform algebra was the existence of nonzero point derivations. For bounded point derivations, Wermer refuted this by constructing a Swiss cheese K (a type of compact planar set) such that $R(K)$, the algebra of complex-valued functions on K

approximable by holomorphic rational functions with poles off K, is a counterexample [23]. Later a construction of Cole showed that even unbounded point derivations can fail to exist [9].

The problems we are aiming to solve concern two ways in which the condition that a bounded point derivation exists can be relaxed. One is to replace point derivations with derivations into an arbitrary commutative Banach module. The other way arises from reformulating the existence of nonzero bounded point derivations in terms of ideals. We then replace the existence of a nonzero bounded point derivation by the weaker condition that there exists a closed ideal that is not an intersection of maximal ideals. The first approach leads to the notion of weak amenability, the second approach to the notion of spectral synthesis.

A commutative Banach algebra A is *weakly amenable* if every bounded derivation from A into a commutative Banach A-module is zero. By a theorem of Bade, Curtis, and Dales [7], it is equivalent to require only that every bounded derivation into the dual A-module A^* is zero. A commutative Banach algebra A has *spectral synthesis* if for every closed set E in the maximal ideal space of A and every closed ideal I in A we have $\text{hull}(\ker(E)) = E$ and $\ker(\text{hull}(I)) = I$.

Objective

Our objective is to answer the following two questions which are over 30 years old:

1. Does there exist a nontrivial uniform algebra that is weakly amenable?
2. Does there exist a nontrivial uniform algebra with spectral synthesis?

Scientific Progress Made

We devoted most of our time at BIRS to investigating possible approaches to the construction of a nontrivial weakly amenable uniform algebra. Since the principal way of constructing nontrivial uniform algebras with no nonzero bounded point derivations is Cole's method of iteratively adjoining square roots, we began by investigating the effect that adjoining square roots to a uniform algebra A has on bounded derivations from A into the dual A-module A^*. We found that the effect on bounded derivations into the dual module is completely different from the effect on bounded point derivations.

If one starts with a uniform algebra A on which there is a nonzero bounded point derivation d, and one adjoins a square root to a suitable function in A, one obtains a new uniform algebra, containing A as a subalgebra, to which the point derivation d does not extend. Furthermore, if one adjoins square roots to a suitable family of functions in A, one obtains a uniform algebra containing A to which none of the bounded point derivations on A extend. There may be new nonzero bounded point derivations on the the new uniform algebra, but by iterating the process of adjoining square roots, and passing to a limit, one eventually obtains a uniform algebra on which there are no nonzero bounded (or even unbounded) point derivations.

Each bounded point derivation on a uniform algebra A induces in a natural way a corresponding bounded derivation into the dual A-module A^*. Surprisingly, we found that the operation of adjoining a square root that eliminates a bounded point derivation does not in general eliminate the corresponding derivation into the dual A^*. Specifically, starting with the disc algebra $A(D)$ and adjoining a square root to the identity function z, the bounded derivation $A(D) \to A(D)^*$ induced by differentiation at the origin *does* extend to a bounded derivation $A(D)[\sqrt{z}] \to A(D)[\sqrt{z}]^*$. (Of course the extended derivation is *not* induced by a point derivation on $A(D)[\sqrt{z}]$.) Furthermore, the norm of the extended derivation equals the norm of the original derivation. The uniform algebra $A(D)[\sqrt{z}]$ is isomorphic to the disc algebra $A(D)$, so we can iterate the process. Finally, we can pass to a limit to obtain a new uniform algebra that is *not* isomorphic to the original uniform algebra. However, because at each step of the extension process the norm of the derivation remains the same, the original derivation extends even to the final limit uniform algebra.

More generally, we found that every bounded derivation $A(D) \to A(D)^*$ extends to a bounded derivation $A(D)[\sqrt{z}] \to A(D)[\sqrt{z}]^*$. Here a curious situation arises. We do not know whether, in general, the norm of the extended derivation exactly matches the norm of the original derivation, but we showed that there is an equivalent norm, arising from an isomorphism of the space of derivations $A(D) \to A(D)^*$ with a function space given by work of Choi and Heath [8], under which the norm of the extension *does* coincide with the norm of the original derivation. Consequently, even after iterating the process and passing to a limit, *every* bounded derivation on the original uniform algebra extends to a bounded derivation on the final limit uniform algebra.

We also looked at the possibility of adapting Wermer's Swiss cheese construction to obtain a nontrivial weakly amenable uniform algebra. Here too the elimination of bounded derivations into the dual module appears to be very different from the elimination of bounded point derivations.

As a result of our investigations we obtained a much greater understanding of the weak amenability problem for uniform algebras. Since the only known methods of eliminating bounded point derivations seem to be incapable of eliminating all bounded derivations, we now think that it is likely that there are *no* nontrivial weakly amenable uniform algebras. Our next step, therefore, will be to dissect the proof that every *amenable* uniform algebra is trivial and attempt to push it further to cover the case of weakly amenable uniform algebras.

We also discussed the spectral synthesis problem. However, we spent only a little time on that since we focused on the weak amenability problem where we found we were making more progress.

We also made use of our time together at BIRS to revise a draft paper we wrote earlier on weak sequential completeness of uniform algebras. Being together at BIRS enabled us to find a more elementary proof that every weakly sequentially complete uniform algebra is finite-dimensional. The paper has now been accepted for publication in the Bulletin of the Irish Mathematical Society [14].

Participants

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Chapter 39

Elliptic Stable Envelopes and R-matrices for Superspin Chains from 3d $\mathcal{N}$=2 Gauge Theories (23rit110)

July 9 - 23, 2023

Organizer(s): Nafiz Ishtiaque (Institut des Hautes Études Scientifiques), Seyed Faroogh Moosavian (McGill University), Yehao Zhou (Kavli Institute for the Physics and Mathematics of the Universe)

Overview of the Field

The complicated nature of physical systems makes it both difficult and desirable to find dynamics that can be completely determined. Known examples of such systems, the integrable models, are usually characterized by rich physical and mathematical structures. Integrable structures have been discovered in many a priori distinct and unrelated physical systems, such as different quantum field theories (QFTs) and string theory. These discoveries have proved exceptionally useful for our understanding of nonperturbative dynamics in QFTs and string theory but remained mostly mysterious in origin. This begs for the development of a unified picture of integrability in physical systems.

The simplest and earliest-known examples of integrable systems are one-dimensional lattice models or spin chains where the sites carry representations of a symmetry algebra $\mathfrak{g}$. These were developed as simple models for magnetic materials with nearest-neighbor interactions. Bethe showed in his pioneering work [5] that these systems are exactly solvable. The Schrödinger equation for the system can be solved by making an ansatz about the eigenvalues of the Hamiltonian and then solving certain constraints on them called the Bethe ansatz equations. It turns out that these algebraic equations define the critical locus of a potential function called the Yang-Yang functional Y studied in the context of one-dimensional Schrödinger operator [22]. Within this framework, known as the coordinate Bethe ansatz, the origin of integrability is obscure. In an attempt to understand this origin, the algebraic Bethe ansatz was formulated in which crucial roles are played by certain operators characterizing the spin chains such as the Lax operator and the R-matrix [10]. Integrability of these spin chains is encoded in certain algebraic braiding relations satisfied by the R-matrix called the Yang-Baxter equations [20, 21, 4, 3].

Another example of integrability appears in supersymmetric gauge theories with 4 or 8 supercharges. When all matters of these theories are massive, the low energy dynamics are controlled by a (twisted chiral) superpotential $\widetilde{W}$. For generic $\widetilde{W}$, such theories are topological and the integrability on the Coulomb branch can be argued based on general grounds [16]. Remarkably, the spin chains and the supersymmetric QFTs encode precisely the same integrable dynamics. By identifying $\widetilde{W}$ with the Yang-Yang functional Y, the Bethe/Gauge Correspondence [17, 18] shows that the gauge theory vacua are in one to one correspondence with the excitations of the spin chains. Creation and annihilation of spin excitations correspond to certain solitonic operators in the gauge theory and the full spin chain Hilbert space can be constructed by taking a collection of gauge theories with varying gauge groups. These gauge theory vacua can be characterized by the equivariant cohomology of the Higgs branch of the theory with respect to the maximal torus of the flavor symmetry group.

Given a Lie group G, its Kac-Dynkin diagram Q, a set $\mathbf{w}$ of its representations, and a complex curve C called a spectral curve, we can define a spin chain with G-symmetry. With the additional information $\mathbf{v}$ about the ranks of guage groups, we can also define supersymmetric gauge theories with 4 or 8 supercharges. Suppose we denote the Higgs branches of these theories by $\mathcal{M}[Q(\mathbf{v},\mathbf{w})]$. Then the vacua of these gauge theories can be identified with $H_T(\mathcal{M}[Q(\mathbf{v},\mathbf{w})])$ where T is the maximal torus of the flavor symmetry of the theory. Then the Bethe/Gauge correspondence can be schematically stated as:

$$\text{Spin chain Hilbert space with } G\text{-symmetry and weights } \mathbf{w} = \bigoplus_{\mathbf{v}} H_T(\mathcal{M}[Q(\mathbf{v},\mathbf{w})])\,. \tag{1}$$

The choice of the spectral curve C determines the infinite dimensional quantum algebra that acts on the spin chain and the space-time dimension of the corresponding gauge theory. The space-time dimension in turns determine the choice of cohomology theory on the r.h.s. of 1. For a G-symmetric spin chain with Lie algebra $\mathfrak{g}$, the possible choices are as follows:

Spectral curve	Quantum algebra	Space-time dimension	Cohomology
$\mathbb{E}_\tau$	Elliptic quantum group, $E_{\hbar,\tau}(\mathfrak{g})$	3	Elliptic cohomology
$\mathbb{C}^\times$	Quantum affine, $U_\hbar(\hat{\mathfrak{g}})$	2	K-theory
$\mathbb{C}$	Yangian, $Y_\hbar(\mathfrak{g})$	1	de Rham

Table 39.1: Choices of spectral curves and the corresponding data in G-symmetric spin chains and gauge theories. $\hbar$ is the quantization parameter, τ is the complex moduli of the elliptic curve $\mathbb{E}_\tau$, and $\hat{\mathfrak{g}}$ is the affine Lie algebra of $\mathfrak{g}$. Depending on whether $\mathfrak{g}$ is $\mathbb{Z}_2$ graded or not, the gauge theories have 4 or 8 supercharges.

All these algebras can be defined via the R-matrix solving the Yang-Baxter equations. The Bethe/Gauge correspondence therefore implies that the same R-matrices must be computable from the gauge theory side as well. From the mathematical studies on the cohomology theories [13, 2], it was discovered that a more fundamental quantity than the R-matrix is the stable envelope. The stable envelopes are triangular matrices acting on the spin chain Hilbert space which provide a Gauss decomposition of the R-matrix. It is thus natural to try to compute the stable envelopes using the gauge theories. In the hierarchy of spin chains from Table 39.1, at the top sits the elliptic spin chain. In that sense, the elliptic stable envelopes for superspin chains are the most general quantities that we can compute, because all other stable envelopes can be derived from them by taking various limits.

Recent Developments and Open Problems

A suggestion for the construction of cohomological stable envelopes from the Bethe/Gauge Correspondence has been laid out by Nekrasov [15, 14], and has been implemented for $\mathfrak{sl}_2$ spin chains in [6]. The notion of cohomological stable envelopes was formalized by Maulik and Okounkov [13]. Later this was generalized to the elliptic version by Aganagic and Okounkov [2, 1]. Recently, a gauge-theoretic construction of elliptic stable envelopes has been given in [9, 7] for SL_n symmetric spin chains. All these constructions concern elliptic cohomology of (complex) symplectic manifolds which appear in physics as Higgs branches of the gauge dual of bosonic spin chains via the Bethe/Gauge correspondence. Extending these constructions to the superspin chase was the main goal of our project. The Bethe/Gauge correspondence assigns to a superspin chain a gauge theory whose Higgs branch is non-symplectic. Our project can thus be decomposed into two main parts:

1. Give a mathematical definition of the elliptic stable envelopes for non-symplectic quiver varieties.

2. Identify proper gauge theoretic quantities corresponding to these stable envelopes and compute them.

To address the second point we rely on a string theoretic perspective on the Bethe/Gauge correspondence. For bosonic spin chains this was perspective was developed by Costello and Yagi [8] and this was later generalized to the superspin case in [11]. The string theoretic construction works for A-type Lie algebras so far, we thus focused on these $\mathrm{SL}_{m|n}$ symmetric Lie algebras for the current project.

Research Project Highlights

On the mathematical side, we have developed the mathematical foundation of the theory of elliptic stable envelopes for certain non-symplectic varieties. Our main example of such varieties were classical Higgs branches of 3d $\mathcal{N} = 2$ supersymmetric gauge theories. In particular,

- We have shown that such Higgs branches always admit a mathematical notion which we dubbed "partial polarization", replacing the usual notion of holomorphic polarization which is a key ingredient in the construction of [2];

- Furthermore, we have proven that for any variety (in particular Higgs branches of 3d $\mathcal{N} = 2$ supersymmetric gauge theories) admitting partial polarization, elliptic stable envelope exists, and it is unique;

- Finally, we have proven that certain limits of (properly normalized) elliptic stable envelopes produces the corresponding K-theoretic and cohomological stable envelopes.

On the physical front, we show that ellitptic stable envelopes of superspin chains correspond to mass Janus partition functions of 3d $\mathcal{N} = 2$ gauge theories interpolating between Higgs branch vacua. This directly generalizes the work of Dedusenko and Nekrasov [9] who showed similar results for bosonic spin chains corresponding to 3d $\mathcal{N} = 4$ gauge theories. As an illustrative example, we study in detail the simplest situation of an integrable spin chain with a Lie superalgebra symmetry, i.e. $\mathfrak{sl}_{1|1}$. To construct elliptic stable envelopes, we start with 3d $\mathrm{U}(N)$ gauge theories with $\mathcal{N} = 2$ supersymmetry on $I \times \mathbb{E}_\tau$, where I is a finite interval and $\mathbb{E}_\tau$ is an elliptic curve with complex moduli τ. These gauge theories are coupled to L fundamental and L anti-fundamental chiral multiples. After identifying the Higgs branches of these gauge theories, we use supersymmetric localization to compute the relevant Janus partition function. Furthermore, we reduce our result to 2d and 1d by taking appropriate limits to construct K-theoretic and cohomological stable envelopes, respectively. In particular, we recover the cohomological stable envelopes constructed recently by Rimańyi and Rozansky using a different method [19].

Workflow at BIRS and Outcomes

Based on online discussions between the participants of the program, various aspects of the project were outlined prior to our meeting at Banff. Most importantly, we had figured out the gauge theory quantities needed to be computed to make sure that our conjectural relation between gauge theory and superspin chains was true. At Banff we went through all the computations in detail during our regular meetings and started writing down the defining properties of elliptic stable envelopes for non-symplectic varieties. Near the end of our stay we finalized the structure of the paper to be written with details to be filled in.

We finished writing down the draft a few weeks after our meeting and the preprint was put on ArXiv in August 2023 [12]. We have received very positive feedbacks from the main figures of the field including some of the authors of [2, 9].

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Chapter 40

Optimal transport for the next generation (23rit011)

July 23 - 30, 2023

Organizer(s): Young-Heon Kim (University of British Columbia), Soumik Pal (University of Washington), Brendan Pass (University of Alberta)

Overview of the Field

Optimal transport is an extremely popular and rapidly developing area of mathematics, with many diverse applications. Although first rate textbooks on this topic exist, including those by Villani [4, 5] and Santambrogio [3], they are now several years old and important topics have emerged since their publication, including, in particular, entropic regularization. Some of these topics are covered in the excellent books of Peyre-Cuturi [2] and Galichon [1]; however, these books focus on particular areas of application (computations in machine learning and economics, respectively). Our goal is to write a general and accessible textbook on optimal transport, covering the core theory, including the newer part, and a selection of important, more specialized topics. The target audience will be graduate students with strong quantitative backgrounds, in field such as mathematics, statistics, computer science and economics, with the main prerequisite being graduate level measure theory or probability theory.

Scientific Progress Made

Although one of us (Pal) was unable to attend the research in teams event in person due to a family emergency, he participated regularly via video conferencing, and we were still able to achieve our goal for the meeting. We hammered out a precise concept for our book: it will develop three disparate, but complementary viewpoints on optimal transport: the static and dynamic formulations of optimal transport, and stochastic aspects. In particular, the dynamic point of view, often an add-on in other presentations of optimal transport theory, will be introduced from the beginning and discussed along side its static counterpart. We believe this can be achieved, without sacrificing accessibility, by starting the book with an in-depth study of optimal transport of “empirical measures”; that is, measures uniformly supported on a fixed number of points. By working in this setting, core concepts of optimal transport can be developed while avoiding more subtle, technical functional analytic issues (these will of course need to be covered, but will be deferred until later chapters). We believe that in this setting, readers will be able to clearly see the emergence of the dynamical formulation of optimal transport as an extension from points to finitely supported measures of the dynamic formulation of distance on Euclidean space. We plan in this introductory chapter to fully develop the theory of optimal transport, Wasserstein distance and the Otto calculus on empirical measures. Stochasticity will be introduced at this stage as well, by considering the Schroedinger bridge between empirical measures, and noting how the static and dynamic formulations are recovered in the appropriate limit. Other stochastic and dynamic aspects of the field, including connections between entropic regularization and the Schroedinger bridge problem, will also be emphasized throughout the book.

We also completed a detailed outline for the book. There will be a core portion (Part 1), suitable for a one semester graduate course, covering the main, fundamental concepts which any researcher working on or with optimal transport will need to know, and a second part covering more specialized topics. After an overview, where static, dynamic and stochastic aspects of optimal transport are presented informally, the chapters in Part 1 will include (the plan is of course still tentative at this point):

1. Our introductory chapter on empirical measures.
2. Static optimal transport.
3. Wasserstein space.
4. Dynamic optimal transport.
5. Entropic regularization and stochastic optimal transport.
6. Further properties (including the geometry and PDE behind optimal transport).
7. Further geometry of Wasserstein space.
8. Gradient flows.

The chapters in Part 2 will be on:

9. Analysis and geometry of optimal transport.
10. Computational optimal transport.
11. A statistical perspective.
12. Multi-marginal optimal transport in physics, economics and operations research.
13. Constrained optimal transport and applications in finance and PDE.

There will also be appendices on important background topics, including measure theory and functional analysis, convex analysis, probability and statistics and partial differential equations; these will give readers convenient access to results from these areas which are required for the exposition in the rest of the textbook.

Outcome of the Meeting

The main anticipated outcome of this meeting will be the eventual completion and publication of our book. Two of us (Kim and Pal) are currently (in Fall, 2023) co-teaching a graduate level optimal transport class. They are using our outline as a template for the course and their notes will eventually be developed into part of the book. This also serves as a sort of dry run for the design vision, as topics and the way we plan to cover them can be adjusted if they do not work well in practice during the course. We plan to continue working on the details through the next year (2024) to have a draft of the book ready by the following year (2025).

Participants

Kim, Young-Heon (University of British Columbia)
Pass, Brendan (University of Alberta)

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Chapter 41

Basis properties of the eigensystem of non-self-adjoint operators (23rit099)

August 20 - September 3, 2023

Organizer(s): Boris Mityagin (Ohio State University, USA), Petr Siegl (Graz University of Technology, AT)

Overview of the Field

Our Research in Teams within non-self-adjoint spectral theory in Hilbert spaces focused on the properties of the eigensystem (eigenvectors and root vectors) of operators T with compact resolvent. In the classical case of self-adjoint or normal T, the eigenvectors form an orthonormal basis. Nevertheless, the basis properties of the eigensystem are a delicate problem in general, even for perturbations of self-adjoint operators. To succeed in applications in differential operators, tools from operator theory are usually combined with technical and precise results on specific eigenfunctions and eigenvalues of the unperturbed operator (e.g. analytic properties of eigenfunctions in the complex plane, their asymptotics or concentration for a large spectral parameter).

As model problems, consider Schrödinger operators in $L^2(\mathbb{R})$ with complex potentials

$$T = -\tfrac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta + iV(x), \quad \beta > 1, \tag{1}$$

having the form $T = A + B$ with $B = iV$ and $A = A^*$, where the perturbation $V : \mathbb{R} \to \mathbb{R}$ is assumed to be "small" in a suitable sense (although typically unbounded or singular). It is known that the eigensystem of T contains a Riesz basis for instance if

$$\exists C > 0,\ \exists \gamma < \frac{\beta - 2}{2}\ :\ |V(x)| \leq C(1 + x^2)^{\frac{\gamma}{2}}, \quad x \in \mathbb{R}; \tag{2}$$

singular perturbations $V \in L^p(\mathbb{R})$ with a suitable $p \in [1, \infty)$ can be included as well, see [9].

This claim is based on the abstract Riesz basis test relying on the form version of the local subordination, cf. [1, 9, 13], where the size of the gaps between eigenvalues of the unperturbed operator and the decay of the "matrix elements" of the perturbation are compared. In more detail, let the eigenvalues $\{\mu_k\} \subset \mathbb{R}_+$ of a self-adjoint operator A with compact resolvent be simple and satisfy

$$\exists \gamma, \kappa > 0\ :\ \mu_{k+1} - \mu_k \geq \kappa k^{\gamma - 1}, \quad k \in \mathbb{N}. \tag{3}$$

If a perturbation B satisfies that

$$\exists M > 0,\ \exists \alpha > \frac{1-\gamma}{2}\ :\ |(B\psi_m, \psi_n)| \leq \frac{M}{(mn)^\alpha}, \quad m, n \in \mathbb{N}, \tag{4}$$

where $\{\psi_k\}$ are normalized eigenvectors of A related to the eigenvalues $\{\mu_k\}$, then the eigensystem of $T = A + B$ contains a Riesz basis, see [9]. It is also known that the condition on α and γ cannot be weakened to $2\alpha = 1 - \gamma$ in general.

The goals of our stay at BIRS were to work on several open problems (see details in Section 41) and start a preparation of a monograph covering the progress in the topic in approximately last two decades.

Recent Developments and Open Problems

During the program, we focused mainly on the following topics and open problems.

Local subordination

While the local form-subordination (4) allowed for a progress in Schrödinger operators (1), it appears not to cover the optimal range of perturbations in general. For instance, only a limited range of singular potentials is included comparing to the results on one-dimensional Hill and Dirac operators obtained by other strategies of the proof in e.g. [2, 4, 12, 8]. Moreover, it requires a very regular (power-like) behavior of the eigenvalues' gaps (3) and "matrix elements" of the perturbation (4).

To simplify the presentation in the following, we formulate open problems and our related observations mostly for the special case of the regular gaps behavior with $\gamma = 1$ in (3), i.e. perturbations of the harmonic oscillator, $\beta = 2$ in (1), in applications.

Less regular behavior

It is open if assumptions on the eigenvalues' gaps (3) and "matrix elements" (4) can be relaxed, see also [13, Sec. 7]. For instance in our special case (the harmonic oscillator), can the condition on the perturbation in (4) be replaced by

$$|(B\psi_m, \psi_n)| \leq \omega_m \omega_n, \tag{5}$$

where $\{\omega_n\}_n$ satisfies some summability condition instead of $\omega_n \leq n^{-\alpha}$ with $\alpha > 0$? Our analysis of the existing proofs suggests that such improvement is possible in the final step (and seemingly the main one), where the Schur test is employed to show the boundedness of the operator in $\ell^2(\mathbb{N})$ introduced through its matrix elements

$$\frac{(B\psi_m, \psi_n)}{m-n}, \quad m \neq n, \tag{6}$$

see [9, 13]. An applicable tool are the Schur multipliers, see [6, Sec. 4]. Nevertheless, the previous (normally simpler) steps in the proofs in [9, 13] still seem to require the regular behavior.

Highly singular perturbations

The form local subordination condition (4) is satisfied for the harmonic oscillator perturbed by a compactly supported $V \in H^{-s}(\mathbb{R})$, $s < 1/2$, see [9]. It is not known if the case $s = 1/2$ is extreme (i.e. if the Riesz basis is absent for $s > 1/2$ in general). The earlier results on $-\partial_x^2 + V$ on $(-\pi, \pi)$ and subject to Dirichlet or periodic boundary conditions do not rely on the local subordination and suggest that a different type of estimate is needed (the extreme case here seems to be $V \in H^{-1}$). In more detail, it is crucial to use the decay of the remainder of the H^{-1}-norm of V (which can expressed as a series) and also to employ the algebraic property of the unperturbed system (like $\psi_m\psi_n = \psi_{m+n}$ for $\psi_n(x) = \exp(inx)$), see e.g. [2]. Hence, to find a new abstract Riesz basis test, which would include the results on $-\partial_x^2$ as well as the perturbations of the harmonic oscillator in case $s = 1/2$, the condition in the form of the product bound in (4) or (5) should be replaced by a more general bound capturing possible Toeplitz/Hankel-type structure of the perturbations. For Schrödigner operators, the exact algebraic property of the exponentials is not present, nonetheless, it can suffice to use a replacement for large m, n employing the oscillatory behavior of the eigenfunctions in the region between the turning points.

Critical case

It an open question whether the restriction on the perturbation (2), obtained from the local subordination, is optimal. For the special case of harmonic oscillator, other methods ([1] based on Katsnelson's theorem or [11]) allow to include also bounded perturbations with a natural restriction on the perturbation's norm. On the other hand, the lower resolvent estimates in [7] for perturbations $iV(x) = i \operatorname{sgn} x|x|^{\delta}$ with $\delta > 0$ would exclude the Riesz basis property if the spectrum of the perturbed operator was localized more precisely (which is an ongoing project). Thus (2) seems to be close to optimal, nonetheless, to treat the extreme cases, an improvement of the local subordination like in (5) or for the singular potentials would be essential.

Absence of basis properties

The mechanisms behind the loss of the basis properties are only partially understood, in particular for differential operators.

Simple eigenvalues

The existing abstract constructions, see [1, 9, 11], provide examples of perturbations of self-adjoint operators with simple eigenvalues for which the eigensystem of the perturbed operators lacks the basis property. Nonetheless, the geometric understanding of this mechanism seems missing. Unlike for perturbations of multiple eigenvalues, where perturbations of arbitrarily small norm can lead to the absence of the basis property, see e.g. [5, 3], the perturbation in case of simple eigenvalues needs to be sufficiently large and of an appropriate form.

In case of the harmonic oscillator (or more general Schrödinger operators like in (1)), specific perturbations arising from a conjugation cause the basis property absence, see [10]. However, the effect of more general perturbations (in particular in potentials) is not known.

Multiple eigenvalues

The existing examples with double eigenvalues suggest that the structure of the perturbation, not its norm, is crucial to loose the basis property, see e.g. [3]. This observation and the result on the Riesz property of the block decomposition for the bounded perturbations in [11] is the first step in a construction of a potential perturbation of multi-dimensional Schrödinger operators (e.g. of the two-dimensional harmonic oscillator) without the basis property. On an abstract level, it seems possible to generalize results on a suitable structure of the perturbation from multiplicity two to an arbitrary one. Nevertheless, in potential perturbations of Schrödinger operators, the parameters of the perturbations are no longer directly accessible and they cannot be chosen (almost) freely.

Other related topics

Other discussed related topics include the basis properties of eigenfunctions of Schrödinger operators in $L^p(\mathbb{R})$ for $p \neq 2$, weighted L^p-estimates of eigenfunctions of Schrödinger operators, estimates on the number of non-real eigenvalues for perturbations with a symmetry (like $\mathcal{PT}$) or completeness of eigensystem of Schrödinger operators with imaginary potentials.

Outcome of the Meeting

Although the meeting was hybrid eventually (one participant in Banff, one in USA), the possibility to fully focus on the planned research led to very fruitful exchanges. The environment and support of BIRS are exceptional and contributed greatly to this post-covid restart of our collaboration.

Several new ideas how to proceed in the open questions arose in our discussions, in particular, in the improvements of the local subordination, see Section 41. The work on these questions is ongoing and it is expected to lead to a new paper.

Regarding the preparation of the monograph, we collected and discussed a number of relevant papers and drafted a detailed outline. The anticipated main chapters comprise: Bases in Hilbert spaces; Operational Calculus and geometric Riesz basis criteria; Perturbations of harmonic and anharmonic oscillators; Perturbations of multiple eigenvalues and Hill operators; Analysis beyond geometry (conjugations, imaginary potentials, completeness of eigensystem).

Participants

Mityagin, Boris (Ohio State University)
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Chapter 42

Correlation chains and quantum hierarchies (23rit012)

September 24 - October 1, 2023

Organizer(s): Jason Crann (Carleton University), Ivan Todorov (University of Delaware), Lyudmila Turowska (Chalmers University of Technology and University of Gothenburg)

Overview of the Field

The past decade has witnessed a burst of interactions between operator algebras and quantum information theory, some of the prime examples of which are the announced resolution in the negative [6] of the Connes Embedding Problem (CEP) in operator algebras and the Tsirelson Problem in theoretical physics, the refutal of the Strong Tsirelson Conjecture [11] and the answer to the Weak Tsirelson Problem [10]. These questions, open until recently, build on Bell's seminar work from 1964, establishing the distinction between classical and quantum correlation sets for two parties participating in a quantum experiment. Due to these recent efforts, it is now known that the classical model, the quantum (or tensor product) model, the approximately quantum model and the quantum commuting model of quantum mechanics give rise to a strictly increasing chain of correlation types.

Viewing classical no-signalling correlations as classical bipartite information channels, and generalising to quantum channels (where the domain and co-domain are tensor products of matrix algebras), two of the members of the team introduced [12] quantum no-signalling correlation types and established a similar inclusion chain on the quantum level. One of the motivations behind this was to study quantum resource theories in the context of non-local games – cooperative games, played by two spacially separated and non-communicating players against a verifier. The different no-signalling correlation types here are used as strategies of the players, each corresponding to a certain type of quantum resource they may possess. The strict inequalities between the game values (that is, the optimal winning probabilities), when different types of resources are available to the players, is at the heart of the work [6]. Values of games with quantum inputs and outputs have been considered in the past (e.g., [3, 7]), and it is of substantial interest to study non-local games whose inputs and/or outputs are states in the quantum commuting, or approximately quantum, models, referred to in the previous paragraph. Indeed, results from [7] were recently leveraged to give a simplified resolution of the weak Tsirelson conjecture [2].

Recent Progress

A crucial body of techniques employed in the groundbreaking work [6] is that of self-testing quantum systems. Roughly speaking, a bipartite conditional distribution (a.k.a. correlation) is a self-test if there are unique local measurements and a bipartite state (up to local equivalence) which realize the conditional distribution. Non-local games whose winning strategies are a self-test can therefore only be won in a unique fashion, providing a certificate for quantum systems.

Very recently, self-testing of classical no-signalling correlations in the tensor product model was characterised in operator algebraic terms [8]. The motivation being to open new pathways towards understanding the work of [6] in an operator algebraic context (given its refutation of CEP). This, together with related results on the weak and strong Tsirelson conjectures [2, 10, 11], warrants the study self-testing of quantum no-signalling correlations.

Progress in the described area will depend on the successful use of operator space methods and quantum game intuition, and on fluency with the quantum commuting model of quantum mechanics. Members of the team have pioneered both the commuting operator approach to quantum informational questions (see [4, 9]) and a formalised quantum non-local game theory (see [1, 12]). More recently, the team members completed a paper on quantum game values [5].

Scientific Progress Made

During the Research in Teams meeting, a formalism for defining and studying self-testing of quantum no-signalling correlations was established using the theory of operator systems. In that formalism, the tensor product model results of [8] were generalised to the quantum hierarchy, and examples of self-testing in that hierarchy were proposed.

A suitable notion of local isometry in the commuting operator framework was also established, together with a related local dilation order. These were used to characterise self-testing in the commuting operator model for both classical and quantum no-signalling correlations, simultaneously completing an avenue left open from [8], and providing a new approach to constructing infinite-dimensional self-tests in the commuting operator framework, which is an open problem.

Outcome of the Meeting

The results of the meeting will produce a novel addition to the growing literature on self-testing, opening many new avenues of scientific exploration at the interface of operator algebras, quantum entanglement, and quantum foundations. We anticipate having a paper ready for submission in early 2024.

Participants

Crann, Jason (Carleton University)
Todorov, Ivan (University of Delaware)
Turowska, Lyudmila (Chalmers University of Technology and University of Gothenburg)

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Chapter 43

Mathematics Teaching & Learning in Rural Tanzania (23rit013)

October 15 - 22, 2023

Organizer(s): Florence Glanfield (University of Alberta), Joyce Mgombelo (Brock University)

GRATITUDE

We express our gratitude for the support of BIRS for the 7 days and for the support of the Centre for Mathematics Science and Technology Education, Faculty of Education, University of Alberta in order for our Research Team to meet together. The team used 6 days at BIRS due to travel arrangements; we arrived on Sunday October 15 and departed Saturday October 21, 2023.

PURPOSE OF RESEARCH IN TEAMS MEETING

The purpose of the one-week research team working session was to make significant progress on a book manuscript that explores the ways in which a Global Affairs Canada development project was designed to build capacity for primary school mathematics teaching and learning in rural areas in Tanzania. The book outlines the ways in which the project was designed and implemented, and reframes how sustainability of such international development projects can be re-conceptualized in terms of the ways in which participants continue to live and enhance their mathematical, professional, and personal learnings.

OBJECTIVES OF THE MEETING

The objectives for the week: concentrated time for review of data from project baseline and endline studies, data from post-project studies, and to make significant progress on a book manuscript.

OUTCOMES OF THE MEETING

This time together focused our work on the data we collected from our studies in relation to how sustainability of such international development projects can be re-conceptualized in terms of how participants continue to live and enhance their mathematical, professional, and personal learnings. The week long meeting time provided the team an opportunity to develop a shared understanding of the audience for the book; revisit data collected and reports written throughout the 5 year project; revisit data collected from post-project data collection; thoroughly discuss table of contents and themes; identify and practice strategies for collaborative writing and reviewing so that the book can be successfully completed; identify potential reviewers and the steps needed to prepare a book proposal; outline a timeline for the work; establish team meeting dates; and draft a table of contents. A description of the proposed table of contents follows.

PROPOSED TABLE OF CONTENTS

PART 1	Will respond to the question, "how was the project planned to build sustained capacity for mathematics teaching in rural and remote communities in Tanzania?"
CHAPTER 1	DESIGN OF THE GLOBAL AFFAIRS CANADA PROJECT - This chapter will focus on the proposed project and will describe the intentions of the Global Affairs Canada (GAC) Project.
CHAPTER 2	EXPERIENCES PRE PROJECT - This chapter will focus on pre-project experiences, prior to the development of the GAC proposal. It will outline the research that had been conducted in Tanzania prior to the development of the GAC proposal and will describe the relationships that were built and the learnings from the experiences.
CHAPTER 3	THEORETICAL PERSPECTIVES - This chapter will focus on the theoretical underpinnings of the work, complex learning systems, Indigenous ways of knowing, being and doing, and a theory of cognition known as enactivism. This will be tied back to chapters 1 and 2 to illustrate the ways that the theories were evident within the proposal and the pre-project experiences.
CHAPTER 4	VALUES & COMMITMENTS - This chapter will conclude part 1 and describe the authors' personal values and commitments in relation to the project; each of the authors were involved in the implementation of the GAC project.
PART 2	EMERGENCE OF HISABATI NI MAISHA/MATHEMATICS IS LIFE: This part will focus on the pragmatics of implementing the GAC project and how it became known as Hisabati Ni Maisha (Mathematics is Life).
CHAPTER 5	PRINCIPLES - This chapter will describe the principles that emerge from the theoretical perspectives that the GAC project implementation team ascribed to - such as embracing emergence; valuing local knowing; valuing multi languages - use of Swahili, English, and different tribal languages; ongoing engagement of project participants.
CHAPTER 6	THE LIVING PROJECT I - This chapter will focus on the organizational, pedagogical, and mathematical dimensions of implementing the project - tying these dimensions back to the theoretical underpinnings, the values, and the principles.

CHAPTER 7	THE LIVING PROJECT 2 - This chapter will focus on the features of the project that were sustained over time in the implementation: e.g. distributed decision making and interaction among participants; the concept of resourcing; and the importance of a unifying slogan / concept.
PART 3	HISABATI NI MAISHA DAIMA (Mathematics is life forever). This section will focus on illustrations of sustained capacity post project implementation.
CHAPTER 8	STATEMENT OF CAPACITY/IMPACT - This chapter will focus on data collected in post project studies to describe the ways that participants have illustrated the impact of the project. We see this notion of capacity / impact in the participants' views as a transition to reframing the notion of sustainability.
CHAPTER 9	SUSTAINABILITY - The chapter will focus on describing sustainability as it is related to through the roles and actions of participants post project and their spheres of influence. This chapter will build on the in depth interviews with select project participants 5 years post project implementation.
CHAPTER 10	ADJACENT POSSIBLE - Evidence of the ways that the project has shaped and continues to influence the Tanzanian mathematics educational system - through the use of project participants and project strategies.
CHAPTER 11	HISABATI NI MAISHA: AGENT OF SUSTAINABILITY - This chapter will draw from chapters 8, 9, and 10 to propose that the implemented project is an agent of sustained capacity for mathematics teaching in rural and remote communities of Tanzania.
PART 4	LESSONS LEARNED - This section will respond to the question, "What lessons did we learn from the development work - what are the key takeaways for sustained capacity?"
CHAPTER 12	This chapter will describe lessons such as mechanisms for building capacity in relation to this project: e.g. the actors involved; the relationship with mathematics; ongoing engagement with and among project participants; and will return to the theoretical perspectives, the values, and the principles described earlier.

Participants

Glanfield, Florence (University of Alberta)
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Chapter 44

Graph Fourier Transform for Samples of Structured Graphons (23rit007)

November 5 -12, 2023

Organizer(s): Mahya Ghandehari (University of Delaware, Department of Mathematical Sciences), Jeannette Janssen (Dalhousie University, Department of Mathematics and Statistics)

Overview of the Field

Due to the large amount of networked data that is being collected, the recent field of Graph Signal Processing (GSP) has attracted the attention of many data scientists (in mathematics and engineering), and has turned into a fast-growing and vibrant research area. In classical signal processing, the Fourier transform and its variants play a crucial role in analysis, modifying, and synthesizing signals. This fundamental concept has been generalized to graph signals using spectral features of the underlying graph [15]. As a result, processing of a graph signal rigidly depends on the underlying network; this is a major drawback, as the underlying graph of a signal may sustain minor variations due to error or the natural evolution of the network. Thus, it becomes vital to establish instance-independent graph signal processing methods.

The theory of graph limits and graphons was introduced only in 2006 [12], and the use of graphons to develop a common scheme for signal processing on large dynamic graphs (i.e. instance-independent graph signal processing schemes) was first proposed in 2020 [14]. A *graphon* is a symmetric measurable function on $[0,1]^2$. Graphons offer a non-parametric approach to network modeling, which is very valuable when studying stochastic networks. Indeed, graphons represent random processes that generate networks, and networks produced by the same graphon have similar large-scale features. For this reason, we think of graphons as the large-scale blueprint of any graphs that they generate. Naturally, to study networks, we only need to study the (underlying) graphon, rather than the individual networks generated by that graphon [2, 11]. For recent applications of graph limit theory in developing novel GSP methods and robustness/consistency analysis of GSP, see for example [1, 3, 10, 13, 8, 9, 16].

Recent Developments and Open Problems

In [14], Ruiz, Chamon and Ribeiro used graphon theory to prove robustness of graph Fourier transform. Namely, they proved that for a sequence of graph signals $\{(G_n, f_n)\}$ that converge to a graphon signal (w, f), the graph Fourier transform of (G_n, f_n) converges to the graphon Fourier transform of (w, f) if the following conditions on w and f are satisfied:

(i) the graphon signal f is bandlimited,

(ii) w is a non-derogatory graphon (i.e. the integral operator associated with w does not have repeated nonzero eigenvalues.)

This important convergence result lays the ground for developing instance-independent graph signal processing, where similarity of graphs is formalized in the context of graph limit theory. We generalized this result in [7], and dropped both conditions which were imposed on w and f, allowing for a convergence theorem applicable to all graphons. The significance of this generalization lies in the fact that many important examples of graphons, including Cayley graphons on non-Abelian groups, have multi-dimensional eigenspaces. In order to obtain the general convergence result, we refined the definition of the graph/graphon Fourier transform. Inspired by Fourier analysis of non-Abelian groups, we replaced the concept of "Fourier coefficients" by projections onto eigenspaces of the shift operator. The results of [14, 7] pave the way for a compelling initiative in the development of important instance-independent GSP tools (such as preferred orthonormal basis, frames, filters, etc.) built upon the limiting graphon. This RIT workshop was focused on developing suitable GSP frameworks for graphs sampled from graphons with particular spatial structures (e.g. Cayley graphons or spatial graphons). Such graphons are usually suitable models for large networks arising from a variety of applications.

Scientific Progress Made

During our RIT week, we focused on the following two problems:

I. Identifying samples of multi-dimensional spatial graphons. A challenging question regarding networks generated according to spatial models is to uncover the hidden spatial layout of the network. The classical *seriation problem* is the 1-dim spatial layout problem for graphs. The objective of the seriation problem is to obtain a *Robinson ordering* for vertices of a spatial graph, i.e., to order vertices so that adjacent ones are precisely those that are placed close to each other. Although the error-free seriation problem is solved in polynomial time, its perturbed version turned out to be very difficult. The goal of our research programme in this area is to use the theory of graph limits for the approximate seriation of *almost Robinson graphs* and their higher dimensional analogues. One can think of almost Robinson graphs as samples of Robinson graphons. These are graphons whose values increase when moving towards the diagonal along any horizontal or vertical ray. In [4, 6], the PIs and collaborators introduced a function Γ on the space of graphons, and showed that it successfully formalizes the Robinson property. Namely, Γ satisfies the following essential properties on graphons:

1. **(Recognition)** $\Gamma(w) = 0$ *iff* w is a Robinson graphon [4].
2. **(Continuity)** If w is $\|\cdot\|_\square$-close to a Robinson graphon then $\Gamma(w)$ is close to 0 [4].
3. **(Stability)** If $\Gamma(w)$ is close to 0 then w is $\|\cdot\|_\square$-close to a Robinson graphon [5].

This is the first known function that allows the recognition of samples of Robinson graphons. That is, for a dense graph sequence $\{G_n\}$ converging to a graphon w, we have $\Gamma(w) \to 0$ *iff* the graphon w is Robinson.

During our stay at BIRS, we managed to extend our 1-dim results to higher-dim cases as follows. Firstly, we obtained a suitable generalization of the concept of 1-dim spatial graphs to higher dimensions. We call this new class of graphs the *spatial graphs on* $([0,1]^n, \|\cdot\|_p)$. Next, we addressed the following central question:

> *Estimate how close a network G is to being spatial on $([0,1]^n, \|\cdot\|_p)$, and find a network R that is spatial on $([0,1]^n, \|\cdot\|_p)$ and approximates G.*

To handle this question, we follow the same philosophy as in the 1-dim case. Namely, we identify the appropriate definition of *n-dim spacial graphons* with respect to norm $\|\cdot\|_p$, and define a graphon parameter Λ that acts as a gauge of spatiality on $([0,1]^n, \|\cdot\|_p)$; i.e., a function that *recognizes* graphons spatial on $([0,1]^n, \|\cdot\|_p)$, is *continuous* on the space of graphons, and provides a *stable* measurement of the spatial property. To be precise, we define Λ as follows:

Definition. Let $w : [0,1]^n \times [0,1]^n \to [0,1]$ be a graphon. For $0 \le d \le 1$, define $U(d) := \{(X,Y) \in \mathbb{R}^n \times \mathbb{R}^n : \|X-Y\|_p \ge d\}$ and $L(d) := \{(X,Y) \in \mathbb{R}^n \times \mathbb{R}^n : \|X-Y\|_p \le d\}$. Define

$$\Lambda(w) = \sup_{0\le d\le 1} \sup_{\substack{A_1\times A_2 \subseteq U(d) \\ B_1\times B_2 \subseteq L(d), \\ |A_1\times A_2| = |B_1\times B_2|}} \left[\iint_{A_1\times A_2} w - \iint_{B_1\times B_2} w \right].$$

Roughly speaking, the parameter Λ takes the largest amount of average violation to the definition of graphon spatiality over blocks (rather than points). Using this parameter, we propose a method for obtaining spatial approximations of graphons. The spatial approximation of a given graphon w will be a graphon τ_w that is spatial on $([0,1]^n, \|\cdot\|_p)$, and is defined using certain block averages of w where the block size is informed by $\Lambda(w)$. We summarize our results below.

New results. Let $1 \le p \le \infty$, and $w : [0,1]^n \times [0,1]^n \to [0,1]$ be a graphon. Then,

(i) **(Recognition)** w is spatial on $([0,1]^n, \|\cdot\|_p)$ a.e. if and only if $\Lambda(w) = 0$.

(ii) **(Continuity)** Λ is continuous with respect to cut-norm.

(iii) **(Stability)** There exists $\alpha > 0$ such that for any graphon $w : [0,1]^n \times [0,1]^n \to [0,1]$, there exists a graphon τ_w, called the spatial approximation of w on $([0,1]^n, \|\cdot\|_p)$, satisfying

$$\|w - \tau_w\|_\square \le \Lambda(w)^\alpha.$$

The significance of (iii) lies in the fact that the error of approximation is bounded in terms of cut-norm, which is the suitable norm when studying large or dynamic networks. Luckily, it turns out that the approximation τ_w is easy to formulate. However, the proof of stability is rather technical, as it needs to combine careful counting arguments with delicate norm estimates. Our results pave the way for addressing the question of how to describe almost spatial graphons in dimension n. To be more precise, we plan to prove a result of the following type:

(Recognition of samples) Suppose $\{G_n\}$ is a sequence of graphs converging to a graphon $w : [0,1] \times [0,1] \to [0,1]$. Then we have the following:

$$\exists \text{ meas.-pres. } \phi \text{ s.t. } w^\phi \text{is spatial on } ([0,1]^n, \|\cdot\|_p) \quad \Leftrightarrow \quad \exists \text{ labeling of } V(G_n) \text{ s.t. } \Lambda(G_n) \to 0.$$

II. Developing signal processing on almost–structured graphs. We dedicated a portion of our time at BIRS to the question of developing GSP methods on samples of spatial graphons. In particular, we investigated how the Robinson structure of the underlying graphon can be exploited to guide the graph signal processing of its samples. This line of research ties together our earlier work in [7] with our new result regarding the recognition of near-spatial networks. Taking the underlying structure of the graphon into account, we proposed new methods for developing signal processing tools (such as filters and frames) for the class at hand. This work is in preliminary stages, and we plan to pursue it in near future.

Outcome of the Meeting

We found our time in BIRS extremely fruitful and productive. We made a substantial progress extending our earlier results on 1-dim spatial graphs/graphons to the multi-dimensional case. In addition, our new results allow the use of any L^p norm for $1 \le p \le \infty$. Currently, we are in the process of writing a paper containing our new results, which we plan to submit to a high-quality journal for publication. In addition, we initiated a new project regarding developing new GSP methods on spatial graphons. Finally, we used our time at BIRS to make further progress on a book manuscript, which will be published by *Applied and Numerical Harmonic Analysis (ANHA)* book series in 2025.

Participants

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Chapter 45

Quantum Damour Equation (23rit020)

November 19 - 26, 2023

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Overview of the Field

Physics on null hypersurfaces is rich of unusual features. Gravity induces constraints on a null hypersurface, which are the projection of the equations of motion. These are the celebrated Raychaudhuri [1] and Damour [2] equations. They describe how the gravitational data on a null hypersurface are constrained as time evolves. In [3], we focused on the Raychaudhuri constraint and proved 3 main results. First, we showed that it can be recast fully intrinsically as the conservation law for a Carrollian stress tensor, reviewing and ameliorating the membrane paradigm [4]. Inspired by [5, 6], we secondly constructed the phase space and charges, proving in particular how the various symplectic sectors of the theory intertwine together in the symplectic structure and Poisson brackets. In particular, the gravitational data induced on a null hypersurface can be conveniently split into a spin-0, spin-1, and spin-2 symplectic pairs, where "spin" here stands for the number of indices of a field on a cut. Lastly, we introduced the notion of dressing time, which is a particular clock with which we dressed the symplectic variables. This time is conjugated to the Raychaudhuri constraint. In a forthcoming paper [7], we will further report on the Raychaudhuri equation, and how quantization of gravity on null hypersurfaces is dramatically different from the quantization on a time- or space-like hypersurface. We regard this as a promising avenue of investigation, where the unexplored and surprising features of gravity on null hypersurfaces could help us unveiling a controlled quantization of the gravitational phase space.

The dynamics induced by the Raychaudhuri and Damour equations inherently follows the split into spin-0, 1, and 2 data. The spin-2 data encode the radiative degrees of freedom sourcing the geometric spin-0 and spin-1 fields. The dynamical equations display a similar pattern: the Raychaudhuri equation dictates how the spin-0 data evolve due to radiation and external matter, while the Damour constraint expresses how the spin-1 data evolve due to these effects. In our analysis prior to this research group meeting, we have consistently restricted our attention to the spin-0 evolution only, that is, to the Raychaudhuri constraint. Clearly, a step further in this direction requires to include the Damour constraint, and thus the spin-1 sector, in our analysis. This presents two challenges: the first one is to study the classical phase space, and dressing apparatus, while the second one is the quantization of the phase space including the Damour symplectic data. It was the primary purpose of our BIRS research group to address these two challenges.

Scientific Progress Made

We made progress in understanding how to incorporate the spin-1 data in the phase space.

We first showed how to parameterize the phase space, such that the three sectors (spin-0, 1, and 2) appear in the canonical symplectic potential on the hypersurface. We then focused on the freedom to shift the (Ehresmann) connection used in the geometric description of null hypersurfaces [8]. We demonstrated that this is indeed a freedom: the stress tensor and equations of motion are independent of this shift. Importantly, we then proved that such a freedom is pure gauge: the charge associated to this transformation identically vanishes. This allows us to use this gauge transformation to couple with the hypersurface diffeomorphisms, such that the Ehresmann connection does not change on phase space. This is useful to introduce a universal notion of time, such that we can construct the Poisson brackets of all the gravitational data. We then discussed how to introduce not only a dressing time but also a dressing shift, such that both the Raychaudhuri and Damour constraints appear in the symplectic structure.

We secondly addressed the challenge of quantization. We briefly touched upon this more far-reaching goal, studying how the brackets can be canonically quantized, and the consequences on the transverse charges, which are intimately related to the Damour constraint. Setting up a perturbative semi-classical scheme brings various choices and a certain level of arbitrariness that we carefully studied. In particular, we focussed on understanding under which conditions the various sectors of the theory can be disentangled and studied separately.

Outcome of the Meeting

We plan to report in two separate manuscripts the results and ideas discussed in this research group.

The first manuscript will collect the classical results: the introduction of the shift symmetry, the analysis of the complete phase space, and the construction of the Poisson brackets including the spin-1 data. It will then display the dressing apparatus, such that the Raychaudhuri constraint is conjugated to the dressing time, while the Damour constraint is conjugated to the dressing shift. The outcome will be a complete description of the classical phase space of gravity induced to a arbitrary null hypersurface. It will in particular encompass horizons, but also more general null hypersurfaces, characterized by non-trivial expansion, shear, and angular momentum.

The second manuscript is more in an embryonal phase. It will discuss aspects of the quantization of the phase space on null hypersurfaces. An important part of it will be devoted to the semi-classical approach. It will then present a more systematic discussion of what we intend to call ”mesoscopic quantum gravity”, which is a scale between semi-classical gravity and more top/down approaches to quantum gravity, such as string theory or loop quantum gravity. At this mesoscopic scale we maintain contact with classical aspects and control over symmetries, but avoid quantum field theoretical divergences arising without imposing the gravitational constraints.

Participants

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