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Contents

Five-day Workshop Reports

Chapter 1

Advances in Computational Scattering (06w5065)

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Scattering is the study of the interaction of waves with obstacles. These obstacles could be anything from gratings, to tumours, to ships; the waves could be electromagnetic, elastic, or acoustic. This is a very well-established field of study, mathematically, but only a limited number of exterior scattering problems can be solved analytically.

In recent years, many engineers, computational scientists and numerical analysts have investigated numerical algorithms to simulate scattering, including (but not limited to) the use of integral equations, finite element methods, series methods, geometrical optics, absorbing layers and spectral methods. These developments have helped to make computational scattering algorithms indispensible in several industries for design purposes. Many deep mathematical questions have also been raised as a consequence of this development. The overarching principle and the central challenge in computational scattering is to approximate the scattered wave as accurately and efficiently as possible. Indeed, one may identify the major open problems in the field as the development of high-frequency, high-accuracy algorithms; efficient and accurate absorbing boundary conditions; and preconditioners for discretizations of exterior scattering problems. Regardless of the specific algorithms one may use to study scattering, these issues must be confronted head-on.

This workshop aimed to bring together experts in computational scattering with a view to cross-fertilization and communication. The format included a few overview-style talks each day, followed by informal discussion periods. These discussions were particularly important since the academics working in computational scattering appear to be evolving a subject in seemingly parallel directions, without much interaction. The hope was that participants would learn about other techniques being used to study exterior scattering, discuss common issues and open problems, and hopefully form cross-discipline collaborations.

The workshop was successful in achieving many of its goals. The talks provided an exciting snapshot of the current state-of-the-art in computational scattering. This forum provided a particularly suitable place for students entering the field to gain an overview of the subject area; one of the workshop highlights was the number of informal introductory lectures given by eminent mathematicians to the graduate students.

Computational scattering: basic ideas and workshop themes

Computational scattering theory is the study of algorithms to approximate wave-obstacle interactions. The governing equations might be the scalar wave equation (acoustic scattering), Maxwell's equations (electromagnetic scattering) or nonlinear PDE such as those arising in gravitational waves; the obstacles under study are either bounded in space, or are akin to diffraction gratings. A typical assumption is that all nonlinearities are compactly supported in space, allowing for simpler physics far from the obstacle. One may be interested in the propagation of the resulting scattered wave in a waveguide or in all of space. The goal is to compute approximations to this scattered wave, given information about the incident wave, the obstacle, and the medium of propagation. Necessarily, one must also study the mathematical properties of the approximation procedure, which in turn cannot be seperated from the PDE at the continuous level.

Frequently scattering problems are posed in a frequency-domain formulation. Looking for time-harmonic solutions, $F(x,t) = e^{i\omega t} f(x)$, one is led to a time-independant PDE. We illustrate the idea in the context of the frequency-domain formulation of Maxwell's equations. Let Ω be a bounded region in $\mathbb{R}^3$. A perfect conductor occupies the region Ω. By taking the Fourier transform of Maxwell's equations (Ampere's law and Faraday's law) we are lead to the system

$$i\omega\epsilon E(x) + curl\, H(x) - \sigma E = 0, \qquad x \in \mathbb{R}^N \setminus \Omega \tag{1.1}$$

$$- i\omega\mu H(x) + curl\, E(x) = F, \qquad x \in \mathbb{R}^N \setminus \Omega. \tag{1.2}$$

Here ϵ and μ are respectively the permittivity and permeability of the medium, which may vary in space. The transform variable is ω, while the (rescaled) fields E and H correspond to the Fourier transforms of the electric and magnetic fields, and F is a source term including information about applied current densities. To close the system, one needs to prescribe boundary conditions on the obstacle Ω, and some conditions at spatial infinity. The latter conditions are referred to as the Silver-Müller conditions. One can eliminate the magnetic field H, and rewrite the system above as a second-order problem. Since we are considering scattering from a perfect obstacle, we can write the Silver-Muller conditions in terms of E^s, the scattered field; the total field $E(x) = E^s(x) + E^i(x)$ for some prescribed incident field E^i. We are finally lead to the following system:

$$\nabla \times (\mu^{-1} \nabla \times E) - k^2 \epsilon E = F, \qquad x \in \mathbb{R}^N \setminus \Omega \tag{1.3}$$

$$E = E^i + E^s, \qquad x \in \mathbb{R}^N \setminus \Omega \tag{1.4}$$

$$E \times \nu = 0, \qquad x \in \partial\Omega \tag{1.5}$$

$$\lim_{r \to \infty} r((\nabla \times E^s) \times \hat{r} - i\omega E^s) = 0, \qquad r \to \infty. \tag{1.6}$$

Even for this simple model, several observations can be made. First, unless Ω is a very simple shape, and the material coefficients are constant, we cannot analytically compute the field $E(x)$, and must resort to numerical simulation. Second, the field $E^s(x)$ occupies an infinite computational region, $\mathbb{R}^3 \setminus \Omega$. This region must be appropriately truncated in order to allow for computations; any truncation strategy must be analyzed for its effect on the accuracy of approximations. While the governing PDE has at most one solution thanks to the Silver-Muller condition, the operator $\nabla \times (\mu^{-1} \nabla \times \cdot) - k^2 \epsilon \cdot$ is not coercive; it is well-known that in the interior of cavities, Maxwell's equations permit resonances. Any truncation strategy must take this possibility into account. The choice of boundary condition one prescribes to truncate the region thus plays an important role in the success of the numerical method. Depending on the nature of the boundary condition employed, we refer to them as non-reflecting, absorbing, perfectly matched layers, etc. For the purpose of this meeting we referred to such conditions collectively as **artificial boundary conditions**. **Xavier Antoine, Thomas Hagstrom, Eric Luneville, David Nicholls, Nilima Nigam and Sergey Sadov** spoke on this topic.

Third, the wave number k sets a length-scale called the *wavelength* $\lambda \equiv \frac{1}{k}$. As the wave number k increases, this characteristic scale, becomes smaller. For instance, to accurately resolve a periodic function in $1-D$, anywhere between 3-10 mesh points per wavelength are required. Now consider the scattering of a 30 Gigahertz incident wave, such as those used in high-speed microwave radio relays. This sets a length scale at the order of millimeters. If the scattering obstacle has a radius 10 meters, with complicated geometrical features or electromagnetic properties, one requires billions of mesh points even just to resolve the scattered wave in a thin layer around the obstacle. A major goal is to avoid resolving such small scales while performing computations involving large, complex obstacles; this is an *extremely* challenging task. Indeed, off-the-shelf packages do not suffice in most such situations, and much work is being done in developing novel discretization methods suited for scattering problems. **Jean-Davide Benamou, Anne-Sophie Bonnet-Ben Dhia, Oscar Bruno, Simon Chandler-Wilde, Joseph Coyle, Leszek Demkowicz, Paul Martin, Peter Monk, Jie Shen, Symon Tsynkov and Tim Warburton** spoke on the topic of **accurate discretizations and high-frequency calculations**.

Next, since the governing PDE is not positive-definite, one should not expect any linear system arising from a discretization process to be positive-definite. In practice one can imagine very large linear systems which need to be solved iteratively; the development of **preconditioning strategies** would significantly impact the size of problems one can attack. **Robert Beauwens, Annalisa Buffa, Matthias Maiscak, and Jean-Claude Nedelec** described recent work in the area.

Finally, the inverse problem related to this model- determining the location of the obstacle, and/or the permittivity and permeability of the medium near the obstacle based on the observed scattered wave - is an ill-posed problem. Since the invention of radar, scientists and engineers have striven not only to detect but also to identify unknown objects through the use of electromagnetic waves. Any success in this direction has potentially huge impact in application areas from medical imaging to seismic exploration. Current progress was reported by **David Colton, Fioralba Cakoni and George Hsiao**

Workshop Themes, recent work and open problems

The workshop was intended to review the current state-of-the-art in computational scattering, and also to discuss future directions for the community to investigate. To focus the discussion, the workshop was organized around three major themes: artificial boundary conditions, high-frequency computations, and preconditioning. Some recent work on inverse scattering was also discussed.

Absorbing and artificial boundary conditions

When finite difference, finite element or spectral methods are used to resolve the scattered wave near the obstacle, the computational region must be restricted to be finite. This truncation is achieved by means of absorbing or exact boundary conditions. These conditions can be implemented in various ways, e.g. by using boundary integral equations, series implementations, or the perfectly matched layer of Berenger. No matter which techniques are used, the goal is to obtain as accurate an approximation to solutions of the original scattering problem, as efficiently as possible. Unfortunately, these are competing requirements. There are significant implementation and/or accuracy issues which remain open problems. The construction of high-accuracy artificial boundary conditions in the time-domain is particularly important for applications. It is, however, a complex endeavour to balance the needs of accuracy in space and time with the requirements of efficiency (memory and computation).

Xavier Antoine reviewed recent developments in the technique of *on-surface radiation conditions* with regards to the challenging problem of simulating high-frequency acoustic and electromagnetic scattering problems. He also discussed the development of accurate and local artificial boundary conditions for smooth geometries and the construction of well-posed and well-conditioned integral equations for the iterative solution of high-frequency scattering problems.

Thomas Hagstrom reviewed the state-of-the-art in the construction, analysis, and application of arbitrarily-accurate radiation boundary conditions for time-domain simulations. Specific topics included: (i.) Experiments with nonlocal boundary conditions employing efficient compressions of the time-domain kernels; (ii.) Reformulated local boundary condition sequences and their use in polygonal domains and stratified and anisotropic media; (iii.) Speculations on potential improvements of the local boundary condition sequences and extensions to inhomogeneous media and nonlinear problems.

In an acoustic waveguide, assumed to be semi-infinite along one propagation axis, one can easily construct from the spectral theory of a simple transverse operator an “exact” transparent condition. More precisely, such a condition is based on an explicit diagonalisation of the Dirichlet to Neuman operator. The situation for Maxwell’s equations is more intricate. Indeed, the operator which associates the electrical field to its derivative (equivalent to a Dirichlet to Neuman operator) is not implemented because the transverse and longitudinal Maxwell operators remains coupled and no explicit diagonalisation may be performed. In a talk by **Eric Luneville**, a new transparent condition in a two dimensional case for the regularized Maxwell equations was proposed. The BC was based on the diagonalization of an operator which involves mixed unknowns, e.g. the coupling of the electric tangential component and the divergence of the electric field, or the coupling of electric normal component and the rotational of the electric field. Unfortunately, this transparent condition requires one to deal with a mixed variational formulation where, for example, the divergence on the transparent boundary appears as a new unknown of the problem. However, this formulation is well-posed and its approximation by Lagrange finite elements is convergent. This approach is an alternative way to other methods such that integral equation or Perfectly Matched Layer techniques. It is of interest to point out that it appears as a theoretical tool in the proof of convergence of PML techniques too. Such transverse decompositions are also related to modal approximation. This approach may also be used for elastodynamic problems. In that case, the spectral theory of the transverse operator is not obvious.

Boundary perturbation methods are among the most classical techniques for approximating scattering returns from irregular obstacles. Despite a history which dates to Rayleigh's calculations in the nineteenth century, their convergence, stability, and capabilities were, for almost a century, misunderstood. The work of Bruno & Reitich not only placed these methods on a secure theoretical foundation, but also provided fast, high-order computational strategies. Subsequent work by **David Nicholls** has further clarified the properties and limitations of these methods, and suggested new algorithms to achieve high-order approximations in a rapid and numerically stable manner. Nicholls gave an overview of these boundary perturbation methods and discussed recent enhancements.

Nilima Nigam presented some recent work on artificial boundary conditions for the scattering of elastic waves from bounded obstacles, including extensions of the boundary perturbation approach of Bruno and Reitich, as well as investigations into an overlapping Schwarz domain decomposition method.

The unique solvability of an exterior Dirichlet problem implies the existence of an operator that maps the Dirichlet data (function on the obstacle boundary) to the normal derivative of the solution (another function on the boundary). The Dirichlet-to-Neumann map thus defined is a boundary pseudodifferential operator of order 1. In 2D problems, the boundary is one-dimensional, usually diffeomorphic to a circle, and the DtN can be exactly (without truncation by order) described by a discrete symbol, which is a function of three parameters: boundary parameter s, Fourier series index (discrete momentum) n, and the wavenumber k. As k goes to infinity, the symbol has a nice asymptotic behaviour uniformly in s and n. This idea was discussed by **Sergey Sadov**, who described a reformulation of this asymptotic property as a microlocal refinement of the Kirchhoff approximation.

High frequency methods and novel discretization techniques

The conditioning and accuracy of most discretization techniques for scattering problems depend crucially on the wave number of the incident wave. In addition, there are algorithms suitable for moderate frequency scattering, and others appropriate for geometrical optics. A key challenge in this field remains the development and analysis of an algorithm which works over a large range of frequencies, and whose performance can be controlled independant of the frequency. In this workshop developments of new high-accuracy methods suitable for a large range of wavenumbers were discussed.

The high frequency asymptotic representation of wavefields (Geometrical Optics in its simple form) is a computationally attractive approach because the discretization is, to a large extent, independent of the frequency. Unfortunately this technique has both theoretical and practical limitations. There has been much work in combining or coupling the usual "frequency aware" (or "full") wavefield and "asymptotic techniques". **Jean-David Benamou** spoke on numerical microlocal analysis, applied to scattering problems. While it is easy to compute a full wavefield representation from its (constructive) asymptotic representation, the opposite extraction (or "analysis") from a given wavefield of its frequency-independent asymptotic representation is far from obvious. He presented a numerical method which, given an analytical or numerical solution of the Helmholtz equation in a neighborhood of a fixed observation point, and assuming that the geometrical optics approximation is relevant, determines at this point the number crossing rays and computes their directions and associated complex amplitudes.

There has been another large body of work on high-frequency methods, which are based on integral equations, high-order integration, fast Fourier transforms and highly accurate high-frequency methods. These can be used in the solution of problems of electromagnetic and acoustic scattering by surfaces and penetrable scatterers — even in cases in which the scatterers contain geometric singularities such as corners and edges. The solvers exhibit high-order convergence, they run on low memories and reduced operation counts, and they result in solutions with a high degree of accuracy. They require, among other tools, accurate representations of obstacle surfaces. A new class of high-order surface representation methods was discussed by **Oscar Bruno**, which allows for accurate high-order description of surfaces from a given CAD representation. These methods are employed in conjunction with a class of high-order, high-frequency methods using integral equations which was developed recently. The talk ended with a description of a general and accurate computational methodology which is applicable and accurate for the whole range of frequencies in the electromagnetic spectrum.

An important aspect of the numerical analysis of scattering algorithms is the precise dependence of their accuracy and conditioning on the frequency. There are many open questions in this direction, some of which have motivated the design of new algorithms. **Simon Chandler-Wilde** gave an overview of recent work on boundary element methods for high frequency scattering problems. He first described what was known about the dependence of the conditioning of boundary integral equations on frequency and on the choice of coupling parameters in combined layer-potential formulations. He next discussed attempts to reduce the number of degrees of freedom by incorporating some of the oscillatory behaviour of the solution in the basis functions used in the boundary element method. His talk contained several open problems.

Often the modeling of the complex physics involved in a scattering problem leads to mathematical challenges. **Anne-Sophie Bonnet-Ben Dhia** described work on acoustic scattering in the presence of a mean flow. This was work motivated by the need to develop noise-reducing technologies for planes, particularly in the neighborhood of the airports. Unfortunately, there exists no satisfactory way to solve the Linearized Euler Equations in the harmonic regime and in unbounded domains; a major effort in this current work involves developing a well-posed model. Bonnet-Ben Dhia's work consists in solving a linearized equation, set on the perturbation of displacement, the so-called Galbrun's equation. An augmented formulation of this process was proposed, which includes a non-local (in space) term, linked to the convection of vortices along the stream lines. This is then combined with a perfectly matched layer to truncate the region.

Finite and spectral elements

The workshop also brought together researchers who used finite element or spectral element techniques in the study of wave propagation. Since the solutions are quite oscillatory, the "standard" strategies are severely limited in terms of efficiency; speakers presented novel discretization techniques which took into account the particular behaviour of scattered waves and which ameliorated some of the difficulties which plague existing techniques.

For scattering by complicated obstacles or in the presence of inhomogenous media, the use of nonuniform meshes can confer many advantages. However, the construction and implementation of hierarchic finite element bases on unstructured tetrahedral meshes poses challenges at the computational and analytical level, especially where a non-uniform order of approximation may be utilized. Enforcing the appropriate conformity properties of the approximation across element interfaces is typically a difficult task in this case, and recent work on this problem was presented by **Joseph Coyle**. He first related the problem to the intrinsic orientation of the edges and faces as well as the global numbering of the basis functions. Observing that an appropriate reordering of the local numbering of the vertices allows any global tetrahedron to be reduced to one of two possible reference tetrahedra that leads the way for the construction of the hierarchic bases where ease of implementation is not sacrificed.

The theory of hp-discretizations for Maxwell problems was reviewed by **Leszek Demkowicz**, who summarized the main points of the projection-based interpolation theory, convergence results for Maxwell eigenvalues and recent results on the existence of polynomial preserving extension operators in H(curl) and H(div) spaces. He then spoke on the subject of goal-oriented hp-adaptivity, presenting an extension of the original, energy-based hp-algorithm and its applications to borehole logging EM simulations. Finally, he discussed the impact of automatic hp-adaptivity in simulations involving the use of PML. The automatic reproduction of "boundary layers" by the hp-adaptivity significantly reduces the tedious design and tuning of PML's.

Although direct scattering problems in cavities and waveguides are typically linear and well-posed, they are difficult to solve numerically because the oscillatory nature of the solution forces the use of large numbers of degrees of freedom in the numerical method, and the resulting linear system defies standard approaches such as multigrid. This is a particular problem at high frequencies when the scatterer spans many wavelengths. In an effort to improve the efficiency of a volume based approach as the frequency increases and to allow the solution of problems at widely different frequencies on a single grid, **Peter Monk** described his recent work in the use of plane waves as a basis for approximating the scattered field. These are used in a discontinuous Galerkin scheme based on a tetrahedral finite element mesh. This method is termed the Ultra Weak Variational

Formulation (UWVF) by its originators O. Cessenat and B. Despres. The use of the Perfectly Matched Layer or Fast Multipole Method to improve the artificial boundary condition needed by the method was also discussed. Interestingly the linear system from the UWVF is easier to solve than the one arising from the finite element method, and this allows a simple parallel implementation of the method. The method has been validated on a variety of problems, and extended to the acoustic-elastic fluid-structures problem.

The use of spectral methods in wave scattering is a very active field of research. **Jie Shen** present an efficient and stable spectral algorithm and their numerical analysis for the Helmholtz equation in exterior domains. The algorithm couples a boundary perturbation technique with a well-conditioned spectral-Galerkin solver based on an essentially exact Dirichlet-to-Neumann operator. Error analysis as well as numerical results were presented to show the accuracy, stability, and versatility of this algorithm.

Recent investigations of the spectral properties of the discrete Discontinuous Galerkin (DG) operators have revealed important connections with their continuous Galerkin analogs. Theoretical and numerical results, which demonstrate the correct asymptotic behavior of these methods and precludes spurious solutions under mild assumptions, were presented by **Tim Warburton**. Given the suitability of DG for solving Maxwell's equations and their ability to propagate waves over long distance, it is natural to seek effective boundary treatments for artificial radiation boundary conditions. A new family of far field boundary conditions were introduced which gracefully transmit propagating and evanescent components out of the domain. These conditions are specifically formulated with DG discretizations in mind, however they are also relevant for a range of numerical methods.

Special techniques

As mentioned earlier, there has been much work recently in the development of specifically tailored techniques for wave scattering problems. Examples of such work include the use of asymptotic formulae derived using classical techniques, and Huygen's principle.

Paul A. Martin provided the first classical derivation of the Lloyd-Berry formula (published in 1967) for the effective wavenumber of an acoustic medium filled with a sparse random array of identical small scatterers. The approach clarifies the assumptions under which the Lloyd-Berry formula is valid. More precisely, an expression for the effective wavenumber was derived, assuming the validity of Lax's quasicrystalline approximation but making no further assumptions about scatterer size. In the limit of vanishing scatterer size it was shown that the Lloyd-Berry formula is recovered. We have also obtained a similar formula in two dimensions. The methods employed should extend to analogous electromagnetic and elastodynamic problems.

Among the well-known challenges that arise when computing the unsteady wave fields is the deterioration of numerical schemes over long time intervals (error buildup) and the unboundedness of the domain of definition. The latter is typical for many applications, e.g., for the scattering problems, when the waves are radiated toward infinity. In the literature, a standard way to deal with the first issue is to increase the order of accuracy (quite independently, paraxial approximations can be employed), whereas the second issue requires truncation of the domain and setting of artificial boundary conditions (ABCs). According to conventional wisdom, exact ABCs for multidimensional unsteady problems are nonlocal not only in space but also in time, and the extent of temporal nonlocality continually increases as time elapses. It turns out, however, that in many cases both types of difficulties can be addressed using a unified approach based on exploiting the Huygens's principle. The propagation of waves is said to be diffusionless, and the corresponding governing PDE (or system) is said to satisfy the Huygens principle, if the waves due to compactly supported sources have sharp aft fronts. The areas of no disturbance behind the aft fronts are called lacunae. Diffusionless propagation of waves is rare, whereas its opposite - diffusive propagation with after-effects is common. Nonetheless, lacunae can still be observed in a number of important applications, including acoustics and electromagnetism. The key idea of using lacunae for computations is that any finite size region falls behind the propagating aft front, i.e., right into the lacuna, after a finite interval of time. In other words, any given feature of the solution will only have a finite predetermined lifespan on any fixed domain of interest. By incorporating these considerations into a numerical scheme, one can make its grid convergence uniform in time. The same considerations facilitate design of exact unsteady ABCs with only fixed and limited (non-increasing) extent of temporal nonlocality. At the workshop, **Symon Tsynkov** described recent progress made in constructing the lacunae-based numerical schemes for the d'Alembert equation, as well as for the linearized Euler equations and the Maxwell equations. He also discussed different physical models from the standpoint of existence of the lacunae and showed in some interesting cases that are technically speaking diffusive, e.g., the propagation of electromagnetic waves in dilute plasma, lacunae can still be identified in the solutions in some approximate sense.

Preconditioning strategies

The efficient solution of the linear systems obtained as a consequence of the discretization of exterior scattering problems is an open problem, since these systems are typically dense. Canned preconditioning techniques have been rather unsuccessful. Part of the difficulty in the preconditioning of frequency-domain problems lies in the indefinite natue of the associated linear systems. **Robert Beuwens** gave an overview of the principles behind complex iterative schemes, used to solve large sparse linear systems of equations. He discussed two kinds of methods, which are often used in combination: preconditioning methods and convergence acceleration methods. Preconditioning methods aim at building an approximate system close to the one to be solved but which is inexpensive to solve both in terms of computing time and memory requirements. Convergence acceleration methods are used to transform slowly converging sequences or even diverging sequences into rapidly converging sequences. Acceleration techniques popular today include the polynomial acceleration or Krylov subspace methods as basic blocks in the building of elaborate preconditioners. The talk concluded with recent developments concerning multi-level and recursive ordering methods on the one hand and the parallelization of preconditioned Krylov subspace methods on the other hand.

Preconditioning techniques have recently been developed for boundary integral equations used in this context; there is a pressing need for a systematic preconditioning strategy for other algorithms as well. The use of Calderon projections in the study of integral equations suggests the use of operator-level preconditioners, where the continuous problem is preconditioned by application of suitable pseudo-differential operators. Discretization is performed only after this preconditioning. The electric field integral equation (EFIE) arises in the scattering theory for harmonic electromagnetic waves. **Annalisa Buffa** described an optimal preconditioning technique for the conforming Galerkin approximation of the EFIE via Raviart-Thomas finite elements. At the continuous level, Calderon formulas provide an explicit representation of the inverse operator of the electric field integral operator up to compact pertubations. A stable discretization of the Calderon formula was presented, and then an optimal preconditioner for the linear system which arises from the Galerkin discretization of the EFIE was shown.

Jean-Claude Nedelec also spoke on preconditioning the Maxwell integral equations using Calderon identities

Starting from the well known combined boundary integral formulations due to Brakhage/Werner and Burton/Miller **Olaf Steinbach** reviewed existing modifications which are needed for the numerical analysis in the correct function spaces. While most of the proposed modifications rely on a compactness argument, the current work involved an alternative approach, which leads to a stable approximation scheme.

The symmetric coupling of finite elements and boundary elements for electromagnetic problems results in highly ill-conditioned linear systems of equations. **Matthias Maischak** presented a block-preconditioner for the GMRES method which is based on domain decomposition methods applied to the “FEM-part” and the “BEM-part” separately and analysed the eigenvalue distribution of the preconditioned system. It was shown that the efficiency of this method only depends on the ratio of coarse grid mesh size and the overlap. Numerical examples for the eddy-current problem underline the efficiency of this method.

Inverse problems

While describing important applications of scattering theory, one is naturally lead to consider inverse problems. Important inverse problems include the reconstruction of biologically relevant information from medical tomography data, the location of hydrocarbons based on seismic imaging information, and the detection of mines. Mathematically, inverse problems in scattering pose severe challenges due to their ill-posed nature.

Fioralba Cakoni spoke on mathematical and computational aspects of inverse Electromagnetic Scattering Problems, specifically as it pertains to synthetic aperture radar (SAR). SAR suffers from limitations arising from the incorrect model assumptions which ignore both multiple scattering and polarization effects. The main theme of this talk was the use of a qualitative method, the linear sampling method, to solve inverse electromagnetic scattering problems. Cakoni first introduced the main mathematical ideas of the linear sampling method for the simple case of electromagnetic scattering by a perfect conductor. She next showed how to use the technique to find both the shape and the surface impedance of a partially coated perfect conductor without knowing a priori whether the obstacle is coated. In the case of an inhomogeneous background, she presented a new method which avoids the need to compute the Green's function of the background media. Numerical examples showed the validity of this approach.

Some recent developments in inverse scattering were described by **David Colton**, who also discussed a major open problem in the field.

Scattering theory in periodic structures has many applications in micro-optics. The treatment of the inverse problem, recovering the periodic structure or the shape of the grating profile from the scattered field, is useful in quality control and design of diffractive elements with prescribed far field patterns. **George Hsiao** discussed an inverse diffraction grating problem to recover a two-dimensional periodic structure from scattered waves measured from above and below the structure. The problem was reformulated as an optimization problem including regularization terms. The solution is obtained as the minimizer of the optimization problem, where the objective function consists of three terms: the residual of the Helmholtz equation, the deviation of the computed Rayleigh coefficients from the measured data, and the regularization term to cope with the ill-posedness of the inverse problem. He then described solvability and parameter sensitivity of the algorithm, and showed some numerical experiments validating the approach.

Presentation Highlights

The workshop brought together experts in a variety of computational techniques, with a focus on exterior scattering problems. In addition, graduate students and postdoctoral fellows were invited, to establish connections with established mathematicians. To optimize research interaction, several different activities were planned:

- 30 minute lectures by experts
- Poster presentations by graduate students and postdocs: the posters were on display for the duration of the workshop in the coffee room area. Since this area was heavily utilized during breaks, the students and postdocs got several opportunities to discuss their work with other mathematicians. We actually recommend this format for poster sessions for future workshops; the younger mathematicians were very appreciative of the extended opportunity to showcase their research.
- Two panel discussions: At the end of Day 2 and Day 4 of the workshop, panel discussions were held on integral equation methods and finite element methods respectively. These lively discussions included presentations of open problems, discussions of key challenges and suggestions for future research.
- Informal lectures: several expert mathematicians volunteered to give informal lectures to the graduate students. Particularly given the range of mathematical expertise at the workshop, this was a very valuable opportunity for the students.

Poster presentations

- **Binford, Tommy** (Rice University)
  Title: *Experiments with a Dirichlet to Neumann Map for High Order Finite Elements*
  For electromagnetic scattering problems, the number of degrees of freedom to acheive a desired accuracy can be prohibitively large depending on the domain. Artificial boundary methods are a powerful tool for treating radiation conditions while preserving the physical behavior with fewer degrees of freedom. Work by Nicholls & Nigam on Dirichlet to Neumann maps has provided a method of handling the radiation condition for perturbed simple geometries such as a circular boundary. In this poster, Binford showed experiments where one applies a high order finite element method in conjunction with a Dirichlet to Neuman map to solve Helmholtz' equation for a right circular cylindrical scatterer with different perturbations of a circular artificial boundary away from the scattering object.
- **Ecevit, Fatih** (Max Planck Institute)
  Title: *High-frequency asymptotics and convergence of multiple-scattering iterations in two-dimensional*

scattering problems
One of the main difficulties in high-frequency electromagnetic and acoustic scattering simulations is that any numerical scheme based on the full-wave model entails the resolution of the smallest wavelength. It is due to this challange that simulations involving even very simple geometries are beyond the reach of classical numerical schemes. Ecevit presented an analysis of a recently proposed integral equation method for the solution of high-frequency electromagnetic and acoustic scattering problems that delivers *error-controllable solutions in frequency-independent computational times*. Within single scattering configurations the method is based on the use of an appropriate ansatz for the unknown surface densities and on suitable extensions of the method of stationary phase. The extension to multiple-scattering configurations, in turn, is attained through consideration of an iterative (Neumann) series that successively accounts for multiple reflections. Here we derive a high-frequency asymptotic expansion of the successively induced currents in this latter procedure and, within this context, we derive an estimate for its convergence rate. As we show, this rate is explicitly computable and it depends solely on geometrical characteristics; in particular, it is independent of the specific incidence of radiation. Numerical results confirm the accuracy of this high-frequency estimate for the case of several interacting structures.

- **Han, Young-Ae** (Caltech)
  Title: *A Continuation Method for high-order parametrization of arbitrary surfaces*
  In this poster, a super-algebraically convergent technique to approximate complicated surfaces in 3-D using locally smooth functions was presented. The method accurately renders geometric singularities such as edges and corners. The approach was based on continuing each smooth branch of a piecewise-smooth function into a new function which, defined on a larger domain, is both smooth and periodic. These “continuation functions” have Fourier coefficients that decay super-algebraically, and thus result in high-order approximations of the given function throughout its domain of definition. Among other benefits, this approach resolves the Gibbs phenomenon. Examples showing the success of this strategy were also shown.

- **Kurtz, Jason** (U. Texas at Austin)
  Title: *Fully-Automatic hp-Adaptivity for Acoustic and Electromagnetic Scattering in 3D*
  Two popular strategies for studying exterior scattering problems are coupled FEM-PML or FEM-Infinite element methods. This work describes an adaptive hp refinement algorithm for both strategies which yields exponential convergence in the energy norm. The hp-adaptive method is ideally suited for scatterers with geometric singularities and/or for discretizations truncated by a perfectly matched layer. Three crucial implementation issues were addressed in the poster: namely, fast integration of element stiffness matrices, a domain-decomposition multi-frontal solver, and a “telescoping” solver for a sequence of locally nested meshes. Computational results were presented for both PML and infinite element truncations.

- **Sifuentes, Josef** (Rice University)
  Title: *GMRES performance in integral equation methods for scattering by inhomogeneous media* Discretizations of integral equation techniques lead to linear systems which are solved iteratively (typically using GMRES). The number of iterations increases considerably with wave number.The poster described recent investigations into the wave-number dependence of the spectrum of the discretized integral operator. This line of research will eventually lead to better preconditioning strategies.

Open problems and future directions

One of the big successes of this workshop was due to the scientific generosity of the participants, who not only provided clear expositions on their work, but also detailed open problems and future directions they believed to be of significance. Some of these were reiterated during the two panel discussions (summarized below) and informal talks.

Over the course of the workshop, the participants identified some major directions for future research. Problems which need theoretical and analytical work include careful investigations into wave-number dependent error analysis of existing algorithms, and preconditioning strategies. At the computational level, the community felt the need to develop benchmark problems to test algorithms, and demonstrate the effectiveness of computational strategies on scattering from complex structures and physics.

Another concern which was shared was the overwhelming effort required in meshing complex geometries. It is estimated that of the total time spent on studying scattering problems in an engineering context, developers spend around 80% of their time on describing the geometry and implementing meshes, and only 20% on the actual simulation. While no consensus emerged on how best to deal with this problem, it became clear that for newer algorithms to become widely applicable, they had to account for this bottleneck.

To get a full flavour of the range of open problems suggested, we encourage the interested reader to look at the website:

`http://www.math.mcgill.ca/nigam/BANFF/front.php`

This website contains many of the talks, and links to participant websites and papers.

Integral equation techniques

- It is well-known that most numerical methods for scattering problems require a mesh which can resolve the incident wave. This means, in particular, that the size of the mesh grows with the wave number k. However, in some situations this may not be necessary. For example, the scattering of a high frequency wave off a convex smooth obstacle should not require such high numerical resolution. An open problem is to characterise the scattering problems for which O(1) discretizations are possible as $k \to \infty$. Does the convexity of the scattering object play an important role, is smoothness of material properties crucial?

- Integral equation techniques rely on the fast and accurate quadrature of oscillatory kernels. This poses interesting problems in the theory of quadrature, not just restricted to scattering. For example, how should one deal with oscillatory integrals, particularly in complex 3-D geometries, in O(1) computational time, without sacrificing accuracy?

- A major open area of investigation remains the hunt for good preconditioners in the twin limit as mesh size $h \to 0$ and wave number $k \to \infty$.

- Geometrical optics is a powerful tool for studying very high frequency scattering. While developing numerical algorithms suitable for a range of frequencies, it would be desirable to incorporate ideas from geometrical optics to deal with the high frequency range. An application would be, for example, acoustic muffling problems, where an integral equation solver may be appropriate for the object, and geometrical optics suffices to capture the large-scale and atmospheric effects.

- An important open area in the numerical analysis of scattering algorithms concerns estimates (above and below) of condition numbers for integral equations for general objects. Some results are known on simple geometries, but these need to be extended.

- A specific question in the numerical analysis of integral equation techniques is whether the Galerkin method is stable for classical Brakhage-Werner integral equations on Lipschitz domains.

- The error analysis of the classical Brakhage-Werner integral predicts a condition number which grows as $O(k^{1/3})$ as $k \to \infty$. This is not reflected in actual computations for a large class of scatterers. Why?

- There exist a profusion of algorithms for scattering, suitable in certain specific frequency regimes. The workshop participants agreed that a key goal is to establish stability for any numerical method uniformly in wave number k.

- Much is known about the physics of wave propagation and interaction in anisotropic and inhomogenous materials. Rather than look for a preconditioner *ab initio*, a fruitful direction of research would involve using knowledge of the physics to design optimal preconditioners.

- While describing a scattering problem in terms of integral equations, one has several choices. Some integral equation formulations are more suitable for computation than others; exploiting this requires a detailed understanding of the spectral properties of various integral equations.

- A valuable contribution from the community would be a set of non-trivial computational examples, showing the efficacy of integral equation based methods. At present, open-source software for boundary integral equations is not as well-developed as it's finite element analog.

- There are some situations where integral equation methods are both natural and more efficient than volumetric discretizations. An important project would be to classify the problems on which one should use integral equation methods.

- Domain decomposition methods are powerful tools which enable parallelization of computation, particularly for large obstacles. Communication between domains occurs via Stekhlov-Poincaré maps, which are accurately described in terms of integral operators. More investigation is needed into optimal combinations of integral equation methods and domain decomposition techniques.

- The use of integral equations of the second kind to solve exterior scattering problems is popular, in particular since the integral operators involved are not singular. Standard boundary element techniques do not always seek approximations in the correct Sobolev spaces. Indeed, integral equation techniques are quite versatile, and performing discretizations appropriately will allow for a wider range of problems to be solved.

- Integral equation methods lead to dense matrices; a lot of attention has been paid recently to operator-level preconditioning to improve the computational efficiency of these methods. Calderon projections offer many possibilities in terms of reformulations of integral equations; these need to be further examined for their computational suitability. Upon preconditioning with these projections, an integral equation of form $Bx = F$ can be transformed to one of type

  $$ABx = (I - K)x = AF.$$

  A closer theoretical investigation of the compact operator K is required for various projection methods. In particular, what is the behaviour of these projections at the discrete level, in the presence of meshes with high aspect ratios?

- At the discrete level, both storage and efficient computation of the linear systems arising from integral equation methods poses challenges. One fruitful direction of work which needs more development is the use of algebraic approximation methods and hierarchical matrices in this context. It is, for example, not obvious how one should precondition a system arising from the use of an adaptive mesh.

Volumetric discretization techniques and artificial boundary conditions

- Multigrid techniques for scattering require that the coarsest grid resolve the wavelength of the incident wave. This is too severe a restriction for this method to be practical at high frequencies; a variant of a multilevel technique which is genuinely independent of frequency is required. Similarly, while domain decomposition techniques are gaining popularity, the dependence of their performance on wavenumber is not clearly described.

- Scattering problems which involve wires or thin structures are notoriously difficult to solve, but applications involving wires and antennae are very important. For example, one may wish to study the electromagnetic fields inside the fuselage and body of an airplane, with the goal of reducing it's signature. In such applications, actually meshing to the level of the wire, while simultaneously capturing the large-scale object, will require either an extremely large mesh or a highly graded one. Existing algorithms need to be tested against benchmark problems involving wires, and we need to develop other algorithms if required.

- Plane-wave time-domain discretization techniques are gaining popularity. Here, one approximates the scattered field using plane wave basis functions. These algorithms need to be rigorously analysed for their convergence and stability properties. It has already been noticed that plane-wave techniques can be cheaper and more accurate than methods reliant on trigonometric or polynomial basis functions, provided one has some a priori knowledge of the direction of the wave to be approximated. The use of other special basis functions, to enable high-order calculations in an inexpensive fashion, also needs to be further investigated.

- In practice, the description of obstacle shapes or the incident wave requires the use of stochastic parameters and shapes. Few high-order methods currently exist for studying stochastic scattering problems; this field provides a wealth of open problems.

- As for the study of Integral Equation based methods, the error analysis of volumetric algorithms rarely includes explicit dependence on the frequency for quantities of interest. A major theoretical undertaking would be to develop tools to evaluate the dependence upon the frequency.

- Volumetric solvers, when coupled with appropriate boundary conditions, can lead to essentially sparse systems, which unfortunately are not positive-definite. A major open problem remains the construction of efficient solution techniques at the discrete level, perhaps using low-frequency or elliptic problems as preconditioners.

- Current convergence and stability results on vector-type finite element techniques for scattering do not extend to highly anisotropic meshes or materials. Since high-contrast and strongly anisotropic materials occur in practice, a careful study of numerical methods in this context is required. Indeed, effective *a posteriori* error estimates are not available, making adaptive meshes difficult to implement.

- An interesting question arises in the study of electromagnetic scattering: since the solutions of Maxwell's equation obey the Gauss, Ampere and Faraday laws. Should finite element approximations obey these at the element level? Is there any room for "fully compatible discretization" of electromagnetic waves?

- hp-adaptive finite element techniques can be very efficient, particularly when the scatterer or the medium has several scales, near-singular geometric features, or strong anisotropies. A rigorous error analysis of such methods for a variety of scattering problems remains an open challenge.

- The perfectly matched layer of Berenger has been very successful in certain contexts. Is there a stable PML for all symmetric hyperbolic systems? What about the PML for anisotropic elastic scattering: Is it stable?

- Exact boundary conditions are exact implementations of the Stekhlov-Poincaré maps on a truncating boundary. Is there a purely local (in space and time) exact boundary condition for the wave equation in the time domain?

This list of open problems by no means exhausts the issues brought up during the workshop; several more technical questions were presented in the actual talks and posters, for which we refer the reader to the associated website.

List of Participants

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Chapter 2

Convex Sets and their Applications (06w5059)

Mar 04 – Mar 09, 2006

Organizer(s): Ted Bisztriczky (University of Calgary), Paul Goodey (University of Oklahoma), Peter Gritzmann (Technische Universität München), Martin Henk (Universität Magdeburg), David Larman (UC London)

Introduction

BIRS-REPORT, T. BISZTRICZKY

The main objective of this Workshop was to bring together in Banff eminent and emerging researchers from the three main branches of Convex Geometry: Discrete, Analytical and Applied. There has not been such a unifying conference in the past fifteen years. The organizers believe that this objective was met during the week of March 4 - 9. First, of the thirty-nine participants, one third represented the current group of emerging researchers in the field; furthermore, five of these thirteen participants were graduate students. Specifically, three (Langi, Naszodi and Papez) from the University of Calgary, one (Jimenez) from the University of Alberta, and one (Garcia-Colin) from the University College, London.

Next, a common feature of many of the lectures was an expository component. This reflected the acknowledgement and approval of the participants of the unifying aspect of the Workshop. The prevailing intent of the lectures was to present the major problems and recent advances of their particular branch of Convexity. Of particular note were the expository lectures on the combinatorics of polytopes, the lectures introducing some of the current topics of interests in linear and convex optimization, and the lectures concerning the various measures associated with convex bodies.

Finally, the consensus of the participants was that such a unifying convexity workshop was not only timely but also overdue. Their enthusiasm for the meeting is well evidenced by the full program of thirty - six lectures, and by a very faithful attendance at these lectures. The smallest number of listeners at any lecture was thirty, and that number was attained only at the last lecture on Thursday.

Abstracts

Iskander Aliev

A sharp lower bound for the Frobenius number

Ferdinand Georg Frobenius (1849–1917) raised the following problem: given N positive integers $a_1, \ldots, a_N$ with $\gcd(a_1, \ldots, a_N) = 1$, find the largest natural number $g_N = g_N(a_1, \ldots, a_N)$ (called the Frobenius number) such that g_N has no representation as a non-negative integer combination of $a_1, \ldots, a_N$.

In the present talk, after a short historical overview, we discuss a geometric approach to the Frobenius problem, based on results of Ravi Kannan, Peter Gruber and Andrzej Schinzel. The introduced technique allows us to give an optimal lower bound for the Frobenius number g_N in terms of the absolute inhomogeneous minimum of the standard $(N-1)$-simplex.

Margaret Bayer

Flag vectors of polytopes: an overview

For a d-dimensional polytope P, and $S = \{s_1, s_2, \ldots, s_k\} \subseteq \{0, 1, \ldots, d-1\}$, $f_S(P)$ is the number of chains of faces $\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset P$ with $\dim F_i = s_i$. The *flag vector* of P is the length 2^d vector $(f_S(P))_{S \subset \{0,1,\ldots,d-1\}}$. This lecture gives a historical overview of the study of flag vectors of polytopes.

The flag vector is an extension of the face vector, or f-vector, which has been the subject of research since Euler. In the cases of 3-dimensional polytopes and simplicial d-polytopes, characterizations of f-vectors are known, and in these cases, the flag vector is determined linearly by the f-vector.

Richard Stanley (1979) studied flag vectors of Cohen-Macaulay posets, a class that contains face lattices of convex polytopes. Bayer and Billera (1985) proved the generalized Dehn-Sommerville equations, the complete set of linear equations satisfied by the flag vectors of all convex polytopes. Kalai (1987) used rigidity theory to show the inequality $f_{02} - 3f_2 + f_1 - df_0 + \binom{d+1}{2} \geq 0$. The flag vectors of 4-dimensional polytopes were studied by Bayer (1987), but a complete characterization of flag vectors of 4-polytopes continues to elude us to this day.

A crucial ingredient in the characterization of f-vectors of simplicial polytopes was the connection with toric varieties. In the nonsimplicial case, the middle perversity intersection homology of the toric variety gives an h-vector, linearly dependent on the flag vector. Results from algebraic geometry translate into linear inequalities on the flag vector (Stanley 1987).

Another main source of linear inequalities is the cd-index of a polytope, discovered by Jonathan Fine (1985). The cd-index is a vector linearly equivalent to the flag vector; it can be viewed as a reduction of the flag vector by the generalized Dehn-Sommerville equations. Stanley (1994) proved the nonnegativity of the cd-index for convex polytopes. Billera and Ehrenborg (2000) strengthened the result by showing that among d-polytopes the cd-index is minimized by that of the d-simplex. This depends on a co-algebra approach to the cd-index developed by Ehrenborg and Readdy (1998).

Two separate techniques enable one to generate new linear inequalities on flag vectors from old. The convolution operation was introduced by Kalai (1988); he also used this to demonstrate a particularly nice basis for the flag vectors of polytopes. Ehrenborg (2005) gives a lifting technique that applies to inequalities on the cd-index.

We are still, apparently, far from a characterization of flag vectors of polytopes. In fact, we do not even know if the closed convex cone of flag vectors is finitely generated. Special classes of polytopes, such as cubical polytopes and zonotopes, have been studied. In addition there are some results on more general classes of partially ordered sets: general graded posets, Eulerian posets, and Gorenstein* lattices.

Károly Bezdek

Short Billiards

The talk is a survey talk on periodic billiards centered around the following theorem and conjecture of the author.

DEFINITION. We say that $\mathbf{b}$ is a k*-sided billiard arc* of the convex body $\mathbf{K} \subset \mathbf{E}^n, n \geq 2, k \geq 1$ if $\mathbf{b}$ is a k-sided polygonal arc in $\mathbf{E}^n$ whose vertices lie on the boundary of $\mathbf{K}$ and whose each angle bisector is perpendicular to a supporting hyperplane of $\mathbf{K}$ passing through the corresponding vertex of $\mathbf{b}$ and finally, whose first (resp., last) segment is perpendicular to a supporting hyperplane of $\mathbf{K}$ passing through the corresponding endpoint of $\mathbf{b}$.

THEOREM. *If the minimum width of the convex body* $\mathbf{K} \subset \mathbf{E}^n, n \geq 2$ *is at least* 1*, then the length of any billiard arc of* $\mathbf{K}$ *is at least* 1.

COROLLARY. Let $X \subset \mathbf{E}^n, n \geq 2$ be a (finite) set of diameter at most 1. Then the length of any billiard arc of $\mathbf{B}[X] := \bigcap_{x \in X} \mathbf{B}^n[x]$ is at least 1, where $\mathbf{B}^n[x] \subset \mathbf{E}^n$ stands for the closed n-dimensional unit ball centered at x.

DEFINITION. We say that $\mathbf{b}$ is a *k-sided billiard polygon* of the convex body $\mathbf{K} \subset \mathbf{E}^n, n \geq 2, k \geq 2$ if $\mathbf{b}$ is a k-sided polygon in $\mathbf{E}^n$ whose vertices lie on the boundary of $\mathbf{K}$ and whose each angle bisector is perpendicular to a supporting hyperplane of $\mathbf{K}$ passing through the corresponding vertex of $\mathbf{b}$.

CONJECTURE. Let $X \subset \mathbf{E}^n, n \geq 2$ be a (finite) set of diameter at most 1. Then the length of any billiard polygon of $\mathbf{B}[X] := \bigcap_{x \in X} \mathbf{B}^n[x]$ is at least 2.

REMARK. The above theorem and conjecture for $n = 2$ follow from a theorem of the author and R. Connelly (1989).

Károly Böröczky, Jr.

Convex bodies of minimal volume, surface area and mean width with respect to thin shells

Given $r > 1$, let us consider convex bodies in E^n that contain a fixed unit ball, and whose extreme points are of distance at least r from the centre of the unit ball, and we investigate how well these convex bodies approximate the unit ball in terms of volume, surface area and mean width. The main results joint with K. Böröczky, C. Schütt and G. Wintsche are as follows: As r tends to one, there are asymptotic formulae for the error of the approximation, and asymptotically the whole boundary of the extremal bodies are covered by faces that are asymptotically regular triangles in E^3.

René Brandenberg

Minimal containment under homothetics

(joint work with Lucia Roth)

Minimal containment problems arise in a variety of applications, such as shape fitting problems, data clustering, pattern recognition or medical surgery. Typical examples are norm maximization, computing the circumball, circumcylinder or the width of a given body or minimal enclosing boxes or ellipsoids. A possible general framework gives the following definition

MINIMAL CONTAINMENT PROBLEM (MCP):

Input: $d \in \mathbb{N}$, $K \subset \mathbb{R}^d$ convex body.
Task: $\min\ \varphi_d(C)$, such that $K \subset C \in \mathcal{C}_d$,

where $\mathcal{C}_d$ usually is the orbit of a given convex body under a group of transformations like homothetics, similarities or affine mappings and φ_d a monotone functional such as the volume or the dilatation factor of C.

In this talk we focus on the MCP under homothetics (MCP$_{Hom}$), which itself has a lot of applications but is also needed as an important subroutine in solving lots of other MCP problems. Besides some negative complexity results the following was shown by Gritzmann and Klee: if C is given by a strong separation oracle and if K is a $\mathcal{V}$-polytope then MCP$_{Hom}$ can be solved in polynomial time using the ellipsoid method.

Because of the bad practical performance of the ellipsoid algorithm much effort has been spend to find better solutions, at least when C is the Euclidean ball. One recent idea are so called core set algorithms. Here the approximation of the circumball of a point set P is reduced to the computation of the circumball of a small subset of P, where 'small' means independent of the size and the dimension of P.

We present a new and easy to implement cutting plane method, based on linear programming, which is dual in nature to the core set idea and very easy to implement. It solves the general MCP$_{Hom}$ up to any given accuracy and because of its adaptive character it also has a good practical performance.

Finally we point out some relations to well known theoretical problems in convex geometry, which play a substantial role not only in the analysis of our method but also in the task to generalize the core set method to non-euclidean containers.

David Bremner

Approaches to facet enumeration under symmetry

Well known theorems of Minkowski and Weyl tell us that every convex polytope is the convex hull of a finite set of points and the bounded intersection of a set of (facet defining) halfspaces. In practice transforming from one representation to the other is often of interest, and usually difficult. One of the obvious difficulties is that the output may be huge with respect to the input size; on the other hand there is typically a symmetry group acting on the polytope, and the practioner may only be interested in equivalence classes of the output under this group.

I will start by giving a brief survey of the state of the art of facet enumeration, including some idea of what kind of inputs on which the known techniques face difficulties.

I will then describe some preliminary experience with a pivoting technique for generating equivalence classes of facets of a convex polytope under the action of an isometry group. I describe connections with previously studied “adjacency decomposition” methods, as well as some of details of invariants, isometry testing for bases, and pruning the search. I discuss the performance of the pivoting method, which depends not just on the degeneracy of the polytope, but on how the symmetry group acts on bases (of facets) of the polytope. This work is joint with Achill Schürmann and Frank Vallentin.

Time permitting, I will mention some work of David Avis that applies the “extend and canonicalize” techniques of Read, McKay, and others to enumerate the entire face lattice up to symmetry.

Jesús A. De Loera

Transportation Polytopes: a twenty-year update

A transportation polytope consists of all multidimensional arrays of nonnegative numbers that satisfy certain sum conditions on subsets of the entries. They arise naturally in optimization and statistics and have also interest for pure mathematics due to the appearance of permutation matrices, latin squares, magic squares, as lattice points of these polytopes. In this talk we present recent advances on the understanding of the combinatorics and geometry of these polyhedra. In particular, we try to give a complete report on the status of a long list of open questions last collected in the 1984 monograph by Yemelichev-Kovalev-Kravtsov and the 1986 survey paper of Vlach.

Richard Ehrenborg

The cd-index, polytopes and Gorenstein* lattices

The f-vector enumerates the number of faces of a polytope according to dimension, that is, f_i is the number of faces of dimension i. The flag f-vector is a refinement of the f-vector which counts flags of faces in the polytope. There are linear relations between the entries of the flag f-vector known as the generalized Dehn-Sommerville relations. Hence it would be interesting to have an explicit basis for the subspace spanned by these relations.

The $\mathbf{cd}$-index, conjectured by Fine and proved by Bayer and Klapper, gives such a basis. It offers an efficient way to encode the flag f-vector of a polytope. In fact, Stanley showed that the $\mathbf{cd}$-index exists for Eulerian poset, namely a poset where each interval satisfies the Euler-Poincaré relation.

Very little is known about the $\mathbf{cd}$-index of a general polytope. Fine conjectured that the $\mathbf{cd}$-index of a polytope has non-negative coefficients. This conjecture was proven by Stanley, in fact, he proved that the cd-index is non-negative for spherical-shellable (S-shellable) complexes.

A poset is Gorenstein* if it is Eulerian and the associated chain complex is Cohen-Macaulay. The most natural example of a Gorenstein* poset is the face lattice of a convex polytope. For Gorenstein* posets Stanley stated two conjectures: (1) The $\mathbf{cd}$-index for Gorenstein* poset is non-negative. (2) The $\mathbf{cd}$-index for Gorenstein* lattice is coefficientwise minimized by the $\mathbf{cd}$-index of the simplex of the same dimension.

A partial step toward Stanley’s second conjecture was taken by Billera and Ehrenborg. They proved the $\mathbf{cd}$-index of a polytope is coefficientwise minimized by the simplex of the same dimension. Their proof uses the geometric fact that polytopes are shellable.

Kalle Karu using techniques from algebraic geometry proved Stanley’s first conjecture, that the $\mathbf{cd}$-index of a Gorenstein* posets.

Recently, Ehrenborg and Karu proved Stanley's second conjecture. I will end the talk by outlining the proof and where it differs from the earlier proof for polytopes.

This is joint work with Kalle Karu.

Ferenc Fodor

Geometric transversals in low and high dimensions

This talk contains results that were achieved jointly with Ted Bisztriczky (Calgary) and Deborah Oliveros (Mexico City), and with Gergely Ambrus (Szeged, Auburn) and András Bezdek (Auburn).

Let $\mathcal{F}$ denote a family of ovals in the Euclidean plane. A line is a *transversal* to a family $\mathcal{K}$ if it intersects every member of $\mathcal{K}$. $\mathcal{K}$ has the property T if it has a transversal. $\mathcal{K}$ has the property $T(k)$ if every at most k-membered subfamily of $\mathcal{K}$ has a transversal. $\mathcal{K}$ has the property $T-k$ if there is a line that meets all members of $\mathcal{K}$ with the possible exception of at most k of them.

In 1989, Tverberg proved that $T(5) \Rightarrow T$ for a disjoint family of translates of an oval, a conjecture of Grünbaum (1958). In general, we know that neither $T(3)$ nor $T(4)$ is enough to guarantee the same. Katchalski and Lewis (1980) proved that there exists a universal constant k_3 such that $T(3) \Rightarrow T - k_3$ for any finite family of disjoint translates of an arbitrary oval. They estimated $k_3 \leq 192\pi$ and conjectured that $k_3 = 2$. It was shown, using a construction with unit disks, by A. Bezdek (1991) that $k_3 \geq 2$. The upper estimate on k_3 was improved by Tverberg (1991) and later by Holmsen (2000). The currently known best upper bound for k_3 is 22, established by Holmsen (2000). Holmsen (2000) constructed examples which show that $k_3 \geq 4$. Holmsen also showed that $k_3 = 4$ for finite families of unit squares whose sides are parallel to the coordinate axes.

Danzer (1963) proved that $T(5) \Rightarrow T$ for a pairwise disjoint family of unit disks. Kaiser (2002) showed that $k_3 \leq 12$ for such a family. Finally, Heppes settled the question in 2004 by proving that $T(3) \Rightarrow T-2$ for unit disks. An example of Aronov, Goodman, Pollack, and Wenger (2000) showed that $T(4) \not\Rightarrow T$ for unit disks. It was proved by T. Bisztriczky, D. Oliveros and F. F. in 2005 that if $\mathcal{F}$ is a finite family of mutually disjoint unit disks with the property $T(4)$, then $\mathcal{F}$ has the property $T-1$.

A family of balls in $\mathbf{R}^d$ is *thinly distributed* if the distance between the centres of any two balls is at least twice the sum of their radii. Hadwiger (1957) proved that for any family of thinly distributed balls in $\mathbf{R}^d$, $T(d^2) \Rightarrow T$. Grünbaum (1960) improved Hadwiger's statement by proving that $T(2d-1) \Rightarrow T$. Holmsen, Katchalski and Lewis (2003) showed that there exists a constant $n_0 \leq 46$ such that $T(n_0) \Rightarrow T$ for any family of pairwise disjoint unit balls in $\mathbf{R}^3$. The constant n_0 was improved subsequently by Cheong, Goaoc and Holmsen (2004) to 11.

G. Ambrus, A. Bezdek and F. F. (2005) improved on the distance condition in Hadwiger's (1960) theorem proving that if $\mathcal{F}$ is a family of unit balls in $\mathbf{R}^d$ with the property that the mutual distances of the centres are at least $2\sqrt{2+\sqrt{2}}$ then $T(d^2) \Rightarrow T$.

We note that Cheong, Goaoc, Holmsen and Petitjean (2005) very recently proved that $T(4d-1) \Rightarrow T$ for disjoint unit balls in $\mathbf{R}^d$.

Natalia Garcia-Colin

On a generalization of a problem of McMullen
regarding the neighborliness in convex polytopes

McMullen proposed the following question. Determine the largest integer $n = f(d)$ such that any set of n points in general position in the affine $d - space$ $\mathbb{R}^d$ can be mapped by a projective transformation on to the vertices of a convex polytope. It is known that

$$2d + 1 \leq f(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil$$

.

In the paper where Larman proved the lowed bound, he also proved that the lower bound is sharp in the cases where d=1,2 and 3. The upper bound was proved by Ramirez-Alfonsin by constructing a family of Lawrence Oriented Matroids where every of its members can be made cyclic by reorienting one element.

Using the techniques developed by Ramirez-Alfonsin, in his paper, we construct a family of Lawrence Oriented matroids that can always be made cyclic by reorienting a subset $S \in X$ of the ground set X (of vertices) with cardinality at most k. This construction gives an upper bound for the following problem:

Determine the largest integer $n = f(d, k)$ such that any set of n points in general position in the affine $d-space$ $\mathbb{R}^d$ can be mapped by a permissible projective transformation on to the vertices of a $k-neighborly$ convex polytope. Namely:

$$d + \left\lfloor \frac{d}{k+1} \right\rfloor + 1 \leq f(d,k) < 2d - k + 1$$

.

Finally, we prove the following related problem in the plane using purely geometric methods:

In $\mathbb{R}^2$ let X be a subset of n point in general position. Let $g(X)$ the largest k such that there exists a subdivision A, B of X such that $conv(A \backslash \{x_1, x_2, ..., x_k\}) \cap conv(B \backslash \{x_1, x_2, ..., x_k\}) \neq \emptyset$. If

$$g(n) = \max_{X \in |X|=n} g(X), \qquad then \qquad \lim_{n\to\infty} \frac{g(n)}{n} = \frac{1}{2}.$$

Paolo Gronchi

Shadow systems

Shadow systems were introduced in 1958 by Rogers and Shephard [6] as families of convex hulls of a given set of points moving with constant speed along a fixed direction. Rogers and Shephard showed that the volume a shadow system is a convex function of the time-like parameter t.

Shephard [7] noted that the elements of such a family can be defined as the projections of a higher dimensional convex body along the direction $z + tv$ onto the hyperplane $z^{\perp}$. This fact enables us to construct shadow systems and also to extend the convexity property of the volume to different quantities. Precisely, a first consequence is that projections, Minkowski sums and convex hulls of shadow systems are still shadow systems. Hence, the brightness along a fixed direction is a convex function of t and, via Cauchy's formula, also the surface area is convex in t. Similarly, taking projections onto 1-dimensional subspaces, we infer that the mean width and the diameter are convex functions of the parameter t. By the same argument, Shephard [7] proved that quermassintegrals and mixed volumes of shadow systems are convex functions of t.

More recently, Campi, Colesanti and Gronchi [1] proved that the Sylvester functional (i.e., the expected value of the volume of a random polytope from a convex body) is a convex function of the parameter of parallel chord movements, a particular kind of shadow systems. Campi and Gronchi [2], [4] proved the same convexity property for the volume of the L^p-centroid bodies and the L^p-zonotopes. Furthermore, they showed [3] that the reciprocal of the volume of the polar body of an origin-symmetric shadow system is a convex function of the parameter. Meyer and Reisner [5] extended such a result to the non symmetric case.

The convexity of a functional along parallel chord movements can be used, via Steiner symmetrization, to characterize ellipsoids as minimizers. The same property leads also to maximizers in special classes. Namely, triangles among two-dimensional convex sets and parallelograms in the symmetric case, parallelotopes among zonoids [4], simplices among d-dimensional polytopes with at most $d + 3$ vertices [5].

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María A. Hernández Cifre

On the minimal annulus of a convex body: some optimization problems
(joint work with P. Herrero)

Let K be a convex body (i.e., a compact convex set) in the Euclidean plane. Associated with K are a number of well-known functionals: the area A, the perimeter p, the diameter D, the minimal width ω, the circumradius R_K and the inradius r_K.

Another interesting functional to be considered for a convex body K is the thick of its *minimal annulus*. The minimal annulus of the body K is the annulus (the closed set consisting of the points lying between two concentric discs –concentric n-balls in $\mathbb{R}^n$) with minimal difference of radii that contains the boundary of K. Of course, the minimal annulus is uniquely determined (Bonnesen, 1929, in $\mathbb{R}^2$ and $\mathbb{R}^3$, and Bárány, 1988, in higher dimension). This object and its properties were studied originally by Bonnesen for planar convex sets in order to sharp the isoperimetric inequality in $\mathbb{R}^2$.

In this talk we intend to present how the minimal annulus of a planar convex body K is related with the six classic geometric measures associated with it. First, we obtain all the possible bounds (upper and lower bounds) for the measures A, p, D, ω, R_K and r_K of a convex body K with fixed minimal annulus. Then, we deal with the relation between the minimal annulus and, either the circumradius, or the inradius of K: we study some properties relating the minimal annulus with both measures, and then we solve the problem of maximizing and minimizing the remaining geometric measures when, either the circumradius and the minimal annulus, or the inradius and the minimal annulus, are given. We prove the optimal inequalities for each of those problems, determining also its corresponding extremal sets.

Markus Kiderlen

Spherical liftings and projections in convex geometry

Let K be a lower dimensional convex body in d-dimensional space containing the origin. The support function of K can be obtained from its support function relative to its affine hull by applying a linear transformation, which we call a spherical lifting. Starting from this motivation, we will introduce more general spherical liftings. Spherical liftings map positive finite measures on the unit sphere of a linear space L to measures on the unit sphere in d-dimensional space. The dual operators, the so-called spherical projections, will also be introduced. We will show that many geometric operations, like projections or translative integrals can conveniently be expressed using spherical liftings and projections. One central result will be that spherical projections preserve convexity, implying in particular a directed version of the observation that the 1-st projection function of a convex body is a support function.

We will then turn to averages of spherical lifted projections, where averaging is understood with respect to the invariant probability measure on all k-dimensional subspaces L. We discuss in how far a convex body is determined by one or several of these averages.

Alexander Koldobsky

Inequalities of Khinchin type and sections of L_p-balls, $p > -2$.

We extend Khinchin type inequalities to the case $p > -2$. As an application we verify the slicing problem for the unit balls of finite-dimensional spaces that embed in L_p, $p > -2$.

David Larman

Determining properties of convex bodies from information about certain sections

The talk centred around partial results to three problems:

- Let K be a convex body in E^d and let p be a point of $\operatorname{int} K$ such that every two section of K through p has a projective centre different from p. Is K an ellipsoid?
- Let K, L be convex bodies in E^d with $L \subset \operatorname{int} K$. Suppose we know the $(d-1)$-volume of every $d-1$ section of K which touches L. Does this determine K uniquely?
- Let K, L be convex bodies in E^d with $L \subset \operatorname{int} K$. Suppose that every $(d-1)$ section of K that touches L is centrally symmetric. Is K an ellipsoid?

Carl Lee

Multiple views of h-vectors

I will give a brief survey of several different ways of looking at h-vectors of polytopes, including combinatorial views (winding numbers, shellings, bistellar operations), and algebraic (the face-ring, stress, weights, the "volume ring"). I will offer reminders of several interesting open problems.

Zsolt Lángi

Isoperimetric inequalities for k_g-polygons

(joint work with Balázs Csikós and Márton Naszódi)

The discrete isoperimetric problem is to determine the maximal area polygon with at most k vertices and of a given perimeter. It is a classical fact that the unique optimal polygon on the Euclidean plane is the regular one. The same statement for the hyperbolic plane was proved by Károly Bezdek and on the sphere by László Fejes Tóth. In the present paper we extend the discrete isoperimetric inequality in the following way.

Let $\Gamma \subset \mathbf{M}$ be a simple closed polygon in $\mathbf{M}$ and let $k_g \geq 0$ be fixed. If $\mathbf{M} = \mathbb{S}^2$, we assume that Γ is contained in an open hemisphere. Take the closed curve P obtained by joining consecutive vertices of Γ by curves of geodesic curvature k_g facing outward (resp. inward). If k_g is the geodesic curvature of a circle of radius r, then Γ is assumed to have sides of length at most $2r$ and the smooth arcs of P connecting two consecutive vertices are assumed to be shorter than or equal to a semicircle. We call P an *outer* (resp. *inner*) k_g*-polygon* with the same set of *vertices* as that of Γ. We call a k_g-polygon with perimeter l a (k_g, l)*-polygon*. An outer (resp. inner) (k_g, l)-polygon is *optimal* if its area is maximal among the areas of outer (resp. inner) (k_g, l)-polygons having the same number of vertices. We prove the following statements.

PROPOSITION. Let $\mathbf{M}$ be $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$. Let $l > 0$ and $k_g \geq 0$ be given. Then the only optimal inner (k_g, l)-polygons in $\mathbb{M}$ are the regular ones.

THEOREM. Let $\mathbf{M}$ be $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$. Let $k_g \geq 0$, $l > 0$ and n be given with the above restrictions. If l is not equal to the perimeter of the circle of geodesic curvature k_g, then the only optimal outer (k_g, l)-polygons in $\mathbf{M}$ are the regular ones. If l is equal to the perimeter of the circle of geodesic curvature k_g, then a (k_g, l)-polygon is optimal if and only if its underlying polygon Γ is inscribed in a circle of geodesic curvature k_g.

Jospeh M. Ling

Non-linear inequalities for 4-dimensional convex polytopes

In this talk, we consider the characterization problem for the f-vectors and the flag f-vectors for 4-polytopes. Four new (infinite) lists of quadratic inequalities for the flag f-vectors of 4-polytopes are presented. These inequalities extend the four inequalities obtained by M. Bayer in 1984. Four cubic inequalities for the flag f-vectors are also presented. Furthermore, the projections of the newly found inequalities onto the f-vectors yields new (infinite) lists of quadratic inequalities for the f-vectors. An application of these include an estimate of the number of edges in terms of the number of vertices and the number of facets.

Alexander Litvak

On the vertex index of convex bodies

We introduce the vertex index of a given d-dimensional centrally symmetric convex body, which, in a sense, measures how well the body can be inscribed into a convex polytope with small number of vertices. This index is closely connected to the illumination parameter of a body, introduced earlier by Karoly Bezdek, and, thus, related to the famous conjecture in Convex Geometry about covering of a d-dimensional body by 2^d smaller positively homothetic copies. We provide asymptotically sharp (up to logarithmic terms) estimates of this index in the general case and discuss extremal cases. More precisely, we show that the vertex index varies between $cd/\sqrt{\ln 2d}$ and $Cd^{3/2}\ln(2d)$, where c and C are absolute positive constants. Here, the lower estimate is sharp (up to a logarithmic term) for crosspolytopes and the upper estimate is sharp (again, up to a logarithmic term) for ellipsoids. Also, we provide precise estimates in dimensions 2 and 3. We conjecture that the vertex index of a d-dimensional Euclidean ball is $2d\sqrt{d}$. We prove this conjecture in dimensions two and three.

Monika Ludwig

Elementary moves on triangulations

(joint work with Matthias Reitzner)

Let P be an n-dimensional polyhedron in $\mathbb{R}^N$, that is, a finite union of n-dimensional convex polytopes. A finite set of n-simplices αP is a *triangulation* of P if no pair of simplices intersects in a set of dimension n and if their union equals P. An *elementary move* applied to αP is one of the two following operations: a simplex $T \in \alpha P$ is dissected into two n-simplices T_1, T_2 by a hyperplane containing an $(n-2)$-dimensional face of T; or the reverse, that is, two simplices $T_1, T_2 \in \alpha P$ are replaced by $T = 3DT_1 \cup T_2$ if T is again a simplex. Triangulations αP and βP are equivalent by elementary moves, $\alpha P \sim \beta P$, if there are finitely many elementary moves that transform αP into βP.

THEOREM. If αP and βP are triangulations of the n-polyhedron P, then $\alpha P \sim \beta P$.

This result is a metric version of the Alexander-Newman theorem for simplicial complexes. As an application the following extension result is obtained.

THEOREM. Every valuation on simplices in $\mathbb{R}^n$ has a unique extension to a valuation on polyhedra in $\mathbb{R}^n$.

Efren Morales-Amaya

A Characterization of ellipsoids

(joint work with J. Jeronimo)

Motivated by a theorem due to Rogers [3], we give a characterization of ellipsoid in the spirit of the Höbinger Problem [1], [2]. Namely, we proved that if $K \subset \mathbb{R}^n$ is a convex body, $n \geq 3$, and for every three for every three parallel hyperplanes A, D and E there exists point $p \in \mathbb{R}^n$ with the following property: for every line l passing through p, the central projections K_A and K_D of K from $l \cap A$ and $l \cap D$, respectively, onto E are homothetics, then K is an ellipsoid.

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Márton Naszódi

Touching Homothetic Bodies and Antipodality

According to Klee's definition (1960), an *antipodal set* in Euclidean d-space is a set X with the property that, through any two points of X, there is a pair of parallel hyperplanes supporting X. In this talk, I present two research topics that are related by the idea of antipodality.

In the first part of the talk, the maximum number of touching positive homothetic copies of a convex body in Euclidean d-space is discussed. According to a conjecture of Károly Bezdek and János Pach, this number is 2^d; which bound, if it holds, is sharp as it is attained by cubes. The previously known bound was 3^d, I improved it to 2^{d+1}. I present the proof of this recent result.

The second part of the talk focuses on the extension of the concept of antipodality to hyperbolic d-space. This is a joint work with Károly Bezdek and Deborah Oliveros. We define antipodality in three different ways, as follows.

Following Klee, we say that a set X in hyperbolic d-space is *p-antipodal* if, through any two points of X, there is a pair of parallel hyperbolic hyperplanes supporting X.

Following Erdős' concept of antipodality (1957), a set X in hyperbolic d-space is *a-antipodal*, if the angle determined by any three points of X is acute.

Finally, an *h-antipodal* set in hyperbolic d-space is a set X with the property that for any $x_1, x_2 \in X$, the set X is contained in the intersection of the horoballs H_1 and H_2, where H_1 is the horoball bounded by the horosphere that passes through x_1, contains x_2 inside and is perpendicular to the hyperbolic line $\overline{x_1 x_2}$, and H_2 is defined similarily.

We find upper bounds on the cardinality of an antipodal set in hyperbolic d-space, according to the different definitions.

Shmuel Onn

Multiway polytopes: universality and convex optimization
(slides are available at

$$http://ie.technion.ac.il/~onn/Talks/multiway_polytopes.pdf$$

A k-way (transportation) polytope is the set of all $m_1 \times \cdots \times m_k$ nonnegative arrays $x = (x_{i_1,\ldots,i_k})$ such that the sums of the entries over some of their lower dimensional sub-arrays (margins) are specified. More precisely, for any tuple $(i_1, \ldots, i_k)$ with $i_j \in \{1, \ldots, m_j\} \cup \{+\}$, the corresponding *margin* $x_{i_1,\ldots,i_k}$ is the sum of entries of x over all coordinates j with $i_j = +$. The *support* of $(i_1, \ldots, i_k)$ and of $x_{i_1,\ldots,i_k}$ is the set $\mathrm{supp}(i_1, \ldots, i_k) := \{j : i_j \neq +\}$ of non-summed coordinates. For instance, if x is a $4 \times 5 \times 3 \times 2$ array then it has 12 margins with support $\{1, 3\}$ such as $x_{3,+,2,+} = \sum_{i_2=1}^{5} \sum_{i_4=1}^{2} x_{3,i_2,2,i_4}$. Given a family $\mathcal{F}$ of subsets of $\{1, \ldots, k\}$ and margin values $u_{i_1,\ldots,i_k}$ for all tuples with support in $\mathcal{F}$, the corresponding k-way polytope is the set of nonnegative arrays with these margins,

$$T_{\mathcal{F}} \;=\; \left\{ x \in \mathbb{R}_+^{m_1 \times \cdots \times m_k} \;:\; x_{i_1,\ldots,i_k} \;=\; u_{i_1,\ldots,i_k}\,,\;\; \mathrm{supp}(i_1, \ldots, i_k) \in \mathcal{F} \right\} \,.$$

In this talk we present the following two remarkable contrasting statements regarding multiway polytopes and discuss some of their many applications:

UNIVERSALITY THEOREM: Every rational polytope $P = \{y \in \mathbb{R}_+^m : Ay = b\}$ is polynomial time representable as an $r \times c \times 3$ multiway polytope of line-sums, that is, with $\mathcal{F} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$,

$$T \quad = \quad \{\, x \in \mathbb{R}_+^{r \times c \times 3} \;:\; \sum_i x_{i,j,k} = w_{j,k}\,,\; \sum_j x_{i,j,k} = v_{i,k}\,,\; \sum_k x_{i,j,k} = u_{i,j} \,\} \,.$$

OPTIMIZATION THEOREM: For any fixed $d, k, m_1, \ldots, m_{k-1}$, and family $\mathcal{F}$ of subsets of $\{1, \ldots, k\}$, there is a polynomial oracle-time algorithm that, given n, arrays $w_1, \ldots, w_d \in \mathbb{Z}^{m_1 \times \cdots \times m_{k-1} \times n}$, margin values $u_{i_1,\ldots,i_k}$ for all tuples $(i_1, \ldots, i_k)$ with support in $\mathcal{F}$, and convex $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by evaluation oracle, solves the corresponding convex integer multiway programming problem,

$$\max\{\, c(w_1 x, \ldots, w_d x) \;:\; x \in \mathbb{N}^{m_1 \times \cdots \times m_{k-1} \times n},\;\; x_{i_1,\ldots,i_k} \;=\; u_{i_1,\ldots,i_k}\,,\;\; \mathrm{supp}(i_1, \ldots, i_k) \in \mathcal{F} \,\} \,.$$

These results, their consequences, applications and extensions appear in several recent papers joint with various coauthors among J. De Loera, R. Hemmecke, U. Rothblum and R. Weismantel, including *Convex combinatorial optimization* (Disc. Comp. Geom. 32:549–566, 2004), *The complexity of three-way statistical tables* (SIAM J. Comp. 33:819–836, 2004), *All rational polytopes are transportation polytopes and all polytopal integer sets are contingency tables* (IPCO 10, LNCS 3064:338–351, 2004), *Markov bases of three-way tables are arbitrarily complicated* (J. Symb. Comp. 41:173–181, 2006), *N-fold integer programming* (submitted), and *Convex integer programming* (in preparation).

Peter Papez

Ball-Polyhedra

This talk outlines the results of a joint paper written by K. Bezdek, Z. Lángi, M. Naszódi and P. Papez. The main goal of this paper is to study the geometry of intersections of finitely many unit balls from the point of view of discrete geometry in Euclidean space. We call these sets *ball-polyhedra*. They have been studied in the past, in particular Reuleaux polygons; although the name ball-polyhedra seems to be a new terminology for this special class of linearly convex sets. In fact, there is a special kind of convexity entering along with ball-polyhedra which we call *lens-convexity* and study as well. This paper is not a survey on ball-polyhedra, instead it lays a rather broad ground work for future study of ball-polyhedra by proving several new properties of them and raising open research problems as well.

In this talk, I first define ball-polyhedra and supporting spheres. The supporting spheres are the objects that play the role of supporting hyperplanes in the theory of polyhedra. Next, we examine a special class of ball-polyhedra called standard ball-polyhedra. This is the family of ball-polyhedra for which the Eüler-Poincare formula holds. We also examine Steinitz' Theorem for the edge graph of standard ball-polyhedra. The talk concludes with a survey of results from diverse areas of geometry related to ball-polyhedra.

Carla Peri

Uniqueness and stability results in geometric tomography

Geometric tomography concerns the retrieval of information about a geometric object via measurements of its sections or projections.

In this talk we consider two types of data for line sections, namely parallel or point X-rays. After reviewing some of the main results on continuous parallel or point X- rays and discrete parallel X-rays, we present recent uniqueness results for discrete point X-rays (see [1]). The discussion will show that, somewhat surprisingly, non- uniqueness results hinge on the existence of arbitrary long arithmetic progressions of relative prime numbers, and on the existence of some geometric incidence structures.

The final part of the talk will concern recent progress in stability estimates (see [2]).

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Rolf Schneider

Intersections of balls in normed spaces

(joint work with José Pedro Moreno)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and B its unit ball. Any positive homothet of B is called a *ball*. For $K \in \mathcal{K}$ (the system of nonempty compact convex subsets of $\mathbb{R}^d$), let $\beta(K)$ denote the intersection of all balls containing K. Thus, $\mathcal{B} := \{K \in \mathcal{K} : \beta(K) = K\}$ is the system of all intersections of balls. The system $\mathcal{M}$ of Mazur sets is defined by the property that $K \in \mathcal{K}$ belongs to $\mathcal{M}$ if and only if to every hyperplane H with $K \cap H = \emptyset$

there exists a ball B' with $K \subset B'$ and $B' \cap H = \emptyset$. Motivated by questions and observations in two papers by Granero, Moreno and Phelps from 2004, we first give examples of norms for which (a) $\mathcal{B}$ is not ball stable (i.e., not closed under the addition of balls), (b) $\mathcal{B}$ is not closed, hence the ball hull map β is not continuous. For polyhedral norms, we show that β is Lipschitz continuous, and we give complete characterizations of the norms having one of the following properties: (a) $\mathcal{B}$ is closed under Minkowski addition, (b) $\mathcal{B}$ is closed under addition of balls, (c) $\mathcal{M} = \mathcal{B}$, (d) $\mathcal{M}$ contains only balls and one-pointed sets.

Carsten Schütt

On the minimum of several random variables

(joint work with Y. Gordon, A. Litvak, and E. Werner)

Let f_i, $i = 1, \dots, n$, be symmetric, identically distributed random variables. We investigate expectations

$$\int_\Omega \left\| \sum_{i=1}^n x_i f_i(\omega) \right\|_M d\mathbb{P}(\omega)$$

where $\| \ \|_M$ is an Orlicz norm. We find out that these expressions are maximal if the random variables are in addition required to be independent.

In case the random variables are independent we get quite precise estimates for the above expectations. In particular, for independent Gauß variables we have for all $x \in \mathbb{R}^n$

$$c_1 \|x\|_M \le \int_\Omega \max_{1 \le i \le n} |x_i f_i(\omega)| d\mathbb{P}(\omega) \le c_2 \|x\|_M$$

where the Orlicz function is $M(t) = e^{-\frac{1}{t^2}}$. This case is of particular interest to us. In a paper on generalized zonotopes these estimates are applied to obtain estimates for volumes of certain convex bodies.

For a given sequence of real numbers $a_1, \dots, a_n$ we denote the k-th smallest one by

$$k\text{-} \min_{1 \le i \le n} a_i .$$

Let $\mathcal{A}$ be a class of random variables satisfying certain distribution conditions (the class contains $N(0,1)$ Gaussian random variables). We show that there exist two absolute positive constants c and C such that for every sequence of positive real numbers $x_1, \dots, x_n$ and every $k \le n$ one has

$$c \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \mathbb{E}\ k\text{-} \min_{1 \le i \le n} |x_i \xi_i| \le C \ \ln(k+1) \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $\xi_1, \dots, \xi_n$ are independent random variables from the class $\mathcal{A}$. Moreover, if $k = 1$ then the left hand side estimate does not require independence of the ξ_i's. We provide similar estimates for the moments of $k\text{-} \min_{1 \le i \le n} |x_i \xi_i|$ as well.

Grzegorz Sójka

Minor illuminations and and the determination of convex bodies by values of $\pm\infty$-chord functions

(joint work with David Larman)

The notion of illuminations is strongly connected with the famous Hadwiger Illumination Conjecture. It says that it should be possible to cover boundary of arbitrary n-dimensional convex body by at most 2^n translates of its interior.

The second notion mentioned in the title are chord functions. In 1998 A. Soranzo generalized the definition of i-chord functions to the case $i = 3D \pm \infty$. For arbitrary convex body K and non-zero vector u he used the following formulae:

$$\rho_{-\infty,K}(u) = 3D \min \{\rho_K(u), \rho_= K(-u)\};$$

$$\rho_{+\infty,K}(u) = 3D \max\{\rho_K(u), \rho_= K(-u)\},$$

where ρ_K denotes the radial function of K.

In 2004 D. Larman and Grzegorz Sójka found a link between this two subjects. They generalized the notion of illuminations to the case when the source is some internal point of the convex body considered. During this presentation we will speak about their observation and related results.

Valeriu Soltan

Homothety classes of convex sets

Let A_H denote the homothety class (i. e., the family of positive homothetic copies) generated by a closed convex set $A \subset \mathbb{R}^n$. We study the conditions under which the Minkowski sum, the Minkowski difference, and the binary intersection, defined, respectively, by

$$\begin{aligned}
B_H + C_H &= \{B' + C' \mid B' \in B_H,\ C' \in C_H\},\\
B_H \underset{n}{\sim} C_H &= \{B' \sim C' \mid B' \in B_H,\ C' \in C_H,\ \dim(B' \sim C') = n\},\\
B_H \underset{n}{\cap} C_H &= \{B' \cap C' \mid B' \in B_H,\ C' \in C_H,\ \dim(B' \cap C') = n\}
\end{aligned}$$

belong to a unique homothety class generated by a closed convex set of dimension n in $\mathbb{R}^n$ (more generally, belong to the union of countably many homothety classes generated by closed convex sets in $\mathbb{R}^n$).

We also study planar sections and projections of homothetic convex sets in $\mathbb{R}^n$. In particular, closed convex sets $B, C \subset \mathbb{R}^n$ (not necessarily compact) are homothetic if and only if either of the following conditions holds: (a) the orthogonal projections of B and C on each 3-dimensional plane of $\mathbb{R}^n$ are homothetic, where similarity ratio may depend on the projection plane, (b) there are points $p \in B$ and $q \in C$ such that such that for every pair of parallel 3-dimensional planes L and M through p and q, respectively, the sections $B \cap L$ and $C \cap M$ are homothetic.

Jozsef Solymosi

Additive Discrete Geometry

One of the most important results in discrete geometry,a theorem of Szemerédi and Trotter [2], gives a sharp bound on the maximum number of incidences between points and lines in the Euclidean plane. In particular it says that n lines and n points determine at most $O(n^{4/3})$ incidences. Let us suppose that an arrangement of n lines and n points defines $cn^{4/3}$ incidences, for a given positive c. It is widely believed that such arrangements, where the number of incidences is close to the maximum, have special structure. However no results are known in this direction. There are numerous proofs of the Szemerédi-Trotter theorem (the most elegant is Székely's [3]) but none of them gives information about the structure of arrangements with many incidences. In this talk we mentioned that if n is large enough and the number of incidences is at least $cn^{4/3}$ then the arrangement contains a triangle. This seemingly obvious statement is quite difficult to prove, the only known proof uses Szemerédi's Regularity Lemma [1]. We gavefurther examples how to analyze extremal point-line arrangements using methods from algebra and number theory.

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Alina Stancu

From a characterization of ellipsoids to the p-affine surface area

Characterizing the ellipsoids as the only convex bodies with sufficiently regular boundary which are homothetic to their illumination, or p-illumination, bodies will lead naturally to an interpretation of the p-affine surface area. As consequences we will discuss a couple of inequalities relating these affine quantities to volumes.

Konrad J. Swanepoel

Explicit upper bounds for edge-antipodal polytopes

A d-polytope P is *edge-antipodal* if for any two vertices x and y joined by an edge there exist two parallel hyperplanes, one through x and one through y, such that P is contained in the closed slab bounded by the two hyperplanes. This notion was introduced by Talata (1999), who conjectured that the number of vertices of an edge-antipodal 3-polytope is bounded above by a constant. Csikós (2003) proved an upper bound of 12, and K. Bezdek, Bisztriczky and Böröczky (2005) gave the sharp upper bound of 8. Pór (200?) proved that the number of vertices of an edge-antipodal d-polytope is bounded above by a function of d. However, his proof is existential, with no information on the size of the upper bound. Our main result is an explicit bound.

THEOREM Let $d \geq 2$. Then the number of vertices of an edge-antipodal d-polytope is bounded above by $(\frac{d}{2}+1)^d$.

This theorem is proved by considering a metric relative of edge-antipodal polytopes that we call subequilateral polytopes. For more detail, as well as references to the literature, see

http://arxiv.org/math.MG/0601638

Roman Vershynin

New convex geometry problems in linear programming

The Simplex Method is the oldest and easiest algorithm in Linear Programming. Nevertheless, it puts the theory of computing in an awkward position. This is not a polynomial time algorithm (counterexamples are known), but in practice it runs in polynomial time. To theoretically explain the strange behavior, Spielman and Teng introduced the notion of the Smoothed Analysis of Algorithms. There, one ”smoothes” an input by a small random perturbation, in hope that this models ”most” practice problems. Spielman and Teng showed that the smoothed complexity of the simplex method is polynomial. Their analysis brings up a variety of new problems in convex geometry. We go one step further to show that the number of steps in the smoothed simplex algorithm is actually polylogarithmic, rather than polynomial, in the number of constraints of the linear program.

Wolfgang Weil

Directed tomographic transforms

(based on joint work with Paul Goodey)

The basic problem in Geometric Tomography is to retrieve information about a compact (convex or star-shaped) set $K \subset \mathbb{R}^d$ from data arising from sections or projections of K. Generalizing classical results on projection or section functions for centrally symmetric bodies, we introduce directed section functions of star bodies and two different types of directed projection functions of convex bodies. These are functions on the flag manifold $\{(L,u)\}$, where L varies among the j-dimensional subspaces $L \subset \mathbb{R}^d$, $1 \leq j \leq d-1$, and u is a variable unit vector in L. These directed section resp. projection functions determine a body K uniquely (resp. uniquely up to a translation). As a more general problem, one can consider the averaged directed section and projection functions (obtained as integrals over all j-dimensional subspaces L containing the direction u) and ask whether even they determine the underlying body. In the main part of the lecture, we study certain of these averaged functions and show relations between them as well as uniqueness results.

It turns out that uniqueness holds for a large range of dimensions d and j, but that there are also infinitely many pairs (j, d) where uniqueness fails. The proofs are based on the fact that the considered tomographic transforms can be described by linear operators on the unit sphere S^{d-1}, which intertwine the action of the rotation group. The injectivity properties of the operators are represented in the non-vanishing of the multipliers w.r.t. spherical harmonics. The explicit behaviour of the multipliers is complicated but recursion formulas leading to the mentioned injectivity results were obtained using Zeilberger's algorithm.

Elisabeth Werner
Spaces between polytopes and zonotopes
(joint work with Y. Gordon, A. Litvak and C. Schütt)

We study geometric parameters associated with the Banach spaces $(\mathbb{R}^n, \|\|_{k,q})$ normed by

$$\|x\|_{k,q} = \left(\sum_{1 \le i \le k} |< x, a_i >|^{*q} \right)^{1/q},$$

where $\{a_i\}_{i \le N}$ is a given sequence of N points in $\mathbb{R}^n$, $1 \le k \le N$, $1 \le q \le \infty$ and $\{\lambda_i^*\}_{i \ge 1}$ denotes the decreasing rearrangement of a sequence $\{\lambda_i\}_{i \ge 1} \subset \mathbb{R}$. In particular, we give estimates on the volume of the unit balls of these spaces.

Jörg M. Wills
On the zeros of the Ehrhart polynomial
(joint work with M.Henk)

The Ehrhart polynomial counts the number of lattice points of the integer multiples nP of a lattice polytope P in $\mathbb{Z}^d$. It can be written as a product

$$G(s, P) = \prod_{i=1}^{d} (1 + \frac{s}{\gamma_i}),$$

where $s \in \mathbb{C}$ is the complex variable and $-\gamma_i \in \mathbb{C}$ the zeros (or roots) of G. For $s \in \mathbb{N}$, $G(s, P)$ counts the lattice points of $s\,P$. The motivation of such investigation comes from the interaction between P and the zeros $-\gamma_i$, i. e., between Geometry and Algebra. In this talk we discuss two topics:

- Relations between the γ_i and Minkowski's successive minima, in particular between their arithmetic and geometric means.
- Polytopes with all zeros $-\gamma_i$ on the line Re $s = -\frac{1}{2}$. In this case the Ehrhart polynomials have some properties in common with the Riemann ζ-function, as Bump et al. (2000) and Rodriguez-Villegas (2002) proved.

We show some basic properties of these polytopes.

List of Participants

Aliev, Iskander (University of Edinburgh)
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Bezdek, Karoly (University of Calgary)
Bisztriczky, Ted (University of Calgary)
Boroczky Jr, Karoly (Alfréd Rényi Institute of Mathematics)
Brandenberg, Rene (Technische Universität München)
Bremner, David (University of New Brunswick)
De Loera, Jesus (University of California, Davis)

Ehrenborg, Richard (University of Kentucky)
Fodor, Ferenc (University of Szeged)
Garcia-Colin, Natalia (University College London)
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Schneider, Rolf (University of Freiburg)
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Chapter 3

Coarsely Quantized Redundant Representations of Signals (06w5078)

March 11 – March 16, 2006

Organizer(s): Sinan Güntürk (Courant Institute of Mathematical Sciences), Thao Nguyen (City College, CUNY), Alexander M. Powell (Vanderbilt University), Özgür Yılmaz (University of British Columbia)

Overview and Scientific Objective of the Workshop

Digital data is the driving force behind much of our modern technology. The Internet and cellular phones are ubiquitous examples of the need to handle information accurately, efficiently, and robustly. The fact that digital signals and data sets can be processed with great precision and speed places high demands on providing accurate conversion between the analog and digital worlds. However, the technology used in the analog to digital (A/D) conversion by necessity involves analog devices which have physical limitations that, at first sight, conflict with these accuracy demands. For example, typical printers are only capable of applying a very limited set of ink tones (which may be as small as a single black tone) to paper for rendering intermediate grayscale levels. Similarly, in audio applications, the inaccuracy of analog circuits in working with binary expansions is a problem that routinely needs to be addressed.

To cope with these problems, engineers have empirically developed special signal processing techniques leading to alternative signal and number representations that are quite different from the standard decimal or binary representations. Typical techniques take advantage of the fact that, although most analog devices used in A/D conversion fail to provide high precision in the amplitude domain, many of them are capable of sampling very densely in the time or the space domain over which the signals are defined. In these techniques, judiciously chosen dense arrangements of a limited set of discrete amplitude values are used to approximate the underlying analog signals. In the early days of printing, a similar task used to be carried out by an experienced halftoning artist. In today's digital world, there are several automated schemes that are implemented efficiently in consumer products, not only for digital halftoning, but also for audio encoding and decoding.

Although some of the very basic examples of such schemes were invented more than 30 years ago and the engineering practice has matured extensively in the meantime [1], a general mathematical theory of coarsely quantized representations did not experience a parallel development. In fact, many of the fundamental questions remained open until recently. In the past few years, a new interdisciplinary research activity was initiated by the breakthrough work of Daubechies and DeVore [2], and was followed by burgeoning interest among other mathematicians and engineers. This new theory has led to fruitful cross-pollination with applications in several disciplines of pure and applied mathematics, such as harmonic analysis, combinatorics, number theory, dynamical systems, approximation theory and information theory.

The BIRS workshop *Coarsely Quantized Redundant Representations of Signals* brought together expert mathematicians and theoretical engineers working on problems related or relevant to signal representations in coarsely grained environments. The strong interdisciplinary nature of the subject meant that mathematicians and engineers who are otherwise less likely to interact due to the distances in their respective fields of expertise had a great opportunity to meet, present their work and start future collaborations in the stimulating atmosphere of BIRS.

Background, Recent Developments and Open Problems

Coarse quantization is emerging as a part of a larger technological trend towards robust and redundant systems which utilize low cost components where the fundamental trade-off is between redundancy and hardware precision. Daubechies and Calderbank introduced the paradigm of "democracy of bits" to discuss this in a general setting [3]. This trend is further illustrated by the recent parallel advances in the sensors community, where dense microsensor networks using large numbers of highly correlated, but inexpensive, sensors are currently a huge area of focus. It is no coincidence that much of today's cutting edge technology, such as Super Audio CD (SACD) systems, contains coarse quantization as a key component.

From an application point of view, the main focus of the workshop has been sigma-delta modulation in analog-to-digital encoding and error diffusion in digital halftoning:

Sigma-delta modulation is an oversampled analog-to-digital (A/D) conversion method for audio signals. A typical approach in obtaining digital representations of audio signals is to take the binary expansion of signal samples taken at the critical Nyquist rate of 44.1 KHz. However, this method suffers from high accuracy demands on the analog circuitry that is to be employed. Sigma-delta modulation, on the other hand, allocates as few as 1 bit for the discretization of signal samples taken at rates as high as 100 times faster than the critical rate. At the heart of this method lies an algorithm that recursively rounds (quantizes) each sample value to one of the few quantization levels in a way that is suited to achieve small global reconstruction error rather than small individual sample error.

The analogous problem in two dimensions is the digital halftoning process employed in image printing. While today's printers can achieve high resolution in space (such as 1200 dpi), most often, only single-tone dots are available to be printed on the corresponding medium. In this case, only 0's and 1's can be used to represent efficiently any shade of the gray-scale. Error diffusion [4] can be thought of the exact analog of sigma-delta modulation in two dimensional signals and achieves this conversion effectively.

Sigma-delta modulation has its roots in the 60's when Inose and Yasuda developed the first unity bit encoding by negative feedback [5]. Similarly, the classical error diffusion algorithm of Floyd and Steinberg was invented in the 70's [9]. In the late 80's, interest in the information theory community was led by Gray who discovered some of the best known theoretical results, e.g., [7, 8]. It was, however, only in the late 90's that an approximation theoretical framework was given to the problem by Daubechies and DeVore [2]. Since then, there has been a rapid development in the theoretical analysis of sigma-delta modulation with emerging connections to other mathematical fields [9, 10, 11].

At the core, the problem of signal quantization is linked with the underlying choice of signal representation. Overcomplete data expansions not only provide robustness with respect to noise and data loss, but most coarse quantization algorithms are, in fact, specifically built around exploiting redundancy. Frame theory [12] provides a mathematical framework for stably representing signals by overcomplete expansions. Work during the past two decades on sampling wavelet frames, Gabor frames, and Fourier frames has greatly advanced this field. More recently, finite frames have gained attention as a piece of this theory which is tailored for applications involving finite, but potentially high dimensional, data. For example, finite frame expansions have recently been proposed as joint source-channel codes for erasure channels [13], for multiple-antenna code design [14], and for modified quantum detection problems [15].

The problem of one-bit quantization is related to many other mathematical disciplines. Some of these would be regarded as classical now, but at the same time many connections are new or relatively unexplored.

- Combinatorics: combinatorial, geometric and linear discrepancy theory form the common grounds for understanding some of the fundamental problems in one-bit quantization, especially in understanding some of the universal lower bounds.

- Dynamical systems: most algorithms that yield efficient one-bit or low-bit representations are based on dynamical systems (sigma-delta modulation and error diffusion both fall into this category). Typically, underlying these dynamical systems are certain non-expanding piecewise affine maps on the Euclidean space. There is a strong connection between understanding the stability properties of these maps in high dimensions and the asymptotic performance of the associated algorithms.
- Functional and harmonic analysis: a natural analytical framework to study oversampled coarse quantization lies within the theory of frames. While the quantization problem in an orthonormal basis is trivial, there is no corresponding theory for redundant representations. Progress in this direction can also have far reaching consequences in compression and denoising.

Presentation Highlights

As noted earlier, one of the highlights of the workshop was the diversity of the scientific disciplines represented by the participants. Ingrid Daubechies and Ronald DeVore, two leading mathematicians in applied mathematics and approximation theory, were joined by experts in various disciplines, including John Benedetto and Yang Wang (harmonic analysis), Peter Casazza (functional analysis, Banach spaces, frame theory), Jan Allebach and Chai Wah Wu (halftoning, image processing), Tomasz Nowicki (dynamical systems), Benjamin Doerr (combinatorics, discrete mathematics), Vinay Vaishampayan, Helmut Boelcskei and Vivek Goyal (information theory, signal processing), Matt Yedlin (electrical engineering, circuit theory). This diversity helped create a productive and intellectually inspiring environment for the workshop.

Below is a list of talks presented at the workshop grouped with respect to subject.

Sigma-Delta Quantization and A/D Conversion

The talks in this category focused on sigma-delta quantization algorithms and their use as an A/D conversion method, emphasizing the mathematical as well as practical aspects.

The workshop started with an expository opening talk by *Ingrid Daubechies*, titled “Mathematical study of coarsely quantized representations of signals: what, why, how – an overview.” In this talk, Daubechies discussed the major practical challenges in A/D conversion, and she formulated the corresponding mathematical problems. Then, she gave a survey of the main results in this direction obtained since 1998, referring to work she has done jointly with Ron DeVore, as well as several results of Güntürk, Thao, Powell, Vaishampayan, and Yılmaz.

The next talk in this category was presented by *Thao Nguyen* and entitled “The Tiling Phenomenon of Sigma-Delta Modulators.” As of now, sigma-delta quantization is the only known and efficient algorithm for the coarse quantization of redundant expansions. While this method has proved successful in practice, its mathematical analysis is difficult. This is due to the fact that sigma-delta modulation is based on a nonlinear dynamical system. In general, one does not know how to derive explicitly the expression of the output of such a system. It was discovered however that the quantizers of an important generic family of sigma-delta modulators possess an outstanding property called “tiling”, which provides substantial algebraic information about their outputs. Nguyen gave a retrospective on this research from its origin to current investigations. He covered several issues including the mathematical origin of tiling and the consequence of tiling for the rigorous prediction of the quantization error.

Matt Yedlin focused on the practical aspects of sigma-delta algorithms in his talk “Industrial Applications of Sigma-Delta Converters.” Yedlin emphasized the major role sigma-delta based A/D conversion plays in modern technological applications including digital audio receivers, sampling music synthesizers, biomedical data acquisition (EEG), seismometers and wireless communication systems. After a historical account of the subject, which included the first patent filed about sigma-delta converters, Yedlin focused on the current industrial applications of sigma-delta A/D converters over a wide range of frequency and resolution settings. Yedlin then profiled a number of currently available commercial products, ranging from the 19 Hz 24 bit AD (Analog Devices) 7783 which is used for pressure and temperature sensing to a high-speed sigma-delta converter, AD 7725 with an input bandwidth of 350kHz, used for radar and sonar data acquisition. Finally, Yedlin illustrated some of the practical issues with a hardware demonstration of the AD7725 sigma-delta converter.

Bin Han, in his talk titled "Time Average MSE Analysis for the First Order Sigma-Delta Modulator," presented improved estimates for the mean square error (MSE) of first-order sigma-delta modulation. The goal of Han's research is to close the gap between the numerically obtained (faster) error decay rates (as a function of the redundancy) and the rigorously obtained (slower) error decay rates. Building upon several results of Daubechies, DeVore and Güntürk, Han proved improved estimates for the MSE under certain additional assumptions on the input signals.

Frame expansions and quantization

A natural analytical framework to study oversampled coarse quantization lies within the theory of frames. While the quantization problem in an orthonormal basis is trivial, the corresponding problem for redundant representations is highly challenging and mostly unresolved. The talks in this category highlighted the recent progress toward a comprehensive theory of quantization for redundant frame expansions.

The first talk was presented by *John Benedetto* on "Sigma-Delta quantization and finite frames". Benedetto focused on first-order sigma-delta schemes in the setting of finite frames, and presented several error estimates that prove that sigma-delta quantizers outperform pulse code modulation (PCM) schemes when the frame is sufficiently redundant. His presentation emphasized the techniques for obtaining refined error estimates based on analytic number theory tools (e.g., [10]). The theory presented in Benedetto's talk and its extension to higher order schemes is a collaboration with Alex Powell and Özgür Yılmaz. The analytic number theoretic portion also includes Aram Tangboondouangjit in the collaboration.

Mark Lammers's talk "Alternate Dual Frames and Sigma-Delta Quantization" focused on a joint work with Alex Powell and Özgür Yılmaz. Lammers investigated the use of alternate dual frames in sigma-delta quantization of finite frame expansions, and proved that reconstruction with alternate dual frames can substantially reduce quantization error in many settings, including pointwise and MSE error estimates for higher order sigma-delta schemes. In particular, Lammers showed that using alternate dual frames is an effective way of dealing with the "boundary term" problem, and allows for k-th order sigma-delta schemes to achieve pointwise error of order N^{-k} when the canonical dual frame may not yield the same result.

Next, *Peter Casazza*, a leading expert in frame design, gave a stimulating talk in which he identified several frame design problems and conjectures for sigma-delta quantization. This talk provoked many questions and comments from the participants, helping the participants organize and isolate various critical issues and relevant problems.

In his talk "Sigma-Delta Quantization and the Traveling Salesman Problem," *Yang Wang* considered sigma-delta quantization in the finite frame expansion setting with the maximal and mean square errors as measures of quantization distortion. Wang showed that this problem is related to the classical Traveling Salesman Problem (TSP) in the Euclidean space. Using the fact that sigma-delta error bounds depend on ordering of frame elements and incorporating some *a priori* bounds for the Euclidean TSP, Wang showed that, in general, sigma-delta modulation is superior to PCM in the quantization error performance. Wang also gave a recursive algorithm for finding an ordering of the frame elements that leads to good maximal error *and* mean square error at the same time.

The next presentation in this category was by *Vern Paulsen* titled "Frame Paths and Sigma-Delta Quantization," a joint work with Bernhard Bodmann. Building on the ideas of Benedetto-Powell-Yılmaz, Paulsen studied the performance of finite frames for the encoding of vectors by applying sigma-delta quantization algorithms to the sequence of frame coefficients. Paulsen's focus was on frame paths which allow the user to systematically increase the redundancy of the frame to compensate for the errors created by quantization. Paulsen showed that the earlier bounds for the worst-case quantization error can be improved. Moreover, he gave lower bounds on the worst-case error. In addition, Paulsen introduced some new frame paths, which can in turn be used to construct finite frames with arbitrarily high redundancy that have some interesting features.

Bernhard Bodmann gave the next talk "Zero-terminated frame paths and second order sigma-delta quantization" on a joint work with Vern Paulsen, which extended the ideas presented in Paulsen's talk to higher order sigma-delta schemes. In particular, Bodmann studied the performance of finite frames for the encoding of vectors by applying standard second order sigma-delta quantization to the frame coefficients, where the frames under consideration are obtained from regular sampling of a frame path in a Hilbert space. In order to achieve error bounds that are comparable to the results for second-order sigma-delta quantization of oversampled bandlimited functions, frame paths that terminate in the zero vector were constructed.

Shidong Li talked about “A Dual Frame Formula and Pseudoframes with Applications.” In his talk, Li obtained a dual frame formula with which optimal duals can be obtained through a parametric sequence of functions. Next, Li considered *pseudoframes for subspaces* (PFFS), an extension of the notion of frames. PFFS are collections similar to a frame for a subspace X in H. Unlike the elements of a frame for X, however, the elements of a pseudoframe for X need not be contained in X. Li discussed characterizations, constructions, and several key properties of PFFS, and gave examples that illustrate the benefits of extending the concept of frames. Using the theory of PFFS, Li constructed compactly supported and/or fast decaying dual Gabor functions as well as arbitrary compactly supported (pseudoframe) biorthogonal duals of a B-spline Riesz sequence. Other highlights of Li’s presentation include the existence of tight pseudo-duals of frames of translates, the illustration of how PFFS can be incorporated in the construction of biorthogonal wavelets so as to obtain results that are more favorable, how PFFS can be exploited in a quite general optimal noise suppression problem, and finally, examples where pseudo-dual or dual frame formula can be used to reduce perturbation and/or quantization noise in a general redundant frame or pseudoframe representation.

In her talk “Fusion Frames: Redundant Representations under Distributed Processing Requirements,” *Gitta Kutyniok* considered a different generalization of the concept of a frame for a Hilbert space. The focus of her collaboration with Peter Casazza and Shidong Li was on applications under distributed processing requirements such as sensor networks or sensorineural systems. Kutyniok noted that for such applications frames can be used locally, but the global structure cannot be handled by using classical frame theory. She then introduced the notion of a fusion frame, which is a sequence of subspaces satisfying a frame-like property. This property was used to ensure that local collections of frame elements linked together by using a fusion frame yield a global frame structure. In this sense the new theory of fusion frames provides the link between distributed and centralized structures. Kutyniok showed that the theory of fusion frames has in fact many similarities with the classical frame theory for sequences of vectors. Moreover, this theory can be used to ease the construction of frames by considering local structures. An application to noise reduction under distributed processing requirements was also discussed.

Helmut Boelcskei’s talk was on “Noise shaping and predictive quantizers of order $N > 1$ for arbitrary frame expansions.” After briefly reviewing the classical paper by Tewksbury and Hallock on “Oversampled, linear predictive and noise-shaping coders of order $N > 1$,” Boelcskei discussed his work with Hlawatsch that extends of the ideas of Tewksbury and Hallock to the case of oversampled filter banks. This work was then used to outline how noise shaping and linear predictive coders can be applied to arbitrary frame expansions. In the last part of his talk, Boelcskei described such an extension and provided a proof of the reconstruction MSE decaying as r^{-2N+1} where r denotes the frame redundancy and N is the order of the noise shaping filter.

“PCM Quantization Errors and the White Noise Hypothesis” by *David Jimenez* was the last talk in this category. The White Noise Hypothesis (WNH), introduced by Bennett approximately a half century ago, assumes that in PCM quantization scheme the errors in individual channels behave like white noise, i.e., they can be modeled as independent and identically distributed random variables. The WNH has been the key in estimating the mean square quantization error in the case of PCM. In this joint work with Yang Wang, Jimenez took a close look at the validity of the WNH. He proved that in a redundant system the errors from individual channels can never be independent, implying that, strictly speaking, WNH is not valid. Jimenez also presented numerical experiments which confirm this theoretical result whenever the quantization is coarse, i.e., the quantizer step size is large. On the other hand, Jimenez showed that with fine quantization, the WNH is essentially valid, in that the errors from individual channels become asymptotically *pairwise* independent and each uniformly distributed in $[-\Delta/2, \Delta/2)$, where Δ denotes the stepsize of the quantization.

Digital Halftoning

While today’s printers can achieve high resolution in space (such as 1200 dpi), most often, only single tone dots are available to be printed on the corresponding medium. In this case, only 0’s and 1’s can be used to represent efficiently any shade of the gray-scale. Error diffusion can be thought of the exact analog of sigma-delta modulation in two dimensional signals and achieves this conversion effectively. One of the goals of this workshop was to provide a platform where researchers in the fields of audio quantization and digital halftoning could meet. The talks listed in this category are on digital halftoning.

The first talk in this category was “Digital halftoning - a model-based perspective” by *Jan Allebach*.

Allebach started his talk with an overview of the basic principles of digital halftoning algorithms, which are used in virtually all printing devices and many display systems as well, in order provide a visually pleasing rendering of continuous-tone images that can be stably reproduced by the output device. He then described the two dominant desktop printing technologies, inkjet and electrophotography, and discussed how the physical characteristics of these technologies impact the preferred choice of halftoning algorithm. Next, he discussed hardware and software architectures used in desktop printing systems, and showed how these architectures impact the computational approaches that can be used for digital halftoning. After this extensive review, Allebach described how model-based approaches can yield high quality halftoning algorithms that perform robustly on the intended platform, subject to the constraints on computational architecture and resources. He first discussed the direct binary search (DBS) algorithm that minimizes a cost function based on visual quality, and represents a gold standard of sorts for dispersed-dot halftoning. Allebach noted that the DBS algorithm is heuristic in nature, however, it is still possible to show that it simultaneously minimizes mean-squared and maximum error. Furthermore, DBS is too computationally demanding to be used in desktop printing applications; but it serves as the basis for the design of other algorithms that are implementable with real systems. Then Allebach focused on two such solutions: tone dependent error diffusion, which achieves nearly the same quality as DBS, and can be found in inkjet products, and the hybrid screen which is suitable for electrophotographic engines. The hybrid screen creates stochastic, dispersed-dot textures in highlights that gradually transform to clustered-dot, periodic textures in mid-tones. Allebach concluded his talk with a discussion of measures for halftone image quality where he proposed a new approach to assessing image quality based on the directional sequency spectrum.

Tomasz Nowicki's talk was titled “On the existence of a minimal attractor in Convex Dynamics,” joint work with Charles Tresser. The talk was on the convex dynamics concerning a family of the piecewise affine maps (translations) which were inspired by some applications in printing and other analog-to-digital (or continuous-to-discrete) coding of sequences of signals which employ error diffusion. Here, the pieces are Voronoi regions of the corners (sets of points for which such a corner is the closest one) of some polytope(s) and the translation vectors are the vectors from the respective corners to arbitrary points of the polytope. The fundamental theorem Nowicki presented was on the boundedness of the dynamics. He showed that there exists a unique minimal bounded invariant set which contains the polytope itself and all the corners of the Voronoi regions.

In his talk “Discrepancy and Digital Halftoning,” *Benjamin Doerr* first gave a brief introduction to the field of discrepancy theory. Doerr explained various notions of discrepancy, such as combinatorial, geometric, and linear discrepancy, and tied these notions with hypergraph colouring problems. He then showed how Asano et al [16] modeled the digital halftoning problem as a linear discrepancy problem. Doerr proposed a novel solution to this problem based on a technique he introduced, which he called “dependent randomized rounding”. In his solution, Doerr is able to obtain probabilistic error bounds as well as deterministic ones.

Chai Wah Wu presented his work during his talk titled “Some techniques for speeding up digital halftoning algorithms.” In the first part of his talk, Wu described some techniques to speed up two high-quality digital halftoning algorithms: Direct Binary Search (DBS) and error diffusion. Noting that DBS operates by considering pairs of halftone pixels to swap and does so when the swap reduces the overall error, Wu showed that one can reduce the number of operations needed to evaluate and update a potential swap of pixels in each step of DBS by extracting some redundancy in the computation. He also considered some heuristics for selecting which pairs of pixels to evaluate that reduce the total number of pairs evaluations. In the second part of the talk, Wu considered the use of lookup tables to speed up the error diffusion algorithm. In particular, he presented an implementation of the Floyd-Steinberg error diffusion algorithm where the computation requires only 1 addition and 3 table lookup operations per pixel. Based on this, Wu concluded that it may be possible to implement error diffusion-like algorithms at speeds similar to screening algorithms.

Sparse expansions and quantization

The role of sparse expansions in signal processing and information theory has been understood recently, with the leading works of Candès, Donoho, Gilbert, Strauss and Tropp. The talks in this category are on the theory of sparse expansions and the related quantization issues.

In his talk “Instance-optimal estimates in Compressed Sensing,” *Ronald DeVore* first gave an extensive review of the recently developing theory of compressed sensing. In DeVore’s words, “Discrete Compressed

Sensing samples a discrete signal $x \in \mathbb{R}^N$ by n linear measurements, each an inner product of x with a vector from $\mathbb{R}^N$. If n is the number of measurements allocated to the sensor then the whole process can be represented by an $n \times N$ matrix Φ. The vector $y = \Phi(x)$ represents the n samples we have about x. A decoder Δ is a mapping from $\mathbb{R}^n \to \mathbb{R}^N$. The vector $\Delta(\Phi(x))$ is the approximation we have to x from the information y." In his talk, DeVore discussed how well such an encoding-decoding scheme can perform given the pair n and N by comparing the error $\|x - \Delta(\Phi(x))\|_{\ell_p}$ to the best k-term approximation error $\sigma_k(x)_{\ell_p}$.

Next, *Vivek Goyal* talked on "Sparsity and Quantization: When Undersampling is Oversampling." Goyal noted that in a variety of applications, sparse solutions to underdetermined linear systems of equations are preferred, and for at least thirty years, practitioners have used 1-norm minimizations to promote sparsity. Recent research has turned this ad hoc procedure into the basis for certain deterministic and probabilistic guarantees, fueling further work in sparsity-based modeling. In his talk, Goyal related sparsity-based modeling to his earlier work on quantized overcomplete expansions, showing that large improvements are obtained by exploiting boundedness of quantization error. Goyal also discussed universality of such source coding techniques.

Holger Rauhut considered the problem of reconstructing a sparse multivariate trigonometric polynomial from few sample values in his talk "Random Sampling of Sparse Trigonometric Polynomials". Rauhut investigated the problem in a probabilistic framework in which he modeled the sampling points as being random and independently distributed according to one of the following two probability models: (a) continuous uniform distribution on the unit cube, (b) discrete uniform distribution on a finite set of equidistant sampling nodes. The latter corresponds to the problem of recovering a sparse vector from few samples of its discrete Fourier transform. Rauhut studied two recovery methods: Basis pursuit (BP), which was studied recently by Candès, Romberg, and Tao in the context of the discrete Fourier transform, and orthogonal matching pursuit (OMP), a greedy algorithm, studied recently by Tropp et al. for the recovery problem in the context of Gaussian and Bernoulli measurements. Rauhut then presented a result that applies simultaneously to both of the probability models (a) and (b) and to both BP and OMP. His theorem states that if the number of samples N is large enough compared to the sparsity (but possibly much smaller than the overall dimension of the underlying space of trigonometric polynomials) then with high probability the polynomial can be recovered both by BP and OMP. Rauhut noted that this includes a previous result of Candès, Romberg and Tao as a special case (with a different proof). Finally, Rauhut provided numerical experiments which confirm the proposed methods.

The last talk in this category was "The diffusion framework: a computational approach to data analysis and signal processing on data sets" by *Yosi Keller*. Keller noted that the diffusion framework is a computational approach to high dimensional data analysis and processing. Based on spectral graph theory, Keller defined diffusion processes on data sets, then noted that these agglomerate local transitions reflect the infinitesimal geometries of high-dimensional datasets, and finally used these observations to obtain meaningful global embeddings. The eigenfunctions of the corresponding diffusion operator (Graph Laplacian) provide a natural embedding of the sets into a Euclidean space. Keller then showed that the eigenfunctions of the Laplacian form manifold adaptive bases, which pave the way for the extension of signal processing concepts and algorithms from n-dimensional Euclidean space $\mathbb{R}^n$ to general data sets. As an example, Keller applied this approach to collaborative filtering and improving the resolution of coarsely quantized signals.

Outcome of the Meeting and Scientific Progress Made

The organization of this meeting was motivated by the increasing interest in the mathematical and electrical engineering community on coarse quantization strategies for redundant representations of signals. The exclusively positive feedback received from the participants emphasized the scientific appropriateness of the meeting as well as the exciting nature of doing interdisciplinary research. The superb location and facilities of the BIRS was also a big bonus. New collaborations have started (e.g., a joint mathematics-circuits initiative between UBC, Princeton, NYU, and Georgia Tech; joint work by Doerr, Güntürk and Yılmaz) and further meeting ideas were developed (e.g., a summer school for mathematicians and circuit engineers). Compared to some of the previous similar meetings and focused research activities on this topic over the past years, this workshop has been one of the largest and most extensive. It is expected that the trend will continue and more meetings will be organized in the near future. (One of the invitees of the meeting, who unfortunately could not attend, is currently planning the organization of an ICASSP special session for 2007 on the same subject.)

List of Participants

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Chapter 4

Reaction-diffusion and Free Boundary Problems (06w5045)

March 18 – 23, 2006

Organizer(s): Peter Constantin (University of Chicago, USA), François Hamel (Université Aix-Marseille III, France), Robert Jerrard (University of Toronto, Canada), Jean-Michel Roquejoffre (Université Paul Sabatier, Toulouse, France), Lenya Ryzhik (University of Chicago, USA)

Introduction and overview of the Field

Reaction-diffusion equations, semilinear diffusion equations and free-boundary problems form an important domain of the theory of partial differential equations that is both very rich and challenging mathematically and is intricately related to numerous applications in physical, chemical and biological sciences.

The purpose of this conference was to bring together researchers in various areas of this field as well as applied mathematicians to highlight the recent developments and discuss the open problems that are of interest both from the mathematical perspective and from the point of view of applications. Due to the enormous activity of the field, it was impossible to cover every topic in reaction-diffusion equations. We have, chosen to lay the emphasis on the following items, that we considered as particularly interesting in view of their mathematical richness, and potential applications. The following subject have particularly been focused upon:

- *Singular perturbations, free boundary problems and reaction-diffusion equations.* This topic is a classical one in reaction-diffusion equations - see for instance Fife [15], but has undergone very important developments in the last years, such as the recent progress in the proof of the de Giorgi conjecture, the description of the Ginzburg-Landau vortices dynamics, the regularity theory of free boundary problems and the dynamics of reaction-diffusion systems.

- *Complex propagation phenomena in reaction-diffusion equations*. Although some mathematical milestones in the theory of reaction-diffusion equations date back to the 1930's, they were mainly concerned with homogeneous situations. More realistic heterogeneous reaction-diffusion equations or systems have been handled only relatively recently. Over the recent years, mathematical results have considerably enriched our understanding of these models and their biological applications. A very partial list of examples of areas where considerable recent progress has been made include the propagation phenomena related to the existence and the dynamical properties of travelling fronts in heterogeneous environments.

- *Homogenization, stochastics and dynamics of reaction-diffusion equations.* Homogenization of reaction-diffusion and Hamilton-Jacobi problems in a periodic medium is by now well understood. However, only recently progress has been made in similar issues for random media. Reaction-diffusion equations are closely connected to the large deviation problems for diffusion processes and weak stochastic perturbations of dynamical systems. Recently much progress has been made in asymptotic theories in this area, including the situations when the underlying dynamical system is itself random.

Recent Developments and Open Problems

Singular perturbations, free boundary problems and reaction-diffusion systems

Phase transitions, geometric methods in elliptic equations, and the de Giorgi conjecture

The equilibrium state of a binary alloy may be described by the celebrated Allen-Cahn equation: if $u(x) \in (-1,1)$ denotes the local proportion of each component, we have

$$-\Delta u = \frac{1}{\varepsilon^2}W'(u), \quad x \in \mathbb{R}^N \tag{4.1}$$

where W is an even potential, having global minima at ± 1. When ε is - at least formally - sent to 0, the limiting solution u of (4.1) takes the values 1 or -1, the interface between the regions $\{u = 1\}$ and $\{u = -1\}$ being separated by a surface Γ with zero mean curvature. The interface equation is not so hard to derive in a formal fashion: assuming that Γ is smooth, and letting $\phi_0(x)$ be the unique solution of

$$-\phi_0'' = W'(\phi_0), \quad \phi_0(\pm\infty) = \pm 1, \ \phi_0(0) = 0,$$

a plausible ansatz for the solution u_ε of (4.1) is

$$u_\varepsilon(x) \sim \phi_0(\frac{d(x)}{\varepsilon}), \quad \text{with } d(x) = \mathrm{dist}(x,\Gamma) \text{ (signed distance)}$$

which yields: $\Delta d = 0$ on Γ. This precisely says that the mean curvature of Γ is zero. A mathematically rigorous derivation of that fact is, of course, much more difficult. Modica and Mortola [31] prove the following version of this fact: a sequence of minimizers $(u_\varepsilon)_\varepsilon$ of the functional

$$u \mapsto \int (\frac{1}{2}|\nabla u|^2 - \frac{1}{\varepsilon}W(u))\, dx$$

converges to a difference of characteristic functions of the form $\chi_E - \chi_{\Omega\setminus E}$; moreover the set $\partial E \cap \Omega$ is a minimal hypersurface.

The de Giorgi conjecture states the following:

(i) (Nonexistence part) Given a potential W as above, analytic in its argument, let $u(x)$ satisfy

$$-\Delta u = W'(u), \ x \in \mathbb{R}^N; \qquad \frac{\partial u}{\partial x_N} \geq 0. \tag{4.2}$$

Then the level sets of u are hyperplanes, at least if $N \leq 8$.

(ii) (Existence part) For $N \geq 9$, there are truly multi-dimensional solutions of (4.2).

This conjecture was motivated by:

- a theorem of J. Simons [38], asserting that any minimal graph, defined over the whole space $\mathbb{R}^{N-1}$, has to be an affine function,

- a theorem of Bombieri, de Giorgi, Giusti [8] asserting that, for $N - 1 = 2m \geq 8$, the Simons cone $\{\sum_{i=1}^{m} x_i^2 = \sum_{i=m+1}^{2m} x_i^2\}$ is minimal.

The de Giorgi conjecture is also deeply related to the study of the level sets of converging sequences of solutions of (4.1): the nonexistence part says that these level sets are uniformly Lipschitz - and that an internal layer expansion is justified.

The nonexistence part of the conjecture was recently proved, in full generality, by Savin [34]. Earlier results were proved by Ghoussoub-Gui [21] ($N = 2$), Ambrosio-Cabré [1] ($N = 3$), Ghoussoub-Gui [22] (particular cases of the dimensions 4 and 5).

Free boundaries in reaction-diffusion equations, and their qualitative properties

A typical instance of the free boundary problems on which the conference focused is the following class of parabolic equations

$$T_t - \Delta T = \frac{1}{\varepsilon^2}(1-T)\exp(\frac{T-1}{\varepsilon}) := (1-T)f_\varepsilon(T), \quad x \in \mathbb{R}^N. \tag{4.3}$$

Such an equation is a - fairly good, and still widely employed for qualitative predictions - model for the propagation of a flame in a combustible mixture; the function $T(t,x)$ represents the temperature of the mixture and the right-hand side accounts for the rate at which the chemical reaction proceeds. The parameter ε is the inverse of the - fortunately large - reduced activation energy. As one may realize, the reaction term $f_\varepsilon(T)$ is concentrated at the value $T=1$, which is here the normalized burnt gas temperature. When ε is sent to 0, the problem can be shown - at least in a formal fashion - to tend to the more singular one:

$$T_t - \Delta T = \delta_{T=1}. \tag{4.4}$$

The space is here separated into two regions: $\{T<1\}$ and $\{T=1\}$, and the normal derivative of the temperature - provided it exists! - undergoes a jump of size 1 at the boundary $\partial\{T=1\}$. Deriving (4.4) formally is not so difficult: it is a classical internal layer analysis; doing it in a mathematically rigorous fashion is once again a hard problem.

Important progress has been made in the treatment of free boundary problems by Caffarelli and his collaborators, especially in the understanding of their regularity. The methods range from potential theory and harmonic analysis to geometric measure theory; see for instance the series [9] - regularity of elliptic FBP's, [2] - regularity for the Stefan problem, [10] - monotonicity formulae implying uniform estimates for problems of the type (4.3); see also [11] where a lot of these ideas are exposed. This wide body of methods and ideas have been applied - to many other types of problems, such as homogenization of free boundary problems, singular perturbations - the proof of the de Giorgi conjecture by Savin is inspired by the ideas of Caffarelli *et al.* -, fully nonlinear reaction-diffusion equations...

The dynamics of reaction-diffusion systems

Reaction-diffusion systems may exhibit complex dynamics, and important hints in their description are provided by singular perturbations. Examples of complex dynamics may already be found by the following slight generalization of equation (4.3): assume that the chemical reaction follows the single-step scheme $A \to B$, and assume that the reactant A does not diffuse in the same fashion as the temperature: a new parameter - the Lewis number, denoted by Le - enters into play. Let $Y(t,x)$ denote the mass fraction of the reactant; equation (4.3) becomes

$$\left\{ \begin{array}{rcl} T_t - \Delta T &=& Y f_\varepsilon(T) \\ Y_t - \dfrac{\Delta Y}{Le} &=& -Y f_\varepsilon(T) \end{array} \right. \tag{4.5}$$

This system has 1D travelling wave solutions, see [7]. A famous computation of Sivashinsky [39] indicates that, as Le gets ε-far from 1, the wave destabilizes into multi-dimensional patterns $(Le<1)$ or into pulsating waves $(Le>1)$; this was proved in a rigorous way in [23]. Of interest if the behavior of the flame front - here, the set $\{T-1 \sim \varepsilon\}$ near the critical parameter; if the front is described by a graph $\{y=\Phi(t,x)\}$, an evolution equation is once again provided by Sivashinsky [39] in the form of the celebrated Kuramoto-Sivashinsky equation:

$$\Phi_t + \Delta^2\Phi + \Delta\Phi + \frac{1}{2}|\nabla\Phi|^2 = 0. \tag{4.6}$$

A lot has already been said on (4.6); due to its universal character - it arises in a lot of interface problems - the subject is still extremely active. Its rigorous derivation from (4.5) seems to be a challenging open problem. Depending on the geometry considered and the values of the Lewis number, the flame front may satisfy extremely diverse types of evolution equations; see for instance [27] for a version of (4.5) with $Le<1$.

A singular perturbation may also occur in a reaction-diffusion system under the form of a small diffusion; a generic presentation for the system would be

$$\left\{ \begin{array}{rcl} u_t - \Delta u &=& f(u,v) \\ v_t - \varepsilon\Delta v &=& g(u,v) \end{array} \right. \tag{4.7}$$

Singular perturbation results for ordinary differential equations date back to the early 60's; however a seminal work of C. Jones, unifying all these results in the framework of geometric theory of dynamical systems, has fostered a large body of works investigating complex wave patterns for (4.7). Stability of travelling waves is an important topic that has been addressed to in the workshop; some important problems of the moment include

- Complex flame models - such as flames in two-phase flows;
- biological models - such as the Gray-Scott or Gierer-Meinhardt model; see the talk of A. Doelman below;
- detonation models. This last topic is particularly challenging: such models include the whole set of gas dynamics equations, plus an equation for the chemistry. The stability of detonation waves is a complex problem, and the introduction of a reduced model for fast waves in porous media, by Gordon-Kagan-Sivashinsky [24] seems to be quite promising.

Complex propagation phenomena in reaction-diffusion equations

Reaction-diffusion equations appear in many different areas of physics and of the life sciences. They are commonly used to describe phase transitions in various contexts in physics and in chemistry. In combustion theory, for instance, these equations arise in models of flame propagation. Equations of this kind play a central role in modeling biological invasions in various situations (population dynamics, physiology, wound healing, tumor growth, etc, see the classical books of Murray [32] and Shigesada and Kawasaki [37]).

The existence of traveling wave like solutions is an essential feature of this class of equations that is relevant for all the models mentioned above. It is strongly related to propagation phenomena that are particularly important and again a common feature in these areas.

As a mathematical subject, the study of reaction-diffusion equations, traveling waves and propagation properties is very active now. Even though, it was first introduced in the homogeneous framework in the late 1930s (see [16, 26]), there has been a profusion of works since the 1970s with results that have profoundly enriched our understanding of these equations. It is only relatively recently that researchers have been able to address propagation and traveling fronts in heterogeneous environments and to take into account other phenomena, such as transport, interaction with environment, singular behavior etc. The recent years have indeed seen much progress on these questions.

H. Berestycki (EHESS) gave two lectures on recent advances in this area. He first reported on several papers with F. Hamel and N. Nadirashvili [3, 4, 5] on existence and qualitative properties of pulsating traveling fronts in periodic media, for reaction-diffusion-advection equations of the type

$$u_t - \text{div}(A\nabla u) + q \cdot \nabla u = f(x,u), \quad x \in \Omega, \tag{4.8}$$

when $A(x)$, $q(x)$ and $f(x,u)$ have the same periodicity in the x-variables as the domain Ω itself. The influence of different phenomena involved – such as transport, diffusion, reaction, geometry of the domain – on the speeds of propagation were discussed. For instance, several well-known facts can be proved rigorously: the perforations slow down the propagation, whereas stirring always speeds up the fronts.

Another key notion involved here is the asymptotic speed of spreading in domains which have no periodicity. The spreading speed in a given direction is defined as the speed of the leading edge of the solution of the Cauchy problem at large times. For the solutions of the equation

$$u_t = \Delta u + f(u) \tag{4.9}$$

in general domains Ω with sub-linear nonlinearities f of the Fisher-KPP type ($0 < f(s) \leq f'(0)s$ for $s \in (0,1)$ with $f(0) = f(1) = 0$), the spreading speeds may depend in general on the domain and on the initial condition, even if the solution is initially compactly supported. Even for this homogeneous equation, very interesting new phenomena appear, due to the complex geometry of the domain. For instance, in very narrow domains, the spreading speed may be infinite.

More complex dynamical behaviors may also occur. Roughly speaking, even for simple models (4.9) and even in dimension 1, when the nonlinearity f is of the combustion type ($f = 0$ on $[0,\theta]$, $f > 0$ on $(\theta, 1)$ and

$f(1) = 0$ with $0 < \theta < 1$) or of the bistable type ($f < 0$ on $(0, \theta)$, $f > 0$ on $(\theta, 1)$ and $f(0) = f(\theta) = f(1) = 0$), then propagation may occur or fail according to the size of the initial condition. For instance, for bistable nonlinearities with positive mass over $[0, 1]$, A. Zlatos [40] recently proved that, when the initial condition at time 0 is the characteristic function of an interval, then there is a critical positive interval size below which the solution will eventually converge to 0 uniformly in $x \in \mathbb{R}$, and above which it will converge to 1 locally, and actually develop into two expanding fronts. For the critical interval size, the solution eventually converges to the unstable non-trivial ground state. Even if the results are not as precise when the equation involves heterogeneous coefficients and in particular a non-constant flow, propagation/quenching issues were addressed recently and special attention has been put on the role played by the profile of the underlying flow (see P. Constantin, A. Kiselev, L. Ryzhik, A. Zlatos [12, 25]).

Further generalizations of the notion of traveling front or wave in general heterogeneous frameworks were recently introduced for general systems of partial differential equations. These new definitions are based on uniform limits far away, with respect to the geodesic distance inside the domain, from some hypersurfaces. These notions extend the previous known cases of periodic or almost-periodic environments. General situations like the propagation in curved tubes, exterior domains, etc can now be considered. The determination of the shape of the leading edge of the fronts and the stability of these new fronts are some of the main goals of future work.

The question of propagation in media which are locally perturbed is an open problem which is one of the most important cases for the applications. Indeed, the same issues of propagation can be asked when the medium is homogeneous (or even periodic) outside a localized zone and the definition of generalized waves is also adapted to this situation. The archetype is the equation (4.8), where the coefficients A, q and f, or the domain Ω, are homogeneous or periodic outside a compact set. This is the case of a tube which has a local stricture. What are the necessary and sufficient conditions to have propagation ?

Another very interesting open problem is to describe the propagation of generalized fronts in media for which some diffusion or reaction coefficients are monotone in the direction of propagation, or more generally when the characteristics of the medium are different far ahead and far behind the front. These questions may depend strongly on the nonlinearity, propagation may fail for bistable nonlinearities whereas, everything else being unchanged, propagation may occur for monostable equations. These problems have concrete applications in combustion or in biological models for instance.

Biological invasions are indeed one of the most common examples of propagation phenomena and it seems fair to say that these are the most widely used equations in ecological and biological modeling (epidemics, epizootics and tumor growths can also be modelled by reaction-diffusion equations). Much progress has been made in the recent years about the mathematical analysis of such models. It helps to have a better understanding of the concrete applications and to be able to make reasonable predictions. For instance, for ecological models of the type

$$u_t = \mathrm{div}(A(x)\nabla u) + (\mu(x) - \nu(x)u)u \tag{4.10}$$

in periodic fragmented environments, light was recently shed on how a spatially diverse environment affects biological invasions or species survival in this context. A less fragmented medium, which means that the favourable and unfavourable regions are more aggregated, is better for species persistence (see [6]).

More complex models can also be used in the applications. As an example, aggregation phenomena for bacteria can be modelled by systems of equations which involve chemotactic terms, meaning that some species tend to diffuse in the direction of positive concentration gradient of a chemical agent (see [32]). In other contexts, nonlocal models can be used to model long-range dispersion and new versions of the maximum principle, which is one of the most powerful tools in reaction-diffusion equations, were recently established.

In mathematical terms, from a dynamical point of view, front propagation can be thought of as the invasion of a more unstable or less stable state by a more stable or less unstable state. Even if most models do not have a variational structure and no Lyapounov functional is available in general, the study of the spectral properties of the linearized equations around the limiting states is crucial. Another important point is to determine the set of all possible limiting states. For instance, for equations as simple as (4.10), the existence of a stationary positive state is not obvious. Indeed, since the equations are set in unbounded domains, to allow propagation, the lack of compactness creates additional complications. Recent progress was made on these questions, for equations more general than (4.10), and new qualitative and Liouville classification results were obtained.

Homogenization, stochastics and dynamics of the reaction-diffusion equations

Reaction-advection-diffusion equations and weak perturbations of dynamical systems

The question of the interplay of a strong advection and weak diffusion is very natural and physically relevant, and the subject has a long history. The passive scalar model

$$\phi_t + u \cdot \nabla\phi = \varepsilon\Delta\phi,$$

is probably one of the most studied PDEs in both mathematical and physical literature. One important direction of research focused on homogenization, where in a certain limit (typically small diffusion) the solution of a passive advection-diffusion equation converges to a solution of an effective diffusion equation. We refer to [29] for more details and references. The corresponding reaction-diffusion models

$$\phi_t + u \cdot \nabla\phi = \varepsilon\Delta\phi + \frac{1}{\varepsilon}f(\phi),$$

and

$$\phi_t + \frac{1}{\varepsilon}u \cdot \nabla\phi = \Delta\phi + f(\phi),$$

have been also extensively studied. Usually, the existence of such a limit requires additional assumptions on the scaling of u (see e.g. [19] for further references). The Freidlin-Wentzell theory [17, 18, 19, 20] studies such problems in $\mathbb{R}^2$ and, for a class of flows, proves the convergence of solutions as the flow strength tends to infinity to solutions of an effective diffusion equation on the Reeb graph of the stream-function. The graph, essentially, is obtained by identifying all points on any streamline. The conditions on the flows for which the procedure can be carried out are given in terms of certain non-degeneracy and growth assumptions on the stream function. Recently this theory has been extended to a class of three-dimensional flows, where the limit problem is formulated on an “open book” rather on a graph. The dynamics is once again described in terms of the slow variables with the fast variations averaged out. Another direction has been taken in [13] – instead of trying to identify a limit problem, the question is what flows are most effective in mixing the solutions as their strength tends to infinity. It turns out that with an appropriate and natural definition of mixing one can provide a sharp classification of such “relaxation-enhancing” flows.

Homogenization of Hamilton-Jacobi and reaction-diffusion equations

Homogenization of the Hamilton-Jacobi equations in a periodic medium has been well understood since the unpublished preprint by Lions, Papanicolaou and Varadhan from the late 1980’s. The problem is to homogenize the (possibly second-order) equation

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\varepsilon}{2}\Delta u_\varepsilon + H\left(t/\varepsilon, x/\varepsilon, \nabla u_\varepsilon, \omega\right) = 0,$$

and find an effective Hamilton-Jacobi problem

$$\frac{\partial u}{\partial t} + \bar{H}(\nabla u) = 0.$$

Here H is a random Hamiltonian and $\bar{H}$ is the deterministic Hamiltonian for the homogenized problem. This problem (as well as a class of related homogenization questions) has been recently studied in a series of papers by P.-L. Lions and P. Souganidis, and independently by E. Kosygina, F. Rezakhanlou and S.R.S. Varadhan. A very interesting and challenging open problem is obtain non-trivial bounds for the homogenized Hamiltonian – this problem remains open even in the periodic case.

Presentation Highlights

Geometric methods for semilinear reaction-diffusion equations

X. Cabré, during his two-hour lecture, presented recent developments on solutions of reaction-diffusion elliptic equations that are related to some classical results in the theory of minimal surfaces. Three results in minimal surfaces theory and their semilinear analogues.

- *Regularity of solutions of elliptic equations in low dimensions.* Inspired by related results for harmonic maps, Cabré discussed semilinear analogues, particularly recent results by Capella and himself on radial solutions of reaction-diffusion equations, including the well-known Gel'fand equation

$$-\Delta u = e^u.$$

In low space dimensions, they lead to the boundedness or regularity of radial solutions in a ball, and to the instability of radial solutions in the whole space.

- *Flatness of minimal graphs in low dimensions.* This item is related to the de Giorgi conjecture. Cabré explained how bounded solutions in the whole space which are monotone in one variable are always local minimizers of the energy

$$E(u) = \int_{B_R} (\frac{1}{2}|\nabla u|^2 - W(u))\, dx.$$

This implies that, in low space dimensions, they are necessarily functions of only one Euclidean variable.

- *Saddle solutions.* Guided by this variational approach, Cabré discussed the following generalisation of an earlier result by Schatzman [35]: in $\mathbb{R}^{2m}$, equation (4.2) has a solution whose symmetries are the same as those of the Simons cone; this solution, which is unique up to translations, is called the saddle solution of the Allen-Cahn equation; moreover, if $m = 1$, this solution is unstable. Cabré explained his results results in this direction: instability of the saddle solutions in dimensions $2m = 4$ and 6, relying on a delicate estimate of Modica: if u satisfies $-\Delta u = W'(u)$, W satisfying the standard assumptions, then

$$\frac{1}{2}|\nabla u|^2 \leq W(u).$$

Would these solutions be stable in higher dimensions - as is suggested by the Bombieri-de Giorgi-Giusti analysis, this would lead to a counterexample de Giorgi Conjecture.

O. Savin - who put an end to the non-existence part of the de Giorgi conjecture - discussed viscosity solutions of fully nonlinear elliptic equations

$$F(D^2u, Du, u, x) = 0$$

for which $u \equiv 0$ is a solution. If F is smooth and uniformly elliptic only in a neighborhood of the points $(0, 0, 0, x)$, then u is smooth in the interior if $\|u\|_{L^\infty}$ is sufficiently small. This result - which uses difficult Caffarelli-type estimates on second order derivatives - has applications to the study of the regularity of free boundary problems; in particular it can help to prove regularity when Lipschitz continuity and nondegeneracy of the free boundary are known.

Free boundary problems and applications

A. Mellet discussed delicate effects in the homogenization of free boundary problems in two cases. First, he considered the scalar thermo-diffusive model for flame propagation

$$T_t - \Delta T = \frac{1}{\varepsilon^2}(1-T)f(\frac{x}{\delta}, \frac{T-1}{\varepsilon});$$

the parameter δ accounting for possible heterogeneities in the medium. Hysteresis phenomena occur: passing to the limit in $\varepsilon \to 0$, then $\delta \to 0$ do not yield the same result as taking the limits in the reverse order. Second, he presented a model for the equilibrium of a sticky drop on a rough surface; this amounts to minimising a - nonsmooth - functional of the characteristic function of the drop, with highly oscillating coefficients. There is a homogenisation limit to this problem, namely the drop is almost spherically spherical, and the limiting radius may be computed from data.

J.-S. Guo reported on a two-point free boundary problem for a quasilinear parabolic equation, mainly arising in the study of the motion of interface moving with curvature. Global and non-global existence of solutions, was discussed; non-global existence may occur only through a finite-time extinction process - in the case of the mean curvature motion, this amounts to a complete curve shortening. The asymptotic profile at extinction, as well as convergence to a self-simlar profile, were discussed.

The talk of *N. Ghoussoub* concerned the nonlinear elliptic problem

$$-\Delta u = \frac{\lambda f(x)}{(1+u)^2}$$

on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage –represented here by λ– is applied, the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates an instability which greatly affects the design of many devices. The challenge is to estimate λ^* in terms of material properties of the membrane, which can be fabricated with a spatially varying dielectric permittivity profile f. When $\lambda < \lambda^*$ (and when $\lambda = \lambda^*$ in dimension $N \leq 7$), there is at least one steady state, while none is possible for $\lambda > \lambda^*$. More refined properties of steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results– are shown to depend on the dimension of the ambient space and on the permittivity profile.

Asymptotic models of reaction-diffusion systems; application to flame propagation models

Three talks were devoted to various aspects of existence and qualitative properties of waves in reaction-diffusion systems. The talk of *K. Domelevo* reported some results on premixed flames models, where the reactant (i.e. gas fuel) is provided through the vaporisation of liquid fuel droplets. The corresponding simplest mathematical model consists in the usual thermo-diffusive system coupled to the equation for the vaporisation of the droplets. Travelling wave profile exist, and asymptotics with respect to the activation energy reveal new features: if the initial droplet radius is below some explicit threshold, the model is totally similar to the classical thermo-diffusive model. Above the threshold, the combustion is driven by the droplet evaporation. The main result in the talk of *P. Gordon* was a singular perturbation approach to a detonation model in porous media, derived by Sivashinsky; he presented uniqueness results for the speed and wave profile when the thermal diffusion coefficient goes to 0. *M. Haragus* reported on holes in reaction-diffusion systems, i.e.: almost planar interfaces for which the angles of the interface at each point, relative to a fixed planar interface, tend to zero at infinity. She applied dynamical systems ideas - popularised under the name of 'spatial dynamics', to convey the idea that one spatial variable is treated as a time - and showed that, in isotropic reaction-diffusion systems, holes bifurcate from stable planar pulsating fronts.

C.-M. Brauner presented a model of flame front dynamics introduced by Frankel, Gordon and Sivashinsky, more tractable than the classical thermo-diffusive model, and which can yield - by the same process as in the thermo-diffusive model - a single integro-differential equation (Q–S). If the flame front, supposed to evolve in the space $I\!R^2$, is a curve with equation $y = \Phi(t, x)$, then

$$\Phi_t + \frac{\Phi_x^2}{2} - \Phi_{xx} + \alpha(I - \partial_{xx})^{-1}\Phi_{xx} = 0.$$

This asymptotic equation has the same qualitative features as the Kuramoto-Sivashinsky (K–S) one; in particular, it can generate chaotic cellular dynamics. The numerical simulations turn out to be quite convincing.

The modelling of spikes was addressed to in two talks. The talk by *A. Doelman* focussed on how to derive, in a rigorous fashion, an ODE modelling the interaction law between to repelling two-pulse, slowly varying solutions of the a regularized Gierer-Meinhardt system. This system is a heuristic model arising in the description of chemical reactors and biological systems; one of its versions writes

$$\begin{aligned} \varepsilon^2 U_t &= U_{xx} - \varepsilon^2 U + f(U)V^2 \\ V_t &= \varepsilon^2 V_{xx} - V + g(U)V^2 \end{aligned}$$

where $x \in \mathbb{R}, t > 0$, $0 < \varepsilon \ll 1$ is a small parameter, and functions f and g are smooth positive functions. The method employed, based on normalisation group ideas, should be applicable to many other situations.

M. Ward discussed an optimization problem for the fundamental eigenvalue λ_0 of the Laplacian in a planar simply-connected domain that contains N small identically-shaped holes, each of radius $\varepsilon \ll 1$. A Neumann boundary condition is imposed on the outer boundary of the domain and a Dirichlet condition is imposed on the boundary of each of the holes. He presented an asymptotic expansion for λ_0 in terms of certain properties of the Neumann Green's function for the Laplacian. This eigenvalue optimization problem is shown to be closely related to the problem of determining equilibrium vortex configurations in the Ginzburg-Landau theory of superconductivity, and also closely related to the problem of determining equilibrium locations of spikes, to multi-dimensional reaction-diffusion systems.

Complex propagation phenomena in reaction-diffusion equations

H. Berestycki (EHESS, France) reported first on some results with F. Hamel and N. Nadirashvili on pulsating travelling fronts for reaction-diffusion-advection equations in general periodic framework. The qualitative and quantitative role of the diffusion, advection and reaction terms was explained. Nonlinear propagation phenomena in general unbounded domains of $\mathbb{R}^N$, for reaction-diffusion equations with Kolmogorov-Petrovsky-Piskunov (KPP) type nonlinearities, were then discussed. General domains were considered and various definitions of the spreading speeds at large times for solutions with compactly supported initial data were given. The dependency of the spreading speeds on the geometry of the domain was explained. Some a priori bounds can be obtained for large classes of domains. The case of exterior domains was also explained in detail. H. Berestycki finally reported on very recent works with F. Hamel about further generalizations of the usual notions of waves, fronts and propagation speed in a very general setting. These new notions involve uniform limits, with respect to the geodesic distance, to a family of hypersurfaces which are parametrized by time.

J. Coville (CMM-Universidad de Chile, Chile) presented some work devoted to the maximum principles holding for some nonlocal diffusion operators and its applications to obtain qualitative behaviors of solutions of some nonlinear problems with sliding methods. As in the classical case, it can be shown that the nonlocal diffusion satisfies a weak and a strong maximum principle. Uniqueness and monotonicity of solutions of nonlinear equations are therefore expected as in the classical case. J. Coville also presented a optimal condition to have a strong maximum for operator $Mu := J \star u - u$.

S. Luckhaus (University of Leipzig, Germany) reported on joint work with L. Triolo [28], and with A. De Masi and E. Presutti [14], about a hierarchy of scalings in a population model for tumor growth. Interacting particle systems modeling the competition of healthy and malignant cells were considered and lateral contact inhibition and difference of mobility were taken into account in a lattice model. A two scale hydrodynamic limit was derived. On longer time scales the solutions are expected to converge to the tumor growth governed by the eikonal equation. This last step in the scaling hierarchy has not yet been shown starting from the original stochastic process.

H. Matano (University of Tokyo, Japan) reported on recent advances in quenching vs. propagation phenomena for bistable-type equations in heterogeneous media. In some domains with non-periodic perforations, propagation may be blocked by stationary solutions.

K.-I. Nakamura (University of Electro-Communications, Japan) talked about front propagation phenomena for a bistable reaction-diffusion equation in an infinite cylinder with periodic boundaries. By using the first 3 terms of asymptotic expansions of the profile and the speed of front solution, he constructed suitable supersolutions and subsolutions to obtain upper and lower bounds for the front speed when the diameter of the cylinder is very small. These bounds enabled him to show that spatial periodicity always slows down the front propagation in bistable diffusive media.

P. Polacik (University of Minnesota, USA) presented a new result on asymptotic symmetry of positive solutions of parabolic equations on nonsmooth bounded domains. A key ingredient in the proof of this result is a theorem on asymptotic positivity of solutions of linear equations with bounded measurable coefficients. Some perspectives on this technical tool were given.

L. Rossi (Universita Roma I, Italy and EHESS, France) discussed on generalized principal eigenavlue of elliptic operators in $\mathbb{R}^N$ and on some applications. He introduced two different generalizations of the principal eigenvalue for linear elliptic operators in the whole space. He discussed how their signs determine the existence and uniqueness of bounded solutions for an associated class of semilinear equations. The two notions do not coincide in general and some inequalities between these eigenvalues in the case of self-adjoint, one-dimensional and limit-periodic operators were derived.

A. Stevens (Max Planck Institute, Leipzig, Germany) discussed on transport equations for cellular alignment and aggregation and their parabolic limits. A widespread phenomenon in moving microorganisms and cells is their ability to orient themselves with respect to each other and in dependence of chemical signals. Kinetic models for this kind of movement were discussed, which take into account a variety of evaluations of the external chemical field and of the neigboring cells. In case of chemotaxis parabolic limit equations can be derived, which relate the microscopic parameters to the macroscopic ones, e.g. the so-called chemotactic sensitivity.

A. Zlatos (University of Wisconsin, Madison, USA) discussed on spreading of reaction in the presence of strong cellular flows with gaps. He considered a reaction-diffusion-advection equation with an ignition-type reaction term and a cellular flow with a periodic array of gaps. He showed that if the initial flame is large enough, it cannot be quenched by such flows, regardless of their strength.

Homogenization, stochastics and dynamics of the reaction-diffusion equations

M. Freidlin (University of Maryland) presented two lectures on asymptotic problems for stochastic processes and RDE's which covered material from the introductory level to the state of the art of the field. He presented some old and new results concerning averaging and large deviations for stochastic processes. These results allow, in particular, to describe motion of wavefronts for a class of reaction-advection-diffusion equations and systems, as well as to consider some homogenization problems for reaction in incompressible fluid.

A. Kiselev (University of Wisconsin) presented a talk on diffusion and mixing in fluid flow. Enhancement of diffusion by advection is a classical subject that has been extensively studied by both physists and mathematicians. In this work, the authors considered enhancement of diffusive mixing on a compact Riemannian manifold by a fast incompressible flow. The main result is a sharp description of the class of flows that make the deviation of the solution from its average arbitrarily small in an arbitrarily short time, provided that the flow amplitude is large enough. The necessary and sufficient condition on such flows is expressed naturally in terms of the spectral properties of the dynamical system associated with the flow. In particular, they find that weakly mixing flows always enhance the relaxation speed in this sense. The proofs are based on a new general criterion for the decay of the semigroup generated by a dissipative operator of certain form. They employ ideas from quantum dynamics, in particular the RAGE theorem describing evolution of a quantum state belonging to the continuous spectral subspace of the hamiltonian (and related to a theorem of Wiener on Fourier transforms of measures).

E. Kosygina (CUNY) presented her work on homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium. Consider a family $\{u_\varepsilon(t,x,\omega)\},\ \varepsilon > 0$, of solutions of the final value problem

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\varepsilon}{2}\Delta u_\varepsilon + H\left(t/\varepsilon, x/\varepsilon, \nabla u_\varepsilon, \omega\right) = 0, \quad u_\varepsilon(T,x,\omega) = U(x),$$

where the time-space dependence of the Hamiltonian $H(t,x,p,\omega)$ is realized through the shifts in a stationary ergodic random medium. For Hamiltonians, which are convex in p and satisfy certain growth and regularity conditions, she shows the almost sure locally uniform in time and space convergence of $u_\varepsilon(t,x,\omega)$ as $\varepsilon \to 0$ to the solution $u(t,x)$ of a deterministic "effective" equation

$$\frac{\partial u}{\partial t} + \bar{H}(\nabla u) = 0, \quad u(T,x) = U(x).$$

The averaged Hamiltonian $\bar{H}(p)$ is given by a minimax formula. This is a joint work with S.R.S. Varadhan.

J. Nolen (University of Texas, Austin) discussed reaction diffusion fronts in temporally inhomogeneous flows. He considered the propagation of fronts that arise from scalar, reaction-advection-diffusion models with the Kolmogorov-Petrovsky-Piskunov (KPP) nonlinearity. For temporally random flows with a shear structure, he established an extension of the well-known variational representation for the front speed, a nonrandom constant. Also, he used this variational representation to analytically bound and numerically compute the speed. The analysis makes use of large deviations estimates for the related diffusion process. The variational principle is expressed in terms of the principal Lyapunov exponent of an auxiliary evolution problem. This is a joint work with J. Xin [33].

A. Novikov (Pennsylvania State University) considered a homogenization approach to large-eddy simulation of incompressible fluids. In the development of large-eddy simulation one makes two primary assumptions. The first is that a turbulent flow can be categorized by a hierarchy of lengthscales. The second assumption states that the small scales have universal properties, characterized by, e.g. a spectral power law. This motivated a number of physical models that attempt to account for the presence of small scales by suitably modifying the corresponding partial differential equations (PDE), the Navier-Stokes equations. Homogenization theory addresses rigorously the issue of modification of PDE in the presence of small scales. The goal of this talk was to apply homogenization methods to LES modeling of fluid flows.

H. Owhadi (Caltech) talked about homogenization of parabolic equations with a continuum of space and time scales. He addressed the issue of homogenization of linear divergence form parabolic operators in situations where no ergodicity and no scale separation in time or space are available. Namely, he considered divergence form linear parabolic operators in $\Omega \subset \mathbb{R}^n$ with $L^\infty(\Omega \times (0,T))$-coefficients. It appears that the inverse operator maps the unit ball of $L^2(\Omega \times (0,T))$ into a space of functions which at small (time and space) scales are close in H^1-norm to a functional space of dimension n. It follows that once one has solved these equations at least n-times it is possible to homogenize them both in space and in time, reducing the number of operations counts necessary to obtain further solutions. In practice they show that under a Cordes type condition that the first order time derivatives and second order space derivatives of the solution of these operators with respect to harmonic coordinates are in L^2 (instead of H^{-1} with Euclidean coordinates). If the medium is time independent then it is sufficient to solve n times the associated elliptic equation in order to homogenize the parabolic equation. (This is a joint work with Lei Zhang.)

J. Quastel (University of Toronto) discussed the effect of noise on KPP traveling fronts. He and co-authors study the effect of small additive Fisher-Wright noise on the speed of traveling fronts in the KPP equation. It had been observed by physicists in the late 90's that the effect is unusually large and Brunet and Derrida have made some very precise conjectures. Quastel described the proofs of some of these. This is joint work with Carl Mueller (Rochester) and Leonid Mytnik (Technion).

M. Soner (Koc University) talked about backward stochastic differential equations and fully nonlinear PDE's. In the early 90's Peng and Pardoux discovered a striking connection between semilinear parabolic PDE's and backward stochastic differential equations (BSDE in short). This connection and the BSDE's have been extensively studied in the last decade and a deep theory of BSDEs have been developed. However, the PDE's that are linked to BSDE's are necessarily semilinear. In joint work with Patrick Cheredito (Princeton) Nizar Touzi (CREST, Paris), Nic Victoir (Oxford), Soner extended the theory of BSDE's by adding an equation for the second order term, which we call 2BSDE in short. Through this extension they are able to show that all fully nonlinear, parabolic equations can be represented via 2BSDE's. He described this theory and possible numerical implications for the fully nonlinear PDE's.

P. Souganidis (University of Texas) presented two lectures on homogenization in random environments and applications to front propagation. In particular, he described recent developments in the theory of homogenization for fully nonlinear first- and second-order pde in stationary ergodic media in his works with L. Caffarelli and P.-L. Lions. He also considered applications to the theory of front propagation in random environments.

Gizburg-Landau vortices

A. Aftalion (Universite Paris VI) presented her work on vortex lattices in fast rotating Bose Einstein condensates. She described experiments on fast rotating Bose Einstein condensates which display vortex lattices: the lattice is almost triangular with a slight distortion on the edges. The mathematical description can be made with a complex valued wave function minimizing an energy restricted to the lowest Landau level or Fock-Bargmann space. Using some structures associated with this space, she studies the distribution of zeroes of the minimizer.

S. Serfaty (NYU University) gave two lectures on her work on the dynamics of the Ginzburg-Landau vortices. She described the known results on vortex collisions and presented her recent work [36] in this area, extending vortex dynamics past the blow-up time.

Outcome of the Meeting

The meeting provded an opportunity for researchers in various sub-areas of the whole domain of elliptic and parabolic partial differential equations to interact with each other. The talks have been devoted to problems ranging from purely mathematical questions such as De Giorgi conjecture to probabilitistic questions, such as stochastic homogenization, and to applied areas including combustion and biology. Nevertheless, the group had a strong core of common interests which held the meeting very coherent and of a high quality. Numerous frutiful discussions have taken place, across the traditional area boundaries. Overall, we believe that the participants found the conference to be very successful and stimulative for their research.

List of Participants

Aftalion, Amandine (Université Paris 6-UPMC)
Berestycki, Henri (EHESS)
Brauner, Claude-Michel (Université Bordeaux 1)
Cabre, Xavier (ICREA and Universitat Politecnica de Catalunya)
Constantin, Peter (University of Chicago)
Coville, Jérôme (CMM-Universidad de Chile)
Doelman, Arjen (Center for Mathematics and Computer Science)
Domelevo, Komla (Université Paul Sabatier Toulouse III)
Freidlin, Mark (University of Maryland)
Ghoussoub, Nassif (BIRS)
Gordon, Peter (New Jersey Institute of Technology)
Gui, Changfeng (University of Connecticut)
Guo, Jong-Shenq (National Taiwan Normal University)
Hamel, François (Université Aix-Marseille III)
Haragus, Mariana (Université de Franche-Comté)
Jerrard, Robert (University of Toronto)
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Lewicka, Marta (University of Minnesota)
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Matano, Hiroshi (University of Tokyo)
Mellet, Antoine (University of British Columbia)
Nakamura, Ken-Ichi (University of Electro-Communications Tokyo)
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Novikov, Alexei (Pennsylvania State University)
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Polacik, Peter (University of Minnesota)
Quastel, Jeremy (University of Toronto)
Roquejoffre, Jean-Michel (Universite Paul Sabatier Toulouse III)
Rossi, Luca (University of Roma I)
Ryzhik, Lenya (University of Chicago)
Savin, Ovidiu (University of California, Berkeley)
Serfaty, Sylvia (Courant Institute of Mathematical Sciences, NYU)
Soner, H. Mete (Koc University)
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Stevens, Angela (Max Planck Institute for Mathematics in the Sciences)
Ward, Michael (University of British Columbia)
Zlatos, Andrej (University of Wisconsin, Madison)

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Chapter 5

Nonlinear diffusions: entropies, asymptotic behavior and applications (06w5039)

Apr 15 – Apr 20, 2006

Organizer(s): José Antonio Carrillo (ICREA, Univ. Autnoma de Barcelona, Spain), Eric Carlen (Georgia Institute of Technology, Atlanta, USA), Jean Dolbeault (Ceremade, Univ. Paris Dauphine, France), Peter Markowich (Univ. of Vienna, Vienna, Austria), Robert J. McCann (University of Toronto, Canada)

Overview of the Field

Scientists have long recognized the importance of diffusion in modelling everything from stock market prices to the spread of environmental pollutants. If feedback mechanisms are present in the system, then the rates of the process may vary from point to point depending on the gradient and concentration of the material diffusing. The system may then exhibit dramatic transitions, nonlinear behaviour, and pattern formation which challenge prediction and analysis. Such models are used in the manufacture of semiconductors, oil recovery, epidemiology of diseases, and the assessments of environmental impact. There is a strong connection between analytical progress in diffusion problems and their applications in the physical, biological, and engineering sciences. As an example we point out the developments in semiconductor drift-diffusion modelling of the last 30 years, which to a great extent was co-responsible for the successful design of many generations of very large scale integrated (VLSI) structures.

Nonlinear diffusion also plays a key role in other physical, biological, and geometric processes, such as fluid seepage, population spreading, pattern formation, chemotaxis, reaction dynamics, curvature flows, thermalization in plasmas, and avalanches in sandpiles. Such processes are modelled using partial differential equations (PDE), whose stability, structure, and large time dynamics are questions of great relevance to understanding the qualitative and quantitative behaviour of the models and the processes they govern. The same differential equations have also emerged at the heart of Perelman and Hamilton's proposed solution to some of the deepest problems in geometry and topology, which include enumerating the ways in which a three-dimensional universe may connect with itself. This workshop brought leading experts together with a younger generation of aspiring and accomplished scientists to explore the state of the art in nonlinear diffusion, and to set the agenda concerning the mathematical and modelling challenges in this vital area of nonlinear PDE.

Many of the equations discussed at this meeting had a variational form

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(m(u)\ \mathrm{grad} \frac{\delta \mathcal{H}(u)}{\delta u} \right) \ , \tag{5.1}$$

where $u(t,x) \geq 0$ is the density of material diffusing, $m : [0,\infty[\longrightarrow [0,\infty[$ determines its *mobility* or response to forcing by pressure, and the strength of the pressure is determined by a *Lyapunov functional* $\mathcal{H}(u)$, which gives a dissipated *energy* or *entropy*. In the simplest, zeroth order-case, $\mathcal{H}(u)$ depends only on the value of $u(x)$, and not on its derivatives. The resulting evolution, like the heat equation, enjoys a comparison principle which is a key tool in analyzing the solutions, though the nonlinearity may still cause vexing problems. If $\mathcal{H}(u)$ depends also on the gradient of u (i.e., is a first order functional), the fourth-order evolution (5.1) which results has no maximum principle and poses an entirely different set of analytical challenges which have only quite recently begun to be explored. Since first-order energies arise in the modeling of semiconductors, thin-films, and statistical mechanical fluctuations, they have both commercial and academic interest.

The fundamental question is to develop an understanding of how the subsequent evolution depends on the initial conditions and boundary data. This understanding can be based on explicit solutions — usually very rare and depending strongly on separation of variables and symmetry in the data and boundary conditions — perturbation theory and asymptotic analysis, local and global inequalities such as a priori estimates, comparison theorems, numerical simulations and laboratory experiments, and finding the right functional spaces and settings to obtain well-posedness results.

Points of contact between diffusion equations, and kinetic type equations, like the spatially homogeneous Boltzmann equation

$$\frac{\partial u}{\partial t} = \mathcal{Q}(u) \ , \tag{5.2}$$

were also discussed. Both types of equations are dissipative. Indeed, a simple formal calculation shows that for a solution $u(t,x)$ of (5.1), the energy $\mathcal{H}(u(t))$ decreases in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u) = -\int m(u) \left| \mathrm{grad} \frac{\delta \mathcal{H}(u)}{\delta u} \right|^2 \mathrm{d}x. \tag{5.3}$$

Of course, one of the oldest results in this direction is Boltzmann's H–Theorem, which asserts dissipativity of the relative entropy with respect to equilibrium for solutions of (5.2). That is, let u be a solution of (5.2) whose initial data is a probability density with a finite second moment. Let M be the Maxwellian density (i.e., Gaussian with isotropic covariance) that has the same first and second moments. The relative entropy of the solution with respect to the equilibrium solution M is

$$\int \ln \left(\frac{u(v,t)}{M(v)} \right) u(v,t) \mathrm{d}v \tag{5.4}$$

where v is often used instead of x for the argument for u because these variables represent particle velocities in the kinetic context. Boltzmann's H–Theorem asserts that this quantity is monotone decreasing along the evolution described by (5.2). It strongly suggests — and in some cases can be used to prove — that all initial data in a suitable class will become more and more Gaussianly-shaped as time progresses.

Still other types of equations were discussed at the meeting, as we shall explain, but let us start here to keep our report concrete. The issues discussed with regard to other PDE can often be related to issues which pertain also to (5.1) and to (5.2).

Recent Developments and Open Problems

The last few years have witnessed remarkable progress on this subject. Despite the antiquity of the H–Theorem, the use of relative entropy inequalities to obtain *a priori* information on the *rate* of convergence to equilibrium is relatively recent. The previous generation of researchers in the subject relied on spectral estimates coming from a linearization that were valid only in a very small neighborhood of the equilibrium.

Entropy dissipation arguments have recently been employed to study nonlinear diffusion by a number of researchers. An important part of much recent progress has been based on a a geometrical point of view on these evolutions introduced by Felix Otto, who highlighted deep connections between nonlinear diffusion and the variational problems of mass-transport theory.

In particular, in the case that the mobility function m in (5.1) is just the identity $m(u) = u$, the equation (5.1) is the gradient flow for the functional $\mathcal{H}$ with respect to the 2–Wasserstein metric coming from mass transport theory. In other words, the dynamics (5.1) is the steepest descent of the energy $\mathcal{H}$, provided the distance between two probability densities is taken to be the square of the average spatial distance required to displace the particles of one to the other. Otto showed that that when $\mathcal{H}$ is a strictly convex function on the "Riemannian manifold" of probability densities, where the Riemanian metric is induced by the 2–Wasserstein metric, then quantitative measures of the strict convexity of $\mathcal{H}$ had quantitative implications for both the rate of convergence problem, and the contractivity problem for solutions of (5.1). Mass transport theory had recently undergone significant development, building on the work of Brenier and McCann in particular, and this provided many new tools, from an unexpected source, to the study of PDE such as (5.1).

Otto applied this geometric perspective to the porous medium equation, in which case one has

$$\mathcal{H}(u) = \frac{1}{p}\int u^p(x)dx \tag{5.5}$$

for $p > 1$. The point is that the strict convexity of $\mathcal{H}$ implies an inequality between $\mathcal{H}$ itself, and its rate of dissipation. Such "entropy–entropy dissipation" inequalites are generally very hard to prove by means of a direct comparrison between $\mathcal{H}$ and the dissipation rate given in (5.3). The geometric framework introduced by Otto has provided a powerful new perspective on how to prove such inequalities.

If one takes $\mathcal{H}$ to be the relative entropy as given by (5.4), then (5.1) becomes the Fokker–Planck equation with M as the equilibrium solution. In this case, the "entropy–entropy dissipation" inequality was already well known; it is the celebrated logarithmic Sobolev ineuqality of Gross. Much progress has since been made in applying mass transport techniques to the study of logarithmic sobolev inequalities in other settings, and to proving other functional inequalities; e.g., Gagliardo–Nirenberg type inequalities.

A fruitful interaction between the mass transport and the PDE communities has grown rapidly since then, mainly due to the interchange of points of view, different techniques, and progress on the open problems in both subjects. New ideas have led to many improvements in the mathematical results concerning rates of convergence, functional inequalities, sharp geometric constants, and links to optimal mass transportation and kinetic theories. Perelman's announced proof of the geometrization conjecture is a spectacular application of nonlinear diffusion (Ricci flow) to fundamental problems in geometry and topology.

The understanding of large time asymptotics is best developed for model problems such as the porous medium / fast diffusion equation and related nonlinear Fokker-Planck equations, which were the initial objects of investigation. However, even these simple models pose open questions concerning rates of decay for restricted initial data, higher order asymptotics, estimates on the location and evolution of free boundaries, perturbations by convex potentials. The adaptation of relative entropy or variational techniques to other PDEs and coupled systems displaying more complicated dynamics is largely a challenge for the future, though very interesting progress has been recently made, and was the subject of much discussion at the meeting. For example, while the porous medium equation and the Fokker–Planck equation arise through the use of zeroth order functionals $\mathcal{H}(u)$, such as those in (5.4) and (5.5), a much more challenging set of problems arise when one considers first order functionals such as

$$\mathcal{H}(u) = \int |\nabla u|^2 \mathrm{d}x \qquad \text{or} \qquad \mathcal{H}(u) = \int \frac{|\nabla u|^2}{u}\mathrm{d}x\ .$$

These give rise, respectively to the thin film equation, and an equation known as the Derida–Lebowitz–Speer–Spohn (DLSS) equation. These are fourth order equations, and hence there is no maximum principle argument to ensure that solutions with non negative initial data stay non negative. Whether this is even true of not depends on the mobility $m(u)$: For the thin film equation, it is known that if this is degenerate enough at $u = 0$, then the evolution "slows down" as when a solution approaches zero, and positivity is maintained. However, for a mobility of the form $m(u) = u^p$, the smallest value of p for which positivity of the initial data implies positivity of solutions is unknown. This open problem was much discussed at the meeting.

The fact that one can view the DLSS equation as coming from gradient flow with respect to the 2–Wasserstein metric does, however, at least provide a way to construct non–negative solutions. There are many technical difficulties, though, in working with these higher order functionals. The most significant is that the are not convex with respect to the 2–Wasserstein metric, and so the contraction property that hold in the presence of convexity cannot be invoked to assert the uniqueness of the solutions constructed by the gradient flow argument. It remains an open problem to deal with this difficulty.

What is known about the positivity of solutions of the thin film equation comes from the analysis of a family of Lyapunov functionals discovered by Bernis and Friedman. Not only is $\mathcal{H}(u) = \int |\nabla u|^2 \mathrm{d}x$ dissipated under the evolution described by the thin film equation, but so are other functions: The ones found by Bernis and Friedman are zeroth order, and other first order functionals have been found by Laugesen. This raises the question: Given a dissipative evolution equation such as (5.1), is there a systematic way to find other Lyapunov functionals (besides $\mathcal{H}(u)$) for its evolution? A framwork for doing this was presented by A. Jungel and D. Matthes. It resulted in much discussion at the meeting, with specific problems to which the method might be applied being identified. However, other natural question seem to stymie this method at present. For example, one would like to know whether there are functionals involving a term of the form

$$\int \frac{|u_{xx}|^2}{u^p} \mathrm{d}x$$

with some $p > 0$ that are monotone decreasing for the thin film evolution. The functionals found by Laugessen have the form

$$\int \frac{|u_x|^2}{u^p} \mathrm{d}x \, .$$

A better understanding of where they come from, and how one might find others, was the subject of much discussion.

Another focus of the meeting was flux-limited diffusion equations in radiation hydrodynamics. These are interesting since they are the source of new developments in the analysis of bounded variation functions. A typical example here is

$$u_t = \nu \, \mathrm{div} \left(\frac{uDu}{\sqrt{u^2 + \frac{\nu^2}{c^2} |Du|^2}} \right) , \tag{5.6}$$

known as the *relativistic heat equation*. This is of a somewhat different form than (5.1), but it can still be derived in the same way as the other $m(u) = u$ examples of (5.1) by means of the Monge-Kantorovich mass transport theory. This was shown by Brenier, who used a cost function that stipulated an infinite penalty for transporting mass too far. Brenier's work was done at a formal level, and much discussion at the conference focused on how to rigorously and numerically handle such equations.

As evidenced at this meeting, there is a healthy interaction between new developments in mass transport theory and the classical tools of PDE, such as self-similarity, maximum principles, L^p contractivity, compactness methods, smoothing effects, dynamical systems arguments. Moreover, the new techniques mentioned above combine with more traditional approaches to impact questions in kinetic theory such as thermal equilibration rates for homogeneous gases, and well-posedness of inhomogeneous models by variational schemes. Several metrics connected to Wasserstein distances have been shown to describe the asymptotics in spatially-homogeneous kinetic problems.

Indeed, an old result of Tanaka is that the evolution of the spatially homogeneous Boltzmann equation (5.2) for so–called "Maxwellian molecules" is contractive is the 2–Wasserstein metric – just as if (5.2) were the gradient flow for some convex functional with respect to the 2–Wasserstein metric. A very interesting open problem discussed at the meeting is whether this contraction property, perhaps with respect to a mass–transport metric based on some other cost functional, can be extended to (5.2) in general, not just for Maxwellian molecules. Good evidence for the robustness of Tanaka's result, and the plausibility of this conjecture, was given in Carrillo's talk, where he extended Tanaka's result to the case of dissipative collisions, such as arise in granular flows.

Presentation Highlights

Presentations at this meeting ranged focused mainly on problems in mathematical analysis, though a smaller number of talks were devoted to computational and theoretical modeling and phenomenology.

Connections between nonlinear diffusion and geometric (Ricci) curvature flows were a frequent theme. State of the art results were reviewed by Daskalopoulos, Ni and Vázquez on this subject. Entropy-type arguments, Li-Yau-Hamilton inequalities, existence of solutions with measures as initial data and loss of mass at infinity for solutions are some of the topics treated representing a step forward in the understanding of these topics. Bennett Chow introduced a discrete model for geometric flow of a triangulated manifold.

Talks devoted to modelling included a spectacular lecture on droplet coarsening rates by Dejan Slepcev, describing his joint work with Otto and Rump. Connections to nearby problems in other contexts as chemotactic models, concentration inequalities in probability, and traffic flow were described by DiFrancesco, Gentil, and Illner. Aronson discussed the geometrical focusing corresponding to fluid wetting and eventually covering over a dry spot. Several talks were devoted to more traditional models in the kinetic theory of gases, including striking progress towards finding regular solutions by Panferov and Gamba. Novel analyses of the long time behaviour using contractive distances and entropy methods were given by Fellner, Cáceres, Carrillo and Dolbeault.

The understanding of the linearization of fast-diffusion equations and its consequences over the improvement of decay rates was another hot topic at this conference. Denzler, Cáceres, and Matthes explored aspects of this topic using different techniques. Denzler pointed out the existence of a family of explicit solutions to the porous medium equations which capture the leading, first, and second order asymptotics of the general evolution. Instead of being radially symmetric, these solutions are self-similar under affine mappings; they capture the competition between different spatial dimensions under the flow for the first time. Improved decay rates were discussed in several other contexts: reaction-diffusion equations, diffusion equations, thin film equations, logarithmic-type equations, etc., by Fellner, Kim, and Arnold. Perhaps most striking was the algebraic method described by Jüngel and Matthes for automating the search for Lyapunov entropies, and corresponding applications in existence theory and decay rates derived from functional inequalities.

Flux-limited nonlinear diffusions were shown to have uniqueness of entropy solutions by Mazón and Andreu. This family contains important examples of application as the Relativistic Heat Equation. Puel showed us how to approximate the same equation using a sequence of optimal transportation problems. Ambrosio gave a masterful lecture on convergence of the resulting Lagrangian maps in such approximation schemes. Chertock and Kurganov disussed a family of mathematical models which lead to new phenomena, such as discontinuities or evolving fronts appeared inside the support of the solution, which they exhibited numerically.

Finally, several talks were devoted to optimal mass transportation methods and geometric inequalities. Nazaret and Agueh described the use of optimal transportation methods for finding sharp constants in Sobolev and Gagliardo-Nirenberg inequalities. Sturm explained how McCann's notion of displacement convexity for a well-chosen entropy functional with respect to a transportation distance, can be used to define a notion of Ricci boundedness of a metric measure space. Such a space then automatically inherits many important analytical and geometrical properties usually associated with smooth Riemannian manifolds, such as doubling conditions, Sobolev inequalities, and Bishop-Myers-Gromov type comparisons. This resolves a major open question in geometry. Finally, Gregoire Loeper described his startling counterexample to the continuity of optimal mappings on manifolds: it shows that Ma, Trudinger, and Wang's condition guaranteeing the regularity of such maps is not only sufficient, but necessary.

Scientific Outcome of the Meeting

This meeting has represented a unique occasion for setting up the actual status of the research in this field and for fostering the interaction between different groups of mathematicians interested in nonlinear diffusion equations. Among these different groups we can mention mathematicians specialized in partial differential equations, geometric analysis, and the calculus of variations, and, in terms of fields of applications, people studying kinetic theory, fluid mechanics, thin films, probabilistic approaches of particle systems, plasma and solid state physics, and exotic materials.

Many new perspectives were discovered, research collaborations fostered, and directions for future investigations determined. In the words of one of the organizers: “My own research benefited tremendously. Not only did I learn the unpublished history of the line of ideas I had been pursuing, but a world of fourth order equations was opened up to me which represented new arenas of application for the second order techniques I had developed. I spent the next three months exploring consequences of what I learned at BIRS.”

This workshop has showcased some of the recent progresses and set the stage for future developments, new collaborations, and cross-pollination between different communities. Certainly, the impact of this event will be measured by relevant publications in the years to come.

Abstracts of Talks

AGUEH, Martial (Victoria). **Sharp Gagliardo-Nirenberg inequalities and optimal transportation theory.** It is known that best constants and optimal functions of many geometric inequalities can be obtained via the optimal transportation theory. But so far, this approach has been successful for a special subclass of the Gagliardo-Nirenberg inequalities, namely, those for which the optimal functions involve only power laws. In this work, we explore the link between Optimal transportation theory and all the Gagliardo-Nirenberg inequalities. We show that the optimal functions can be explicitly derived from a specific nonlinear ordinary differential equation, which appears to be linear for a subclass of the Gagliardo-Nirenberg inequalities or when the space dimension reduces to 1. In these cases, we give the explicit expressions of the optimal functions, along with the sharp constants of the corresponding Gagliardo-Nirenberg inequalities.

AMBROSIO, Luigi (Pisa). **Convergence of iterated transport maps and nonlinear diffusion equations.** We analyze the asymptotic behaviour of iterated transport maps arising in the implicit time discretization of nonlinear diffusion equations, modelled on the porous medium equation. This analysis allows to answer affermatively to a question raised in a recent paper by Gangbo, Evans and Savin, in connection with gradient flows of a class of polyconvex energy functionals.

ANDREU Fuensanta (Valencia). **Renormalized and Weak Solutions for a Degenerate Elliptic-parabolic Problem with Nonlinear Dynamical Boundary Conditions.** We are interested in the following degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

$$P_{\gamma,\beta}(f,g,z_0,w_0) \qquad \begin{cases} z_t - \text{div}\, A(x,Du) = f,\ z \in \gamma(u), \ \text{ in } Q_T :=]0,T[\times\Omega \\ \\ w_t + A(x,Du)\cdot\eta = g,\ w \in \beta(u), \ \text{ on } S_T :=]0,T[\times\partial\Omega \\ \\ z(0) = z_0 \ \text{ in } \Omega,\ w(0) = w_0 \ \text{ in } \partial\Omega. \end{cases}$$

The nonlinear elliptic operator div $A(x, Du)$ is modeled on the p-Laplacian operator

$$\Delta_p(u) = \text{div}\,(|Du|^{p-2}Du),$$

with $p > 1$, γ and β are maximal monotone graphs in R^2 such that $0 \in \gamma(0)$ and $0 \in \beta(0)$. Particular instances of this problem appear in various phenomena with changes of phase like multiphase Stefan problem and in the weak formulation of the mathematical model of the so called Hele Shaw problem. Also, the problem with non-homogeneous Neumann boundary condition is included.

Under certain assumptions on γ, β and A, we prove existence and uniqueness of renormalized solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ for data in L^1, and also that these renormalized solutions are weak solutions if the data are in $L^{p'}$.

ARNOLD Anton (Vienna). **Improved decay rates for the large time behavior of parabolic equations.** It is well known that the solution to heat equation behaves for large time like the Gaussian with the same moments of order 0, 1, and 2. And the ”distance” from the solution to this Gaussian can be measured or estimated conveniently in terms of the relative entropy. Surprisingly, the known estimates can be improved by using the relative entropy of the initial function with respect to a Gaussian with a smaller second moment.

ARONSON Donald G. (Minneapolis). **Some aspects of the focusing problem for the porous medium equation.** I will discuss the role of self-similar solutions in the analysis of the focusing problem for the porous medium equation.

CÁCERES Maria José (Granada). **Long time behavior of linearized fast diffusion equations using a kinetic approach.** We study the long time behavior of linearized fast diffusion equations showing that their rate of convergence towards the self–similar solution can be related to the number of moments of the initial datum that are equal to the moments of the self–similar solution at a fixed time. As a consequence, we find an improved rate of convergence to self–similarity in terms of a Fourier based distance between two solutions.

The key idea to prove the results is the asymptotic equivalence of a collisional kinetic model of Boltzmann type with a linear Fokker-Planck equation with nonconstant coefficients, for which the recovering of the rate of decay in terms of the Fourier based distance is immediate. (Joint work with G. Toscani)

CARRILLO José A. (Barcelona). **Tanaka Theorem for Inelastic Maxwell Models.** We show that the Euclidean Wasserstein distance is contractive for inelastic homogeneous Boltzmann kinetic equations in the Maxwellian approximation and its associated Kac-like caricature. This property is as a generalization of the Tanaka theorem to inelastic interactions. Even in the elastic classical Boltzmann equation, we give a simpler proof of the Tanaka theorem than the ones by Tanaka (1978) and Villani (2002). Consequences are drawn on the asymptotic behavior of solutions in terms only of the Euclidean Wasserstein distance.

CHERTOCK Alina (Rayleigh). **Strongly Degenerate Parabolic Equations with Saturating Diffusion.** We first consider a nonlinear diffusion equation used to describe propagation of thermal waves in plasma or in a porous medium, endowed with a mechanism for flux saturation, which corrects the nonphysical gradient-flux relations at high gradients. We study the model both analytically and numerically, and discover that in certain cases the motion of the front is controlled by the saturation mechanism. Instead of the typical infinite gradients, resulting from the linear flux-gradients relations, we obtain a discontinuous front, typically associated with nonlinear hyperbolic phenomena. We prove that if the initial support is compact, independently of the smoothness of the initial datum inside the support, a shock discontinuity at the front forms in a finite time, and until then the front does not expand.

Adding a nonlinear convection enhances the conditions for a breakdown. In fact, the most interesting feature is the effect of criticality, that is, unlike small amplitude solutions that remain smooth at all times, large amplitude solutions may develop discontinuities. This feature is easily seen via the analysis of traveling waves: while small amplitude kinks are smooth, in large amplitude kinks part of the upstream-downstream transition must be accomplished via a discontinuous jump (subshocks). Thus induced discontinuities may persist indefinitely since the traveling waves represent a forced motion. Unlike the classical Burgers case, here, due to the saturation of the diffusion flux, the viscous forces have a bounded range. When the inertial forcing exceeds a certain threshold, the disparity between the inertial and dissipative forces is resolved by formation of a discontinuity.

CHOW Bennett (San Diego). **Combinatorial Curvature Flows.** We will discuss geometric flows of both simplicial surfaces and polygons in the plane. The combinatorial Ricci flow of surfaces takes triangulated surfaces which are piecewise hyperbolic, euclidean, or spherical and tries to make the curvatures at the vertices constant. It is related to Thurston's circle packing metrics. At the moment, very little seems to be known about combinatorial flows of planar polygons. We start with a linear equation which can be analyzed.

DASKALOPOULOS Panagiota (New York). **Type II collapsing of maximal Solutions to the Ricci flow.** We consider the initial value problem $u_t = \Delta \log u$, $u(x,0) = u_0(x) \geq 0$ in R^2, corresponding to the Ricci flow, namely conformal evolution of the metric $u(dx_1^2 + dx_2^2)$ by Ricci curvature. It is well known that the maximal solution u vanishes identically after time $T = \frac{1}{4\pi} \int_{R^2} u_0$. We provide upper and lower bounds on the geometric width of the solution and on the maximum curvature. Using these estimates we describe precisely the Type II collapsing of u at time T: we show the existence of an inner region with exponentially fast collapsing and profile, up to proper scaling, a soliton cigar solution, and the existence of an outer region of persistence of a logarithmic cusp. This is the only Type II singularity which has been shown to exist, so far, in the Ricci Flow in any dimension.

DENZLER Jochen (Knoxville). **Delocalized source type solutions for fast diffusion and porous medium.** We describe a family of explicit solutions to the porous medium and fast-diffusion equations, which are not radially symmetric, and we study their asymptotic behavior. Simarly as for the Barenblatt solution, there is reason to hope that these new solutions shed light on the asymptotics for general solutions to PME and FDE. This is joint work with Robert McCann.

DI FRANCESCO Marco (Aquila). **The Keller-Segel model for chemotaxis with prevention of overcrowding: linear vs nonlinear diffusion..** We shall discuss the effects of linear and nonlinear diffusion in the large time asymptotic behavior of the Keller-Segel model of chemotaxis prevention of overcrowding. In the linear diffusion case we provide several sufficient condition for the diffusion part to dominate and yield decay to zero of solutions. We also provide an explicit decay rate towards self–similarity. Moreover, we prove that no stationary solutions with positive mass exist. In the nonlinear diffusion case we prove that the asymptotic behavior is fully determined by whether the diffusivity constant in the model is larger or smaller than the threshold value $e = 1$. Below this value we have existence of non-decaying solutions and their convergence (along subsequences) to stationary solutions. For $e > 1$ all compactly supported solutions are proved to decay asymptotically to zero, unlike in the classical models with linear diffusion, where the asymptotic behavior depends on the initial mass.

DOLBEAULT Jean (Paris). **Nonlinear diffusions as diffusion limits of kinetic equations with relaxation collision kernels.** At the kinetic level, it is easy to relate the parameters with simple physical quantities, but the price to pay is the high dimensionality of the phase space. On the other hand, hydrodynamical equations or parabolic models are in principle simpler to compute, but their direct derivation is far less intuitive. This motivates the study of hydrodynamic or diffusion limits and in our approach, local or global Gibbs states will be considered as basic input for the modeling. This is a very standard assumption for instance in semiconductor theory when one speaks of Fermi-Dirac distributions, or when one considers polytropic distribution functions in stellar dynamics. It is the purpose of this work to provide a justification of nonlinear diffusions as limits of appropriate simple kinetic models.

Let us mention that in astrophysics, power law Gibbs states are well known (see, *e.g.*, [1], and [7] for some mathematical properties of such equilibrium states).

In our approach [2], [3] we say nothing about the physical phenomena responsible for the relaxation towards the local Gibbs state and, on the long time range, towards the global Gibbs state. We introduce at the kinetic level a caricature of a collision kernel, which is simply a projection onto the local Gibbs state with the same spatial density, thus introducing a local Lagrange multiplier which will be referred to as the pseudo Fermi level.

We prove existence and uniqueness of solutions to the kinetic model under the assumption of boundedness of the initial datum and prove the convergence to a global equilibrium. With the parabolic scaling we rigorously prove the convergence of the solutions to a macroscopic limit using compensated compactness theory. Most notably, we are able to reproduce non-linear diffusion equations $\partial_t \rho = \Delta(\rho^m) + \nabla \cdot (\rho \nabla V)$, ranging from porous medium equation to fast diffusion, $0 < m < \frac{5}{3}$, as macroscopic limits by employing the appropriate energy profiles.

In the mathematical study of diffusion limits for semiconductor physics, more results are known, starting with [4],[5]. Other reference papers are [6] and [8].

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FELLNER Klemens (Vienna). **Entropy Methods for Systems Combining Diffusion and Nonlinear Reaction.** Reaction-diffusion systems and coagulation and fragmentation of polymers are examples of models which combine diffusion and nonlinear reactions in terms of an entropy (free energy) functional. We present entropy methods (i.e. the idea how a functional inequality relates the entropy relative to equilibrium with the entropy-dissipation accounting the conserved quantities) to study global existence and long-time behaviour.

In a first part, we discuss in particular a reaction-diffusion system modelling four chemical substances with individual diffusion coefficients, which react by reversible mass-action kinetics within a bounded domain. In this case - up to our knowledge - global L^∞ bounds are unknown, but for which, at least in 1D, a polynomially growing L^∞ bound can be established due to the decay of the entropy. We improve the existing theory in 1D by getting 1) almost exponential convergence in L^1 to the steady state via a precise entropy-entropy dissipation estimate, 2) an explicit global L^∞ bound via interpolation of a polynomially growing H^1 bound with the almost exponential L^1 convergence, and 3), finally, explicit exponential convergence to the steady state in all Sobolev norms.

In a second part, we present work in progress on the Aizenman-Bak model of coagulating and fragmenting polymers with non-degenerate size-dependent diffusion coefficients. Again in 1D, we prove a-priori estimates which show immediately smoothing in time and space while in size-distribution solutions are decaying faster than any polynomial. Moreover, we are very positive to be able to establish a sharp enough entropy entropy-dissipation estimate, which will imply - similar to the strategy above - explicit exponential convergence towards the steady state.

GAMBA Irene (Austin). **Self-similar asymptotics for generalized non-linear kinetic Maxwell models.** We study long time dynamics to solutions of initial value problems to a rather general multi-linear kinetic models of Maxwell type which may describe qualitatively different processes in applications, but have many features in common. In particular we focus in the existence, uniqueness and asymptotics to self-similar (or dynamical scaling) solutions. We use a relationship of spectral properties of the problem in Fourier space to the existence and asymptotic behavior of the solution of the original initial value problem as well as the characterization of the domain of attraction to self-similar states. In particular we show that the self-similar asymptotic dynamics imply that the solutions of these type of problems evolve to ”infinitely divisible“ process from the probabilistic viewpoint, where the tails and time decay laws are classified from the spectral properties related to the original problem.

Examples are models of Maxwell type in classical space homogeneous, elastic or inelastic Boltzmann equation, and the elastic Boltzmann equation in the presence of a thermostat, all with finite or infinite initial energy, as well as Pareto distributions models in economy, and Smoluckowski type of equations.

This is work in collaboration with A. Bobylev and C. Cercignani.

GENTIL Ivan (Paris). **About modified logarithmic Sobolev inequalities and applications to concentration inequalities**

GUALDANI Maria Pia (Austin). **Discontinuous Galerkin method for dissipative quantum models.** The motion of a particle ensemble interacting with an environment can be described with a Wigner approach, where the interaction mechanisms are taken into account by a Fokker-Planck scattering term. Solutions to such kind of models are characterized by an oscillatory behavior; a modified Discontinuous Galerkin method based on non-polynomial function space is used for the numerical approximation to this problem. The choice in the scheme of trigonometric functions for the finite element space allows for a better approximation to the highly oscillatory solutions.

ILLNER Reinhard (Victoria). **Modelling traffic flow using kinetic theory**

JUENGEL Ansgar (Mainz). **Algorithmic derivation of entropy-entropy dissipation inequalities by solving polynomial decision problems.** The proof of analytical and numerical properties of solutions to nonlinear evolution equations is usually based on appropriate a priori estimates and monotonicity properties of Lyapunov functionals, which are called here entropies. These estimates can be shown by subtle integration by parts. However, such proofs are usually skillful and not systematic. In this talk a systematic method for the derivation of a priori estimates for a large class of nonlinear evolution equations of even order in one and several variables with periodic boundary conditions is presented. This class of equations contains the thin-film equations, for instance.

The main idea is the identification of the integrations by parts with polynomial manipulations. The proof of a priori estimates is then formally equivalent to the solution of a decision problem known in real algebraic geometry, which can be solved algorithmically. The method also allow us to prove the non-existence of entropies and to derive new logarithmic Sobolev inequalities.

KIM Yong Jung (Taejon). **Potential comparison and long time asymptotics of convection, diffusion and p-Laplacian in one space dimension.** Recently a potential comparison technique has been developed for solutions to a nonlinear diffusion equation. This method can be applied to other problems after a suitable modification. In this talk this technique will be discussed for the cases in the title. The convergence order of the magnitude of the solution itself is shown in L^1 norm when the three terms are together. Convergence order $1/t$ is shown when only one of them exists under extra conditions for the initial value.

KURGANOV Alexander (Louisiana). **Effects of Saturating Diffusion.** I will talk about strongly degenerate parabolic PDEs with a saturating diffusion flux. The simplest model is:

$$u_t = Q(u_x)_x, \qquad (1)$$

where Q is a bounded increasing function. Such a nonlinear diffusion is "weaker" than the linear one present in the "standard' heat equation,

$$u_t = u_{xx}.$$

The effect of the saturating diffusion in (1) is manifested in a possible "delayed diffusion" phenomenon: initial discontinuities may be smeared out only after a certain (finite) time.

In the past 10 years, Philip Rosenau (Tel-Aviv University) and I together with several of collaborators of ours have been studying various effects of saturating diffusion on convection-diffusion equations,

$$u_t + f(u)_x = [u^n Q(u_x)]_x, \quad n \geq 0,$$

porous media type equations,

$$u_t = [u^n Q(u_x)]_x, \quad n > 0,$$

and reaction-diffusion equations,

$$u_t = Q(u_x)_x - f(u).$$

We have obtained several interesting, sometimes rather surprising results, and I will present some of them, including the most recent ones.

LAURENÇOT Philippe (Toulouse). **Convergence to steady states for a one-dimensional viscous Hamilton-Jacobi equation with Dirichlet boundary conditions.** The convergence to steady states of solutions to the one-dimensional viscous Hamilton-Jacobi equation $\partial_t u - \partial_x^2 u = |\partial_x u|^p$, $(t,x) \in (0,\infty) \times (-1,1)$ with homogeneous Dirichlet boundary conditions is investigated for $p \in (0,1)$. For that purpose, a Liapunov functional is constructed by the approach of Zelenyak (1968). Instantaneous extinction of $\partial_x u$ on a subinterval of $(-1,1)$ is also shown for suitable initial data.

LEE Ki Ahm (Seoul). **Geometric properties in elliptic and parabolic problems.** In this talk, we are going to discuss the geometric properties in parabolic flows, for example porous medium equations, parabolic p-Laplace equations, and free boundary problems. And the study of the asymptotic behavior of these flows will give us another promising method to find the geometric properties of solutions in elliptic problems

LOEPER Grégoire (Lyon). **Regularity of maps solutions of optimal transportation problems.** Given two probability measures μ, ν and a cost function $c(x, y)$, one seeks to minimize

$$\int c(x, T(x)) d\mu(x)$$

among all maps T that push forward μ onto ν. This work is concerned with the continuity of the minimizers. Based on the Monge-Kantorovitch duality, the minimizers are expressed though the gradient of a "c-convex" potential ϕ (c-convexity being the appropriate generalization of convexity for general cost c instead of $c(x, y) = |x - y|^2$). This potential will solve a Monge-Ampère type equation of the form $\det(M(x, \nabla\phi) + D^2\phi) = f(x, \nabla\phi)$. Ma, Trudinger and Wang found a sufficient condition on the cost function so that for smooth positive measures, the optimal T is smooth. I will show that this condition is actually a necessary condition for regularity, and that it is equivalent to the connectedness of the c-subdifferential of c-convex functions. Finally, I will show that when the Ma, Trudinger and Wang condition is satisfied in a strict sense, one can obtain continuity of the optimal T (i.e. C^1 regularity for the potential ϕ) under lower requirements than what is needed for the usual Monge-Ampère equation $\det D^2\phi = f$.

MATTHES Daniel (Mainz). **Two applications of an algebraic method for entropy construction.** This short presentation outlines two recent extensions and applications of the algebraic method for the construction of entropy functionals as introduced by Jüngel and the speaker.

First, a variant of the method is used to estimate the rate of entropy dissipation in the logarithmic fourth order (DLSS) equation in arbitrary space dimensions.

Second, a particular (linear) Fokker-Planck equation is considered. Although the Bakry-Emery-criterion fails in this example, the algebraic approach still yields entropy dissipation estimates. These estimates give rise to a family of Beckner-type interpolation inequalities. Explicit values for the appearing constants are calculated.

MAZON RUIZ Jose M. (Valencia). **Finite Propagation Speed for Limited Flux Diffusion Equations.** To correct the infinite speed of propagation of the classical diffusion equation Ph. Rosenau proposed the tempered diffusion equation

$$u_t = \nu \operatorname{div} \left(\frac{uDu}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}} \right). \tag{5.7}$$

Equation (5.7) was derived by Y. Brenier by means of Monge-Kantorovich's mass transport theory and he named it as the *relativistic heat equation*. We prove existence and uniqueness of entropy solutions for the Cauchy problem for the quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, Du), \tag{5.8}$$

where $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$ and f being a function with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions. In particular, this class includes the relativistic heat equation (5.7) and the flux limited diffusion equation

$$u_t = \nu \operatorname{div} \left(\frac{uDu}{u + \frac{\nu}{c}|Du|} \right) \tag{5.9}$$

used in the theory of radiation hydrodynamics.

We study the evolution of the support of entropy solutions of relativistic heat equation. For that purpose, we give comparison principles between sub-solutions (or super-solutions) and entropy solutions of the Cauchy problem and then using suitable sub-solutions and super-solutions, we establish the following result.

"Let C be an open bounded set in $\mathbf{R}^N$. Let $u_0 \in (L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))^+$ with support equal to $\overline{C}$. Assume that given any closed set $F \subseteq C$, there is a constant $\alpha_F > 0$ such that $u_0 \geq \alpha_F$ in F. Then, if $u(t)$ is the entropy solution of the Cauchy problem for the equation (5.7) with u_0 as initial datum, we have that

$$\operatorname{supp}(u(t)) = \overline{C} \oplus \overline{B_{ct}(0)} \quad \text{for all } t \geq 0."$$

NAZARET Bruno (Paris). **Optimal Sobolev trace inequalities on the half space.** Using a mass transportation method, we study optimal Sobolev trace inequalities on the half space and prove a conjecture made by Escobar in 1988 about the minimizers.

PANFEROV Vladislav (Hamilton). **Strong solutions of the Boltzmann equation in one-dimensional spatial geometry.** We study the nonlinear Boltzmann equation in the setting of one-dimensional (plane wave) solutions, in the assumption of bounded microscopic collision rate, satisfying certain cutoffs. Using the estimates of the relative entropy and of the quadratic functional introduced by Bony we show that the "strong" bounds ensuring L^1 stability propagate globally in time.

PUEL Marjolaine (Toulouse). **A mass transport approach for a relativistic heat equation.** We present in this talk a discrete scheme for a relativistic equation obtained following the method of Jordan Kinderlehrer Otto.

SLEPCEV Dejan (Los Angeles). **Coarsening in thin liquid films.** Thin, nearly uniform, layers of some liquids can destabilize under the effects of intermolecular forces. After the initial phase, the liquid breaks into droplets connected by an ultra-thin liquid film. As the droplets interchange mass, the configuration of droplets coarsens over time. The characteristic distance between droplets and their average size grow, while their number is decreasing.

This physical process can be modeled by an equation for the height of the fluid — the thin-film equation. The evolution is a gradient flow, that is the steepest descent in an energy landscape. I will describe how information on the geometry of the energy landscape yields a rigorous upper bound on the coarsening rate.

The mass exchange between droplets can be mediated by two mechanisms: exchange through the connecting ultra-thin layer and droplet collisions. I will discuss the relative importance of the two mechanisms.

This is joint work with Felix Otto and Tobias Rump.

STURM Karl-Theodor (Bonn). **Optimal Transportation and Ricci Curvature for Metric Measure Spaces.** We introduce and analyze generalized Ricci curvature bounds for metric measure spaces (M, d, m), based on convexity properties of the relative entropy $Ent(.|m)$. For Riemannian manifolds, $Curv(M, d, m) \geq K$ if and only if $Ric_M \geq K$ on M. For the Wiener space, $Curv(M, d, m) = 1$.

One of the main results is that these lower curvature bounds are stable under (e.g. measured Gromov-Hausdorff) convergence.

Moreover, we introduce a curvature-dimension condition CD(K, N) being more restrictive than the curvature bound $Curv(M, d, m) \geq K$. For Riemannian manifolds, CD(K, N) is equivalent to $\text{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ and $\dim(M) \leq N$.

Condition CD(K, N) implies sharp version of the Brunn-Minkowski inequality, of the Bishop-Gromov volume comparison theorem and of the Bonnet-Myers theorem. Moreover, it allows to construct canonical Dirichlet forms with Gaussian upper and lower bounds for the corresponding heat kernels.

VÁZQUEZ Juan Luis (Madrid). **Log-diffusion of measures.** We discuss the diffusion of Dirac measures surrounded by a locally integrable distribution according to the log-diffusion equation in two space dimensions. The point masses trickle into the medium at a rate of 4π units per unit time and mass location.

List of Participants

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Chapter 6

Schrödinger Evolution Equations (06w5030)

April 22 – 27, 2006

Organizer(s): James Colliander (Toronto), Jared Wunsch (Northwestern)

Introduction

The (linear) Schrödinger wave equation, first formulated by Erwin Schrödinger in 1925, provides a description of the time evolution of the wavefunction of a nonrelativistic quantum particle; for a free particle of mass m, it reads

$$i\hbar \frac{\partial}{\partial t} u + \frac{\hbar^2}{2m} \Delta u = 0.$$

From the outset, mathematicians and physicists have been concerned with the ways in which solutions to this equation can be associated to classical particle motion, and the ways in which they cannot: the classical dynamical behavior of particles is intermixed with *dispersive spreading* and interference phenomena in quantum theory.

More recently, *nonlinear* Schrödinger equations have come to play an essential role in the study of many physical problems. Nearly monochromatic waves with slowly varying amplitude occur frequently in science and technology. Second order expansions of physical models of wave phenomena around such waves lead naturally to the cubic nonlinear Schrödinger equation. Thus, the nonlinear Schrödinger equation is a *canonical wave model* since it emerges ubiquitously in the study of waves. Nonlinear Schrödinger (NLS) equations appear in such diverse fields as nonlinear optics, superconductivity, oceanography, and quantum field theory. The main themes of research discussed at this workshop concern the Cauchy or initial value problem for Schrödinger equations. Theoretical and applied aspects of nonlinear Schrödinger equations are nicely surveyed in the textbooks [4], [17].

Nonlinear Schrödinger evolutions involve a dynamical balance between *linear dispersive spreading* of the wave and *nonlinear self-interaction* of the wave. Generalizations of the physically relevant equations with nonlinear and dispersive parameters have been introduced to probe the interplay between these effects. For example, the semilinear initial value problem

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u \\ \quad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \end{cases} \tag{6.1}$$

may be viewed as a nonlinear generalization of the cubic problem on $\mathbb{R}^d$ corresponding to $p = 3$. The Laplacian term Δu generates the dispersion in this evolution equation. The cubic nonlinearity represents self-interaction of the wave. The choice of sign corresponds to (-) *focusing* and *defocusing* (+) nonlinearities. Some overlapping themes of the research discussed at the workshop may be outlined in the setting of (6.1):

- **Optimal Well-posedness**
  What are the minimal regularity assumptions on the initial data u_0 for which the initial value problem (6.1) may be solved locally in time? What happens to rougher initial data? Are the local-in-time solutions in fact global-in-time? Spaces with norms which are invariant under the scaling symmetry of solutions play a crucial role in anwers to these questions.

- **Nonlinear dispersive systems**
  Are the recent methods for solving (6.1) robust enough to also apply to systems of equations? Nonlinear dispersive systems, such as the Zakharov system and the Maxwell-Schrödinger system, more accurately model physical phemomena than the closely related cubic nonlinear Schrödinger equation. In certain regimes of physical parameters, some nonlinear dispersive systems are expected to be well-approximated by simpler problems. How do solutions of nonlinear dispersive systems behave as physical parameters are pushed toward extreme values?

- **Long-time behavior**
  What happens? This is the main question to be addressed when considering an initial value problem. Fantastic progress over the past two decades has led to a nearly complete well-posedness theory for the equations in (6.1). Beyond existence and uniqueness, the issue is to provide qualitative descriptions of the evolution. For defocusing problems on $\mathbb{R}^d$, the expected behavior is similar to that expressed by the linear evolution. On compact domains, persisting nonlinear interactions are expected to generate oscillations on smaller and smaller scales. For focusing problems, the expected behavior includes the emergence of nonlinear coherent structures such as solitons and finite time explosions.

- **Linear equations**
  Among the essential tools for recent progress into nonlinear Schrödinger evolutions are the Strichartz and dispersive smoothing estimates for linear Schrödinger equation. These and related estimates have thus been subjects of geat interest both on their own and for the light they shed on nonlinear phenomena. Under what conditions do the fundamental linear Schrödinger estimates extend to the setting of variable coefficients?

Talks at the workshop described significant progress in each of these four directions.

Optimal Well-posedness

When considering an initial value problem, such as (6.1), some basic questions arise: Does a solution exist? For which initial data does a solution exist? If a solution exists, is it unique? For how long does the solution last? How does the solution depend upon the initial data? Do smoothness properties of the intial data persist during the evolution? Answers to these and related questions are provided by the well-posedness theory for the initial value problem. For certain classes of initial value problems, such as (6.1), striking progress over the past two decades has culminated into a satisfactory local-in-time well-posedness theory. Over the past decade, methods for showing ill-posedness have emerged which have revealed the optimality of various known local-in-time well-posedness results and new ideas for establishing global-in-time well-posedness have been developed. Some of the talks at the workshop contributed in these directions.

Burq described a simplified proof, obtained in joint work with Gerard and Ibrahim, of ill-posedness results due to Lebeau [13] and Christ-Colliander-Tao [3]. An anisotropic scaling of a rather explicit solution provides a clear view into one low regularity mechanism causing havoc for the Cauchy problem. For cubic problems, Carles described an inspired geometrical optics based approach to proving similar results.

Gérard spoke about the cubic NLS in four dimensions, in various settings. See also the survey [9]. On S^4, together with Burq and Tzvetkov, he has obtained rather complete results on well-posedness: the equation is well-posed on H^s for $s > 1$, ill-posed for $s < 1$, and ill-posed in H^1, with the flow map failing to be continuous, even on small data. By contrast, in more recent work with Pierfelice, Gérard has shown that in the H^1 case, slightly relaxing the nonlinearity results in a qualitative change: if we replace $|u|^2u$ by certain homogeneous quadratic polynomials $q(u, \overline{u})$, then there is global well-posedness in H^1 provided that the *ODE* $i\partial_t u = q(u)$ does not blow up.

When the conserved quantities imply an apriori H^1 upper bound, standard local well-posedness results for H^1-subcritical initial value problems may be iterated to obtain global well-posedness for H^1 initial data. Whether initial data of lower regularity for which local-in-time well-posedness holds evolves globally in time has been a topic of intense study over the past eight years.

In the case of the initial value problem for the L^2-critical equation

$$iu_t + \Delta u - |u|^{4/d} u = 0,$$

one can hope for global well-posedness and scattering all the way down to L^2. Staffilani announced joint work [16] with De-Silva, Pavlovic, and Tzirakis, in which the periodic initial value problem is studied in one dimension with data in H^s, and global well-posedness is shown for $s > 4/9$. This talk led to a very interactive question period, with discussion among Staffilani, Planchon, Burq, and Gérard on the relationship between bilinear estimates and local well-posedness.

For the defocusing L^2-critical NLS in four dimensions with radial data,

$$iu_t + \Delta u = |u|u,$$

Visan announced a breakthrough proof, obtained in joint with with Tao, that demonstrates global well-posedness and scattering. While the local theory dates back to the work of Cazenave-Weissler, the global theory is not yet well understood. Visan also described and compared this work with results she, her collaborators and others have recently obtained (see [21] and references therein) in the $\dot{H}^1$-critical case,

$$iu_t + \Delta u = |u|^{\frac{4}{d-2}} u$$

in dimension $d \geq 3$, for arbitrary data. The key ingredients here are an induction on energy strategy due to Bourgain [2] and a new interaction Morawetz inequality [5], [6]. The developments announced by Visan forecast profound improvements in our understanding of the L^2-critical nonlinear Schrödinger equations.

Nonlinear dispersive systems

The methods developed for scalar Cauchy problems like (6.1) have also been applied to more complicated, and more physically accurate, nonlinear dispersive systems. Dispersive systems lack certain simplifying features, such as scaling invariance, enjoyed by (6.1). Adaptations and innovations of the scalar techniques have recently been under investigation. Progress in this direction has demonstrated that the key insights are robust and extend to the setting of nonlinear dispersive systems.

Bejenaru recently proved [1] a global well-posedness result for the Schrödinger map problem posed on $\mathbb{R}^d, d \geq 3$, for small initial data in a scaling invariant Besov norm. A similar result has recently been obtained by Ionescu-Kenig. The proof relies upon structural properties of the nonlinearity and delicate bilinear estimates in (Besov variants of) $X_{s,b}$ spaces. This result is the Schrödinger analog of a celebrated result [19] of Tataru on wave maps.

Grillakis' talk discussed the evolution of a curve by binormal curvature flow and its relationship, under the Hasimoto transformation, to the cubic nonlinear Schrödinger equation on $\mathbb{R}$. He then derived a generalization to a curvature driven surface evolution. Establishing well-posedness for the surface evolution problem appears to be a difficult problem.

Ibrahim discussed recent work with Biryuk and Craig towards establishing that various approximation schemes converge to weak solutions of the Navier-Stokes system. A lively discussion following the talk hinted at the possibility that a decay property required for improvements to the convergence results is linked with spectral cluster estimates like those discussed by Smith.

In a rather different setting, Koch discussed joint work with Saut concerning the local smoothing and Strichartz estimates for a very wide class of third order dispersive equations in two dimensions (which include the linear parts of several equations describing surface gravity waves at various approximations). He obtains local smoothing and Strichartz estimates with a derivative gain, for the generic cases of these third order equations. This work demonstrates that the well-posedness theory based on dispersive estimates is robust and applies to a wide variety of model equations appearing the the applied mathematics and physics literature.

Nakamura described joint work [14] with Wada which established a low regularity local well-posedness result for the Maxwell-Schrödinger system in the Coulomb gauge. The result is based on ideas stemming from work of Koch and Tzvetkov. Estimates of energy type were also discussed which imply that the local solutions in fact extend globally in time.

Nakanishi's talk described joint work with Masmoudi which established that finite energy solutions of the Zakharov system converge to solutions of the focusing cubic nonlinear Schrödinger equation in the subsonic limit. Earlier work on this limit required regularity assumptions beyond finite energy. The Zakharov system models plasma in certain physical regimes. A sound speed parameter in the system moves toward infinity as the mass of the ions converges to the mass of the electrons. This result about the subsonic limit opens up the possibility that fine blowup properties of NLS and the Zakharov system can now be compared in the setting of finite energy solutons.

Tzirakis described a new method, developed with Colliander and Holmer [20], for globalizing certain nonlinear dispersive systems. The method exploits L^2 conservation on one of the system components and an almost conservation property for the other component. The method has been applied to the Zakharov system on $\mathbb{R}$ and to the Klein-Gordon-Schrödinger system on $\mathbb{R}^3$ to prove that the best known local-in-time solutions extend globally in time.

Long-time behavior

Aspects of the maximal-in-time behavior of solutions of nonlinear Schrödinger evolution equations were reported upon at the workshop. Besides the scattering and long-time existence results described previously, new results concerning the asymptotic behavior of soliton solutions and periodic-in-space solutions were discussed.

Holmer described work [12], with Marzuola and Zworski, which explains the behavior of fast solitons in the one dimensional cubic nonlinear Schrödinger equation interacting with a repulsive Dirac-mass singularity. Slow solitons will spend more time in the interaction region than fast solitons when passing through the Dirac singularity. Intuitively, slow solitons will have more back-reflected mass than fast solitons. Also, extremely fast solitons should have barely any back-reflected mass. The result described validates this intuition by showing that, in the high speed limit, the bulk of the soliton mass moves past the potential. However, the upper bound on the size of the back-reflected mass is larger than conjectured in the high speed limit.

In the other direction, Zhou's talk considered the behavior of a soliton trapped by a potential. Zhou discussed recent work [23] with Sigal which establishes asymptotic stability results for soliton solutions of NLS in the presence of an external potential. Under certain assumptions, their result shows that trapped solitons oscillate in a potential well and slowly shed excess energy to spatial infinity, eventually relaxing to an asymptotic equilibrium inside the well. The proof involves a clever application of normal forms reduction to rigorize intuition related to the Fermi golden rule.

Another new result giving insight into the qualitative behavior of global-in-time solutions was discussed by Tao. In joint work, Colliander-Keel-Staffilani-Takaoka-Tao have obtained results on cubic defocusing NLS on the two-torus which are a step toward the "weak turbulence conjecture," describing the movement of energy from low to high Fourier modes. In particular, this group has shown that for all $s > 0$, $\epsilon > 0$, $M \gg 1$ there is $u_0 \in \mathcal{C}^\infty(S^1 \times S^1)$ and $T > 0$ such that $\|u_0\|_s \leq \epsilon$ and $\|u(T)\|_s \geq M$, where u is the solution with initial data u_0. The idea is to convert to an immense system of ODE by expanding onto resonant Fourier modes, and then to make a rather delicate combinatorial construction.

Linear equations

A major recent thrust of work on linear Schrödinger equations has been to understand precisely the regularity of solutions and to obtain associated estimates, usually dispersive smoothing estimates and Strichartz estimates, that are of use in tackling nonlinear problems. For instance, it has been known since the work[1] of Constantin and Saut, Sjölin and Vega, that if $u(t)$ is a solution to the linear Schrödinger equation on $\mathbb{R}^n$, we

[1]The background results briefly described here are surveyed more completely in the textbook [4] of Cazenave.

have the "(local) dispersive smoothing estimate"

$$\int\int_{|x|<R} |\Delta_x^{1/4} u(t,x)|^2\,dx\,dt \leq C_R \|u_0\|_{L^2}^2. \tag{6.2}$$

In other words, we locally find that u is half a derivative smoother than its initial data, when averaged in time. Further refinements of this estimate are possible that are global in space (with a weight) and global in time. A microlocal version of it was first given, for variable coefficient equations, by Craig-Kappeler-Strauss[7].

Perhaps the key estimate on the linear equation for proving well-posedness is the *Strichartz* estimate. Again for the linear equation on $\mathbb{R}^n$, Strichartz deduced estimates of the form

$$\|u\|_{L_t^q L_x^q([0,1]\times M)} \leq C\|u(0)\|_{L^2(M)},$$

for an appropriate exponent q. Later, these estimates were extended to include norms of mixed Lebesgue exponent of the form

$$\|u\|_{L_t^q L_x^r([0,1]\times M)} \leq C\|u(0)\|_{L^2(M)},$$

for appropriate exponents q, r.

Much recent work has gone into generalizing the dispersive smoothing and Strichartz estimates to apply to variable coefficient operators, involving (possibly singular) potentials, and non-Euclidean metrics. One major tool for obtaining such results is *commutator estimates*, perhaps best considered as microlocal energy estimates, i.e., energy estimates localized in phase space. Another tack is to try to find a parametrix for the Schrödinger propagator e^{-itH} directly, and then obtain the estimates from its explicit form.

Tataru [18] presented new results on Strichartz estimates mostly following the latter approach. These estimates, generalizing the local in time variable coefficient estimates on asymptotically conic spaces obtained by Robbiano-Zuily [15] and Hassell-Tao-Wunsch [11], are global in time, and require only extremely weak estimates on the metric, much weaker than the usual "short-range" assumption. They also require very little differentiability. It should be noted that all of these constructions (and indeed, the validity of the usual Strichartz and dispersive smoothing estimates) rest on a crucial geometric assumption: the metric must be *non-trapping*, i.e. geodesics must approach spatial infinity as time goes to infinity.

One upshot of the parametrix construction of Hassell-Wunsch [10], described in its latest refinement in Hassell's talk, is a propagation theorem, describing the formation of singularities of $e^{-itH}u_0$ in terms of oscillatory behavior of u_0. Nakamura presented a different approach to such propagation results, giving a characterization of WFu_0 in terms of u_0 that rests on scattering-theoretic methods, furthermore yielding a generalization to long-range metrics. Doi also discussed new propagation theorems, yielding a very precise description of WF$e^{-itH}u_0$ in the case $H = (1/2)\Delta + V + W$ with V a harmonic oscillator potential and W a perturbation term. There are three different regimes, depending on the size of W: if W is of order $|x|^\rho$, and $\rho < 1$, then the perturbation is irrelevant to the formation of wavefront set. If $\rho = 1$ then there is a finite-speed correction to the propagation of singularities for the harmonic oscillator (see [8]). If $1 < \rho < 2$ then there is an infinite-speed correction. This generalizes a classic result of Zelditch [22], who took $\rho = 0$.

Robbiano reported on results generalizing the weighted, global in space dispersive smoothing estimates to a very broad class of operators, which most notably allows for potentials of any order (provided that the resulting operator has a self-adjoint extension).

New research directions

This workshop revealed new insights into the dynamical balance between nonlinear self-interaction and dispersive spreading of waves. It is of course impossible to predict precisely future research developments. However, recent significant developments and transparent gaps in the theory suggest two emergent research themes.

1. **Scattering at critical regularity.** Bourgain's induction on energy strategy [2] and subsequent developments [6], [21] in the energy critical semilinear Schrödinger problem forecast profound developments for the global-in-time theory of the defocusing problem (6.1) in the energy subcritical case. An optimal (at least in L^2-based Sobolev spaces) local well-posedness theory for (6.1) is now in place. The extant global well-posedness theory has relied upon a priori estimates inferred from energy conservation.

Global results based on energy conservation (or almost conservation) have required regularity assumptions beyond what is necessary for local well-posedness. The talk by Visan hints that the induction strategy and virial or Morawetz-type estimates may perhaps be adapted to prove scattering and global well-posedness results for the defocusing L^2-critical version of (6.1) (with $p-1=\frac{4}{d}$). Speculating further leads to the prospect that further development of these ideas may establish that optimal $H^{s_c}(\mathbb{R}^d)$ initial data for the defocusing version of (6.1), with $0 \leq s_c = \frac{d}{2} - \frac{2}{p-1} \leq 1$, evolves globally in time and scatters. Another outstanding open problem is to establish global well-posedness and scattering in the energy supercritical setting where $s_c > 1$.

2. **Variable coefficient nonlinear problems.** Strichartz and other dispersive linear estimates were first established in the constant coefficient setting. Bilinear and multilinear versions of the linear constant coefficient estimates underpin much of the recent progress on well-posedness and qualitative behavior. A thrust of recent work described at the workshop demonstrates that linear dispersive estimates are robust and extend to the variable coefficient setting. Further development of analogous bilinear and multilinear estimates will open up the study of many new problems in wave phenomena, and strengthen links with geometry, science and engineering.

If the recent past is a reasonable guide into the near future, there will be a continued rapid development of this field of research. The workshop at BIRS provided a splendid forum for the exchange of ideas and questions and contributed toward an improved understanding of Schrödinger evolution equations.

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Chapter 7

Analytic and Geometric Theories of Holomorphic and CR Mappings (06w5015)

April 29, 2006 – May 4, 2006

Organizer(s): John Bland (University of Toronto), Herve Gaussier (Université de Provence), Kang-Tae Kim (Pohang University of Science and Technology), Steven G. Krantz (Washington University), Finnur Larusson (University of Western Ontario), Junjiro Noguchi (University of Tokyo)

Overview of the Field

Several complex variables is a modern and dynamic part of mathematics. Spawned in the early twentieth century by William Fogg Osgood (United States), Henri Poincaré (France), Fritz Hartogs (Germany), and others, the subject has always had an international flavor. Certainly our workshop reflected that flavor, as we had participants from Korea, Japan, Australia, Russia, Iceland, the U.S.A., France, and India. All of the participants already knew each other—at least professionally—but it was a special pleasure to meet face-to-face and to exchange ideas spontaneously and in real time. Many of the participants are young mathematicians on the cutting edge of current research. It is especially important for a broad cross-section of workers in the field to be able to interact with these important researchers. That is what our workshop achieved.

The two seminal results proved at the inception of this subject are these:

Theorem (Poincaré): *Let*

$$B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

be the unit ball in $\mathbb{C}^2$ *and let*

$$D^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$$

be the bidisc. Then there is no biholomorphic mapping

$$\Phi : B \to D^2 .$$

This theorem tells us that any obvious guess at a Riemann mapping theorem in several complex variables will fail—just because the two most obvious candidates for the "default" domain are not equivalent.

Theorem (Hartogs): *Let*

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 2\} \setminus \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\} .$$

Let f be a holomorphic function on Ω. Then there exists a holomorphic function F on

$$D^2 = \{(z_1, z_2) : |z_1| < 2, |z_2| < 2\}$$

such that $F\big|_\Omega = f$.

The result of Hartogs says that Ω is *not* the natural domain of definition of any holomorphic function; for any holomorphic function on Ω analytically continues to a strictly larger domain. This theorem should be contrasted with the situation in $\mathbb{C}^1$: For *any* domain $\Omega \subseteq \mathbb{C}^1$, there is a holomorphic function f on Ω that cannot be analytically continued to a larger domain. A domain in any $\mathbb{C}^n$ which is the natural domain of definition of some holomorphic function is called a *domain of holomorphy*.

Thus the focus of several complex variables for the past century has been on holomorphic mappings, and on domains of holomorphy. A vast array of techniques has been developed for coming to grips with these two circles of ideas. Today several complex variables interacts profitably with harmonic analysis, partial differential equations, differential geometry, commutative algebra, algebraic geometry, one complex variable, real variable theory, and even formal logic.

Among the methodologies that have been developed to tackle the questions that have been described are

(i) Sheaf theory (J. Leray and K. Oka);

(ii) Partial differential equations, notably the $\overline{\partial}$ problem (J. J. Kohn [13] and L. Hörmander [12]);

(iii) Methods of differentiable geometry, notably Kähler manifold theory and Stein manifold theory (S. S. Chern, H. Grauert, R. Narasimhan, H. Remmert).

These powerful pieces of mathematical machinery have proved to be useful in a variety of mathematical contexts. They continue to be developed today. Furthermore, more recondite techniques such as the Monge-Ampère equation, dynamical systems, Banach algebras, function algebras, operator techniques, and many other widespread ideas continue to develop alongside the principal streams of thought. All these branches of the subject interact profitably, and produce a colorful and productive melange of mathematical work.

Recent Developments and Open Problems

In the past twenty years, the subject of several complex variables has blossomed in a variety of new directions. Among these are

- **The theory of dynamical systems in several complex variables.** Of course dynamical systems found their genesis in the work of H. Poincaré on questions of celestial mechanics. But in fact it was quite early in the twentieth century that Fatou and Julia saw the relevance of these new ideas to complex function theory (in *one complex variable*). In the hands of Hubbard, Douady, Mandelbrot, and many others, one-variable dynamical systems has blossomed into a vigorous part of modern mathematics. The growth of dynamical systems in the several-variable setting is considerably newer.

  And the questions are quite a lot harder. Deep ideas from pluripotential theory, differential geometry, partial differential equations, and many other parts of mathematics must be brought to bear in order to achieve any progress at all. Major workers in the area include Bedford (present at the workshop), Buzzard, Fornaess, Sibony, Lyubich, and Douady.

- **The theory of automorphism groups of domains.** It has already been noted that, because of Poincaré's theorem, we can expect no version of the Riemann mapping theorem in several complex variables. In fact Poincaré's stunning result has been enhanced and refined in the ensuing years. Work of Chern-Moser [7], Burns-Shnider-Wells [6], and Greene-Krantz (organizer of this conference) [10], [11] has shown that the phenomenon that Poincaré discovered is in fact generic in a variety of senses.

But Poincaré himself gave us some guidance as to how to come to grips with the phenomenon. He proved his result by analyzing the automorphism groups of the two domains D^2 and B. Here, if $\Omega \subseteq \mathbb{C}^n$ is a domain, then the automorphism group $\mathrm{Aut}(\Omega)$ is the collection of biholomorphic self-maps of Ω. This set forms a group under the binary operation of composition of mappings. And in fact, if we equip the automorphism group with the topology of uniform convergence on compact sets (equivalently, the compact-open topology), then (at least for a bounded domain Ω) the automorphism groups turns out to be a real Lie group. Thus powerful bodies of machinery may be brought to bear on the study of automorphism groups.

For a given domain Ω, one may study algebraic, topological, and analytic properties of $\mathrm{Aut}(\Omega)$ and determine how they reflect the complex geometry of Ω (and vice versa). This study, which has been particularly vigorous in the past twenty-five years, has become an *ersatz* for the Riemann mapping theorem: it gives us a device for differentiating and comparing domains. Major workers in the field include Bedford (present at the workshop), Kim (present at the workshop, and an organizer), Krantz (present at the workshop, and an organizer), Pinchuk, Greene, Berteloot (present at the workshop), and Kodama (present at the workshop).

- *CR* **geometry, analysis of hypersurfaces, and normal forms.** Poincaré advocated a program of studying the equivalence and inequivalence of domains in $\mathbf{C}^n$ by **(i)** first proving that any biholomorphic mapping of domains will extend smoothly to the boundaries and **(ii)** then constructing differential invariants on the boundaries. It turned out that the mathematical techniques were not yet available to carry out step **(i)**, and it awaited Charles Fefferman to (in 1974 [8]) prove that a biholomorphic mapping of strongly pseudoconvex domains continues smoothly to the boundaries. His work was quickly followed by work of Chern and Moser on constructing the anticipated differential boundary invariants. Fefferman later used these invariants to develop a classification theory for strongly pseudoconvex domains [9]. Bell and Ligocka [], [] were able to extend and simplify Fefferman's theorem, and to turn it into a powerful and versatile tool for our subject. Today the study of biholomorphic mappings, and the cognate idea known as Condition R, is a vital part of several complex variables.

  Work continues on developing these ideas for broader classes of domains. The methods involve commutative algebra, differential geometry, and partial differential equations. Principal workers in the field today include Gong, Hayashimoto (present at the workshop), Isaev (present at the workshop), Ezhov, and Schmalz (present at the workshop).

- **Holomorphic functions and mappings in infinite dimensions.** The idea of studying holomorphic functions and mappings on Banach spaces is more than fifty years old. Certainly the functional calculus and other natural questions of operator theory begged for such a development. But it must be said that little substantial progress was made in the subject until relatively recently. The difficulty, it seems, is that people were trying to study holomorphic functions and mappings on *any* Banach space. What we have learned in the past ten years—thanks to work of Lempert and others—is that considerable progress can be made if we restrict attention to *particular* Banach spaces. There are now a theory of domains of holomorphic, a theory of normal families, a theory of the $\overline{\partial}$ problem, and many other important cornerstones of function theory in the infinite-dimensional setting. Among the experts in this subject area are Cima, Graham (present at the workshop), Kim (present at the workshop), Krantz (present at the workshop), and Lempert.

- *CR* **functions and mappings** In the 1960s, Kohn and Rossi introduced the concept of a *CR* function. This is a function on a real hypersurface in $\mathbf{C}^n$ that is in effect the "trace" of a holomorphic function. One desires a definition of such an object that is intrinsic, and makes no reference to holomorphic functions; this is achieved by way of partial differential equations. Thus are defined the *CR* functions. Also *CR* mappings are defined similarly. In the intervening forty years, CR functions and mappings have assumed a prominent role in the subject. They are useful for studying the $\overline{\partial}$ and $\overline{\partial}_b$ problems, for studying holomorphic mappings, and for developing function theory. Certainly this subject area was one of the focuses of our conference. Several of the talks were about CR functions and mappings, and they generated vigorous discussions and collaborations.

Presentation Highlights

There were a total of twenty presentations made at our workshop. Each one was 45 minutes in duration, followed by a lively discussion period. A sketch of the presentations follows:

Eric Bedford, Indiana University Currents in Complex Dynamics

Abstract: This will concern some results where the study of complex dynamics leads to interesting currents.

Rational mappings are of the form

$$f = \left[\frac{p_0}{q_0} : \ldots : \frac{p_k}{q_k}\right] : \mathbf{P}^k - - \to \mathbf{P}^k$$

where $p_0, \ldots, p_k; q_0, \ldots, q_k$ are polynomials. Such a map is defined in a Zariski open subset of $\mathbf{P}^k$. Birational maps are the rational maps such as f above that admit another rational mapping $g : \mathbf{P}^k - - \to \mathbf{P}^k$ such that $f \circ g = id$ and $g \circ f = id$ except for Zariski closed sets. This work studies the dynamics of the iterates $f^m = f \circ \ldots \circ f$ of a birational map f.

The first important invariant to study is the degree of f, which usually explains the complexity of the dynamics. While the topological degree for the birational maps is generically 1, one should look for the validity of the definition for the dynamic degree

$$\delta(f) = \lim_{n\to\infty} (\deg(f^n))^{1/n}$$

which is not in general well-defined here.

It turns out that this can be made sense if the birational map f has a finite period. In fact, by resolving the singularities by birational blow-up, one can turn this situation into a holomorphic dynamics.

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{f}} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow[f]{} & X \end{array}$$

$$\begin{array}{ccc} V_1 \xrightarrow{\tilde{f}} E_1 \xrightarrow{\tilde{f}} E_2 \xrightarrow{\tilde{f}} V_2 & \subset & Y \\ & & \downarrow \pi \\ V_1 \xrightarrow{f} \cdot \xrightarrow{f} \cdot \xrightarrow{f} V_2 & \subset & X \end{array}$$

This eventually leads to a dynamics in the Picard group $H^{1,1}(\mathbf{P}^k)$ by the linear mapping $f^* : H^{1,1}(\mathbf{P}^k) \to H^{1,1}(\mathbf{P}^k)$, the set of hyperplanes in $\mathbf{P}^k$. It has turned out that, in such a case

$$\delta(f) = \text{Spectral radius of } f^*.$$

This work is related to some researches in Theoretical Physics.

François Berteloot, Université Paul Sabatier Are proper holomorphic self-maps of smoothly bounded domains automorphisms?

Abstract: The title question has been a long standing question in Complex Analysis, which produced several prominent results in the past.

Besides positive answers (strictly pseudoconvex domains, pseudoconvex domains with real analytic boundaries, Reinhardt or circular domains...) the speaker has described a class of circular domains in $\mathbf{C}^2$, whose boundaries are spherical outside a finite number of circles and which do admit branching proper holomorphic self-maps. He has discussed a new approach which reduces the problem to some special cases and, for complete circular domains, leads to a positive answer.

This approach relates two facts. One is a quantified dilation property for *CR* mappings on strictly pseudoconvex hypersurfaces. The other is a contrast between the dynamics of the map (its topological entropy is zero) and the dynamics of the induced map on the boundary (its topological entropy is greater than the log of its degree). This new viewpoint seem rather promising toward a better understanding of the proper mappings in general.

Alexander Brudnyi, University of Calgary L_2-holomorphic functions on coverings of strongly pseudoconvex manifolds

Abstract: I will talk about L_2-holomorphic functions on coverings of strongly pseudoconvex manifolds and show how to solve some problems posed by Gromov, Henkin and Shubin (including some Hartogs type theorem for *CR*-functions). Techniques will include partial differential equations, especially generalizations of the important technique of Hörmander.

Michael Eastwood, University of Adelaide Complex methods in real integral geometry

Abstract: Here, real integral geometry means the Radon transform, the X-ray transform, and variants thereof. I shall indicate how some standard machinery of complex analysis (cohomology, direct images, and so on) can be used to figure out the range and kernel of the real transforms. This is joint work with Toby Bailey and Robin Graham. The work has far-reaching applications to the study of complex differential invariants, with applications to biholomorphic mappings.

Peter Ebenfelt, University of California at San Diego Rigidity and complexity of *CR* structures and their mappings

Abstract: Among strictly pseudoconvex *CR* manifolds, the simplest one is (arguably?) the sphere. *CR* mappings of the sphere into itself are well understood. On the other hand, mappings of a sphere into a higher dimensional sphere can be very wild, but, in some sense, the complexity of such mappings are controlled by the codimension of the mapping (i.e. the difference between the *CR* dimensions of the spheres). For instance, if the codimension is strictly less than the *CR* dimension of the source sphere, then the mapping is linear (up to automorphisms of the target sphere). If M is a strictly pseudoconvex manifold whose "complexity" is suitably close to that of the sphere, then analogous rigidity properties hold for sphere embeddings of suitably low codimension. In this talk, I will make this more precise and discuss some recent results along these lines.

Buma Fridman, Wichita State University Discrete fixed point sets for holomorphic maps

Abstract: The talk will examine the cardinality and configuration of isolated fixed point sets of holomorphic self-maps of complex manifolds. As a consequence of one of our observations (time permitting) a construction of a bounded domain in $\mathbf{C}^n$ with no non-trivial holomorphic retractions will be presented, and some open questions posed. This is joint work with Daowei Ma, and relates to other joint work with Daowei Ma, Kang-Tae Kim, and Steven G. Krantz.

Ian Graham, University of Toronto The Cartan-Caratheodory-Kaup-Wu theorem in an infinite-dimensional Hilbert space.

Abstract: By using a notion of normality appropriate to the study of geometric problems in infinite-dimensional but separable spaces, as well as a triangularizability assumption, we show how the theorem in the title can be generalized to Hilbert spaces.

Theorem: Let Ω be a bounded convex domain in a separable complex Hilbert space. If a holomorphic mapping $f : \Omega \to \Omega$ satisfies the following three conditions:

(i) $f(p) = p$ for some $p \in \Omega$,

(ii) $f'(p)$ is triangularizable, and

(iii) the spectrum $\sigma(f'(p))$ is contained in the unit circle,

then f is a biholomorphism.

In addition to the fact that this is a generalization of a well-known pivotal theorem in several complex variables to an infinite dimensional space, the methods used in its proof demonstrate a new aspect toward the analysis of dynamics/iterations of holomorphic mappings in the infinite dimensions, which is a territory that needs to be explored in the future.

C.-K. Han, Seoul National University Symmetry algebra for even number of vector fields

Abstract: In the 2004 September Conference on *CR* geometry, Levico, A. Koranyi proposed the problem of determining the dimension of infinitesimal automorphism of the multi-contact structure given by two independent vector fields in $\mathbf{R}^3$ whose bracket is transversal. This presentation features a complete answer to this question for even number $(2n)$ of vector fields and discuss the existence for the case $n = 1$. This talk is a classic example of PDE presentation of Cartan's prolongation method that has been made explicit.

Adam Harris, University of New England Asymptotic behaviour of J-holomorphic curves near a Reeb orbit of elliptic type

Abstract: The idea of introducing pseudoholomorphic maps into a contact manifold cross the real line was originally used by Hofer as a tool for addressing the Weinstein conjecture, concerning the existence of periodic orbits of the Reeb flow. A more detailed study of the asymptotics of these mappings was undertaken in the late 90s by Hofer, Wysocki and Zehnder. While the existence of these mappings is guaranteed under quite mild conditions, relatively little is known explicitly about them (eg., how to write them down in specific cases). I will discuss some recent joint work with Wysocki in this direction, giving conditions under which they may be represented as a straightforward generalisation of the parametrization of plane algebroid curves.

Atsushi Hayashimoto, Nagano National College of Technology Normal forms for a class of finitely nondegenerate hypersurfaces in $\mathbf{C}^4$

Abstract: Consider real analytic hypersurfaces in $\mathbf{C}^4$ through the origin. Assume that they are finitely nondegenerate and the diagonal components of the Levi forms are -1, 1, 0 at the origin. For such a class of hypersurfaces, we construct normal forms of their defining functions. The normal form is an outgrowth of the founding ideas of Poincaré about biholomorphic mappings and the classification of domains up to biholomorphic equivalence. The method of constraction is an analogy of that of P. Ebenfelt papers which appeared in *Indiana Univ.Math. J.* (1998) and *J. Diff. Geom.* (2001).

Alexander Isaev, Australian National University Proper Holomorphic Maps between Reinhardt Domains

Abstract: Most proper holomorphic maps between bounded Reinhardt domains are elementary algebraic. In dimension 2, we identify all pairs of domain for which there exists a map which is not elementary algebraic, and obtain a complete description of all proper holomorphic maps. The work is joint with N. Kruzhilin.

Akio Kodama, Kanazawa University A characterization of complex manifolds admitting effective actions of the direct product of unitary groups by biholomorphic automorphisms

Abstract: The presentation was designed to lead the audience to some "analytic" natures of complex manifolds M under some "topological" plus "Steinness" conditions on M. The speaker took the problem of characterizing Stein manifold M whose holomorphic automorphism group is topologically isomorphic to that of the product of k-dimensional ball and the ℓ-dimensional complex Euclidean space $\mathbf{C}^k$.

The upshot of such an example is that it admits a unitary action. The speaker made a very clever use of centralizers and normalizers of the torus actions in the automorphism group (techniques developed by S. Shimizu in his study of tube domains) and showed in the end that indeed M is biholomorphic to the product domain specified above.

Loredana Lanzani, University of Arkansas A Real Analysis approach to the d-bar problem

Abstract: In the first part of this talk the speaker gave an overview of joint work with E. M. Stein concerning L^r-estimates of the Hodge system for forms in $\mathbf{R}^N$. In the second part she discussed the following question: can these results be used to obtain new estimates of the d-bar problem for (p, q)-forms in $\mathbf{C}^n$?

The motivation came from the following theorem

Theorem (Bourgain-Brezis 2004, Van Shaftingen 2004/05) If $f, g \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ are vector fields with compact support satisfying the equation

$$\begin{cases} Curl\ Z = f \\ Div\ Z = 0 \end{cases}$$

then it holds that

$$\|Z\|^{L^{n/(n-1)}}(\mathbf{R}^n) \leq C\|f\|_{L^1(\mathbf{R}^n)}.$$

In comparison of the earlier theorem by Gagliado-Nirenberg, the author, in a collaboration with E.M. Stein, has put this theorem in a perspective, by reformulating the problem in the context of the Hodge-de Rham complex:

$$0 \to \Lambda_0 \to \Lambda_1 \to \cdots \to \Lambda_n \to 0$$

where Λ_p is the L^2 completion of the set of all compactly supported smooth L^2 forms of degree p. Then the above PDE can be translated to

$$\begin{cases} d_\ell Z = f \\ d_\ell^* Z = 0 \end{cases}$$

for instance (the above theorem concerns the case $\ell = 0$.

More generally, the speaker considers

$$\begin{cases} d_\ell Z = f \\ d_\ell^* Z = g \end{cases}$$

and then obtains, in a collaboration with Stein, the following theorem:

Theorem: For $n \geq 2$, for every ℓ with $2 \leq \ell \leq n-2$, the solutions for the preceding equation satisfy the estimate

$$\|Z\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leq C\left(\|f\|_{L^1} + \|g\|_{L^1}\right).$$

Moreover,

$$\begin{aligned} \ell = 0: &\quad \|Z\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leq C\|f\|_{L^1} \\ \ell = n: &\quad \|Z\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leq C\|g\|_{L^1} \\ \ell = 1: &\quad \|Z\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leq C\left(\|f\|_{L^1} + \|g\|_{H^1}\right) \\ \ell + n - 1: &\quad \|Z\|_{L^{n/(n-1)}(\mathbf{R}^n)} \leq C\left(\|f\|_{H^1} + \|g\|_{L^1}\right), \end{aligned}$$

where

$$\|g\|^{H^1} = \|(Pg)^*\|_{L^1(\mathbf{R}^n)}.$$

Here $(Pg)^*$ denotes the non-tangential maximal function of the harmonic extension of g to the upper half space of $\mathbf{R}^{n+1}$.

The speaker has also mentioned that a similar result should hold for the $\bar{\partial}$ complex (Cauchy-Riemann complex), and expects to obtains definite results soon.

Steven Lu, Université de Quebec á Montreal Brody curves in logarithmic varieties with surjective log-albanese map.

Abstract: We give conditions for the algebraic degeneracies of Brody curves in logarithmic varieties with surjective logarithmic Albanese map and hence conditions for such varieties to be hyperbolic. These questions relate to fundamental questions of complex algebraic geometry.

Alip Muhammed, York University On the Riemann-Hilbert-Poincaré Problem for the inhomogeneous Cauchy-Riemann equation on **C**

Abstract: The inhomogeneous Riemann-Hilbert-Poincaré problem with general coefficient for the inhomogeneous Cauchy-Riemann equation on the unit disc is studied using Fourier analysis. It is shown that this problem is well posed only when the coefficient is holomorphic. In the other cases poles or essential singularities have to be dealt with and hence only the Robin boundary condition is well posed for the inhomogeneous Cauchy-Riemann equation.

Stefan Nemirovski Domains and their coverings

Abstract: This talk discussed the relationship between the global geometry of the universal covering of a bounded domain and the local geometry of its boundary. The typical theorems are:

Theorem 1. Let D, D' be Stein, bounded strongly pseudoconvex domains with a real analytic (C^ω) boundary in a complex manifold. Let Y, Y' be their universal coverings. Then, Y is biholomorphic to Y', if and only if ∂D is locally biholomorphic to $\partial D'$ at some points $p \in \partial D$ and $p' \in \partial D'$.

There has been several impressive results by Poincaré, Alexander, Chern-Ji, Burns and others about the domains with spherical boundaries. In this regard the following theorem has also been presented.

Theorem 2. Let D be a Stein, bounded strongly pseudoconvex C^2-smooth boundary. Then, $D \simeq B/\Gamma$ (a quotient of the ball) if and only if ∂D is spherical (everywhere locally CR diffeomorphic to the standard sphere).

The speaker also points out that this puts the following two conjectures on perspective:

Conjecture 1 (Ramadanov Conjecture): If the Fefferman's asympototic expansion formula of the Bergman kernel for a strongly pseudoconvex domain

$$B_D(z) = \frac{\varphi(z)}{(\rho(z))^{n+1}} + \psi(z)\log\rho(z)$$

has the coefficient $\psi(z)$ of the logarithmic term vanishing to the infinite order, then the boundary of the domain is spherical.

Conjecture 2 (Cheng's Conjecture): For a strongly pseudoconvex domain, if its Bergman metric is also Kähler-Eistein, then the domain is biholomorphic to the ball.

The speaker demonstrated that in complex dimension 2 the first conjecture implies the second.

Gerd Schmalz, University of New England Cartan connection for Engel *CR*-manifold

Abstract: Engel *CR* manifolds are 4-dimensional manifolds with a 1-dimensional *CR* distribution. Their structure algebra is not semi-simple, therefore the standard methods do not work. The speaker, in a joint work with Beloshapka and Ezhov, presented a simple explicit construction which demonstrates the basic ideas of Cartan connections. Of particular interest, the speaker mentioned the result that they were able to distinguish the most essential 4 components among 30, in the curvature of the Cartan connection for Engel 4-manifold that determines the obstruction entirely.

Rasul Shafikov, University of Western Ontario Extension of holomorphic maps between real hypersurfaces of different dimensions

Abstract: It is shown that a germ of a holomorphic map from a real analytic hypersurface M in $\mathbf{C}^n$ into a strictly pseudoconvex compact real algebraic hypersurface M' in $\mathbf{C}^N, 1 < n < N$ extends holomorphically along any path on M. This result has important applications to analytic continuation of, and uniqueness of holomorphic mappings. It intersects with a number of the other talks presented at this workshop.

Berit Stensones, University of Michigan Plurisubharmonic Polynomials

Abstract: We shall study some properties of plurisubharmonic polynomials in two variables. The questions we address is motivated by the so called peak point problem on pseudoconvex domains with real analytic boundaries. A first step towards constructing peak functions is to show that these domains can be bumped to a desired order. We shall show some positive results in that directions. Peak points have important applications in functions algebras, in the study of holomorphic mappings, and particularly in the study of invariant metrics. They are a technical but incisive part of the function theory of several complex variables.

Kaushal Verma, Indian Institute of Science Smooth isometries of invariant metrics

Abstract: It is known that biholomorphisms are isometries of invariant metrics such as the Kobayashi and Caratheodory metrics. One can ask the converse question: are all smooth isometries of these metrics biholomorphic (or anti-biholomorphic)? This talk will report on some work in this direction along with some applications. Our results give a new way to formulate the fundamental theorem of Bun Wong and Rosay. This result in turn has been foundational for the theory of automorphism groups and our understanding of the geometric analysis of domains in complex space.

Scientific Progress Made

One of the delights of a scientific meeting such as this one is the discovery of mutual interests with mathematicians with whom one has never previously communicated. At our workshop, Kim and Krantz found mutual interests with Verma. Lina Lee found mutual interests with Berit Stensones. Buma Fridman found mutual interests with Stefan Nemirovski. Alexander Isaev found common interests with Hayashimoto. There are many other examples.

We know already that some of our discussions are leading to new results and new papers. We anticipate that communications among us will continue, and new directions in the geometric function theory of several complex variables will be charted as a result.

In fact it seems natural that more meetings will be a natural outgrowth of this one. One of us (Krantz) was already approached by several participants with the idea that we should have further and regular meetings. This type of recurrent activity is frequently the basis for major mathematical progress and also for profound effects on the infrastructure of the subject. We might hope to conduct some of our future meetings in Banff.

Outcome of the Meeting

The meeting was a convivial and productive one. We were fortunate that there were attendees from the Steklov Institute (Nemirovski), from Australia (Harris, Ezhov, Eastwood), from Japan (Noguchi, Hayashimoto, Kodama), and others whom we in the West do not frequently encounter. Many new ideas were exchanged, and some new collaborations initiated.

Fridman, Kim, and Krantz came away from the meeting with the start of two new papers. Krantz's graduate student Lina Lee was fortunate to be able to meet with several experts in her subject area (notably Bedford, Berteloot, and Stensones) and come away with many ideas that will be useful in her thesis. Fridman, Isaev, and Nemirovski (all Russians) had several useful discussions.

Today more than half of all published mathematics papers are collaborative. This is in stark contrast to the situation one hundred years ago—when virtually no mathematical work was collaborative. The change is a product of the development of the subject—results are now more difficult to come by. It is also a product of considerable cross-pollination among fields. Finally, it is the product of the proliferation of mathematics institutes and productive meetings like the one that we just completed at BIRS.

Of course the venue of Banff—nestled in the Canadian Rockies—is a very special one. It served as a great attraction for several of our participants to travel a great distance (some traveled for 24 hours!) so that they could participate. The staff at BIRS is a delight, the food is very good, and the accommodations are very appealing. We took an afternoon off so that we could enjoy the natural surroundings; some of us hiked up tunnel mountain and others visited Lake Louise. In all, this was a delightful event for all concerned and we cannot wait to return.

List of Participants

Bedford, Eric (Indiana University)
Berteloot, Francois (Universite Paul Sabatier)
Bland, John (University of Toronto)
Bos, Len (University of Calgary)
Brudnyi, Alex (University of Calgary)
Dwilewicz, Roman (University of Missouri)
Eastwood, Michael (University of Adelaide (Australia))
Ebenfelt, Peter (University of California at San Diego)
Fridman, Buma (Wichita State University)
Gauthier, Paul (Universite de Montreal)
Graham, Ian (University of Toronto)
Han, Chong-Kyu (Seoul National University)
Harris, Adam (University of New England)
Hayashimoto, Atsushi (Nagano National College of Technology)
Isaev, Alexander (Australian National University)
Kim, Kang-Tae (Pohang Institute of Science and Technology, Korea)
Kim, Kyounghee (Indiana University)
Kodama, Akio (Kanazawa University)
Krantz, Steven G. (Washington University in St. Louis)
Lanzani, Loredana (University of Arkansas)
Larusson, Finnur (University of Western Ontario)
Lee, Lina (Washington University, St. Louis)
Levenberg, Norm (Indiana University)
Lu, Stephen (Universite de Quebec a Montreal)
Mohammed, Alip (York University)
Nemirovski, Stefan (Steklov Institute)
Noguchi, Junjiro (University of Tokyo)
Schmalz, Gerd (University of New England)
Shafikov, Rasul (University of Western Ontario)
Stensones, Berit (University of Michigan)
Verma, Kaushal (Indian Insitute of Science)

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Chapter 8

Forests, Fires and Stochastic Modeling (06w5062)

May 6 – May 11, 2006

Organizer(s): John Braun (University of Western Ontario), Charmaine Dean (Simon Fraser University), Fangliang He (University of Alberta), David Martell (University of Toronto), Haiganoush Preisler (USDA Forest Service)

Statisticians have an important role to play in the study of various aspects of forestry. The objective of the BIRS workshop was to facilitate interaction between statisticians and researchers that study forest fires and forest ecology.

One theme that emerged in several of the keynote talks as well as in the roundtable discussions was the importance of melding science with statistics. In some of the talks, physically reasonable differential equation models as well as other types of deterministic models were augmented to incorporate the natural variability inherent in some of the systems observed (e.g. animal trajectories, weather, fire behaviour). It is likely that advances in forestry science and statistics will be made rapidly, if these types of approaches are emulated in other situations.

As this report will indicate, the outcome of this meeting was enhanced collaboration among these groups of researchers and an increase in energy and enthusiasm to solve open forestry-related statistical problems.

This report begins with a brief overview of the field (forest fire and forest ecology research). The next section gives summaries of the main presentations. The subsequent section highlights some of the progress made during the workshop; summaries of four roundtable discussions are contained there. This report concludes with an outline of the collaborations that are emerging as a result of this highly inter-disciplinary workshop. A brief bibliography is given at the end; a more extensive bibliography can be obtained from `http://www.stat.sfu.ca/~dean/forestry/BIRSBibliography.pdf`.

Overview of the Field

Forest fires are a natural component of many of Canada's forested ecosystems but they also pose threats to public safety, property and forest resources. Every year, forest fires cause millions of dollars worth of damage and force the evacuation of some communities. Such problems will be exacerbated as people establish more homes and cottages in and near forested areas and climate change alters forest vegetation and weather.

The forest fire research community has made steady progress over the past four decades, in increasing our understanding of the nature of forest fires. Mathematical models for predicting fire occurrence have been developed. Deterministic fire spread models are being implemented for planning purposes and for use in computer simulations to aid in prediction of the future behavior of existing and potential fires. Such models are used in conjunction with queueing models in the strategic management of fire-fighting resources such as aircraft and fire fighters. Although much has been learned about the interactions of weather and fuel-types and their effects on fire spread and intensity, a large number of questions remain. For example, how can jump-fires ignited by burning bark and other firebrands carried by the wind, in advance of a spreading fire, be modelled as a stochastic process? How can the fire hazard in particular areas be assessed reliably? What are the potential impacts of climate change on fire regimes and fire management systems? What are the effects of multiple interacting threats (e.g., insects attacks or pollutions levels) on fire risk?

The development of methods for mapping species abundance and diversity is important for forest management and conservation. It is important to understand how the conversion of old-growth forests to managed forests affects the structure, function, and species diversity of ecosystems and whether converted forests will eventually recover to a state that mimics many of the traits of old-growth stands. Methods for the development and dynamics of the ecosystems from a newly disturbed to an old-growth stand and the development of tools for predicting vegetation succession are of importance. In the short term, reliably projecting some specific important stand features is of importance. However, evolution to a substantial scale which will investigate stand dynamics for exploring environmental changes and for forest management is of essence.

Presentation Highlights

To ensure that there was sufficient time for formal and informal discussions of emerging and ongoing research collaborations, the number of long formal presentations was kept to a minimum. These talks are summarized in this section. It should be noted that there were also some shorter talks which were a part of the roundtable discussion. Some of the material presented in those talks is discussed in that section.

A stochastic space-time model for annual precipitation extremes – Jim Zidek

This talk began with a discussion of what is extreme. The answer depends on the context: engineering of bridges and dams might have a different notion than the EPA which monitors particulate concentrations. In all contexts, the notion of 'return value' is important. Return values are effectively extreme percentiles. For example, a 100-year return value corresponds to the 99th percentile.

The talk then centered on a Coupled Global Climate Model (CGCM) which combines ocean and atmosphere models, taking in various greenhouse gas scenarios as input. The response of interest is annual maximum precipitation levels in Canada (gridded at a resolution of 312 cells) based on data simulated from the CGCM. A spatially coherent distribution over the grid is required.

Data consisted of three independent simulation runs of hourly precipitation (mm/day) in 21-year windows (to look for trends):1975-1995, 2040-2060, and 2080-2100. This gives $21 \times 3 = 63$ annual precipitation maxima per grid cell.

Two approaches were considered to analyze these data: using multivariate extreme value theory; and using hierarchical Bayes.

For a single grid cell, the Fisher-Tippett generalized extreme-value distribution could be used to model precipitation extremes. The generalized Pareto was also suggested as a possible model. Other alternatives were also considered, but some were dismissed as unduly complex. All of these extreme value models possess shortcomings for this application. In particular, extending Fisher-Tippett to the multiple cell case leads to a large class of possible limit distributions, and extremes must be asymptotically dependent for large return periods. A point process approach was also considered but rejected because of difficulties in extending to the multiple cell case.

The hierarchical Bayes approach showed more promise, though it would not be effective for very extreme data. The idea was to approximate the joint distribution of (transformed) cell maximum precipitation data by a multivariate t-distribution. Specifically, the log-transformed data were assumed to have a multivariate normal distribution, conditional on their mean vector and variance-covariance matrix. Prior distributions were placed on these parameters: a multivariate normal for the mean vector and an inverted Wishart for the variance-covariance matrix, giving rise to a t-distributed posterior.

A cross-validation technique was used to assess the appropriateness of the model, in particular, the normality assumption. Using the hierarchical Bayes technique, contour plots of 10-year return values could be obtained, for example.

On Using Expert Opinion in Ecological Analysis: A Frequentist Approach – Subhash Lele

Many ecological studies are characterized by a paucity of hard data. Statistical analysis in such situations leads to flat likelihood functions and wide confidence intervals. Expert knowledge about the phenomenon under study is often available. Such expert opinion may be used to supplement the data in these situations.

One approach to incorporate expert opinion in statistical studies is via the Bayesian framework. This approach, aside from subjectivity in choosing prior distributions, faces operational problems. For example, it is difficult to formulate a precise quantitative definition of what characterizes an expert.

A frequentist approach to incorporating subjective expert opinion in statistical analyses can be taken. It may be easier to elicit data than to elicit a prior. Such elicited data can then be used to supplement the hard, observed data to possibly improve precision of statistical analyses. The approach suggested here also leads to a natural definition of what constitutes a useful expert. A useful expert is one whose opinion adds information over and above what is provided by the observed data. This can be quantified in terms of the change in the Fisher information before and after using the expert opinion. One can, thus, avoid the real possibility of using an expert opinion that adds noise, instead of information, to the hard data.

This approach was illustrated using an ecological problem of modeling and predicting occurrence of species. An interesting outcome of this analysis is that statistical thinking helped discriminate between a useful expert and a not so useful expert; expertness need not be decided purely on the basis of experience, fame or such qualitative characteristics.

A process approach to predicting tree mortality in surface fires – Sean Michaletz and Ed Johnson

This talk was concerned with predicting tree mortality using a heat transfer model of crown scorch. The model commonly used is due to Van Wagner (1973) which is an empirically-based model. It predicts that necrosis (death) height should be related to the $2/3$ power of the fireline intensity.

By considering physical characteristics of plume buoyancy and employing a lumped capacitance heat transfer model, a new heat transfer model can be derived which scales with fireline intensity in the same way as Van Wagner's model, but with no reliance on empirical data for its derivation.

Validation of components of this new model was demonstrated using empirical data from several experiments including wind tunnel measurements.

The proposed model is an example of how process models can be used to improve upon logistic models, which are empirically-based.

Synthetic Plots – David Brillinger

This talk began with the assertion that science needs appraisal methods. The idea of the synthetic plot goes back to Neyman who used it to assess the appropriateness of various models for the distribution of galaxies.

The synthetic plot is based on simulated data. Such a plot is set beside a plot of the observed data. Differences highlight problems with the model.

Sometimes a visual comparison is not adequate. It is preferable to base the comparison on some relevant statistic or set of statistics.

A first example involved Saugeen River (Ontario) monthly flow data to which a first order periodic autoregressive model was fit. The synthetic plot indicated some differences, but a spectral ratio revealed the difference more concisely.

Monthly numbers of fires in subregions of Oregon were next studied using a spatial logit model. Synthetic plots were then visually inspected. Nearest neighbour distance distributions for both the simulated and observed data were then compared to see any discrepancies.

Final examples concerned trajectory tracking for seals and elk. In these examples, Newtonian differential equations were used:

$$dr(t) = v(t)dt$$

where r denotes location and v denotes velocity.

$$dv(t) = -\hat{a}v(t)dt - \hat{a}\nabla H(r,t)dt.$$

$H(r,t)$ denotes a potential function and $\hat{a}$ is a coefficient of friction.

$$dr = -\nabla H(r,t)dt = i(r,t)dt$$

if $\hat{a} >> 0$.

Different choices of H can be used to model attraction or repulsion.

By appending a diffusion term, these equations can be made stochastic. Using an Euler scheme, they can be solved, and an approximate likelihood can be set up. Again, synthetic plots could be used to compare the observed data with simulated data from these models. The bagplot of Rousseuw et al (1999) was used to assist in the comparison.

The Use of Deterministic and Stochastic Fire Growth Models in Alberta – Cordy Tymstra

This talk was an introduction to the Prometheus project which is centered on a deterministic fire spread model which has been programmed and used to make predictions for fire growth in the Canadian Boreal forest as well as in other parts of the world.

The model is based on a set of differential spread equations which are derived from Huygens' principal of wavefront propagation. The equations are solved using a discrete Euler approximation where discretization points define certain ellipses; the envelope containing these ellipses defines the advancing fire front.

Mathematical difficulties involving choice of time step and vertex crossing were demonstrated. These difficulties were the basis for several discussions held after the talk (see the roundtable discussion section below, as well).

Empirical Modeling of Insect Wildfire Interactions in the Forests of the Pacific Northwest, USA – Haiganoush Preisler

In 2005 over 3 million hectares of U.S. federal lands were lost to wildfires, and there was a large amount of insect damage as well.

The combination of drought and bark beetle infestations in the southwest may make the 2006 fire season particularly devastating.

The hypothesis that insect-caused mortality increases fire risk has been around for almost a century (Hopkins 1909). It is not without controversy. Fleming (2002) suggested that more large fires occur within 3 to 9 years after a Spruce budworm outbreak. Bebi (2003) showed that areas affected by 1940 Spruce beetle showed no higher susceptibility to subsequent fires. Lynch (2004) showed that areas affected by Western Spruce budworm showed significant decrease in risk of forest fires.

This talk explored the relationship between insect infestation and fire size and frequency in the U.S. Northwest. A spatial multinomial distribution was used to model fire size class. Multinomial cell probabilities (at each spatial location) were estimated using semiparametric likelihood methods where covariates such as temperature, Palmer drought severity index and spruce budworm and/or bark beetle infestation levels. Maps of cell probability estimates were displayed for each of three size classes.

A multinomial model with ordinal categories was also considered, where the latent variable can be thought as a critical level of flammability. This technique seems to indicate that the probability of fires in the two larger size classes is related to area defoliated by insects; recent bark beetle infestations seem to increase the probability, while budworm infestations in 3-9 years in the past seem to have the opposite effect.

Modeling wildfire probability and impacts in British Columbia – Steve Taylor

This talk began with a review of the use of logistic regression in studying fire occurrence and its relation to several covariates including fuels, topography, weather and ignition sources.

Fires at the wildland urban interface were also discussed, using a modified Buffon's needle problem as a simplified model for interface fire risk.

This talk highlighted several issues and opportunities for statisticians to engage with the forest fire research community.

Introduction to Point Processes – Bruce Smith

Point processes are recognized as an important vehicle for beginning to understand and predict lightning and fire ignitions.

Within the group at BIRS were experts in the area of point processes. These processes can be used as models for fire events in time and space as well as for covariates such as lightning strikes. The key quantity that needs to be physically modelled is the point process intensity which is a generalization of the hazard function. Once this is specified, it is possible to use likelihood methods to fit point process models to data. Using a simple transformation involving the integrated intensity (compensator) leads to a useful diagnostic tool for assessing model adequacy.

There is interest in modelling compensators with covariates. Applying point process techniques to fire data is a goal of several of the participants.

This talk introduced the basics of point processes. It was pitched to the graduate students in the audience as well as the forestry researchers who might benefit from knowledge of some of the key concepts. The focus was on homogeneous and inhomogeneous Poisson processes and the Meyer-Papangelou transformation. The use of this transformation in assessing goodness of fit was demonstrated.

Fire History Evidence from Natural Recorders – Rick Routledge

Recent uncontrollable fires in such places as the Okanagan Valley in British Columbia and the mountains above San Diego, California have cast doubt on the wisdom of universal suppression of forest fires. By attempting to eliminate fire from the landscape, we may simply be allowing the fuel supply to build to the point where a major conflagration becomes virtually inevitable. It is widely believed that minor fires were once common in dry forests, and that aboriginal populations may have used deliberate broadcast burning to manage the landscape. Yet quantifiable evidence on fire history is elusive.

Natural recorders, specifically fire-scarred trees and lake sediments, in generate evidence on the fire history of the dry ponderosa pine forests of western North America. Statistical challenges in interpreting this evidence were highlighted.

Statistical concepts and methods in forest fire history studies – Bill Reed

This talk began with a review of some of the fire history concepts and definitions, noting ambiguities and contradictions in some cases and revising certain definitions. The use of scar data (from surface fires) and time-since-fire-map data in estimating historical fire interval was discussed. Change point identification and testing was also considered.

Concepts considered were hazard of burning which corresponds to the usual hazard rate at a given location; the fire interval which is the expected time between fires at a point (reciprocal of the hazard if constant); annual percent burn; and fire cycle – the time to burn an area equal to that of a study area.

The notions of fire cycle and fire interval are frequently interchanged in the literature, but this is due to deterministic thinking. The two concepts are really quite different. The annual percent burn is a random variable; thus, the time to burn an area equal to that of the study area, is also a random variable. Hence, fire cycle not well defined.

An attempt to rescue the fire cycle definition by means of its expected value was considered. Other revised definitions were an area-wide hazard of burning

$$\Lambda = \lim_{\Delta\to 0} \frac{1}{\Delta} P(\text{fire in the study area in } (t, t+\Delta))$$

A temporal homogeneity is required for this definition to make sense, i.e. to be independent of t. A local hazard of burning is defined as

$$\lambda(x) = \lim_{\Delta\to 0} \frac{1}{\Delta} P(\text{fire at location } x \text{ in } (t, t+\Delta))$$

Assuming spatial homogeneity, one can write

$$\text{FI} = \frac{1}{\lambda}$$

for the definition of the fire interval (FI).

Assuming a Poisson process $N(t)$ for the times of fires in the study area, the time to burn an area equal to that of the study area A is

$$\min(t : S_t \geq A)$$

where $S_t = X_1 + \cdots + X_{N(t)}$. The X's are assumed to independent and identically distributed fire areas. Applying Wald's theorem, it can then be shown that

$$E[T] \geq FI.$$

Note that $E[T]$ is the revised definition of the expected fire cycle (EFC).

For exponential fire sizes, it can be shown that

$$EFC = FI1/\Lambda$$

which means that if fires are fairly infrequent (small Λ) the expected fire cycle (time to burn an area equal to study area) could be considerably larger than the expected fire interval (time between fires at any location). Thus, fire cycle and fire interval do not coincide except for constant sized fires.

The upshot of this discussion is that the notion of fire cycle is not well-defined and should be abandoned. Notions of local hazard of burning and its reciprocal, the fire interval, are preferable measures of fire frequency.

The second part of the talk demonstrated quasilikelihood estimation of a fire interval using scar data, assuming a constant area-wide hazard of burning. Using an overdispersed binomial model for the number of scarred trees (stratified by time epoch), it is possible to set up an EM algorithm to estimate the fire interval. The method was demonstrated on data from Mexico.

The third part of the talk was devoted to considering what happens when the local hazard of burning is clearly not temporally homogeneous. In that case, piecewise constant local hazards were considered within a penalized likelihood framework, in which the Bayesian Information Criterion was used. The method was applied to several data sets from Alberta and Montana.

Scientific Progress Made

A unique feature of this workshop was the extensive use of the roundtable format. There were roundtable discussions on four themes: fire ignition and spread, forest management, forest inventory data and ecology.

Fire Ignition and Spread

Specific Projects

1. Work has begun on a 2-D renewal process which was proposed by Ivanoff and Merzbach (2005). A statistics graduate student has begun to simulate the process for simple cases using the algorithm described in a recent paper of Ivanoff (2006). Her next goal will be to fit this model to various data examples using a likelihood approach. Ultimately, a flexible model may be fit using local likelihood methods.

2. Some statisticians are considering a simple Poisson cluster model for lightning. It involves a nonlinear curve which specifies a lightning track and point events which are located according to a bivariate Gaussian distribution centered at Poisson locations on the track. The track can be estimated using smoothing methods, and the bivariate distribution can be estimated from the residuals. The model has been applied to a small amount of data from the Province of Ontario with mediocre results (using the synthetic plot idea of Neyman, and updated by Brillinger). More meteorological information is required to improve upon this model. Forestry researchers are starting to collaborate on this project and will provide additional lightning data as well as covariate data (e.g. 500 mb measurements).

Fire Perimeter Modelling

Work is proceeding on fluid queue approximations to fire perimeters. These approximations are convenient and are potentially operationally useful. In particular, relations between perimeter predictions and resource allocation will be studied in future.

Fire Spread – Prometheus

Cordy Tymstra is the project leader of the Prometheus fire growth model. This is a deterministic model based on differential equations derived from Huygens' principle of wave propagation. Points of an ellipse are propagated forward according to covariate information including wind direction and speed.

There are a number of issues connected with this model which need to be addressed:

1. **incorporating stochasticity.**

   There are a number of proposals here to be followed up on.

   (a) Conversion of DEs to SDEs.

   (b) Using computer modelling experiments to calibrate the model and estimate standard errors (or their analogue in this setting).

2. **generating spotfires.**

3. **an algorithmic geometry problem.**

   The movement of vertices on the ellipses can sometimes cause awkward cross-overs which can lead to unrealistic patterns. Handling these anomalies is related to a classic problem in geometry related to winding numbers and a point in polygon problem. Cordy Tymstra is giving Tanya Garcia instruction on the background of this problem. This problem will be presented at the PIMS Industrial Problem-Solving Workshop in Vancouver at the end of June. A graudate student in statistics will pursue this problem as part of her MSc project this summer.

Fire Spread – Markovian Lattice Model

A group of statisticians are working on a stochastic spread model based on a continuous time Markov chain. Neighbours 'infect' each other with fire. The principal advantage to this approach is the more natural way of allowing for fire spotting.

Model validation is required. One way that this will be done is through comparisons with simulations coming from Prometheus.

The spotting mechanism will be refined upon using data from Cordy Tymstra and his group.

Incorporating some of the ideas from the lattice model to Prometheus to achieve stochasticity is an emerging goal from this workshop.

Fire Spread – Percolation Model

There was a suggestion that a subcritical percolation model for fire spread may be useful for studying certain ecological questions. These models are time-independent which means that they cannot be used for real-time spread prediction. Their value would be in providing a 'mean-type' measure of fire activity for a given fire.

Opposition to this idea stemmed from the idea that such a model would not be of management interest. Process-driven models may be ultimately of more use in scientific work, but the percolation model may be easier to fit to existing data.

[Note: There was considerable discussion regarding the potential uses of such models. There was also warning that these models are not necessarily accurate across varying time-scales.]

Prescribed Burns

Causal modelling of prescribed burns is a difficult problem. An ignorability assumption is required in order for the methodology to work; this assumption is likely not reasonable. There are likely unmeasured confounders, but what are these?

Propensity score analysis provides a possible way around these problems.

Forest Management

The focus of this discussion was on problems in estimating the economic value of fire suppression, though other aspects of management were addressed as well. There were three short presentations followed by a general discussion.

The first presentation was concerned with spatial planning incorporating uncertainty. Spatial harvest scheduling presents a tough combinatorial problem. The solution presented was to use a stochastic programming model to generate scenarios. Even a deterministic approach is difficult.

The second presentation was concerned with resource-sharing among government provincial, federal and territorial fire fighting agencies. Resources include equipment and personnel. Risk sharing models were discussed; a balance is sought between sharing and keeping fire fighting resources at the home location. Balancing mitigation and suppression is also an objective.

The third presentation focussed on fire suppression. An important question to study: for the fires that we know happened, what would have happened if fire suppression had not been practiced? Observations made during the presentation:

1. Considering arrivals (fires detected and reported) vs year (about 1968-1998), there is an increasing trend, but with extremes and high annual variability.

2. Fire suppression effectiveness in Alberta: The relation between escape probability versus year appears to have a discontinuity at 1983 due to a change in strategy involving a specialized technique for allocation of resources (crews located in areas of high risk). Generally, there is a decrease over time due to increased effort.

3. Estimating the area preserved: there appears to be a strong positive linear relationship between log(area burned) and log(# escaped).

Discussion centered around questions of whether the record was long enough to take climate change into account and whether covariates would help in answering the guiding question. In particular, the surface water temperatures in the North Pacific would help to account for the Pacific decadal oscillation (dragged out El-Nino effect). More analyses could be done.

In evaluating the effect of fire suppression, the role of the type of area where suppression is practiced and the role of other covariates were further considered. For example, different strategies of fire suppression are practiced in Northern and Southern Saskatchewan. Studies in a homogeneous region of Ontario which straddles the intensive/extensive fire suppression boundary will help to clarify the question about the role of suppression since confounding variables do not come into play. A reminder was given of the importance of thinking about how fires get started, how they go out, the use of extinction models, and taking into account covariates.

Other questions to consider

If we were to look afresh at the specific topic for the roundtable we might ask questions such as: What are the indicators of economic value? How should the high and low priority areas that are an integral part of suppression strategies be taken into account? If a higher number of fires is part of the scenario of climatic change, does this change the definition of priority area?

The role of statistics education

Impediments to progress for each of the two groups (broadly classified) who are participating in the workshop were stated to be:

- An impediment for statisticians working on these problems is data related. There is a need for a reasonably good set of data to use in developing models. This could be seen as providing the opportunity to develop a suite of techniques that could be broadly applied.
- An impediment in forestry is the question of how to link analyses provided by the models to the real questions of interest. The answer to this would seem to be networking. Partnerships are important. Statisticians working alone cannot ask the right questions for formulating the models. The discussion took several directions: Developing the collaborative ethic.
- One model for getting the interaction needed are the workshops run at NCAR which brings young investigators together, since early in the career the scientist is provided with a collaborative experience.
- Encourage Statistics PhDs to do Post Docs in other areas, when appropriate, jointly supervised by a specialist in the chosen area and a statistician.
- The idea of a new training environment, bringing together epidemiology and statistics, is being put forward in health area and has been received positively by CIHR and NSERC. Comments on the issues related to teaching statistics:
- Sometimes it is the way the course is taught. We would like to have scientists excited about statistics and statisticians excited about science.
- There has been a move towards statistical science, a broader definition of the discipline.
- A proper mathematical statistics background is still important. The message should be that you cannot understand statistics without understanding the math and you can understand the math.
- The sciences, including applied science, are having difficulty keeping quantitative methods in the curriculum.
- Math is the problem. Without it students cannot go beyond the first introductory course.

Two big gaps in representation of the disciplines in forest fire research were identified: statisticians and social scientists.

Use of Forest Inventory Data for Assessing Historical Fire Regimes

Most of the discussion here centered on the objective of assessing historic fire regimes so as to inform management decisions regarding attempts to generate a ‘natural’ mosaic of stand ages in the forest landscape.

Forest inventory data can provide relevant information, but there are major concerns associated with inaccuracies in estimates of stand age from aerial photo interpretation. Not only are the ages of stands often substantially in error, but the stands themselves can be incorrectly delineated. These difficulties could cause major errors in estimates of historical fire regimes.

Ideas for developing solutions:

- Supplement the inventory data with spot checks for accuracy.
- Rely more heavily, possibly exclusively, on random sampling for spot data.
- Adapt methodology for misclassification errors developed elsewhere, e.g. in epidemiology.
- Make particular use of more reliable data on locations of more recent fires.

Several participants questioned the appropriateness of the goal. The ‘natural’ landscape has evolved over time, and a failure to recognize this has led to confusion. In addition, this misconception could well lead to inappropriate targets since changes to uncontrollable factors such as the climate may render the problem of attaining such a goal ill-posed.

Further discussion centered on the value and achievability of the alternative goal of managed forests with a uniform distribution of stand ages.

Ecology

Ecology problems under investigation include modelling of forest succession, species occurrence including interacting disturbances: fire, mountain pine beetle, timber harvest and climate change.

A number of researchers are investigating the problem of Mountain Pine Beetle infestation in BC and Alberta. A number of questions are under study:

- How can future infections be mitigated?
- How have the structure and value of wood changed after infestation?
- How does the fire hazard change over time?
- How does the beetle spread?
- What are the important factors determining spread?
- What underlying mechanisms determine spread?

It was noted that planting of trees outside their natural distribution boundaries is being considered carefully in Germany with regard such infections; it was also noted that there is some evidence with regard to long-distance dispersal by wind, and fair evidence that climate is a relevant factor; small-particle dispersal models may be helpful.

Another important problem under investigation is that of modelling beetle/fire interactions.

A major challenge in the statistical modelling of forestry ecology data is the use of different spatial resolutions when data have been collected. Work is proceeding on the spatio-temporal modeling when response and covariate data are at different spatial scales.

Outcome of the Meeting

The goal of the meeting was to foster collaborations among forestry researchers and statisticians. The following is a partial list of emerging collaborative initiatives:

- Lightning tracking. Potential collaborators: Cordy Tymstra, Mike Wotton, Rolf Turner, Doug Woolford, and John Braun. A follow-up meeting was held in Toronto in July, 2006.
- Fire spread. Potential collaborators: Cordy Tymstra, Mike Wotton, Reg Kulperger, Dave Stanford and John Braun are studying possible linkages between the Prometheus fire growth model and the stochastic growth model. A follow-up meeting has been scheduled for Toronto in June, 2007.
- Related to the project above is a mathematical problem which has been submitted to the PIMS Industrial Problem Solving Workshop to be held in June, 2007 at Simon Fraser University.
- Wildfire Analysis: Dave Martell and Haiganoush Preisler.
- Modeling of fire and insect risks and modeling of dual risks of fire following insect outbreaks. Potential collaborators Steve Taylor, Subhah Lele, Charmaine Dean, Haiganoush Preisler and Farouk Nathoo.
- Spatial covariance analysis of historic fire weather, Potential collaborators Steve Taylor, Sylvia Esterby.
- Morphometric modeling for forest fire spread. Potential collaborators: Subhash Lele, John Braun.
- Boreal Bird-Habitat Study. Potential collaborators: Steve Cumming, Charmaine Dean, Subhash Lele, Laurie Ainsworth, Jason Nielsen, Farouk Nathoo. A followup meeting will be held in the Fall, 2006.

List of Participants

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Chapter 9

Analytic Methods for Diophantine Equations (06w5101)

May 13 – May 18, 2006

Organizer(s): Michael Bennett (University of British Columbia), Chantal David (Centre de recherches mathématiques (CRM)), William Duke (University of California, Los Angeles), Andrew Granville (University de Montreal), Yuri Tschinkel (Courant Institute NYU and University of Goettingen)

Scientific Results

Some of the oldest questions in mathematics stem from the desire to find integer solutions to equations. From the equation in Pythagoras' theorem, toFermat's last theorem, Waring's problem, the abc-conjecture and Manin's conjecture, professional and amateur mathematicians alikeare thrilled in trying to prove that there are no solutions, or to determine solutions, or to count solutions. With such a venerable topic it is not surprising that there are many competing approaches to such questions, somewhose time has already come, some that are very hot methods right now, and some whose time is yet to come. At this meeting at BIRS therewere participants from many of the different schools of thought in this fascinatingsubject; it was an interesting opportunity for them to come togetherand find common ground.

During the last academic year two of the world's major research institutes, the Centre de Recherche en Mathematiques in Montreal, and the Mathematical Sciences Research Institute in Berkeley, have hosted semester long programs on different aspects on these questions. It was decidedto get together at the end of the academic year for a joint meeting to discuss issues that arise at the thematic programs at each institute. Thus the participants were primarily people who had been at one special year or the other, though perhaps a third were other researchers who are expert in Diophantine equations.

Perhaps the most consistent theme of this meeting was the topic of counting points on higher dimensional varieties, particularly Manin's conjecture. We heard a highly motivating survey by Yuri Tschinkel (Gauss chair at Goettingen), exciting new research from a geometric perspective by Par Salberger (Chalmers, Sweden), from a perspective of automorphic forms by Ramin Takloo-Bighash (Princeton) and from a perspective closer to Diophantine approximations by Jeff Thunder (Northern Illinois U).

There were exciting and controversial new perspectives on Manin's conjecture on K3 surfaces from Arthur Baragar (Nevada) and Ronald von Luijk (who was a CRM and MSRI postdoc this year, and will be a PIMS postdoc next year).

To understand Manin's conjecture on del Pezzo surfaces we heard an explanation of a basic example by Michael Joyce (Tulane) and saw a representation theoretic approach to universal torsors by Alexei Skorobogatov (Imperial College), and a direct approach to these torsors by Ulrich Derenthal (Gottingen).

Among new results was one announced by de la Breteche (Orsay) who showed that a specific height zeta functions (for a toric cubic surface) cannot be analytically continued to the whole complex plane (it has a natural boundary), so that the ”Riemann Hypothesis” is not, in general, even a sensible question.

To count points on higher dimensional varieties one can also proceed by the classical circle method. Roger Heath-Brown (Oxford) told us about his recent major breakthrough on counting points on cubic hypersurfaces (reducing the number of variables in Davenport’s famous result), the extension to quartic varieties was discussed by Tim Browning (Bristol). Trevor Wooley (Michigan) explained his idea to prove that the local-global principle works almost always and discussed what he has shown to date.

Noam Elkies (Harvard) showed how root numbers in families of elliptic curves, in combination with heuristics, could be used to predict surprising behavior regarding uniform boundedness of ranks of elliptic curves over number fields, and to contradict a well-known conjecture on the topology of rational points. Andrew Granville (Montreal) explained his new conjectures on the distribution of rational and integral points on curves and specifically how they impact in a provocative way on the question of ranks of elliptic curves. Aaron Levin (MSRI/Brown) developed techniques of Vojta to bound the number of rational points on curves of genus 1 over fields of bounded degree; and Jordan Ellenberg (Wisconsin) gave impressive new upper bounds, from his work with Akshay Venkatesh, on the heights of points of curves of genus 1, breaking through what had seemed to be a difficult barrier from the work of Heath-Brown.

There were also several talks on related questions: Noriko Hirata-Kohno (Nihon) improved Evertse’s theorem giving good bounds on the total number of solutions to certain Fermat-type Diophantine equations. Preda Mihailescu (Gottingen) showed that techniques in the theory of cyclotomic fields could be used to bound solutions to certain Ljunggren-Nagell type equations. Valentin Blomer (Toronto) improved the error term in the known approximations for representations by ternary quadratic forms using his recent work on convexity-breaking. Pietro Corvaja (Udine) explained how to show that there are large prime factors of any Markov pair, Patrick Ingham (UBC) showed that multiples of integral points on elliptic curves cannot themselves be integral, except in certain obvious cases. Jean-Louis Colliott-Thelene (Paris Sud) presented an extension of the Brauer-Manin obstruction to integral points (instead of rational points), and showed how it explained recent results on integral quadratic forms. Hershy Kisilevsky (Concordia) showed how points on cubic twists give rise to points on certain K3 surfaces; combining this with work of the Dokshitzers one discovers surprising families of surfaces which must contain rational points. Finally Harald Helfgott (Montreal) conjectured that the only extreme examples in the large sieve are the images of points from a finite set of curves, and indicated how he proved this, with Akshay Venkatesh, in two dimensions.

All participants seemed to have greatly enjoyed the meeting. It was an interesting ”coming together” of different approaches to important questions, and most speakers tried to be accessible, so a lot was learned. There were several new collaborations formed during the meeting, and even some results proved, while in Banff.

The meeting was well situated. The lecture hall and the rooms were appropriate, the local BIRS staff was excellent (particularly Brenda Shakotko), as well as of the Banff center. The weather could not have been better and everyone went home having enjoyed the mathematics and re-invigorated by the mountain surroundings.

List of Participants

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Chapter 10

Interfacial Dynamics in Complex Fluids (06w5047)

May 27 – June 1, 2006

Organizer(s): James J. Feng (University of British Columbia), Chun Liu (Pennsylvania State University)

Overview of the Field

Complex fluids refer to those with internal microstructures whose evolution affects the macroscopic dynamics of the material, especially the rheology [7]. Examples include polymer solutions and melts, liquid crystals, gels and micellar solutions. Such materials often have great practical utilities since the microstructure can be manipulated via processing flow to produce outstanding mechanical, optical or thermal properties. A good example is main-chain liquid-crystalline polymers (LCPs). Their molecular backbone is rodlike, with a degree of rigidity, such that the polymer assumes an anisotropic orientational order due to spontaneous alignment of the molecules. This order, further enhanced by extensional flows, leads to exceedingly high strength and modulus in the Kevlar fiber, a commercially successful product of du Pont.

An important way of utilizing complex fluids is through composites. By blending two immiscible components together, one may derive novel or enhanced properties from the composite, and this is often a more economical route to new materials than synthesis. Moreover, the properties of composites may be tuned to suit a particular application by varying the composition, concentration and, most importantly, the interfacial morphology. Take polymer blends for example [11]. Under optimal processing conditions, the dispersed phase is stretched from drops into a fibrillar morphology. Upon solidification, the long fibers act as in situ reinforcement and impart great strength to the composite. The effect is particularly strong if the fibrillar phase is liquid crystalline [2]. The dispersed phase may also be solid as in colloidal dispersions, or gas as in thermoplastic foams. From a scientific viewpoint, the essential physics in all such composites is the coupling between interfacial dynamics and complex rheology of the components.

Despite their practical importance, our current knowledge of two-phase complex fluids is very limited. The main difficulty is that these materials have a myriad of internal boundaries, which move, deform, break up and reconnect during processing. This leads to a seemingly intractable mathematical problem, and also hampers experimental observation and measurement. A secondary difficulty is that the rheology of each component alone is highly complex, with the internal microstructure coupled with the flow field. Thus, these materials feature dynamic coupling of three disparate length scales: molecular conformation inside each component, mesoscopic interfacial morphology and macroscopic hydrodynamics. An understanding of the interfacial dynamics in complex fluids is a major fundamental challenge as well as a significant practical need. The problem involves several traditional disciplines: mathematical modeling, numerical computation, soft-matter physics, fluid mechanics, material science and engineering. An objective of the workshop is to explore new research directions in the context of multi-disciplinary interactions.

Recent Developments and Open Problems

To date, mathematical modeling and numerical simulations of two-phase flows have followed two distinct approaches, each with its own advantages and limitations. In the first, and conceptually straightforward, approach, an interface is treated as a sharp boundary of zero thickness on which boundary conditions are matched to couple the two phases. The interfaces are handled by employing a moving mesh that has grid points on the interfaces, and deforms according to the flow on both sides of the boundary [4]. A practical difficulty of this approach is to reconcile the Eulerian framework naturally suitable for the bulk flow and the Lagrangian description of the moving boundaries. Therefore, keeping track of the moving mesh entails a computational overhead, and large displacement of internal domains causes mesh entanglement as happens when one drop overtakes another. Besides, topological changes during interfacial rupture and reconnection cannot be handled in a rational way.

The second approach seeks to regularize the interface so the problem can be solved in a purely Eulerian framework on fixed grids. These include the volume-of-fluid method, the front-tracking method, the level-set method and the phase-field method [6, 8, 10, P1]. Instead of formulating the flow of two domains separated by an interface, these methods represent the interfacial tension as a body-force or bulk-stress spread over a narrow region covering the interface. Then a single set of governing equations can be written over the entire domain, and solved on a fixed grid in a Eulerian framework. The computational bottleneck is usually resolution of the interfacial structure. So far, the application of these methods has been mostly limited to Newtonian fluids.

Therefore, although a general mathematical formalism for treating two-phase complex fluids is not yet available, the basic elements for constructing such a methodology have emerged in recent years. The open problems are to integrate such elements into versatile and efficient computational algorithms, and to apply these to interfacial flows of complex fluids that are of interest to physicists and engineers. The latter may include, for instance, impact of a liquid on an interface, coalescence and rupturing of interfaces, the effects of surfactants and macroscopic manifestations of mesoscopic morphology in emulsion rheology.

Highlights of the Scientific Program

The workshop attracted the mathematical authorities on each class of methods as well as leading physicists and engineers known for their expertise in complex fluids and interfacial dynamics. In the meantime, we have also included a significant number of younger researchers, with the aim of providing them with essential tools and techniques that they would not normally encounter within their own institutions and research groups. Aside from the formal presentations and discussions, attendants took advantage of the many opportunities for informal interactions, which many considered a major benefit of the workshop over the usual scientific conferences. The presentations fall into six topical areas, within each we give a brief description of one especially interesting talk, with references where possible.

Drop dynamics in complex fluids

The major numerical methods for computing drop deformation were represented at the workshop: volume-of-fluid, level-set, phase-field and the sharp-interface boundary-integral method. The presenters discussed the start-of-the-art of these methods and simulations in drop deformation, coalescence and breakup in viscoelastic, liquid crystalline and surfactant-laden fluids. In addition, experimental observations were reported on coalescence between drops and interfaces, and comparisons with numerical simulations were made for the partial coalescence cascade.

As a highlight of this group of talks, Jianjun Xu discussed his work on *A level-set method for interfacial flows with surfactant* [13]. Using a level-set representation for fluid interfaces with surfactants, the method is based on an Eulerian formulation and couples a semi-implicit discretization of the surfactant equation with the immersed interface method for the flow-solver. 2D simulations reveal the effects of surfactants on single drops, drop-drop interactions and interactions among multiple drops in Stokes flow under an applied shear.

Microfluidics in complex fluids

Microfluidic devices manipulate small amounts of liquids through micron-sized channels, and have attracted a great deal of interest for their applications in analytical chemistry and biological analysis [1]. The talks in this area examined the interfacial instabilities that cause jet breakup and formation of uniform-sized micro-droplets in various flow geometries, as well as control strategies for rheologically complex fluids. For example, Boris Stoeber described *Visco-thermal flow instabilities of thermally responsive fluids* in microfluidic channels [9]. Using thermally responsive polymers that undergo reversible gelation as a result of viscous heating, the author observed unusual flow instabilities in pressure-driven flows through a microchannel. The reversible phase change of these thermally responsive fluids and their particular rheological behavior promise novel microflow control applications including active valving and passive microflow control based on viscous heating within the flow.

Mathematics and physics of the moving contact line

The moving contact line may be one of the most challenging problems in two-phase flows. In the conventional Navier-Stokes sharp-interface formulation, the contact line constitutes a non-integral singularity. In physical reality, the flow in the vicinity depends on the interplay between microscopic molecular forces and continuum-level hydrodynamics. The contributions on this topic ranged from experimental and theoretical studies of short-range interfacial forces to large-scale numerical simulations of forced wetting and two-fluid displacement in channel flows.

David Jacqmin discussed *Phase-field calculations of wetting failure and instabilities*, which employ a phase-field model to coarse-grain the microscopic physics and integrate it into a continuum simulation [5]. The model eliminates the apparent singularity at the wetting line and also captures the gross energetics of wetting. Two- and three-dimensional phase-field-Navier-Stokes calculations of liquid-liquid systems show wetting failure through tipstreaming and splitting instabilities. It is hypothesized that tipstreaming can be understood in part as a quasi-two-dimensional phenomenon.

Structural evolution in polymers and micelles

This topic attracted some ten contributions that dealt with liquid crystalline polymers, wormlike micellar solutions, block copolymers, surfactant solutions and monolayers. The methods of investigation included constitutive modeling, numerical simulation of flow-structural coupling, and experimental measurements. A particularly interesting talk came from Pam Cook entitled *Wormlike micellar solutions: A model and its predictions* [12]. Those so-called "living polymers" continuously break and reform their structure in solutions and exhibit a characteristic breakage time in addition to the network relaxation time. As a result, their rheology features a plateau in the steady state flow curve associated with the development of spatial inhomogeneities in the flow. The talk outlined a network-based constitutive model that incorporates the breaking and reforming of the micelles. Each micellar species is modeled as a nonlinear bead-spring dumbbell. Predictions of the model in transient and steady state shear and in extension are shown to agree with experiments.

Modeling of membranes and other biological systems

Theoretical models and numerical simulations were presented on the mechanical behavior of electro-elastic and biological membranes, charge-selective vesicles and white blood cells passing through the capillary network. The phase-field model, with a properly designed energy functional, was shown to yield a promising description for elastic membranes. In his talk titled *Diffusive interface modeling and numerical simulation of lipid membranes* [3], Qiang Du presented a series of works on the phase-field modeling and simulations of vesicle membrane deformations under the elastic bending energy. These include studies of full three dimensional energy minimizing configurations, effect of spontaneous curvature, interaction with background fluid flows, and multicomponent and open membranes. In addition, he also introduced an ingenious technique for retrieving topological information within the diffusive interface framework which may have broad applications.

Advances in numerical algorithms

This series of talks covered the latest progress in numerical algorithms for computing interfacial motion as well as the bulk flow of microstructured fluids. On the former, several types of methods have been discussed above in relation to simulating drop dynamics. The use of adaptive meshing to resolve diffuse interfaces has produced impressive results. Besides these, several lecturers expounded on techniques based on a mixed Eulerian and Lagrangian formulation. For the latter, the contributions ranged from a study of the well-posedness of viscoelastic flows as related to the high-Weissenberg number problem to closure approximation for the FENE fluid.

A highlight of this group of talks is perhaps *A compromise between the Eulerian and ALE approach to free boundary problems* by Peter Minev (http://www.math.ualberta.ca/ pminev/). His work attempts to produce a scheme that comprises the advantages of the two main approaches for discretization of free boundary problems in fluid dynamics: the Eulerian and the Arbitrary Lagrangian-Eulerian (ALE) approaches. His scheme resembles an ALE scheme because at each time step the free boundary is aligned with finite element faces. On the other hand, the structure of the grid remains unchanged. The algorithm uses a fixed background grid, but at each time step it finds the closest to the free boundary grid points and projects them onto the boundary without changing the structure of the grid. The time discretization is done by means of a velocity correction scheme which utilizes non-conforming elements for the projection step. The mass conservation is implemented by means of one Lagrange multiplier per each distributed phase. Finally, the scheme is validated on several well-known free boundary flows.

Outcome of the Meeting

The problem of interfacial dynamics in complex fluids is multi-disciplinary. But the work done by mathematicians, engineers, physicists and material scientists has largely been independent of one another. Researchers across the disciplines have so far seldom had the chance to interact and learn of each other's work. The greatest achievement of this workshop may have been its facilitating the exchange of ideas among researchers across traditional disciplines. More specifically, we can list the outcomes of the meeting as follows.

(1) A survey of the state of the art of "Interfaces in Complex Fluids" that highlights it as an important emerging area in mathematics and physical science. Experimentalists have described recent observations and measurements on processes such as moving contact-lines, interfacial rheology, Marangoni flows and self-assembly on interfaces. Theoreticians and numerical analysts have summarized the predictive capabilities of their models, including fixed- and moving-grid numerical methods, incorporating continuum and molecular rheological theories.

(2) Identification of the most pressing scientific issues. Experimentalists have listed phenomena that cannot be readily rationalized or explained, including interfacial rupture, thermally and chemically modulated tip-streaming and surface-directed self-assembly. Modelers and simulators have highlighted the need for multi-scale modeling of singular events such as moving contact-lines and topological changes.

(3) Facilitating inter-disciplinary exchange and collaborations. Out of the broad and thorough discussions of interfacial dynamics in complex fluids came a more comprehensive context for the individual problems each researcher has been tackling. In particular, younger scientists have benefited from interactions with more established experts and from the opportunity to integrate or distinguish themselves in this exciting multi-disciplinary community. Collaborations will develop, we hope, between researchers with complementary skills and expertise.

List of Participants

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Chapter 11

Modular Forms and String Duality (06w5041)

Jun 03 – Jun 08, 2006

Organizer(s): Charles Doran (University of Washington), Helena Verrill (Louisiana State University), Noriko Yui (Queens University)

The workshop was a huge success. Altogether thirty-seven mathematicians and physicists converged at the BIRS for the five day workshop. There were 26 one hour talks presented. Some were introductory lectures by mathematicians designed to prepare physicists in modular forms, quasimodular forms, modularity of Galois representations, and toric geometry. Vice versa, introductory lectures by physicists were intended toward educating mathematicians about some aspects of mirror symmetry, string theory in connection with number theory. These introductory lectures were scheduled in the mornings of early days of the workshop. Research talks were scheduled in the afternoons and later days. They covered the recent advances on various aspects of modular forms, differential equations, conformal field theory, topological strings and Gromov–Witten invariants, holomorphic anomaly equations, motives, mirror symmetry, homological mirror symmetry, construction of Calabi–Yau manifolds, among others. More detailed descriptions of scientific activities will be reported on in Section 4.

Though number theorists and string theorists have been working on modular forms, quasimodular forms and more general modular forms in their respective fields, there have been very little interactions between the two sets of researchers with few exceptions. In other words, both camps have been living in parallel universe. This workshop brought together researchers in number theory, algebraic geometry, and string theory whose common interests are modular forms. We witnessed very active and intensive interactions of both camps from early mornings to late nights. We all felt that all things modular have came together at BIRS from both sides: number theory and string theory. At the end of the workshop, there was a strong urge of having this kinds of workshops more frequently. Accordingly, a follow-up of this workshop is in the planning in the year 2008 at BIRS! with the current organizers plus Sergei Gukov from String Theory and Don Zagier from Mathematics.

The Proceedings of the workshop is currently under negotiation with Cambridge University Press, most likely to be published in the London Mathematical Society Lecture Notes Series.

Contents

8. Appendix

1. Organizers:

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2. Press Release: Number Theory and String Theory at the Crossroads

Modular forms have long played a key role in the theory of numbers, including most famously the proof of Fermat's Last Theorem. Through its quest to unify the spectacularly successful theories of quantum mechanics and general relativity, string theory has long suggested deep connections between branches of mathematics such as topology, geometry, representation theory, and combinatorics. Less well-known are the emerging connections between string theory and number theory the subject of next weeks workshop 06w5041: Modular Forms and String Duality at the Banff International Research Centre, June 3 - 8, 2006. Mathematicians and physicists alike will converge on the Banff Centre for a week of both introductory lectures, designed to educate one another in relevant aspects of their subjects, and research talks at the cutting edge of this rapidly growing field. The event also coincides with the introduction of a new journal, "Communications in Number Theory and Physics" (http://www.intlpress.com/CNTP) published by International Press, which will provide a venue for dissemination of results at this crossroads well into the future. An expository proceedings for the workshop itself is under consideration for publication by Cambridge University Press.

3. Summary of scientific and other objectives

Physical duality symmetries relate special limits of the various consistent string theories (Types I, II, Heterotic string and their cousins, including F-theory) one to another. By comparing the mathematical descriptions of these theories, one reveals often quite deep and unexpected mathematical conjectures. The best known string duality to mathematicians, Type IIA/IIB duality also called **mirror symmetry**, has inspired many new developments in algebraic and arithmetic geometry, number theory, toric geometry, Riemann surface theory, and infinite dimensional Lie algebras. Other string dualities such as Heterotic/Type II duality and F-Theory/Heterotic string duality have also, more recently, led to series of mathematical conjectures, many involving elliptic curves, K3 surfaces, and modular forms. Modular forms and quasi-modular forms play a central role in mirror symmetry, in particular, as generating functions counting the number of curves on Calabi–Yau manifolds and describing Gromov–Witten invariants. This has led to a realization that time is ripe to assess the role of number theory, in particular that of modular forms, in mirror symmetry and string dualities in general.

One of the principal goals of this workshop is to look at modular and quasi-modular forms, congruence zeta-functions, Galois representations, and L-series for dual families of Calabi–Yau varieties with the aim of interpreting duality symmetries in terms of arithmetic invariants associated to the varieties in question. Over the last decades, a great deal of work has been done on these problems. In particular it appears that we need to modify the classical theories of Galois representations (in particular, the question of modularity) and modular forms, among others, for families of Calabi–Yau varieties in order to accommodate "quantum corrections".

As dictated by the research interests of the participating members, the research activities will be focused on the following themes:

(A) Arithmetic of Calabi–Yau varieties defined over number fields: Arithmetic of elliptic curves, K3 surfaces, Calabi–Yau threefolds, and higher dimensional Calabi–Yau varieties defined over number fields in connection with string dualities. These will include the following topics and problems: Interpretation of string duality phenomena of Calabi–Yau varieties in terms of zeta-functions and L-series of the varieties in question, the modularity conjectures for Calabi–Yau varieties, the conjectures of Birch and Swinnerton-Dyer for elliptic curves and Abelian varietes, the conjectures of Beilinson-Bloch on special values of L-series and algebraic cycles, and intermediate Jacobians of Calabi–Yau threefolds. Calabi-Yau varieties of CM (complex multiplication) type and their possible connections to rational conformal field theories and, in the elliptic fibered case, behavior under F-Theory/Heterotic string duality.

(B) Mirror symmetry for families of Calabi–Yau varieties: Characterization of mirror maps in connection with the mirror moonshine phenomenon and, via Fourier-Laplace transform, the classification of Q-Fano threefolds. In particular, differential equations associated to modular and quasi-modular forms related to GKZ-hypergeometric systems and, more generally, to Picard-Fuchs differential systems will be investigated.

(C) Modular and quasi-modular forms in string duality: Modular forms and quasi-modular forms have appeared frequently in mirror symmetry contexts, e.g., in the generating functions counting the number of simply ramified covers of elliptic curves with marked points, in Gromov–Witten invariants, and also as mirror maps. The appearance of modular and quasi-modular forms in string dualities, e.g., in the Harvey-Moore conjectures of Heterotic-Type II duality, will be investigated. Understanding why modular and quasi-modular forms play central roles in string dualities is one of our goals.

4. Summary of scientific activities

Talks presented at the workshop may be classified into not clearly disjoint sets of the following seven subjects. They are:

(a) Modular, quasimodular, bimodular forms, and their applications
(b) Topological string theory and modular forms
(c) Modularity, and arithmetic questions
(d) Mirror symmetry: various versions
(e) Toric geometry
(f) Differential equations
(g) Miscellaneous topics

The workshop's kick-off talk was delivered by Don Zagier about modular forms and differential equations. This talk sat the tone of the entire workshop. Modular forms (of one variable) are the key players of the workshop. The standard references on modular forms are the classical four proceedings volumes (I,II,III,IV) of the Antwerp conference (LNM 320, 349, 350 and 476) [B1], plus the volumes (V, VI) (the proceedings of the Bonn conference) (LNM 601, 627) [B2], entitled "Modular Functions of One Variable". The most recent lecture notes are the proceedings of the Nordfjordeid Summer School on "Modular Forms and their Applications", where they cover modular forms of one variable, Hilbert modular forms, and Siegel modular forms. (This is yet to appear [B3].)

The most recent references on String Theory and Mirror Symmetry are the Clay Math. Monographs *Mirror Symmetry* [B4] which covers recent advances on mathematics and physics about mirror symmetry, *Mirror Symmetry V*, AMS/IP Advanced Studies in Math. [B5] (this is the proceedings of the BIRS workshop 03w5061, December 2003), and *Calabi–Yau Varieties and Mirror Symmetry*, the Fields Institute Communications [B6].

These references served as the cornerstone for the rapidly developing topics discussed at the workshop.

(a) Modular, quasimodular, bimodular forms, and their applications

Zagier's lecture started with the definition of modular forms of one variable, and then quickly formulated the "Problem" that the derivative of a modular form is NOT a modular form, and addressed the strategy for attacking the problem. Let f be a modular form of weight $k \in \mathbf{Z}$ for a subgroup $\Gamma \subset SL_2(\mathbf{R})$, and let Df denote the derivative of f, i.e., $Df := f' = \frac{1}{2\pi i}\frac{df}{dz}$ (with $z \in \mathbf{H}$ where $\mathbf{H} := \{\, z = x+iy,\ y > 0\,\}$ denotes the upper-half complex plane). Method 1: Change the definition of modular forms (which led to *quasimodular* *forms*); Method 2: Change the definition of derivative (which led to the new derivative $\partial f := Df - \frac{k}{4\pi y}f$ and *almost holomorphic modular forms*); Method 3: Eliminate the problem (which led to the first *Rankin–Cohen brackets*); Method 4: Avoid the problem (which led to *Eichler integrals, theory of periods*); Method 5: Enjoy the problem (which led to *differential equations of modular forms*). **Theorem**: *(1) A modular form f of weight k satisfies an autonomous differential equation of order 3, that is, there is a polynomial P such that $P(f, f', f^{(2)}, f^{(3)}) = 0$. (2) A modular form f of integral weight $k > 0$ satisfies a linear differential equation of order $k+1$ with respect to a meromorphic modular function t. That is, write $f = \phi(t)$ locally, then $L\phi = 0$ for some linear differential operator L of order $k+1$ with polynomial coefficients.* His second lecture discussed in detail quasimodular forms, differential equations, Rankin–Cohen brackets and related algebraic structures, as enjoyments! The structure of Rankin–Cohen algebra as a commutative associative algebra was one of the main points of discussion. Also bimodular forms are introduced at the end of his lecture. The vector space of quasimodular forms is contained in that of bimodular forms.

Kaneko's talk was a continuation of Zagier's lectures, and covered modular forms and quasimodular forms and their applications. He reported on joint works with M. Koike, and D. Zagier. Consider an order-two differential equation of the form $f^{''}(z) - \frac{k+1}{6}E_2(z)f^{'}(z) + \frac{k(k+1)}{12}E_2^{'}(z)f(z) = 0$ where $k \in \mathbf{Q}$, $z \in \mathbf{H}$ and $' = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$ with $q = e^{2\pi i z}$. Solutions are explicitly described which depend on congruence conditions of k. Some of these solutions appear also in physics literature (e.g., conformal field theory). The generating function of the (weighted) number of simply ramified covers of genus g over an elliptic curve with marked points is shown ([1]) to be expressed in terms of quasi-modular forms. This work provided a mathematical proof to Dijkgraaf's "theorem" [2] about mirror symmetry for elliptic curves.

Both Zagier and Kaneko were concerned with modular and quasimodular forms for congruence subgroups. **Ling Long**'s talk was about modular forms for *noncongruence* subgroups $\Gamma \subset SL_2(\mathbf{Z})$. The Fourier coefficients of modular forms for noncongruence subgroups have unbounded denominators. Long reported on her joint work with Atkin, W. Li and Z. Yang [3] about a refinement of Atkin and Swinnerton-Dyer congruence property satisfied by the Fourier coefficients of modular forms concentrating on some examples of noncongruence subgroups.

J. Stienstra reported on his joint work with Zagier on bimodular forms and holomorphic anomaly equation. Let $f_1, f_2, \cdots, f_n$ be quasimodular forms and ∂ be the derivation on quasimodular forms introduced by Zagier's talk. A holomorphic anomaly equation (HAE) is the differential equation $\partial f_n = -\frac{n}{2}\sum_{i=1}^{n-1} f_i f_{n-i}$. The main result is that solutions to HAE are given by bimodular forms.

(b) Topological string theory and modular forms

E. Scheidegger reported on a joint work with A. Klemm, M. Kreuzer and Riegler [4]. Let Σ_g be a Riemann surface of genus g and X be a smooth Calabi–Yau threefold. The problem addressed here is the enumerative properties of holomorphic maps from $\Sigma_g \to X$, which are concocted into the generating functions of the Gromov–Witten invariants (the genus g topological string amplitude on X). There are various approaches to the problem; this talk concentrated on the interpretation of the generating functions in terms of modular forms for Calabi–Yau threefolds X with specific $K3$-fibrations. Assuming that X are complete intersections in toric varieties with $h^{1,1} = 2$ with $K3$ fibrations and that $H_2(X, \mathbf{Z}) = i_*M \oplus [\beta]\mathbf{Z}$ where $M =< 2n >$, $n = 1, 2, 3, 4$, the authors construct 29 topologically inequivalent $K3$ fibrations, and show that the generating functions are weakly holomorphic modular forms of weight $k = -3/2$ on $\Gamma_0(8n)$.

A. Clingher discussed the correspondence between elliptically fibered $K3$ surfaces with section and elliptic curves endowed with certain flat connections. The correspondence is purported to describe the dualities in string theory, the F-theory/heterotic duality in eight dimensions. He reported on a joint work with C. Doran about geometric ways of exhibiting the above correspondence. Let $G = (E_8 \times E_8) \rtimes \mathbf{Z}_2$ be a Lie group where $\mathbf{Z}_2$ switches E_8. Elliptic curves with G-flat connections are classified. Moving to the other side, elliptically fibered $K3$ surfaces X with section are shown to correspond in one-to-one manner to H-polarizations $H \subset Pic(X)$ (H being the rank 2 hyperbolic lattice). The moduli space of elliptically fibered $K3$ surfaces with section is known by Dolgachev to be isomorphic to a type IV symmetric domain (the quotient of the period domain of X by the group Γ of isometries). The correspondence underlying the duality has to do with a partial compactification of this space, e.g., Mumford's partial compactification. The talk ended with examples of elliptically fibered $K3$ surfaces such that $Pic(X) \supset H \oplus E_8 \oplus E_8$, which correspond via Shioda–Inose structure to pairs of elliptic curves $\{(E, E')\}$.

A. Klemm discussed topological strings (TS) and modular forms. He used the holomorphic anomaly equation to solve the gravitational corrections to Seiberg–Witten theory. He constructed propagators that give recursive solution in the genus modulo a holomorphic ambiguity. The gravitational corrections can be expressed in closed form as quasimodular function on $\Gamma(2)$. (Cf. his recent preprint [5] for part of his talk.)

V. Bouchard reported on his joint work with A. Klemm and M. Aganagic in progress. The philosophy behind their work is to use symmetries to solve the topological strings. Steps involved are mirror symmetry, quantum mechanics (which led to modular forms) and modular forms (via monodromy groups). Gromov–Witten invariants arise from A-model on a Calabi–Yau threefold, which via mirror symmetry, correspond to holomorphic anomaly equations from B-model side. B-model side is easier computation wise. The inverse propagators are claimed to have modular property that they should be almost holomorphic modular forms of weight 0 on some subgroup Γ of $SL_2(\mathbf{Z})$. They show, depending on the choice of polarization, the genus g topological string amplitude is either a holomorphic quasimodular form, or an almost holomorphic modular form of weight 0 on Γ. (See [17] for details.)

(c) Modularity, and arithmetic questions

Ron Livné, in his first lecture, gave an overview on modularity of Galois representations. He defined the Galois representations associated to algebraic varieties over $\mathbf{Q}$, and those arising from modular forms. Several strategies (Faltings, Serre, Livné) to establish the modularity of a Galois representation were discussed. The modularity of 2–dimensional Galois representations are illustrated by examples, elliptic curves over $\mathbf{Q}$, singular $K3$ surfaces defined over $\mathbf{Q}$ and rigid Calabi–Yau threefolds over $\mathbf{Q}$.

In his second talk, **Livné** addressed the question of how to determine defining equations explicitly for abelian varieties of dimension > 1. This is really a challenging problem and there are no methods known to tackle this problem. He reported on a joint work with A. Besser about the explicit construction of universal families of Kummer surfaces over Shimura curves over $\mathbf{Q}$.

Sergei Gukov discussed in two lectures about interactions between number theory and physics illustrating with several examples. (In fact, his second lecture had to be scheduled in haste on strong demands by mathematicians.) The main point of his talks was that various partition functions arising in physics have modular properties. Example 1: Conformal Field Theory. Index/Elliptic genus $Z(\tau, z)$ ($\tau \in \mathbf{H}$, $z \in \mathbf{C}$ is a topological invariant of a Calabi–Yau threefold, and under modular transformation of $\Gamma_1 = SL_2(\mathbf{Z})$, it is a weak Jacobi form of weight 0 and index $k = \hat{c}/2$ where $\hat{c}$ is the central charge. Example 2: Guage Theory. Consider a smooth 4-manifold X with Chern characters $c_1 = 0,\ c_2 = k$. The electro-magnetic duality implies that the partition function Z^X_{VW} of Vafa and Witten is a modular form for an index 2 subgroup of $SL_2(\mathbf{Z})$. Example 3: Black Hole Attractors. Let X be a Calabi–Yau threerfold. Given charge $\gamma \in H^3(X, \mathbf{Z})$, look for solutions to the attractor equations. Solutions are isolated points in the moduli space. According to Greg. Moore, attractor Calabi–Yau threefolds are defined over number fields. In particular, rigid Calabi–Yau threefolds are attractive. Example 4: Rational Conformal Field Theory (which are defined as exactly solvable Conformal Field Theory). Conjecture (Friedan–Shenker): RCFTs are dense. Conjecture: RCFT if and only if Calabi–Yau manifold X has complex multiplication (CM). Complex tori and Fermat Calabi-Yau orbifolds have CM. Example 5: The Chern–Simons Theory. Let M be a 3-manifold, and A a connection on principal G-bundle. The Chern–Simons invariant is defined by $CS(A) := \frac{k}{4\pi}\int_M Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$. The goal is to compute the partition function $Z(M,G) = \int \mathcal{D}Ae^{-CS(A)} = exp(\sum_{\ell=0}^{\infty}(\frac{2\pi}{k})^{\ell-1}S_\ell) = \exp(\frac{k}{2\pi}S_0 + S_1 + \sum_{n\geq 1}(\frac{2\pi}{k})^n S_{n+1})$ where the coefficients are interesting invariants. When G is compact and $\ell > 1$, then $S_\ell \in \mathbf{Q}$. However, if $G = SL(2,\mathbf{C})$ (non-compact) and $\ell > 1$, then S_ℓ are not of finite type, in particular, not necessarily in $\mathbf{Q}$. When M is a hyperbolic 3-manifold of finite type, e.g., $M = \mathbf{H}^3/\Gamma$ (where $\Gamma \subset PSL(2,\mathbf{C})$, then S_ℓ are contained in the number field, $\mathbf{Q}(tr(\gamma))$, $\gamma \in \Gamma$. If $M = \mathbf{S}^3 \setminus K$ where K is a knot, then Z is related to a colored Jones polynomial. Example 6: (Zagier–Lawrence and Hikami and others.) Here $M = \mathbf{S}^3/\Gamma$ with $\Gamma \subset SU(2)$ (finite subgroup). Then $Z(M, SU(2))$ is expressed in terms of the Eichler integral of a modular form of weight $3/2$.

Noriko Yui introduced motives to describe topological mirror symmetry for certain classes of Calabi–Yau threefolds constructed from Fermat hypersurfaces in weighted projective 4-spaces by orbifolding. Motives which are invariant under mirror maps are determined. Also one-to-one correspondence between motives and monomial classes is established at the Fermat (the Landau–Ginzburg) point in the moduli space. These results are obtained in the paper [7]. Modularity question (of Galois representations) becomes more manageable at motivic level, and the modularity of many motives are established. There appear rank 4 motives, which are conjecturally modular in the sense that there should be Siegel modular forms of weight 3 on some congruence subgroups of $Sp(4, \mathbf{Z})$ that determine the L-series of such motives. How can one determine the modular groups in the conjecture? For this, consider the 14 Calabi–Yau threefolds whose Picard–Fuchs differential equations are of hypergeometric type. In a joint work with Yifan Yang and his student [8], it is shown that the monodromy group is contained in the congruence subgroup $\Gamma(d_1, d_2) \subset Sp(4, \mathbf{Z})$ of finite index. (However, it is still open if the monodromy group itself is of finite index in $Sp(4, \mathbf{Z})$.)

Rolf Schimmrigk addressed the modularity of three Calabi–Yau threefolds over $\mathbf{Q}$, focusing on their connection to physics of the string worldsheet. He discussed three specific examples of Calabi–Yau threefolds, which are built up from lower dimensional Calabi–Yau manifolds, e.g., elliptic curves, and $K3$ surfaces. Motives arising from his examples are already discussed in Yui's talk. Also he constructed explicitly mirror maps for certain mirror pairs of Calabi–Yau orbifolds in weighted projective 4-spaces.

(d) Mirror symmetry: various versions

Shinobu Hosono gave an introductory talk on "What is the mirror symmetry?" SYZ's version: Every Calabi–Yau manifold has a T^3-fibration, up to some singular fibers. Konstevich's version: There exists an equivalence of categories $D^b(coh(X))$ and $D^b(Fukaya(Y))$ for a mirror pair (X, Y) of Calabi–Yau manifolds. He explained the two versions of mirror symmetry focusing on the quintic Calabi–Yau threefold and its mirror partner. Picard–Fuchs differential equations for these mirror pair are explicitly determined. (Originally, **Bong H. Lian** was scheduled to give an introduction to mirror symmetry. But due to unforeseen development, he had to cancel his participation and his talk. Bong Lian sent in his lecture notes [B9] about mirror symmetry. At the last minutes, Hosono kindly agreed to give an introductory talk about mirror symmetry.)

Johannes Walcher reported on his recent work [8] on the determination of the number of holomorphic disks of degree d ending on the real Lagrangian L in the quintic Calabi–Yau threefold. For $d = 1$, 30 such disks, 1530 for $d = 3$, and 1088250 for $d = 5$ and so on. These numbers are coming from a generalized Picard–Fuchs differential equation. String theory motivates study of $2d$ field theories on worldsheets of higher genus, with boundary and possibly unoriented. Pick boundary conditions (D-branes). On A-model side, one has Lagrangian submanifold $\subset X$, and on B-model side, one gets holomorphic submanifold with holomorphic vector bundle on the mirror Y. Tension of the domain wall is the topic of his talk. Computing the open string instanton expansion (normalized appropriately), one reach at the numbers listed above. This is an open analogue of the paper [9] where the numbers of rational curves of degree d on the quintic Calabi–Yau threefold are computed using mirror symmetry.

M. Aldi reported on a joint work with E. Zaslow [10] about homological mirror symmetry. The construction of Seidel's mirror map for abelian surfaces and Kummer surfaces is discussed computing explicitly twisted homogeneous coordinate rings from the Fukaya category of symplectic mirrors. The computation of Seidel's map, however, depends on a symplectomorphism representing the large complex structure monodromy. Several examples of mirror maps are presented for abelian surfaces, and Kummer surfaces.

(e) Toric geometry

Helena Verrill gave an introductory talk on toric geometry focusing on the Batyrev–Borisov construction of toric Calabi–Yau hypersurfaces and complete intersections. For general references about toric varieties are Fulton [B7] and Oda [B8]. Batyrev [11] starts with a polytope Δ and associates a family $\mathcal{F}(\Delta)$ of hypersurfaces. A special class of polytopes that play the key role in Batyrev–Borisov mirror symmetry is the class of reflexive polytopes. A polytope Δ is *reflexive* if the origin is in Δ and Δ and its dual Δ^* are lattice polytopes. In dimension 4, there are $473, 800, 776$ reflexive polytopes. Associated to a pair of reflexive polytopes (Δ, Δ^*), there correspond families $(\mathcal{F}(\Delta), \mathcal{F}(\Delta^*))$ of Calabi–Yau hypersurfaces. Batyrev gave formulas for the Hodge numbers $h^{2,1}(\Delta)$ and $h^{1,1}(\Delta^*)$. and then showed that that $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta^*)$ are mirror pairs in the sense of topological mirror symmetry, that is, the Hodge numbers $h^{2,1}(\Delta) = h^{1,1}(\Delta^*)$ and $h^{2,1}(\Delta) = h^{1,1}(\Delta^*)$. The Batyrev–Borisov construction produces singular Calabi–Yau manifolds, and one needs to consider MPCP (maximal projective crepant partial) desingularization (i.e., triangulation of polytopes). Verrill illustrated the Batyrev–Borisov construction by a number of examples. She remarked that most Calabi–Yau families can contain rigid Calabi–Yau threefolds.

C. Doran reported on his joint work with J. Morgan [12] on the computation of integral homology, the topological K-theory, and the rational Hodge structure on cohomology of Calabi–Yau hypersurfaces and complete intersections in toric varieties. The Doran–Morgan approach to these problems is purely topological. The geometric representatives of integral cohomology classes for H^2 and H^3 are explicitly described. The rational Hodge structures of weight 3 are determined from the geometric representatives.

(f) Differential equations

G. Almkvist reported on his joint work with W. Zudilin and D. van Straten [13]. (Incidentally, both were invited to the workshop but could not take part by respective reasons). The talk was concerned with the Apéry like limits arising from recursions of Calabi–Yau differential equations of order 4. Let $\sum_{n=0}^{\infty} A(n)x^n$ be the analytic solution of a Calabi–Yau differential equation of order 4. The $A(n)$ satisfy a recursion formula with polynomial coefficients with $A(n) = 0$ for $n < 0$ and $A(0) = 1$. Suppose that $B(n)$ satisfy the same recursion but with $B(n) = 0$ for $n \leq 0$ and $B(1) = 1$. The main concern of his talk was what happens to the limit $B(n)/A(n)$ when n tends to ∞. This is a generalization of the Apéry limit for a third order differential equation (discussed in Zagier's first talk). These limits are conjectured to be expressed by special values of L-functions at $s = 2$ or $s = 3$. Several examples in support of the conjecture are presented.

Masahiko Saito reported in his recent work about Painlevé property of ordinary differential equations. In particular, he and his collaborators analyzed movable branching points and gave a necessary condition for Painlevé property to hold by means of geometry of logarithmic symplectic varieties.

Ahmed Sebber reported on differential theta relations and Galois theory for Riccati equations. He discussed several types of ordinary differential equations: Riccati equation, algebraic hypergeometric equation, Chazy equation, and non-linear differential equations satisfied by theta functions. The Riccati equation is of the form: $\frac{du}{dx} + u^2 = q(x)$. The problem addressed here was : For which potential q, the Riccati equation has algebraic solutions? That is, u satisfies an algebraic equation of degree n over some fixed (differential) field. There happened to be only four values for n, namely, $n = 3, 4, 6, 12$. Some applications to physics were mentioned at the end of his lecture.

(g) Miscellaneous topics

Nam-Hoon Lee discussed a method of constructing Calabi–Yau manifolds using the method of smoothing normal varieties developed by Kawamata and Namikawa [14]. This construction yielded 22 new examples of Calabi–Yau threefolds with Picard rank 1 [15]. Also he gave a counterexample to the conjecture of Tyurin that very Calabi–Yau manifold is constructible or is birational to a variety that is a deformation of constructible Calabi–Yau manifolds. Such a counterexample is a Calabi–Yau threefold $X(10) \subset \mathbf{P}(1,1,1,2,5)$ in the weighted projective 4-space with weight $(1,1,1,2,5)$.

J. Stienstra discussed a recent new interpretation of mirror symmetry, namely, AdS_5/CFT theory. Classically, string theory lives on 10-dimensional space (Minkowski 4-space $\times$ Calabi–Yau threefold). Recently, Anti-de-Sitter 5-space AdS_5 and Sasaki–Einstein 5 manifold came into the mirror symmetry picture. The boundary at infinity of AdS_5 is Minkowski 4-space. Sasaki–Einstein 5 manifold is a manifold Y whose metric cone $CY := \mathbf{R}_{\geq 0} \times Y$ is Kähler and Ricci-flat, i.e. CY is a singular and non-compact Calabi–Yau threefold. Then AdS_5/CFT correspondence is interpreted as the correspondence between 3-dimensional Calabi–Yau singularity and quiver gauge theory (cf. McKay correspondence). Several examples of Sasaki–Einstein 5-manifolds and their metric CY cones are presented. Dimension 3 CY singularities are constructed from toric data, e.g., multi-grid.

C. Herzog addressed the following question: What is the low energy gauge theory description of a set of D-branes at a Calabi–Yau singularity? Let X be a singular Calabi–Yau manifold. Claim: a gauge theory can be constructed from a set of objects in the derived category $D^b(X)$ of coherent sheaves, called fractal branes. An exceptional collection of objects in $D^b(X)$ is defined as an ordered set of sheaves satisfying special mapping properties, and it gives a convenient basis of D-branes. It is conjectured that a simple exceptional collection always exists for any Deligne–Mumford stack with coarse moduli space which is "mildly" singular variety with positive curvature. Some examples of such collections are constructed. This talk reported on his joint work with R. Karp ([16]).

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6. Particpants

We had in total 37 participants for the workshop, ten were either graduate students or postdoctoral fellows, and two were official observers. We had five last minutes cancellations (Terry Gannon (Alberta, Canada), Amer Iqbal (Washington), Bong H. Lian (Brandeis, USA), John McKay (Concordia, Canada), and Andrey Todorov (Santa Cruz, USA and MPIM Bonn, Germany) by various reasons.

List of Participants

Aldi, Marco (Northwestern University)
Almkvist, Gert (Lunds Universitet)
Bakhova, Maiia (Louisiana State University)
Ballard, Matthew (University of Washington)
Bouchard, Vincent (Mathematical Sciences Research Institute)
Clingher, Adrian (Stanford University)
Doran, Charles (University of Washington)
Elliott, George (University of Toronto)
Frechette, Sharon (College of the Holy Cross)
Gukov, Sergei (California Institute of Technology)
Herzog, Christopher (University of Washington)
Hosono, Shinobu (University of Tokyo)
Judes, Simon (Columbia University)
Kadir, Shabnam (University of Hannover)
Kaneko, Masanobu (Kyushu University)
Klemm, Albrecht (University of Wisconsin)
Lee, Edward (University of California, Los Angeles)
Lee, Nam-Hoon (Korea Institute for Advanced Study)
Livne, Ron (Hebrew University of Jerusalem)
Long, Ling (Iowa State University)
Lu, Stephen (Universite de Quebec a Montreal)
Ng, Richard (Iowa State University)
Papanikolas, Matthew (Texas A&M University)

Roth, Michael (Queen's University)
Saito, Masahiko (Kobe University)
Scheidegger, Emanuel (Universitaet Augsburg)
Schimmrigk, Rolf (Indiana University South Bend)
Sebbar, Abdellah (University of Ottawa)
Sebbar, Ahmed (Université Bordeaux 1)
Stienstra, Jan (Utrecht University)
Tsutsumi, Hiroyuki (Osaka University of Health & Sport Sciences)
Verrill, Helena (Louisiana State University)
Walcher, Johannes (CERN)
Whitcher, Ursula (University of Washington)
Yu, Jeng-Daw (Harvard University)
Yui, Noriko (Queens University)
Zagier, Don (Max-Planck-Institut fuer Mathematik)

7. Titles and Abstracts of Talks at the Workshop

JUNE 4, 2006

9:00am–10:00am **Don Zagier** (MPIM Bonn and College de France)

Introduction: Modular Forms and Differential Equations

10:15am–11:15am **Ron Livné** (Hebrew and IAS Princeton)

Modularity of Galois Representations: Overview

Given a two dimensional Galois representation of geometric origin, over a field, it is conjectured, and often known, that it arises from a modular form for GL(2). Even when general theorems do not exist (yet), it is sometimes possible to prove a given instance of the conjecture by making a guess and verifying it. In this survey talk we will give a miscellany of results in these directions.

11:30am–12:30pm **Helena Verrill** (Louisiana State)

An Introduction to the Batyrev–Borisov Construction of Toric Calabi–Yau Varieties

This talk will give an overview of the Batyrev-Borisov construction of toric Calabi-Yau hypersurfaces and complete intersections. It will start with a quick summary of the definition of toric varieties and reflexive polytopes, discuss properties of toric Calabi-Yaus, and Kreutze and Skarke's method of enumerating reflexive polyhedra. Some examples will be discussed, which have been used to find new modular Calabi-Yau threefolds, though the topic of modularity will be left to another speaker. This talk will not introduce new results, and is particularly aimed at graduate students.

3:00pm–4:00pm **Shinobu Hosono** (Tokyo)

Introduction to Differential Equations in Mirror Symmetry

4:15pm–5:15pm **Emanuel Scheidegger** (TU Vienna)

Topological Strings on $K3$ Fibrations and Modular Forms

We explain that motivated by heterotic-type II duality, certain Gopakumar-Vafa invariants (and hence, conjecturally, Gromow-Witten invariants) for Calabi-Yau manifolds that admit a K3 fibration can be collected in a generating function. This function is in general an automorphic form determined by the topology of the fibration. In the class of K3 fibrations in toric varieties in which the Picard lattice of the fiber has rank one, we show how this automorphic form can be determined explicitly.

5:30pm–6:30pm **Jan Stienstra** (Utrecht)

From multi-grid to multi-helix; remarkable geometries from AdS/CFT

A 1-grid with parameters a, b, c is the system of parallel equidistant lines in the plane described by the equation $ax+by+c = \mathrm{integer}$. An N-grid is a system of N 1-grids. The 'dual' of an N-grid can be presented as a tessellation of the plane by rhombi. We are interested in grids with integral (a, b)-parameters. These give rise to periodic rhombus tilings. Moreover, viewing (a, b) as a vector the tiles can be equipped with extra pictures. There are simple 'braiding rules' to transform a given rhombus tessellation into a new one. It turns out that by repeated application of the braiding rules one can reach a tiling in which the extra pictures form an interesting pattern, like the projection of a link with a spanning Seifert surface and embedded into the Seifert surface is a bi-partite graph. In the talk this will be illustrated with many pictures. We will also indicate how this relates to the AdS/CFT correspondence, Mirror Symmetry and the McKay correspondence.

JUNE 5, 2006

9:00am–10:00am **Don Zagier** (MPIM Bonn and College de France)

Quasimodular forms, Rankin–Cohen brackets and related algebraic structures

10:15am–11:15am **Masanobu Kaneko** (Kyushu)

Modular and Quasimodular Forms and their Applications

I shall review works with Don Zagier on "mirror symmetry in dimension one" and with Masao Koike on modular and quasimodular solutions of certain differential equation, with a brief mention to their possible connection to conformal field theory.

11:30am–12:30pm **Johannes Walcher** (IAS Princeton)

Opening Mirror Symmetry on the Quintic

Aided by mirror symmetry, we determine the number of holomorphic disks ending on the real Lagrangian in the quintic threefold. The tension of the domainwall between the two vacua on the brane, which is the generating function for the open Gromov–Witten invariants, satisfies a certain extension of the Picard–Fuchs equation governing periods of the mirror quintic. We verify consistency of the monodromies under analytic continuation of the superpotential over the entire moduli space. We reproduce the first few instanton numbers by a localization computation directly in the A-model, and check Ooguri–Vafa integrality. This is the first exact result on open string mirror symmetry for a compact Calabi–Yau manifold.

2:30pm–3:30pm **Gert Almkvist** (Lund)

Apéry-like limits connected with Calabi-Yau differential equations

Let $\sum_{n=0}^{\infty} A(n)x^n$ be the analytic solution of a Calabi-Yau differential equation (4-th order). Then $A(n)$ satisfies a recursion formula with polynomial coefficients with starting values $A(n) = 0$ for $n < 0$ and $A(0) = 1$. Let $B(n)$ satisfy the same recursion with $B(n) = 0$ for $n <= 0$ and $B(1) = 1$. Very often the limit of $B(n)/A(n)$ exists when $n \to \infty$. Usually the limit is a rational linear combination of values of L-functions at $s = 2$ or $s = 3$. This is a joint work with van Straten and Zudilin.

4:00pm–5:00pm **Christopher Herzog** (Washington)

How Exceptional Collections Stack Up

I would attempt to give a broad overview of my two recent papers with Robert Karp, hep-th/0507175 and hep-th/0605177. The papers advance the program of using exceptional collections of objects in the derived category of coherent sheaves to understand the low energy gauge theory description of a D-brane probing a Calabi-Yau singularity.

5:15pm–6:15pm **Nam-Hoon Lee** (KIAS)

Constructing Calabi–Yau Manifolds

A smoothing theorem for normal crossings to Calabi-Yau manifolds was proved by Y. Kawamata and

Y. Namikawa. This talk is about a study of the observation that the Picard groups and Chern classes of these Calabi-Yau manifolds are constructible from the normal crossings in such smoothings. Various applications will be discussed, including the construction of many new examples of Calabi-Yau 3-folds with Picard number one. With this construction as a starting point, I hope to convince audience that smoothing normal crossings is a promising method of constructing Calabi-Yau manifolds. This talk is based on my recent preprint (math.AG/0604596).

8:00pm–9:00pm **Ling Long** (Iowa State)

Modular Forms for Noncongruence Subgroups

Majority of finite index subgroups of the modular groups are noncongruence subgroups. In the 1970, Atkin and Swinnerton-Dyer have pioneered the investigation on modular forms for noncongruence subgroups. Some of their important observations have been verified by Scholl. Despite that, modular forms for noncongruence subgroups still remains to be very mysterious. In this talk, we will discuss some recent results on the arithmetic properties of modular forms for noncongruence groups.

JUNE 6, 2006

9:00am–10:00am **Adrian Clingher** (Stanford)

Geometry underlying the F-Theory/Heterotic String Duality in Eight Dimensions

One of the dualities in string theory, the F-theory/heterotic string duality in eight dimensions, predicts an interesting correspondence between two seemingly disparate geometrical objects. On one side of the duality there are elliptically fibered K3 surfaces with section. On the other side, one finds elliptic curves endowed with certain flat connections. I will discuss the basic Hodge theoretic framework underlying the duality as well as its consequences in algebraic geometry and number theory.

10:15am–11:15am **Sergei Gukov** (CalTech)

Strings, Fields, and Arithmetic, Part I

String theory and quantum field theory are known to have many deep connections and applications to geometry and topology. In recent years, new connections between string/field theory and number theory started to emerge. Examples include a relation between elliptic genera and Jacobi forms, complex multiplication and black hole attractors/RCFTs, etc. In this talk, I will review some the known relations and in the end present new ones.

11:30am–12:30pm **Noriko Yui** (Queen's)

Motives, Mirror Symmetry and Modularity

We consider certain families of Calabi–Yau orbifolds and their mirror partners constructed from Fermat hypersurfaces in weighted projective spaces. We use Fermat motives to interpret the topological mirror symmetry phenonemon. These Calabi–Yau orbifolds are defined over $\mathbf{Q}$, and we can discuss the modularity of the associated Galois representations. We address the modularity question at motivic level. We give some examples of modular Fermat motives. We then formulate a modularity conjecture about rank 4 Fermat motives that there exist Siegel modular forms on some congruence subgroups of $Sp(4, \mathbf{Z})$.

8:00pm–9:00pm **Sergei Gukov** (CalTech)

Strings, Fields, and Arithmetic, Part II

JUNE 7, 2006

9:00am–10:00am **Albrecht Klemm** (Wisconsin)

Modular, Quasimodular Forms and Gromov–Witten Invariants

10:15am–11:15am **Ron Livné** (Hebrew and IAS Princeton)

Explicit descriptions of universal $K3$ families over Shimura curves

It is quite hard to give explicit algebraic description of abelian varieties of dimension > 1. However, the Kummer surfaces of abelian surfaces - and sometimes related $K3$ surfaces - do have useful projective models. In joint work with A. Besser we give some instances where this can be done universally over Shimura curves over $\mathbf{Q}$. While the universal families of abelian surfaces exist only when rather high level is added to the moduli problem, our universal $K3$ fibrations exist over very low level Shimura curves - often level < 1, allowing particularly simple descriptions.

11:30am–12:30pn **Chuck Doran** (Washington)

Algebraic Topology of Calabi–Yau Threefolds in Toric Varieties

We compute the integral homology (including torsion), the topological K-theory, and the Hodge structure on cohomology of Calabi-Yau threefold hypersurfaces and complete intersections in Gorenstein toric Fano varieties. The methods are purely topological. This is joint work with John Morgan.

2:30pm–3:30pm **Marco Aldi** (Northwestern)

Twisted homogeneous coordinate rings of abelian surfaces via Mirror Symmetry

Seidel's mirror map reconstructs the homogeneous coordinate ring of a given projective CY or Fano variety in terms of Lagrangian intersection data on its mirror. We discuss the computation of Seidel's mirror map for abelian and Kummer surfaces and related work on integrality and noncommutative geometry.

4:00pm–5:00pm **Masahiko Saito** (Kobe)

Painlevé Property of ODEs and Deformation of Logarithmic Symplectic Varieties

We will analyze movable branching points of algebraic ordinary differential equations, and give a necessary condition for Painlevé property by means of geometry of logarithmic symplectic varieties. The result establishes the reason why the condition of Okamoto-Painlevé pairs is necessary for Painlevé equations. Furthermore, we can give a very simple proof of a result of Fuchs', Poincaré, Malmiquist and M. Matuda

5:15pm–6:15pm **Rolf Schimmrigk** (Indiana, South Bend)

String Modular Ω–Motives and Aspect of Mirror Symmetry

The purpose of this talk is to describe some new extensions of recent string modularity results for elliptic curves and $K3$ surfaces to higher dimensional varieties of both Calabi-Yau and Fano type. The resulting examples establish that it is possible to construct Calabi–Yau varieties from the physics of the string worldsheet in all physically interesting dimensions. These constructions also provide arithmetic checks for ideas in mirror symmetry. Implications for the mirrors of rigid Calabi–Yau threefolds, as well as elliptic curves, will be discussed. The results concerning special types of Fano varieties provide checks for a conjecture of Serre concerning the type of modular forms associated to generalized Calabi–Yau Hodge structures.

8:00pm–9:00pm **Ahmed Sebber** (Bordeaux)

Differential theta relations and Galois theory for Riccati equations

JUNE 8, 2006

9:00am–10:00am **Jan Stienstra** (Utrecht)

Bimodular Forms and Holomorphic Anomaly Equation

10:15am–11:15am **Vincent Bouchard** (MSRI/Perimeter)

Topological Strings, Holomorphic Anomaly, and (Almost) Modular Forms

8. Appendix

Several people had to cancel their participation to the workshop at the last minutes by various reasons. They were

• Terry Gannon (Alberta): He and his wife had twins on May 28th. Accordingly, Terry had to cancel his participation in the workshop and his talk.

The Monster, Modular Functions and RCFT

Rational conformal field theory 'explains' the modularity of lattice theta series, affine Kac-Moody algebra characters, Monstrous Moonshine functions, etc. We try to identify the essence of this argument, and use this to speculate on the nature of a more conceptual second proof of the Monstrous Moonshine conjectures.

• Amer Iqbal (Washington): He was not able to obtain visa to enter Canada.

• Bong H. Lian (Brandeis): He had to go to Singapore for an urgent matter on June 2nd. He had to cancel his participation and his talk. He sent in a pdf file of his intended talk.

Introduction to Mirror Symmetry

This will be a mathematical survey of both the history and development of mirror symmetry beginning with the early suggestions from physics. Topics may include
0) Two dimensional super conformal field theory
1) Basic examples of mirror manifold constructions
2) Mathematical predictions of mirror symmetry
3) Fourier-Mukai transforms

• John McKay (Concordia): He got sick and in hospital.

• Andrey Todorov (Santa Cruz/MPIM Bonn): He arrived at Calgary International Airport on June 3rd. Upon his arrival, he found out that his uncle in Bulgaria died, and had to take a flight back immediately to Europe. He promised to send in notes of his intended talk.

Regularized Determinants of CY Metrics and Applications to Mirror Symmetry

We will prove the existence of the analogue of the Dedekind Eta Function for CY threefolds and K3 surfaces with unimodular Picard Group. We will discuss the combinatorial properties of the Generalized Dedekind Eta function in the A and B model and its relations with Harvey–Moore–Borcherds Product Formulas.

9. Feedbacks

Here are some comments from participants.

Rolf schimmrigk: netahu@yahoo.com

Dear Noriko, Charles, and Helena,
I wanted to thank you for the very interesting and enjoyable BIRS meeting. The mixture of themes and people was very successful, and I certainly learned quite a bit in discussions with several of the participants, in particular Don. I'm sure I'll learn even more from him when I take him up on his invitation to the MPI in Bonn, which I plan to do in the not so distant future.
Best regards,
Rolf

Sergei Gukov: gukov@theory.caltech.edu

Dear Noriko,
Thank you very much for your kind letter and I have to say that myself I enjoyed the workshop very much (we should certainly have more of these in the future!) and it was very productive for me too! Ironically,

even the fact that Don missed my second talk turned out to be a very good thing since we stayed till very late discussing one of the problems I mentioned and I learned a lot of interesting things from Don (it turns out he has thought about closely related questions). So, for all of this, I want to thank you very much for your hard work and for making it possible!

Having said this, I would certainly be happy to be an organizer of the next workshop. Please, let me know what I can help with at various stages.

With best regards,

Sergei.

Jan Stienstra : stien@math.uu.nl

Dear Noriko,

I hope you had a good journey back to Bonn. My return trip went smooth and completely on schedule. On friday at lunch you said that we should formulate our complaints about the extra housing at Banff Centre in an e-mail to you, so that you can pass this on to the BIRS director. I think that it would also be interesting to ask the early leavers whether their decision to leave early had something to do with the housing issue.

Incidentally, I enjoyed the workshop itself very much and want to thank you (and the other organizers) once again.

with best regards,

Jan

About the housing issue: BIRS offers 5 day workshops. However, its housing policy for the nights before and after the worshop seems not very favourable for having really effective 5 days to work. I had to repeat my request for one extra night before and one extra night after the workshop, three times before it was answered. For the night after the workshop I was referred to an expensive room (CAD 139) elsewhere in the Banff Centre, while Corbett Hall was completely empty that night. I am afraid that this housing policy will encourage late arrivals and early departures, and thus lead to an erosion of the 5 day workshops to actually 4 days or less. In the workshop I attended almost 50% of the participants had left before the lectures of the last day. If this continues, I am afraid that soon many participants (from Europe and Asia) spend almost as much time on travelling and jet lag as they can spent effectively on the workshop and for this reason will reconsider their participation.

yours sincerely,

Jan Stienstra

Ron Livné rlivne@ias.edu

Dear Noriko,

I enjoyed very much the workshop, too bad we did not get to talk more.

Best Regards, and thanks,

Ron

Chapter 12

Creative Writing in Mathematics and Science (06w5091)

Jun 17 – Jun 22, 2006

Organizer(s): Chandler Davis (University of Toronto), The Writing & Publishing Department (The Banff Centre), Marjorie Senechal (Smith College), Jan Zwicky (University of Victoria)

Like its two predecessors, the third BIRS workshop in Creative Writing in Mathematics and Science brought together twenty participants from a range of backgrounds: a mix of mathematicians and scientists with active vocational interests in writing, and writers with active intellectual interests in science and mathematics. Again we included writing in all genres, and required that participants bring work in progress, rather than published work. In a departure from previous practice, this time only ten places were filled by invitation; the other ten were chosen by The Banff Centre method, adjudicated application. The Banff Centre handled all aspects of the application process, from advertising to sending applications to the organizers for adjudication, to notifying the applicants of the results. We received thirty-three applications for these spaces, more than for many programs that draw from a strictly literary clientele through the Banff Centre. Most were of outstanding quality.

By all criteria of evaluation, the workshop was, in the phrase used by several of the participants, 'wildly successful'. Several used the word 'transformative'. Many have already reported exceptional creative activity coming out of the workshop, on revisions and on new pieces. In fact, play that was revised in the light of comments received at the workshop has already been accepted, in its revised version, for a public reading in San Francisco.

Here's how the workshop proceeded. We met for five forty-five minute sessions each day, drafts having been circulated between two weeks and a day in advance. Without exception, participants came to the sessions well-prepared to comment on their colleagues' efforts; everyone spoke in nearly every session. In some cases, the session consisted of feedback on a proposal – reactions to the ordering of chapters, to the sketched content of chapters, sometimes to the shape of a whole manuscript. In one case, we spent the session putting together a reading list for a theatre project on the nature of time. In cases where we were presented with drafts of completed work, the sessions tended to focus on two elements, often simultaneously: the success of the piece as writing (often very detailed suggestions were offered) and the accuracy of the science or mathematics that was in or behind the piece. Writers found the presence of working mathematicians and scientists extremely helpful in this regard. Mathematicians and scientists found the presence of writers (perhaps especially poets, with their awareness of craft) illuminating. The testimonials attached to this report show how valuable the workshop was to many; and underline the many different ways in which it was valuable. The on-line recording of the final reading will convey some sense of the quality and range of the work. Please go to http://www.banffcentre.ca/programs/program.aspx?id=438. This audio record will be supplemented by the anthology that the organizers are currently putting together. Work is being solicited from participants in all three workshops, and editing will occur over the coming months. The much-respected Canadian literary journal, *The Fiddlehead*, has expressed interest in devoting a special issue to the anthology. The Banff Centre has also expressed interest in publishing the anthology.

Two of the organizers, Chandler Davis and Marjorie Senechal, are mathematicians (and co-editors of *The Mathematical Intelligencer*). We agree that cooperation with The Banff Centre was also critical to this workshop's resounding success because it brought to the mix of participants outstanding literary professionals who were able to assist the mathematicians in their writing far more effectively than we could have assisted one another.

The third organizer, Jan Zwicky, has been a faculty member of many workshops and literary arts programs, including The Banff Centre's May Studio, Saskatchewan's *In the Field* residencies, UBC's Booming Ground, The Kootenay School of Writers, Campbell River's Writing on the Water Workshop, and the Maritime Writers Workshop. She has also taught Creative Writing at the University of New Brunswick, and edited for Brick Books for many years. She reports: "My experience with this workshop was unique: I have never before encountered such an intensely focussed, nor such an *adventurous*, group of writers. There was a tremendous sense of communal effort, of generosity mixed with deep respect for the need to get the work right. Criticism was often stringent, but was never offered, nor taken, personally. Was this just the result of a lucky mix of people? I don't think so, though I am sure we were lucky. I believe these features of the workshop were due, in significant measure, to the clientele: people interested in art with a specific, often difficult, intellectual focus. The writers who came were not, could not possibly be, anti-intellectual; the mathematicians and scientists who came were not, could not possibly be, impervious to aesthetic questions. It is this mix that, I believe, accounts for the extraordinary success of the workshop. Collaborative, cross-disciplinary intellectual-aesthetic interests were of the essence."

Other features also contributed to the workshop's success: calling for and adjudicating applications (new for a BIRS workshop); the collaborative process in the sessions themselves, the absence of a designated 'senior' writer leading the workshop (new for a Banff Centre writing workshop); the length of the sessions; the decision to circulate work in advance; the scheduling of two round table sessions for the discussion of *ideas* relevant to writing, science, and mathematics; the length of the workshop (five days); and the provision of time each afternoon for writing. It was also important that assistance with funding was made available to participants: we are grateful to BIRS for its generous support of the participants' program fees, and equally grateful to The Banff Centre for covering administrative costs and the costs of the final reading and reception. We are also grateful to The Banff Centre for its promotion of the program via its website and Summer Festival promotional material. We urge that all these features be retained in future workshops of this nature. Additionally, we suggest that *all* places be filled by adjudicated application; and hope that future workshops will be advertised through BIRS' website as well as The Banff Centre's

But, most importantly, we urge that there *be* other workshops of this nature. No other creative writing workshops anywhere are devoted to the shape of mathematical and scientific content. The success of this workshop points to the importance of future collaboration between BIRS and The Banff Centre, the pooling of resources and expertise to facilitate genuine and exciting cross-fertilization.

Respectfully submitted,
Chandler Davis, University of Toronto
Kim Mayberry, The Banff Centre
Marjorie Senechal, Smith College
Jan Zwicky, University of Victoria

List of Participants

Abate, Marco (Universita di Pisa)
Anand, Madhur (Laurentian University)
Bonny, Sandy (University of Alberta)
Burgess, Sarah (University of Victoria)
Chapman, Robin (University of Wisconsin, Madison)
Cipra, Barry (freelance)
Davis, Chandler (University of Toronto)

Desjardins, Sylvie (University of British Columbia, Okanagan)
Diacu, Florin (University of Victoria)
Dickinson, Adam (York University)
Dunn, Katharine (freelance writer)
Elmslie, Susan (Dawson College (and, Winter 2006, McGill University))
Gunderson, Lauren (Freelance Writer)
Holmes, Nancy (University of British Columbia, Okanagan)
Kasman, Alex (College of Charleston)
Maddow, Ellen (The Talking Band)
Senechal, Marjorie (Smith College)
Wedin, Randall (Wedin Communications)
Zimet, Paul (Smith College)
Zwicky, Jan (University of Victoria)

Chapter 13

Statistics at the Frontiers of Science (06w5073)

June 24 – June 29, 2006

Organizer(s): Gemai Chen (The University of Calgary), David R. Brillinger (University of California at Berkley), Jianqing Fan (Princeton University), Jun Liu (Harvard University), James O. Ramsay (McGill University), Keith J. Worsley (McGill University)

Overview of the Field

Statistics may be broadly defined as the theory and methods of collecting, organizing and interpreting data for solving real world problems. The problems may be formulated as testing of a hypothesis, choosing between alternative hypotheses, estimating an unknown quantity, predicting a future event, and in general, making a decision under uncertainty with minimum risk or loss.

Historically, the ideas and methods of statistics developed gradually as society grew interested in collecting and using data for a variety of applications. The earliest origins of statistics lie in the desire of rulers to count the number of inhabitants or measure the value of the taxable land in their domains. As the physical sciences developed in the 17th and 18th centuries, the importance of careful measurements of weights, distances, and other physical quantities grew. Astronomers and surveyors striving for exactness had to deal with variation in their measurements. Many measurements should be better than a single measurement, even though they vary among themselves. How can we best combine many varying observations? Statistical methods that are still important were invented in order to analyze scientific measurements.

By the 19th century, the agricultural, life, and behavioral sciences also began to rely on data to answer fundamental questions. How are the heights of parents and children related? Does a new variety of wheat produce higher yields than the old, and under what conditions of rainfall and fertilizer? Can a person's mental ability and behavior be measured just as we measure height and reaction time? Effective methods for dealing with such questions developed slowly and with much debate.

As methods for producing and understanding data grew in number and sophistication, the new discipline of statistics took shape in the early part of the 20th century. Ideas and techniques that originated in the collection of government data, in the study of astronomical or biological measurements, and in the attempt to understand heredity or intelligence came together to form a unified ”science of data”. As huge computing power has become more and more accessible in the past two decades or so, the complexity of our society has increased dramatically, the amount of relevant information has exploded, and statistics has become more and more essential in reaching a scientific decision.

Most importantly, what we can see from the above overview is that it is the application that gives new life to statistics, and at the same time, statisticians have been helping scientists in solving current problems and formulating new theories that will lead to advancement of knowledge.

The Objectives of the Workshop

The workshop has invited active statistical researchers involved in various areas of scientific research to meet and exchange ideas. Our objectives are: (1) to bring the inspiring excitement of frontier scientific research to statistical community to strengthen the current statistical research, and (2) to create opportunities for new and/or deeper collaborative research.

Five group leaders have participated in the invitation, organization and running of the workshop. They are

- Professor David R. Brillinger of University of California at Berkley responsible for **Time Series and Stochastic Processes** (Due to other commitment, Professor Brillinger was not able to attend the workshop after organizing his group. Professor Bruce Smith took over the responsibility during the workshop.)
- Professor Jianqing Fan of Princeton University responsible for **Financial and Risk Analysis**
- Professor Jun Liu of Harvard University responsible for **Bio-Medical Research**
- Professor James O. Ramsay of McGill University responsible for **Functional Data Analysis**
- Professor Keith J. Worsley McGill University responsible for **Random Field and Image Analysis**

The above five areas represent a sample of the areas where statisticians are working closely with frontier scientific researchers to attack some of the most challenging problems today. Taking a function or a spectrum as input instead of a single numerical datum, functional data analysis methods have evolved from science and are now helping scientists to research from a broader and more realistic perspective. Advances in geometry and random fields driven by our desire to understand human being better have made it possible to quantify some topological changes, such as those in brain shape. The ups and downs of the financial market have definitely stimulated the statistical research to come up new models, such as the two proposed by the two 2003 Nobel Prize winners in economics. The great efforts made to understand the origin of life and develop new and more effective drugs to battle diseases have changed the way statistical research is usually done, offered a whole range of challenging problems, and made statistics an integral part of the biological and medical research. The innovations and comebacks of new and old techniques in time series and stochastic processes have merged statistics into many scientific endeavors that are changing the face of science day after day.

The various annual statistical meetings do include the above groups in the programs, but the busy schedules usually prevent the different groups to sit together and share ideas. This workshop has provided such an opportunity. More importantly, the seemingly unrelated groups actually have much in common from a methodological point of view, and good opportunities have been taken for transferring, borrowing and strengthening already developed methodologies, and for creating new research directions.

Presentation Highlights

In the following, we will highlight the various presentations given during the workshop according to the 5 different topics.

Financial and Risk Analysis

The presentation of Professor Per Mykland (University of Chicago, mykland@galton.uchicago.edu) titled "A Gaussian Calculus for Inference from High Frequency Data" is aimed at providing a rigorous theory for some of the existing methodology and possible new tools for financial analysis. In the econometric literature of high frequency data, it is often assumed that one can carry out inference conditionally on the underlying volatility processes. In other words, conditionally Gaussian systems are considered. This is often referred to as the assumption of "no leverage effect". This is often a reasonable thing to do, as general estimators and results can often be conjectured from considering the conditionally Gaussian case. This presentation is to try to give some more structure to the things one can do with the Gaussian assumption. It is argued that there is a whole treasure chest of tools that can be brought to bear on high frequency data problems in this case. In particular, approximations involving locally constant volatility processes are considered, and a general theory for this approximation is developed. As applications of the theory, an improved estimator of quarticity, an ANOVA for processes with multiple regressors, and an estimator for error bars on the Hayashi-Yoshida estimator of quadratic covariation are proposed.

Professor Yazhen Wang (University of Connecticut, yzwang@stat.uconn.edu) delivers a talk on "Heterogeneous Autoregressive Realized Volatility Model". Volatilities of asset returns are pivotal for many issues in financial economics such as asset pricing, portfolio allocation and risk management. The availability of high frequency intraday data may allow people to estimate volatility more accurately. In practice, realized volatility is often used to estimate integrated volatility. To obtain better volatility estimation and forecast, some autoregressive structure of realized volatility is proposed in the literature. The use of a heterogeneous autoregressive model for realized volatility is explored and a nonparametric multiscale statistical technique is developed to construct noise resistant realized volatility for estimating integrated volatility.

"Statistical Approaches to Option Pricing and Portfolio Management" delivered by Professor Jianqing Fan (Princeton University, jqfan@princeton.edu) addresses the fundamental issues of finance. The existing financial mathematical models provide useful tools for option pricing. These physical models give us a good first order approximation to the underlying dynamics in the financial market. However, their power in option pricing can be significantly enhanced when they are combined with statistical approaches, which empirically learn and correct pricing errors through estimating the state price densities. Two new semiparametric techniques are proposed for estimating state price densities and pricing financial derivatives. Empirical studies based on the options of SP500 index over 100,000 tests show that the two new semiparametric techniques outperform the ad hoc Black-Scholes method and significantly so when the latter method has large pricing errors.

A related issue is to find a good estimation of the high-dimensional covariance matrix for portfolio allocation and risk management. Motivated by the Capital Asset Pricing Model, a factor model is proposed to reduce the dimensionality and to estimate the covariance matrix. The performance of the new estimate is compared with the sample covariance matrix. Situations under which the factor approach can gain substantially in performance and the cases where the gains are only marginal are demonstrated and identified. Furthermore, the impacts of the covariance matrix estimation on portfolio allocation and risk management are studied. The theoretical results are convincingly supported by a thorough simulation study.

An interesting feature of the talk is that traditionally, people seek to have the best, the most, or the perfect. But in applications involving complicated phenomena such a finance, it is more beneficial to seek a better, a useful, or an improved result.

Professor Cheng-Der Fuh (Academia Sinica, stcheng@stat.sinica.edu.tw) gives an interesting and deep talk on "Efficient Likelihood Estimation in State Space Models". Likelihood principle is the most used principle is statistical theory and applications. Motivated by studying asymptotic properties of the maximum likelihood estimator (MLE) in stochastic volatility (SV) models, likelihood estimation in state space models is investigated. It is first proved that under some regularity conditions, there is a consistent sequence of roots of the likelihood equation that is asymptotically normal with the inverse of the Fisher information as its variance. With an extra assumption that the likelihood equation has a unique root for each n, then there is a consistent sequence of estimators of the unknown parameters. If, in addition, the supremum of the log likelihood function is integrable, the MLE exists and is strong consistent. Edgeworth expansion of the approximate solution of likelihood equation is also established. Several examples, including Markov switching models, ARMA models, (G)ARCH models and stochastic volatility (SV) models, are given for illustrations.

An interesting feature of the talk is that the essence of the well known maximum likelihood method is given a much simplified discussion and one can now see what needs to be done to use this method regardless of the specific details.

Time Series and Stochastic Processes

"Structural Break Detection in Time Series Models" are addressed by Professor Richard A. Davis (Colorado State University, rdavis@stat.colostate.edu). Much of the recent interest in time series modeling has focused on data from financial markets, from communications channels, from speech recognition and from engineering applications, where the need for non-Gaussian, non-linear, and nonstationary models is clear. With faster computation and new estimation algorithms, it is now possible to make significant in-roads on modeling more complex phenomena. In this talk, Professor Davis develops estimation procedures for a class of models that can be used for analyzing a wide range of time series data that exhibit structural breaks. The novelty of the approach taken here is to combine the use of genetic algorithms with the principle of minimum description length (MDL), an idea developed by Rissanan in the 1980s, to find "optimal" models over a potentially large class of models.

This methodology is demonstrated in a number of applications including piece-wise AR models, segmented GARCH models, slowly varying AR models, linear models with dynamic structures and state space models. In addition to fitting piece-wise autoregressive models, which works well even for local stationary models that are smooth, extensions to piece-wise nonlinear models including stochastic volatility and GARCH models are also considered.

Professor R.H. Shumway (University of California at Davis, rhshumway@ucdavis.edu) talks about "Mixed Signal Processing for Regional and Teleseismic Arrays". Successful monitoring of a proposed Comprehensive Nuclear Test-Ban Treaty (CTBT) ultimately rests on interpretations of time series that are produced on seismic and infrasound arrays as well as on auxiliary information from other sources such as satellites and radionuclide sampling. Underground events such as earthquakes and explosions generate plane waves propagating across arrays of seismometers and proper use of this information is critical to the successful detection, location and identification of the source phenomenon.

When simultaneous events occur or when propagating noises are present at an array, mistakes can be made in locating an event as well as in reading the magnitude-related variables that are critical for discriminating between classes of events. The performance of conventional high-resolution estimators such as MUSIC for two typical mixtures of signals is examined and an alternate approach using a combination of nonlinear stepwise regression and model selection techniques is developed. The new method yields the correct number of signals on two typical mixtures and allows deconvolution of the component signals.

Peter Buhlmann (ETH Zurich, buhlmann@stat.math.ethz.ch) takes on the challenging "DNA Splice Site Detection with Group-penalty Methods for Categorical Predictors". DNA splice sites detection has been pursued, among other approaches, by non-Markovian time series models. As an alternative, Professor Buhlmann uses logistic regression with (short) DNA sequence as categorical predictors. When including higher-order interaction terms, such models become very high-dimensional (e.g. 1'000 - 16'000 predictors). He proposes to use the group-penalty and new modifications thereof for hierarchical model fitting and presents efficient algorithms which are particularly suited for high-dimensional problems. He shows that the proposed methods are statistically consistent for sparse but high-dimensional problems where the number of predictor variables may be much larger than sample size. Despite the generality of the new approach, it performs surprisingly well for the DNA splice site detection problems which he has analyzed.

Professor Ian McLeod (University of Western Ontario, aim@stats.uwo.ca) addresses the workshop with "My Current Research in Time Series". He starts with his (1) recent work on statistical algorithms for time series including subset autoregressive modeling, faster ARMA maximum likelihood estimation and automatic Brillinger monotonic trend test. He then moves to (2) improved portmanteau diagnostic checks for univariate and vector ARMA time series, followed by (3) applications of time series in bio-informatics. Finally he discusses a new diagnostic check for lack of statistical independence which is applicable to a wide variety of statistical models including regression and generalized linear models.

This time series and stochastic processes group ends with the talk by Wai Keung Li (The University of Hong Kong, hrntlwk@hku.hk) on "Least Absolute Deviation Estimation for Fractionally Integrated Autoregressive Moving Average Time Series Models with Conditional Heteroscedasticity". In order to model time series exhibiting the features of long memory, conditional heteroscedasticity and heavy tails, a least absolute deviation approach is considered to estimate fractionally autoregressive integrated moving average models with conditional heteroscedasticity. The time series generated by this model is short memory or long memory, stationary or nonstationary, depending on whether the fractional differencing parameter $d \in (-1/2, 0)$ or $(0, \infty)$, $d \in (-1/2, 1/2)$ or $(1/2, \infty)$ respectively. Using a unified approach, the asymptotic properties of the least absolute deviation estimation are established. The large sample distribution of residual autocorrelations and absolute residual autocorrelations is also derived and these results lead to two useful diagnostic tools for checking the adequacy of the ftted models. Some Monte Carlo experiments were conducted to examine the performance of the theoretical results in finite sample cases. As an illustration, the process of modeling the absolute return of the daily closing Dow Jones Industrial Average Index (1995-2004) is also reported.

Functional Data Analysis

In an overview address titled "Functional Data Analysis: Where it's been and where it might be going?", Professor Jim Ramsay (McGill University, ramsay@psych.mcgill.ca) reviews the history and current trends in functional data analysis: where the field tends to fit in with respect to other methods looking a distributed data, what the more important accomplishments have been over the last decade, current work on modeling dynamic systems, and what the future might hold.

For example, over the past two years Professor Ramsay's research has been aimed at taking functional data analysis into the world of dynamic systems modeling. This means developing the capacity to estimate the parameters defining a system of differential equations, either linear or nonlinear, from noisy data, often taken from only a subset of the variables in the system. This has worked out very well, and involves using the DIFE to define a roughness penalty, and then using a two- or three-stage estimation procedure in which:

- coefficients of basis functions expansions for variables are defined as functions of parameters defining the system, and are estimated conditional on system parameters by optimizing a fit measure plus roughness penalty

- defining system parameters as functions of smoothing or other model complexity parameters, and each time these latter are changed, optimizing a measure of fit without regularization and, finally

- optimizing an estimate of mean square error such as GCV with respect to smoothing parameters.

This three-stage process involves what Professor Ramsay has come to call a "parameter cascade" in which parameters are segmented into levels, with the lowest level being "local parameters", the next level "global parameters", and the highest level "complexity parameters." It was this perspective that finally really made our many successful estimates of DIFE systems possible.

The parameter cascade idea has immediate application to many other situations. In particular, it deals with the famous Neyman-Scott problem very nicely, applies neatly to linear and nonlinear mixed effects or multi-level models, works well with psychometric problems involving estimating examinee ability and item characteristics, and so on. In effect, it is a Bayesian like framework that leads to much faster and more direct methods than MCMC that also have easily computable interval estimates. Unlike MCMC, it is easy to deploy to users.

In an coordinated talk titled "Functional Data Analysis: Tools and issues", Professor Hans-Georg Mueller (University of California at Davis, Mueller@wald.ucdavis.edu), Professor Daniel Gervini (University of Wisconsin at Milwaukee, gervini@uwm.edu), Professor Fang Yao (Colorado State University, ffyao@stat.colostate.edu) and Professor Gareth James (University of Southern California, gareth@usc.edu) explore a range of topics in functional data analysis.

Professor Mueller opens the talk by discussing the characteristics of functional data: High-dimensional (infinite-dimensional) with a topology characterized by order, neighborhood and smoothness, and various warping methods—curve alignment or registration. He also addresses the issues of functional convex calculus and weak convergence.

Professor Gervini picks up functional principal component analysis which is a generalization of the usual principal component analysis for multivariate data. From Karhunen-Lóeve expansion, to asymptotic properties, to functional longitudinal sparse data analysis, and to future studies (spacial and image analysis, non-Gaussian longitudinal data, high-dim matrices) Professor Gervini gives a complete summary of functional principal component analysis.

Professor Fang Yao discusses various functional regression models including multivariate regression, functional linear model, functional response model, and generalized functional linear model.

Professor James reports on the challenges when developing and applying functional data analysis methodologies (increasingly complex functional regression models, inference and asymptotics in new settings, survival analysis (joint modeling), ecology, financial time series, and gene expression time courses).

A feature of the above talks in this group is that it is firmly believed that the field will be driven forward by new applications.

Random Field and Image Analysis

Among the 5 topics of the workshop, this topic seems to be the most abstract, but the applications presented are just astonishingly concrete.

Professor Robert J. Adler (Technion - Israel Institute of Technology, robert@ieadler.technion.ac.il) gives the first talk on "Rice and Geometry". The classic Rice formula for the expected number of upcrossings of a smooth stationary Gaussian process on the real line is one of the oldest and most important results in the theory of smooth Gaussian processes. It has a multitude of applications, and has been generalized over the years to non-stationary and non-Gaussian processes, both over the real and over more complex parameter spaces, and to vector valued rather than real valued processes.

Over the last few years, a new Rice"super formula" has been developed, which incorporates effectively all the (constant variance) special cases known until now. More interesting, however, is that this new formula shows that all of the related formulae have a deep geometric interpretation, giving a version of the Kinematic Fundamental Formula of Integral and Differential Geometry for Gauss space.

Later on Professor Adler outlines a proof of the new formula and says: "There are two choices. One is to use Riemann metric and geometry; the other is to rediscover Riemann metric and geometry."

Professor Keith Worsley (McGill University, keith@math.mcgill.ca) speaks on behalf of Dr. Jonathan Taylor (Stanford University, jonathan.taylor@stanford.edu) on "Deformation Based Morphometry, Roy's Maximum Root and Recent Advances in Random Fields". He starts with a study of anatomical differences between controls and patients who have suffered non-missile trauma. He then employs a multivariate linear model at each location in space, using Hotelling's T^2 to detect differences between cases and controls. If covariates are further included in the model, Roy's maximum root is a natural generalization of Hotelling's T^2. This leads to the Roy's maximum root random field, which includes many special types of random fields: Hotelling's T^2, T, and F, so, in effect the Roy's maximum root random field "unifies" many different random fields. The geometric interpretations of this "unified" random field theory is explored.

Professor Jiayang Sun (Case Western Reserve University, jiayang@sun.STAT.cwru.edu) continues to talk about "Three Statistical Imaging Problems". First, she describes two neuron-imaging problems ((i) ctivities in spinal cord: Fos expression, involving cats and measured with spatial counts of neurons activated by stimulated nerves; (ii) changes in brain activities: force, EMG (3), EEG (64) involving human and measured with structured time series (fat and short)) that challenge the status quo, namely, statistical models assumed for typical neuronal data. These problems offer opportunities for new modeling and statistical inference, including multiple comparisons arising from a negative binomial random field, which is generally applicable for analyzing data from over-dispersed Poisson regression models. The third imaging problem that led to her research on identifying the pixels that most likely correspond to the false discoveries from a FDR procedure is also touched.

In the talk titled "Granger Causality on Spatial Manifolds: Applications to Neuroimaging", Professor Pedro A. Valds-Sosa (Cuban Neuroscience Center, Ciudad Habana, Cuba, peter@cneuro.edu.cu) combines economics with neuroscience. The (discrete time) vector Multivariate Autoregressive (MAR) model is generalized as a stochastic process defined over a continuous spatial manifold. The underlying motivation is the study of brain connectivity via the application of Granger Causality measures to functional Neuroimages. Discretization of the spatial MAR (sMAR) leads to a densely sampled MAR for which the number of time series p is much larger than the length of the time series N. In this situation usual time series models work badly or fail. Previous approaches, involve the reduction of the dimensionality of the MAR, either by the selection of arbitrary regions of interest or by latent variable analysis. An example of the latter is given using a multi-linear reduction of the multichannel EEG spectrum into atoms with spatial, temporal and frequency signatures. Influence measures are applied to the temporal signatures giving an interpretation of the interaction between brain rhythms. However the approach introduced here is that of extending the usual influence measures for Granger Causality to sMAR by defining "influence fields", that is the set of influence measures from one site (voxel) to the whole manifold. Estimation is made possible by imposing Bayesian priors for sparsity, smoothness, or both on the influence fields. In fact, a prior is introduced that generalizes most common priors studied to date in the literature for variable selection and penalization in regression. This prior is specified by defining penalties paired with a priori covariance matrices. Simple pairs of penalties/covariances include as particular cases the LASSO, Data Fusion and Ridge Regression. Double pairs encompass the recently introduced Elastic Net and Fussed Lasso. Quadruples of penalty/covariance combinations are also possible and used for the first time. Estimation is carried out via the MM algorithm, a new technique that generalized the EM algorithm and allows efficient estimation even for massive time series dimensionalities. The proposed technique performs adequately for a simulated “small world” cortical network with linear dynamics, validating the use of the more complex penalties. Application of this model to fMRI data validate previous conceptual models for the brain circuits involved in the generation of the EEG alpha rhythm.

The last talk in this group given by Professor Emery N. Brown (Massachusetts Institute of Technology, brown@neurostat.mgh.harvard.edu) on “Large Scale Kalman Filtering Solutions to the Electrophysiological Source Localization Problem—An MEG Case Study” extends the use of the famous Kalman filter to a new application. Computational solutions to the high-dimensional Kalman Filtering problem are described in the setting of the MEG inverse problem. The overall objective is to localize and estimate dynamic brain activity from observed extraneous magnetic fields recorded at an array of sensor positions on the scalp and to do so in a manner that takes advantage of the true underlying statistical continuity in the current sources. To this end, one can use inverse mapping procedures that combine models of current dipoles with dynamic state-space estimation algorithms. While these algorithms are eminently well-suited to this class of dynamic inverse problems, they possess computational limitations that need to be addressed either by approximation or through the use of high performance computational resources. A High Performance Computing (HPC) solution to the Kalman filter is found and its applicability to the Magnetoencephalography (MEG) inverse problem is demonstrated.

Bio-Medical Research

This is a very dynamic group attacking various bio and life science challenges.

Professor Jun Liu (Harvard University, jliu@stat.harvard.edu) delivers a talk on “A Bayesian Method for Detecting Disease-Related Genetic Interactions”. In case-control association studies, it is of interest to detect multi-locus interactions called Epistasis. More specifically, given genotypes at multiple loci for both cases and controls, one would like to locate most likely positions where a disease-related mutation may have occurred. Various parametric and nonparametric methods are reviewed and it is noted that the existing methods are either of low power or computationally infeasible when facing a large number of markers. An alternative method using MCMC sampling techniques is developed which can efficiently detect interactions among thousands of markers. The issues of statistical significance and how to adjust multiple comparisons are discussed (much of these are conjectures, though).

“Using Genetic Linkage to Inform Positional Cloning”, Professor Mary Sara McPeek (University of Chicago, mcpeek@galton.uchicago.edu) addresses an delicate issue in genomics. The first step in mapping a gene for a trait often involves using linkage analysis to identify a region on a chromosome where a gene of interest may lie. Linkage disequilibrium mapping may sometimes be used to further refine the region. At this point, one may be able to identify one or several genes within the region. Even if only a single gene lies in the region, it may contain a large number of polymorphic sites (base pairs of DNA that vary across individuals), and a question of interest is to determine which site or combination of sites influence the trait. Ultimately, only well-designed biological studies can establish that particular variation influences susceptibility. However, one can address the question of whether a particular set of polymorphisms can fully “explain”, in the statistical sense, the observed linkage to the region. Suppose that many tightly-linked SNPs have been identified and genotyped in affected relatives in a region showing strong linkage with a binary trait. It is noted that if a particular set of SNPs contains all the sites in the region that influence the trait, then conditional on the genotypes at those SNPs, there should be no excess sharing in the region among affected individuals. This idea is used to develop a statistical test of the null hypothesis that a particular SNP or pair of SNPs can explain all the evidence for linkage and to develop a confidence set of individual SNPs and pairs of SNPs that has appropriate coverage probability of the causal SNP or pair of SNPs assuming that there are no more than 2 in that region. Arbitrary genetic model (including epistasis with other unlinked susceptibility loci) and linkage disequilibrium are allowed and appropriate methods to take into account the uncertainty in haplotype frequencies re developed. Discussions on approaches to the problem of adjusting for the selection of the region based on the linkage results in the same sample of individuals are offered.

Professor Hongyu Zhao (Yale University, hz27@email.med.yale.edu) entertains a challenging issue in microbiology to talk about “Protein Interaction Predictions Through Integrating High-throughput Data From Diverse Organisms”. Predicting protein-protein interactions is critical for understanding cellular processes.

Because protein domains represent binding modules and are responsible for the interactions between proteins, several computational approaches have been proposed to predict protein interactions at the domain level. The fact that protein domains are likely evolutionarily conserved allows us to pool information from data across multiple organisms for the inference of domain-domain and protein-protein interactions. Professor Zhao presents his results on estimating domain-domain interaction probabilities through integrating large-scale protein interaction data from three organisms, yeast, worms, and fruit flies. The estimated domain-domain interaction probabilities can be then used to predict protein-protein interactions in a given organism. Based on a thorough comparison of sensitivity and specificity, and other analyses, the proposed approaches are shown to have better performance due to their ability to borrow information from multiple species. The estimated domain-domain interaction probabilities can also be informative in predicting protein-protein interaction in other organisms.

Recent advances in life science have allowed scientists to “follow” the movement of a molecule. Professor Samuel Kou (Harvard University, kou@stat.harvard.edu) introduces the audience to this fascinating topic with a talk titled “Stochastic Modeling and Inference in Nano-scale Biophysics”. With the progress made in nanotechnology, scientists now can follow a biological process on an unprecedented single molecule scale. These advances also raise many challenging stochastic modeling and statistical inference problems. First, by zooming in on single molecules, recent nano-scale experiments reveal that some classical stochastic models derived from oversimplified assumptions are no longer valid. Second, the stochastic nature of the experimental data and the presence of latent processes much complicate the statistical inference. Professor Kou uses the modeling of subdiffusion phenomenon in enzymatic conformational fluctuation and the inference of DNA hairpin kinetics to illustrate the statistical and probabilistic challenges in single-molecule biophysics.

Professor Ying Nian Wu (University of California at Los Angeles, ywu@stat.ucla.edu) moves on to discuss an important technology in bio industry with a talk on “ChIP-chip: Data, Model, and Analysis”. ChIP-chip (or ChIP-on-chip) is a technology for isolation and identification of genomic sites occupied by specific DNA binding proteins in living cells. ChIP-chip data can be obtained over the whole genome by tiling arrays, where a peak in the signal is generally observed at a protein binding site. Professor Wu presents a probability model for ChIP-chip data. Then he proposes a model-based computational method for locating and testing peaks for the purpose of identifying potential protein binding sites and presents a non-parametric method for identifying and representing peaks in multiple resolutions.

Cancer research is not new but is one of the most important and challenging bio-medical researches. Professor Volker Schmidt (UniversitŁt Ulm, volker.schmidt@uni-ulm.de) presents a talk on “Model-Based Analysis of Keratin Filament Networks in Scanning Electron Microscopy Images of Cancer Cells”. The keratin filament network is an important part of the cytoskeleton in epithelial cells. It is involved in the regulation of shape and viscoelasticity of the cells. In-vitro studies indicated that geometrical network characteristics, such as filament cross-link density, determine the biophysical properties of the filament network.

Scanning electron microscopy images of filaments were processed by a skeletonisation algorithm based on morphological operators to obtain a graph structure which represents individual filaments as well as their connections. This method was applied to investigate the effects of the so-called transforming growth factor alpha (TGF-alpha) on the morphology of keratin networks in pancreatic cancer cells. By estimating geometrical network characteristics, like the length and orientation distributions of the keratin filaments, and by fitting random tessellation models, a significant alteration of keratin network morphology could be detected in response to TGF-alpha.

Professor Murray D. Burke (University of Calgary, burke@math.ucalgary.ca) gives the last talk of the workshop on “Semiparametric Regression Models with Staggered Entries and Progressive Multi-Stage Censoring”. In his talk, Professor Burke studies a class of semiparametric regression models when subjects enter the study in a staggered fashion. A strong martingale approach is used to model the two-time parameter counting processes. It is shown that well-known univariate results such as weak convergence and martingale inequalities can be extended to these two-dimensional models. Strong martingale theory is also used to prove weak convergence of a general weighted goodness-of-fit process and its weighted bootstrap counterpart. If three progressive multi-stage censoring schemes are considered, where the experimenter purposely censors a given number of individuals under study at fixed time points, it is also possible to incorporate this censoring into the above models.

Scientific Progress Made

Although there are 5 groups of different researchers attending the workshop, both within the groups and among the groups, participants have had a wide range of (long) discussions.

The important role that statistics can play in financial and risk analysis is clearly demonstrated and enhanced. Finance theory based on (stochastic) partial differential equations can be used to set up the market and make it run. Statistical methodologies can validate/help adjust the financial theory, help us to understand financial volatility better, and help run the market more effectively.

To treat today's time series, new methods are clearly needed, especially in dealing with multivariate series involving both time and space. On the other hand, existing methods can be made more useful through combining them appropriately by identifying the structural breaks in the series.

Science nowadays offers more and more challenging problems and human beings have been able to take more and more complicated measurements. In just over a dozen years, functional data analysis has quickly become a powerful methodology for scientists to attack various old and new problems. This is well reviewed and illustrated in the workshop. If one wants to start a career in statistics, functional data analysis should be seriously considered.

Abstract and concrete are two extremes. When both are sophisticated, it is intimidating to try to understand them. However, when the two are coherently connected, it creates stunning beauty. The random field and image analysis group did just that. On the one side is the boundary crossing in Gaussian random space. On the other side is the change of brain tumors. Successful statistical procedures have been developed to quantify the change of brain tumors using the fundamental results from boundary crossing.

Life takes different forms at different levels. It therefore offers challenges of various kinds. Observations or measurements we can take all contain errors or uncertainties. With errors or uncertainties present, how to detect disease-related genetic interactions, how to use genetic linkage to inform positional cloning, how to predict protein interactions, how to model subdiffusion phenomenon in enzymatic conformational fluctuation, how to isolate and identify genomic sites occupied by specific DNA binding proteins in living cells, how to create a graph which represents individual filaments as well as their connections in cancer cells, and how to allow and handle censoring in medical research? All of these are directly related to whether we can make progress in understanding life. Statistical thinking and methods have been shown to be valuable for the above and other efforts.

Throughout the workshop, it has been demonstrated again and again and in one field after another that science offers new challenges to statisticians and statisticians can help make genuine and significant progress in science.

Outcome of the Meeting

There is no doubt that the workshop is a great success. The relaxed atmosphere and the plenty of time for discussion have allowed participants to share and exchange ideas, discuss issues of mutual interest, and team up to workshop on existing and new problems.

The discussion part has really stood out: in each of the 5 discussion periods for the 5 different groups, there are always participants from different groups to attend and the discussions are always interesting and entertaining.

Thanks to the wonderful and efficient management of BIRS, the workshop runs very smoothly. Here are some of the feedbacks:

"Thank you very much for all the work you did in organizing this retreat and conference. Professionally, it was the best workshop or conference I have ever attended. This was due to the relatively small number of participants, the beautiful location and the large number of social acitivities that you and the others organized. It was a wonderful experience."

"I just came back, and the first thing I want to do is to sincerely thank you and your colleagues for organizing such a nice workshop. I learned a lot, and I also thoroughly enjoyed Banff."

"Many thanks for organizing a fantastic event."

"Thanks so much for taking care of everything and running such an excellent conference."

"Thanks very much for organizing the workshop. I had a very good time and was able to make contact with some researchers previously admired only from afar."

"I have returned home safe and sound. Many thanks again for the invitation and the organising. It was a great workshop and nice environment. I certainly have learned a lot."

"Thank you for organizing the conference. It was a wonderful success. Thanks also go to your hospitality. I enjoy very much the wonderful Banff."

"Thank you very much for having organized this great workshop: it has been very interesting (talks and discussions), it gave me the opportunity to talk "in depth" with many people and I also enjoyed the outdoors around Banff."

"BIRS is better than Oberwolfach in environment, food and management."

List of Participants

Adler, Robert (Technion - Israel Institute of Technology)
Brown, Emery (Massachusetts Institute of Technology (MIT))
Buhlmann, Peter (Swiss Federal Institute of Technology in Zurich (ETHZ))
Burke, Murray (University of Calgary)
Chen, Gemai (University of Calgary)
Chen, Rong (University of Illinois at Chicago)
Davis, Richard (Colorado State University)
Fan, Jianqing (Princeton University)
Fuh, Cheng-Der (Academia Sinica)
Gervini, Daniel (University of Wisconsin at Milwaukee)
Hooker, Giles (McGill University)
James, Gareth (University of Southern California)
Jank, Wolfgang (University of Maryland)
Kneip, Alois (University of Bonn)
Kou, Samuel (Harvard University)
Li, Wai. K. (University of Hong Kong)
Linton, Oliver (London School of Economics)
Liu, Jun (Harvard University)
Lockhart, Richard (Simon Fraser University)
McLeod, Ian (University of Western Ontario)
McPeek, Mary Sara (University of Chicago)
Mueller, Hans-Georg (University of California, Davis)
Mykland, Per (University of Chicago)
Ramsay, Jim (McGill University)
Rohani, Farzan (McGill University)
Schmidt, Volker (University of Ulm)
Shumway, Robert (University of California, Davis)
Smith, Bruce (Dalhousie University)
Sun, Jiayang (Case Western Reserve University)
Valds-Sosa, Pedro A. (Cuban Neuroscience Center)
Wang, Yazhen (National Science Foundation)
Worsley, Keith (University of Chicago)
Wu, Yingnian (University of California, Los Angeles)
Yao, Fang (Colorado State University)
Zhao, Hongyu (Yale University)

Chapter 14

Statistical inference Problems in High Energy Physics and Astronomy (06w5054)

Jul 15 – Jul 20, 2006

Organizer(s): James Linnemann (Michigan State University), Louis Lyons (University of Oxford), Nancy Reid (University of Toronto)

Introduction

In the analysis of data collected in Particle Physics experiments, the use of the best statistical techniques can produce a better quality result. Given that statistical computations are not expensive while accelerators and detectors are, it is clearly worthwhile to invest some effort on the former. The PHYSTAT series of Conferences and Workshops has been devoted to just this topic. It started at CERN in January 2000 with a Workshop on Confidence Limits - what one can say about the maximum possible strength of a hypothesised signal when no effect is seen in the data. The latest Workshop was held at Banff in the Canadian Rockies in July this year. This was also the culmination of the Workshop which had taken place earlier in the year at SAMSI (Statistical and Applied Mathematical Sciences Institute) in the North Carolina Research Triangle Park.

The Workshop was attended by 33 people, of whom 13 were statisticians, the remainder being mostly experimental Particle Physicists, with Astrophysicists making up the total. There were 3 graduate students. The Workshop concentrated on 3 specific topics: a) Upper Limits, in situations where there are systematic effects (”nuisance parameters”). b) Assessing the significance of possible interesting effects, in the presence of nuisance parameters. The subject of significance will hopefully be relevant for experiments at the Large Hadron Collider (LHC) at CERN, where the exciting discoveries that may be made include the Higgs boson, supersymmetric particles, leptoquarks, pentaquarks, free quarks or magnetic monopoles, extra spatial dimensions, technicolour, the substructure of quarks and/or leptons, mini-black holes, etc. In all cases it will be necessary to distinguish among peaks which are merely statistical fluctuations, goofs and genuine signals of new Physics. c) The separation of events which are interesting signal from those due to boring background. This classification process is required in almost every statistical analysis performed in High Energy Physics. For each of these topics there was a Physics Co-ordinator and a Statistics one.

Of course, the 3 topics do interact with each other. Searches for new physics will result in an upper limit when no or little effect is seen, but will need a significance calculation when a discovery is claimed. The multivariate techniques are generally used to provide the enriched subsample of data on which these searches are performed. Just as for limits or significance, nuisance parameters can be important in multivariate separation methods too.

As this was a Workshop, participants were encouraged to be active in the weeks before the meeting. Reading material was circulated as well as some simulated data, on which participants could run computer programmes incorporating their favourite algorithms. This enabled all participants to become familiar with the basic issues before the start of the meeting. The Workshop started with two introductory talks (on "Brief Introduction to Particle Physics and typical statistical analyses" and "Monte Carlo Experiments in High Energy Physics"). These were primarily to describe for Statisticians the terminology used, the sort of physics issues that experimentalists try to investigate, what our statistical problems are and how we currently cope with them, etc. Jim Linnemann took the opportunity to publicise a new web site, `www.phystat.org`, which provides a repository for software useful in statistical calculations for physics. Everyone was encouraged to contribute suitable software, which can range from packages suitable from general use, to the code specifically used in preparing a physics publication.

General Summary

This section gives a brief summary of the main ideas touched on in the subsequent talks and discussion meetings. Summaries of individual talks are provided in the Appendices, and for most talks there is a link on the conference web page to the slides or a relevant paper.

The discussion about limits ranged from a variety of Bayesian techniques, via profile likelihood to pure frequentist methods. An interesting suggestion from statisticians was that hierarchical Bayes might be a good approach for a search for new physics in several related physics channels. There was a lively discussion about the relative merits of the possible approaches, and even of what were the relevant criteria for the comparison. After a late evening session, it was decided that data would be made available by the limits convenor Joel Heinrich, for participants to try out their favourite methods; and Heinrich would compare the results. This work is expected to continue until November.

The significance issue was discussed in the context of Particle Physics and several Astrophysics ones too. Indeed it arises in a wide range of subjects where anomalous effects are sought. The Banff Physics convenor on significance, Luc Demortier, detailed 8 separate ways in which nuisance parameters can be incorporated in these calculations, and discussed their performance. This is going to be a crucial issue for new particle searches at the LHC, where some of the backgrounds will be known only approximately. Demortier also addressed the issues of whether it is possible to assess the significance of an interesting effect, which is obtained by physicists adjusting selection procedures while looking at the data; and why Particle Physics usually demands the equivalent of a 5 standard deviation fluctuation of the background before claiming a new discovery (The probability of obtaining such a large fluctuation by chance is below 1 part in a million).

The multivariate signal- background separation sessions resulted in very positive discussions between physicists and statisticians. Byron Roe explained the various techniques used for separating signal from background. For the MiniBooNE experiment, Monte Carlo studies showed that the 'boosted decision trees' approach yielded good separation, and was capable of coping with over 100 input variables. An important issue was assessing the effect on the physical result, in this case neutrino oscillation parameters, of possible systematic effects. One of the conventional methods for doing this is to vary each possible systematic effect by one standard deviation, and to see how much this affects the result; and then the different sources are combined. Roe pointed out that there is much to recommend an alternative procedure where the effect on the result is investigated of varying all possible systematic sources at random simultaneously.

This was a theme that was taken up in more detail by Toronto statistician Radford Neal, who also emphasised the need for any statistical procedure to be robust against possible uncertainties on its input assumptions. One of Neal's favourite methods uses Bayesian Neural Nets. He also described graphical methods for showing which of the input variables were most useful in providing the separation of signal and background.

Ilya Narsky gave a survey of the various packages that existed for performing signal-background separation. These included R, WEKA, MATLAB, SAS, S+ and his own StatPatternRecognition. Narsky suggested that the criteria for judging the usefulness of such packages should include their versatility, ease of implementation, documentation, speed, size and graphics capabilities. Berkeley Statistician Nicolai Meinshausen gave a useful demonstration of the statistical possiblities within R.

The general discussion in this sub-group covered topics such as the identification of variables that were not too useful, and whether to remove them by hand or in the programme; the optimal approach when there are several different sources of background; the treatment of categorical variables; and how to compare the different techniques. This last issue was addressed by a small group of participants working one evening using several different classifiers on a common simulated data set. Clearly there was not the time to optimise the adjustable parameters for each classification method, but it was illuminating to see how quickly it was possible to be able to use a new approach, and also to produce comparative performance figures. The results were presented by Reinhart Schweinhorst.

Conclusion

As far as the Workshop as a whole was concerned, it was widely agreed that it was extremely useful having Statisticians present to discuss new techniques, to explain old ones, and to point out where improvements could be made in analyses. It was noted, however, that while Astrophysics has been successful in involving statisticians in their analyses to the extent where their names appear on experimental papers, this is usually not the case in particle physics. Several reasons have been put forward to explain this. One is that statisticians like analysing real data, with all its interesting problems. But particle physics experimental collaborations tend to be very jealous about their data, and are unwilling to share it with anyone outside the collaboration until it is too old to be interesting. This results in particle physicists asking statisticians only very general questions, which the statisticians regard as unchallenging and boring. If we really do want better help from statisticans, we have to be prepared to be far more generous in what we are ready to share with them. A second issue might be that in other fields scientists are prepared to provide financial support to a statistics post-doc, to devote his/her time and special skills to helping with the analysis of the data. In particle physics this is at present very unusual.

There was unanimous agreement among those there that the Banff meeting had been both stimulating and useful. The inspiring location and environment undoubtedly contributed to the dynamic interaction of participants. Not only were the sessions the scene of vigorous and enlightening discussion, but the work continued late into the evenings, with many participants learning new techniques, which they would be taking back with them to their analyses. There was real progress in understanding practical issues involved in the three topics discussed, and everyone agreed that it would be very useful and enjoyable to return to Banff for another Workshop in the future.

Appendix: Summaries of individual talks

Plenary Introductory Talks

Louis Lyons: Brief Introduction to Particle Physics and typical statistical Analyses

In this talk I gave an overview of the workshop organization and goals, and a brief introduction for statisticians of some of the main HEP experiments of interest, including the Tevatron, LHC and K2K experiments. I described three typical analyses; the first emphasizing estimation of unknown parameters, the second looking for an interesting signal, which of course must include a discussion of how to separate ‘real peaks’ from statistical fluctuations, and the third concerning the use of event variables and training data to determine how to separate background events from events of interest.

Jim Linnemann: Monte Carlo Experiments in High Energy Physics

My talk was an introduction to the use of Monte Carlo by particle physicists and a couple of other pieces of terminology. Monte Carlo simulations are used by particle physicists to estimate backgrounds and to calculate efficiencies for proposed new physics processes. The simulations consist of event generators with specific input physics and detector simulators describing how particles are observed in the apparatus.

The latter are often particularly slow, up to a minute per event. Both event generators and detector simulators have settings whose uncertainties generate systematic errors in the simulation results. I explained that cross sections are proportional to interaction probabilities, gave simple examples of kinematic quantities we cut on, and gave examples of cuts. Cuts are typically used to reduce data samples to a more manageable size, or remove regions which are difficult to simulate so as to reduce systematic errors. Physicists also use Monte Carlo methods for measuring performance of statistical methods, as do statisticians. Finally, the site phystat.org is now available for contributions to its code repository, and will link to the Banff workshop permanent web site.

Byron Roe: Setting the scene for multivariate signal/background separation

A number of modern multivariate classification are briefly described, with some emphasis on boosted decision trees and related methods. It is difficult to make comparisons in general. For limited tests with MiniBooNE Monte Carlo data, boosted decision trees performed as well or better than an other method tested. It is noted that some hundreds of feature variables can be handled by the methods. Methods of reducing the number of variables are described. The use of simulations to estimate systematic errors varying parameters one at a time, or all together, is briefly noted. For some experiments, it is possible to estimate systematic errors while doing a fit for physics parameters. However, this method can have a problem if there are more systematic errors than data bins. A problem is noted in evaluating errors when doing log-likelihood fits in a region in which the usual analogy to chi-squared cannot be used.

Radford Neal: Statistian's view of the above

My talk tried to focus in on a series of questions, with some tentative thoughts on potential answers. First, it is helpful to be as specific as possible about the questions:

- What is the problem?
- What data is available?
- How can the data answer the questions?

Since Monte Carlo simulations play a large role, we need to consider several important issues related to this, including:

- How do we do inference with this?
- What form does the result of this take?
- How do we create PID variables?
- How can we detect and handle flaws in the models?
- How to run the MC simulation?

My view is that inference would normally be based on the likelihood function, and that a plot of the likelihood function is a very useful summary. I showed how to convert the original likelihood for a classification problem into something that depends only on the properties of the classifier, not on extraneous parameters. This explains why multivariate classifiers seem to be the right thing to do.

Now can we multiply these together to form an likelihood for N events. Is this wise? Possibly not, because our trained classifier is not perfect. A robustness problem may enter at this point. It is also possible that this is less robust to problems in the original formulation. This motivates a kind of thresholding; which leads then to the simplified version of the problem of Poisson background and Poisson signal events.

Note that although this is not a statistical classification problemstatistically motivated classifiers may be very useful here; boosting is an example of this.

Joel Heinrich Setting the scene for limits and nuisance parameters

In this talk I sketched out what is needed in order to compare various proposed methods for computing limits, described some proposed methods for computing limits, and set out some parameter values that seem reasonable for initiating a systematic comparison. During the workshop this developed into a project for a definitive comparison of limits that will continue through the fall of 2006.

Luc Demortier: Setting the scene for p-values, including nuisance parameters

This talk gave an overview of the methods proposed for accommodating nuisance parameters the calculation of p-values. An introduction to the statistical theory of p-values was provided, and a summarization of their properties and their role as a measure of evidence. A large number of methods are available in the literature for incorporating the effects of nuisance parameters, and these were reviewed.

David van Dyk: Statistician's view of the above

I outlined the definition and interpretation of confidence intervals, and gave illustrations where the summary of the experiment by a confidence limit was not very informative. In many cases it is preferable to plot the likelihood function or the posterior distribution. I described in some detail my work with colleagues in high-energy astro-statistics.

Xiaoli Meng: Dealing with Nuisances: Principled and Ad Hoc Methods

My talk had three parts, all on dealing with nuisance parameters in hypothesis testing, particularly in testing the existence of emission lines in high energy astrophysics. The first part was a quick review of the posterior predictive approach, and the second part about how to create a useful pivotal quantity by introducing a ”working” alternative model. The third part was on the idea of using moment methods, instead of maximization, to construct an approximation to profiled likelihood, a method that could be potentially useful when the usual approach of maximizing a likelihood provides unstable results.

Parallel Session Specialized Talks

Limits and Significance

Roger Barlow: Significance and Likelihood ratio and confidence limits

I discussed whether Delta chi squared could be used as a measure of significance when comparing models with data, specifically for adding bumps to histograms. The conclusion is that you can’t. Luc talked about this too the next day. Turns out there are papers in Biometrika that made all this clear long ago. I also asked whether my procedure to rank multichannel results for p-value purposes was sensible. There was no direct response, but the multichannel part of Joel’s challenge will show the answer.

Anthony Davison: p-value functions

I was asked to talk briefly about significance functions. The starting-point is the observation that if Y is a continuous scalar random variable whose distribution function $F(y;\theta)$ depends upon a scalar parameter θ, then $U = F(Y;\theta)$ has the $U(0,1)$ distribution and is therefore a pivot: a function that depends on both data and parameter and whose distribution is known and does not depend on the parameter value. Confidence limits for θ based on an observed value y of Y may therefore be read off as the solutions in θ to the equations $F(y;\theta) = \alpha, 1-\alpha$; the resulting $1-2\alpha$ confidence interval has limits (θ_-, θ_+). There are obvious changes for upper and lower $1-\alpha$ intervals.

In practice we need to deal with three issues: we must replace Y with some general function of a data set; we must deal with nuisance parameters; and the data may be discrete. I discuss these in turn.

When Y represents a set of continuous data with log likelihood $\ell(\theta)$ and maximum likelihood estimator $\hat\theta$, then under mild regularity conditions on the underlying distribution we find that the likelihood root $r(\theta) = sign(\hat\theta-\theta)\left[2\left\{\ell(\hat\theta)-\ell(\theta)\right\}\right]^{1/2}$ has an approximate standard normal distribution, at least to first order; this means that if θ is the true parameter value, then

$$\{r(\theta) \le z\} = \Phi(z) + O(n^{-1/2}), \quad z \in Reals,$$

where n is an index of sample size, and Φ is the $N(0,1)$ distribution function. This implies that $r(\theta)$ is an approximate pivot, and that $\Phi\{r(\theta)\}$ may be treated as a significance function from which confidence limits for θ may be obtained as solutions to $\Phi\, r(\theta) = \alpha, 1-\alpha$, as above. The resulting two-sided confidence interval typically has error of order $1/n$, while the one-sided intervals have error of order $1/\sqrt{n}$; an asymmetry term cancels from the expansions when the two-sided interval is used. Typically such intervals have better properties than those based on the score or Wald statistics. A third-order correct interval is obtained by replacing $r(\theta)$ in the above discussion with the modified likelihood root

$$r^*(\theta) = r(\theta) + r(\theta)^{-1}\log\left\{\frac{q(\theta)}{r(\theta)}\right\},$$

where $q(\theta)$ depends on the problem; often it is either a score or a Wald statistic, but a fairly simple general form is available from work by Fraser and Reid. In this case the error for one-sided intervals drops to $O(n^{-3/2})$, and in many cases where exact computations are available or where simulations have been performed the error seems in fact to be numerically negligible.

When $\theta = (\psi, \lambda)$, where ψ is a scalar interest parameter and λ a possibly vector nuisance parameter, the log likelihood in the computation of the likelihood root is replaced by the profile log likelihood

$$\ell_{\mathrm{p}}(\psi) = \max_\lambda \ell(\psi, \lambda),$$

giving

$$r(\psi) = sign(\hat{\psi} - \psi)\left[2\left\{\ell_{\mathrm{p}}(\hat{\psi}) - \ell_{\mathrm{p}}(\psi)\right\}\right]^{1/2},$$

where $\hat{\psi}$ is the overall maximum likelihood estimator of ψ. Likewise $q(\theta)$ is replaced by a similar quantity $q(\psi)$ readily computed in many cases. Confidence intervals based on the resulting modified likelihood root $r^*(\psi)$ have again been found to be extremely close to exact ones, where these are available.

If the underlying data are discrete, then the discussion above applies with small modifications: Davison, Fraser, and Reid (2006, Journal of the Royal Statistical Society, series B) show that the error committed in taking the appropriate $r^*(\psi)$ will be of order $1/n$ rather than the $1/n^{3/2}$ seen in the continuous case, but that the numerical error is typically very small.

The overall implication is thus that excellent frequentist confidence limits can be obtained by using $\Phi\{r^*(\psi)\}$ as a significance function in both continuous and discrete cases. It turns out also that minor modifications produce also Bayesian confidence intervals. More details and numerous examples are given in Brazzale, Davison and Reid (2006, Applied Asymptotics: Case Studies in Small-Sample Statistics, to be published by Cambridge University Press). Other accounts of this theory can be found in the books by Barndorff-Nielsen and Cox (1994, Inference and Asymptotics, Chapman & Hall), and Severini (2000, Likelihood Methods in Statistics, Oxford University Press).

Joel Heinrich: Stopping times and likelihood, a query

In certain situations, a physicist may be induced by observing events to publish right away, rather than wait for a pre-specified time. This is a definite change in procedure, the physicist's behavior is different than in the usual case. This leads to an altered frequentist ensemble, where the observed quantity is the waiting time t to the nth event, where n is a pre-specified constant. The parameter of interest s then becomes a scale parameter for the continuous (gamma) distribution of t. This is an easier problem from the frequentist perspective, and (in the b=0 case) there is an exact probability matching prior 1/s which yields perfect agreement between frequentist coverage and Bayesian credibility. From the subjective Bayesian point of view, however, the likelihood remains the same despite the change in the stopping rule, so nothing needs to be modified. More work will be done to fully explore the implications.

Tom Junk: Hypothesis Testing in HEP with Uncertain Nuisance Parameters, and an observation on Odd p-value Behavior

The multi-channel problem of testing for the presence or absence of a new particle is discussed. Typically searches produce histograms of data passing some optimized selection requirements, arranged in bins of some variable for which the signal to background ratio varies from bin to bin of the histogram. Sophisticated discriminant variables are often used, and there are many sources of uncertainty in the rates and shapes of these histograms. Often the bins of the histogram where little signal is expected serve as a calibration of one or more uncertain backgrounds. If the signal distribution shape is similar enough to the background shape, the effectiveness of the technique of fitting the sidebands weakens. Often the predictions in each bin of the signal and the background suffer from limited Monte Carlo samples, which introduces another source of uncertainty and a set of nuisance parameters in each bin.

One can address the problem very similarly to the approach Kyle Cranmer proposed at PHYSTAT03 – to maximize the likelihood separately with respect to the nuisance parameters for the null hypothesis and the test hypothesis, and then apply known techniques for computing limits or confidence belts. I have a bias towards testing two hypotheses at a time and not many more, since the acceptance or exclusion of a test hypothesis should not depend on other test hypotheses considered or not considered.

One pitfall to avoid in doing a Bayesian limit with marginalization over the unknown nuisance parameters is double-use of the data sidebands to constrain backgrounds. The integration over nuisance parameters of the likelihood times the prior is effectively fitting the background shape to the data, as only those values of the background normalization nuisance parameters which best reproduce the data will be represented with significant weight in the integral. To use the data sidebands to construct the prior for the background (a Gaussian constraint), and then to use the same data in the likelihood function would double-count the effect of this data on the background uncertainty.

Software is available at

www.hep.uiuc.edu/home/trj/cdfstats/mclimit_csm1/index.html

It calculates both Bayesian upper limits and CLs ones. P-values are also computed for comparing the data to the null and test hypotheses.

An odd feature of p-values in low-statistics single-channel analyses is a manifestation of the discontinuous coverage curves. Over coverage is unavoidable, and particularly noticeable for channels with few expected events. If a second, much weaker channel is combined with a strong channel with little expected background (say s1=3, b1=1, s2=0.1, b2=2), then the observed limit can jump sharply when the second channel is added. Part of the over coverage of the single-channel case is now recovered by dividing the probability of each strong-channel's outcome into sub-outcomes indexed by the weak channel's outcome. If a continuous spectrum of signal and background can be constructed instead of a single counting experiment, then the distributions of the expected outcomes will not only be more optimal because of the extraction of more information from the data, but also will suffer less from the Poisson over coverage problem.

Toby Burnett: Detection of gamma rays ”Finding point sources in the gamma-ray sky”

The main point here is that it is conventional in astrophysics to quote source discoveries with a ”5 sigma” threshold, but without an analysis to demonstrate that the probability of a false positive is really at the level of a 5-sigma Gaussian. This is probably evident in the number of unidentified sources ”found” by EGRET, and certainly demonstrated by the recent GLAST data challenge.

I see it as a clear message to those performing such analyses that the null hypothesis does not depend on the position, so that the p-value must be determined empirically.

James Bueno: Bayesian upper limits I described the software I have written in Root C++ to calculate Bayesian upper limits for the Poisson problem with a choice of informative priors for background and efficiency via numerical Gaussian quadrature, and illustrated it on some examples.

Luc Demortier: Reference analysis I described the use of reference posterior distributions as a means of making inference about a parameter of interest in the presence of nuisance parameters and summarized their properties.

Eric Marchand: On the behaviour of Bayesian credible intervals for some restricted parameter space problems. This is recent work with Bill Strawderman. For estimating a positive normal mean, Zhang and Woodroofe (2003) as well as Roe and Woodroofe (2000) investigated HPD credible sets associated with priors obtained as the truncation of noninformative priors onto the restricted parameter space. They established the attractive lower bound of $(1-\alpha)/(1+\alpha)$ for the frequentist coverage probability of these procedures. W. Strawderman and I established that their lower bound is applicable for a substantially more general setting with underlying distributional symmetry. We showed that the lower bound still applies for certain types of asymmetry (or skewness), and we extended results obtained by Zhang and Woodroofe (2002) for estimating the scale parameter of a Fisher distribution. There is a wide scope of applications, including estimating parameters in location models and location-scale models, estimating scale parameters in scale models, estimating linear combinations of location parameters such as differences, estimating ratios of scale parameters, and problems with non-independent observations.

Gunther Zech: Likelihood vs coverage Coverage intervals versus Likelihood ratio intervals An example was constructed such that a coverage interval and a likelihood interval disagree by a large extent. It was demonstrated that the likelihood interval is at least intuitively much more attractive. The reason for this behavior is the fact that coverage intervals accept all parameter values which are compatible with the measurement whereas likelihood ratio intervals take into account the fact that only one and not several parameters can be true. As a consequence of this caveat of the coverage intervals, which is related to a violation of the likelihood principle, one should not require coverage for likelihood ratio or Bayesian intervals while coverage intervals which exclude relatively high likelihood ratios are very problematic.

Giovanni Punzi: Frequentist limits

I briefly described a fully frequentist method for incorporating nuisance parameters (systematic uncertainties) in the evaluation of confidence intervals, by means of a direct Neyman construction in multi-dimensional space. Thanks to an appropriate choice of ordering algorithm, results were obtained with good general properties: strict coverage, small overcoverage, and continuous behavior in the limit of small systematic uncertainty. The algorithm allows for both 1-sided and 2-sided limits (F-C or central) to be obtained, and puts no requirements on the distribution of the subsidiary measurements (related to the systematics), which may even be unavailable. Some results and comparisons for the benchmark problem of group A were presented.

Conrad, Jan and Cranmer, Kyle: Profile likelihood for marked Poisson processes

Bodhisattva Sen: Confidence intervals with nuisance parameters We discussed the Hybrid Resampling Method for dealing with nuisance parameters. We also proposed an extension of the Feldman and Cousins Unified method to include nuisance parameters. The Expectation-Maximization (EM) algorithm in relation to the signal plus noise model with marks (auxiliary variable associated with each event) is proposed. We also sketch a possible Bayesian hypothesis testing procedure for testing discovery in this scenario.

Multivariate

Stephen Bailey: Signal/background separation of supernova events in Supernova Factory images The Supernova Factory is developing a Support Vector Machine (SVM) approach to replace its current cut-based approach. The SVM works considerably better than cuts at 25% signal efficiency but performs about the same at 50% signal efficiency. The primary difficulty is a time varying background; participants provided several useful suggestions for training classifiers under under such conditions by including background parameters in the classification variables. At the suggestion of the participants, the Supernova Factory will also try Random Forest and Boosted Decision Tree classifiers which might work better with the noise and outliers in the dataset.

Radford Neal: Bayesian neural nets + robust classifiers

I described a neural network analysis of Byron Roe's data, which has 50 PID variables, to be used to classify events into one of two types. The NN has two hidden layers which connections from all inputs to all hidden units. Each connection has a weight attached to it, a 'bias' constant, and a tanh activation function. The output for each hidden unit is

$$tanh(b + \Sigma_i w_i V_i)$$

and this is converted at the end to a binary classifier by the logistic function.

This can be parameterized, by treating the weights and biases as parameters. It can be shown that even with just one hidden layer any function can be approximated with enough units.

In principle the parameters could be estimated by maximum likelihood, but this is not usually a good idea, since the data can be fitted exactly with enough units. NonBayesian methods incorporate either regularization or a naive method called 'early stopping', or an ensemble version of that analogous to cross-validation. This version of early stopping actually does quite well.

It turns out that he number of hidden units is not so crucial; early stopping basically corrects for this.

A Bayesian version works better, by integrating over the space of weights in the network, with a combination of prior and likelihood. This integration is carried out by MCMC. The Metropolis algorithm is very slow, but a hybrid MC version works well. The result is a probability for classification, but this is done on a large number of networks, and averaged (at the last step). The Bayesian version can incorporate a hyper-prior for the parameters on some of the parameters in the network. This allows some improved classification if the data turns out to be very predictable.

You can show that there is no statistical necessity to constrain the number of hidden units. However the computations do get slow as the number gets too large.

The next step is adding some boosting to this, to concentrate on the items that are hard to classify, but of course apply larger weights to the easy-to-classify items so that the output remains unbiased. This hopefully will reduce the computation time.

Toby Burnett: Decision trees

I made two points:

- Classification analysis can be used, not only to distinguish rather different entities, but to improve resolution. GLAST uses classification trees to help characterize gammas that have well-measured energy and direction.
- It is productive to study misclassified background events, in order to discover new variables that can be used in a new classification. Many of the GLAST variables had such an origin.

Ilya Narsky: Multivariate classification Software for multivariate classification

Various software packages for multivariate classification have been used in HEP and elsewhere.

R, a popular tool among statisticians, implements many methods for classification and exploratory analysis of data. It is easy to install and use, has built-in graphics for display of data, and is generally well documented. However, R tends to be slow, especially on large data sets. R is a high-level interpreter suited for interactive analysis but hardly a reasonable choice for analyzing large amounts of data through batch jobs. WEKA, an object-oriented Java package, and MATLAB extensions implement many classification methods and are used by researchers in academia. SAS and S-plus offer software suites for industry but are less likely to be used in academia due to the high cost of their licenses.

HEP researchers have been using various implementations of neural networks for the last two decades. For example, Stuttgart Neural Network Simulator (SNNS) and JETNET have become popular at BaBar. Plenty of other implementations are available.

Several packages have been developed recently within the HEP community to implement advanced classifiers such as boosted decision trees, random forest and others. These include Byron Roe's m-boost, Narsky's StatPatternRecognition and TMVA (now available in Root). StatPatternRecognition, for instance, implements decision trees, boosting, arc-x4, random forest, a bump hunting algorithm (PRIM), linear and quadratic discriminant analysis, and interfaces to two SNNS networks.

At present, HEP analysts can choose software from a great number of available packages. Unfortunately, a HEP researcher working on specifics of a physics analysis, typically a grad student or a post-doc, often knows little about multivariate classification in general and even less about available software to make an informed choice. It would be very useful for the community to survey existing software packages and publish results of this survey online or in any other form available to HEP researchers. The proposed categories for software comparison are listed here: " versatility and the scope of implemented methods " ease of installation and use " quality of manuals and documentation " CPU speed and memory consumption; how these quantities scale versus data size and dimensionality, both for the training cycle and for post-training classification; also maximal sample size and dimensionality that can be handled by the package " types of inputs that can be handled by the package (real, integer, categorical, mixed etc) " quality and convenience of the graphics interface, both for input and output " suitability for interactive analysis and/or batch jobs " ease of integration in the C++ framework Although it would be interesting to compare the predictive power of various implementations of the same method, this task would require a non-trivial amount of manpower. Because implementations of the same method vary among packages, such a comparison would not be possible without careful adjustment of input parameters for each implementation of the classifier, which is a time-consuming effort.

The proposed comparison would be a nice project for one or two undergrads or graduate students and a useful service to the community.

Blobel, Volker: Systematics for goodness of fit Dealing with systematics for chi-square and for log-likelihood goodness of fit

Systematic errors are at the origin of the unsatisfactory situation when data from many experiments are used in a global analysis and parameter estimation, and when attempts are made to determine uncertainties of predictions from parton distributions. Often the profile chi-square for single parameters and functions of parameters appear to be much to narrow.

The different error contributions in HEP experiments and the methods, to incorporate these error contributions in the log-likelihood expression are discussed. Often this is not done in an optimal way.

The normalization factor for all data of a single experiment is a product of many factors and its distribution can often be described by a log-normal distribution. In the log-likelihood expression the factor should be applied to the theoretical expectation, not to the experimental value, to avoid a bias.

Recent experiments publish a lot of data about the contributions to the overall covariance matrix from various systematic effects, and this information has to be used in parameter estimation. The contribution to the covariance matrix describing the statistical errors is usually given as a diagonal matrix. However due to finite experimental resolution. corrections for the bin-to-bin fluctuations have to be applied; correlations between data points are often neglected and the given statistical errors are often too optimistic.

Nicolai Meinshausen: Using R for classification problems

The R-Project web page (www.r-project.org) has the code, all the various packages, a set of manuals, and so on. An advantage of R is that it is easy to play around with the data, fairly quickly, and there are packages written for many many statistical routines. A disadvantage is that it is quite as 'portable' as C++, and it is maybe not so easy to translate R fitted objects to another language. Also it can be slow as it is an interpreted program.

Variable Selection: With approximately 200 variables, one suggestion is that they all be kept in the classifier, and only deleted if they demonstrably don't have any impact on the classifier. Of course the variables in this category may be different between classifiers, but in general the set of really important ones "should" be consistent from one classifier to the next. Nicolai showed some plots available from random forests that identify variable importance according to a few measures related to classification.

Raoul LePage: Comments on Classification

I described how multivariate Gaussian based confidence regions may be obtained directly from scaled bootstrap plots with sufficient blocking to improve normality. This will apply to plots of a kind often seen in talks at this meeting, although not to all of them. The other part of my talk outlined a possible approach to the problem of developing useful classifiers for signal vs noise. The idea is to exploit the capabilities of modern methods of solving linear inverse problems by using them to directly obtain a density over the events ”e” space that is interpreted at p(signal — e) for a particular choice of the probability p of signal (i.e. blending signal and noise events in the proportions p, 1-p). The idea also turns on telling the solver that integrals of some specified functions X(i) ¿ 0 (these are used by the inverse method to build the density) are identically one. Such functions can be chosen from any convenient class thought to be capable of building a good density for the purpose and would be normalized according to their integrals on the training set. In the present context favorite choices might include decision trees with specified parameters, logistic models, or any combination of these.

Neal, Radford: Handling Systematic Errors in Simulations

Reinhard Schwienhorst: Multivariate classification Comparisons

I gave a talk summarizing the work of several people, comparing different classifiers for the data sets that had been sent out before or during the meeting. We used the statistical data analysis package R and software written by Ilya Narsky (one of the workshop attendees) and data sets from MiniBoone, Glast, Babar, and the D0 single top quark search. We found that Bayesian Neural networks, boosted decision trees and random forests did about as well as classifiers tuned within each experiment, even when using these classifiers out-of-the-box, without any special tuning. We also learned that installing and running R is straightforward and results can be obtained in a few hours, at least with the help of an expert

Summary of discussions

Limits with Nuisance Parameters

convenors David van Dyk (statistics) Joel Heinrich (intervals),

After listing the main methods that have been proposed to solve the upper limits problem, an attempt was made to collectively construct a matrix that listed their properties. This resulted in considerable discussion, and the addition of a few more methods. After the break, it was clear that the matrix was only reasonably complete in the column marked coverage, and that only for a single channel. As all the methods had reasonable coverage properties, more information was needed for a prospective user to decide which to use.

A collaborative project to supply the necessary information about each method was therefore initiated. Proponents of each method agreed to supply their resulting intervals for a common set of test cases. This would permit direct comparison of frequentist coverage, interval lengths, and Bayesian credibility. It was decided that 1 channel and 10 channel cases would be investigated. Those doing the work would have until the end of October to complete the task. Joel Heinrich agreed to generate the test cases and process the results.

The statisticians proposed adding a hierarchical Bayesian method to the list, and described in detail how this worked. This was accepted, after some discussion—physicists had been reluctant in the past to employ this strategy, but were persuaded by the statisticians that it would be worth trying.

Significance

convenors David van Dyk, Luc Demortier

Luc to provide based on notes xxxxxxxxxxxxxxxxxxxxx

Multivariate Problems

convenors David van Dyk, Joel Heinrich (intervals), Luc Demortier (p-values) Group C: Nancy Reid and Byron Roe

The following list of questions was compiled, and to some extent discussed, in the classification group:

1. go over classification methods: what can be said about applicability in different situations
2. Models not perfect; what can we say about robustness or other flaws in the model.
3. How do we get an intuitive feeling for the classification methods? Graphical methods?
4. Methods for variable selection? How to find the 'best' set of variables? Why do we need to reduce the number of variables?
5. How many data-sets have been looked at by different people.
6. Unisims, multisims: estimating systematic uncertainties.
7. Uniform framework? (Should we all be re-inventing the wheel?)
8. Question: is subjectivity adding to the quality of the analysis or not?
9. Can we learn about statisticians' methodology: Raoul, Nicolai

A graphical display was suggested by Radford Neal: Plot $\Delta\ log(p/(1-p))$ which is the change in the classifier output when variable 27 is changed by ϵ in the training case, vs variable 27. If the plot is straight, variable 27 affects the logit linearly; if the plot is curved then the variable isn't linear, but it is additive (no interaction). If the plot is scattered, then there are interactions going on with the other variables. (The computation of Δ"$prediction$" adjusts for the presence of the other 49 variables.)

Note that one plot needed for each PID, and that each plot has as many points as there are events in the training sample. There could be an 'ensemble' of such plots, one for each of the networks that go into the bagging, OR, it could be based on the averaged or bagged prediction.

Jim Linneman had the following thoughts

Variable preparation: If you have multiple backgrounds, you could consider training either separate classifiers for each; or training a multiple class classifier (even though all background classes would eventually be lumped into "non-signal"). Might be easier to diagnose that way.

You could consider removing from the training on a particularly resistant background which is truly nearly indistinguishable from your signal, rather than confusing the classifier by calling very similar events background in one case and signal in another. But as above, a multi-classifier might also address that issue differently.

For trees, no need to modify variables since they are invariant under uniform monotone transformations (as are cuts).

For nets, they like to have mean zero std deviation = 1 if you have no better idea. If you do so, then for the Bayes nets, you can use the same hyper-priors (the weight scales start out the same). If you have widely varying variables, say ones covering a wide range with a wide range of frequency, you might for example log-transform them. Ideally, if you have real reason to know some variable should be a good separator, you would ideally want it such that one unit of input change would correspond to one unit of output change, so an important variable might be given a larger initial standard deviation than one; that way similar weights would start out be causing bigger responses to this variable. But: such selection is not always obvious: a variable which by itself (1-d) shows little separation between signal and background can nonetheless be important in classification if it interacts strongly with other variables.

Terminology: physicists confuse the term "correlated or non-independent variables" with "interacting variables": I believe the issue is that correlation or independence would be a property of simply the signal or background distributions individually, while interaction has to do with the pair of variable's needing to be considered together to classify (predict a third variable, such as class membership); so it would be more a property of the ratio of the signal and background pdf's]

Non-ordered categorical variables. Say you have inputs which fall naturally in 3 classes, but which don't have any inherent ordering. For example, you have observations with 3 different kind of electronics, but don't really want to claim they like in a "good better best" relationship. It's better to classify them into the 3 groups with similar characteristics if you can, than asking the classifier to figure it out from, say, the 100 different serial numbers. And the good way to encode those is to have 3 categorical input variables each with 0 or 1 values, so that only one of them is on for a given input. On the other hand, if there is a natural ordering relationship like ”good better best” it may not be so bad to encode them as 1, 2, or 3 (though the response might be nonlinear–little difference between good and better, but big improvement for best). Or if you really have a quantitative number, it’s better to use instantaneous beam intensity as a continuous variable than classifying them as low, medium, and high intensity.

List of Participants

Bailey, Stephen (Lawrence Berkeley National Lab)
Barlow, Roger (Manchester University)
Blobel, Volker (DESY Lab Hamburg)
Bueno, James (University of British Columbia)
Burnett, Toby (University of Washington)
Conrad, Jan (Royal. Inst. Technology (KTH))
Cranmer, Kyle (Brookhaven Nat Lab)
Davison, Anthony (Ecole Polytechnique Fdrale de Lausanne (EPFL))
Demortier, Luc (Rockefeller University)
Fraser, Don (University of Toronto)
Heinrich, Joel (University of Pennsylvania)
Jin, Zi (University of Toronto)
Junk, Tom (University of Illinois Urbana-Champaign)
LePage, Raoul (Michigan State University)
Linnemann, James (Michigan State University)
Lockhart, Richard (Simon Fraser University)
Lyons, Louis (University of Oxford)
Marchand, Eric (University of Sherbrooke)
Meinshausen, Nicolai (University of California, Berkeley)
Meng, Xiao-Li (Harvard University)
Narsky, Ilya (California Institute of Technology)
Neal, Radford (University of Toronto)
Punzi, Giovanni (University of Pisa)
Reid, Nancy (University of Toronto)
Roe, Byron (University of Michigan)
Rolke, Wolfgang (University of Puerto Rico)
Sartori, Nicola (University of Venice)
Schwienhorst, Reinhard (Michigan State University)
Sen, Bodhisattva (University of Michigan)
Siemiginowska, Aneta (Harvard-Smithsonian Center for Astrophysics)
Vachon, Brigitte (McGill University)
Van Dyk, David (University of California, Irvine)
Zech, Gunter (University of Siegen)

Chapter 15

Measurable Dynamics, Theory and Applications (06w5079)

Aug 05 - Aug 10, 2006

Organizer(s): Chris Bose (University of Victoria), Pawel Gora (Concordia University), Brian Hunt (University of Maryland), Anthony Quas (University of Victoria)

Brief overview of the field

The central aim of measurable dynamics is to apply modern mathematical techniques, including measure and probability theory, topology and functional analysis to study the time-evolution of complex evolving systems.

The fact that many simple models in the natural sciences may lead to classically intractable mathematical problems was already observed in the 19th century by H. Poincaré during his investigations into the orbits of celestial bodies. At about the same time, the formal development of thermodynamic theory alerted scientists to a major shift in the mathematical modelling paradigm that was about to take place. Since then, researchers have coined terms like *chaos* and *strange attractor* to describe the perplexing properties observed by Poincaré and others, and we now know that these systems, rather than being isolated curiosities are, in fact, increasingly likely to be encountered once one leaves the familiar territory of standard mathematical models derived from classical Physics, Chemistry or Engineering.

While the origins of the field are rooted in application, the mathematical development in the next century embraced both theoretical and applied approaches. In fact, for the first half of the 20th century, it is fair to say the former dominated as mathematicians struggled to develop new tools to describe the complex systems they were encountering. The celebrated ergodic theorems of Birkhoff and von Neumann, the development of a complete theory of measurable entropy (Rohlin, Kolmogorov, Shannon etc.) and a rudimentary structure theory for such systems (Halmos, von Neumann Hopf, and others) are all examples of powerful theoretical developments on which countless modern applications are built. In some sense the first modern ’application’ of measurable dynamics was its role in formalizing the theory of stochastic processes in the first few decades of this century (the work of Kolmogorov, Khinchine and Doob for example).

All of this theoretical development took a sharp turn with the appearance of computing machinery, whereby, one of the most intractable parts of the dynamical model – the repeated, *infinite* iteration of a single mapping applied to a point to produce an orbit, became one of it’s most accessible features. In the 60’s and 70’s there was an explosion of experimental mathematics focused on the use of computers to study dynamical systems. Fractals and other fractional dimensional objects, Julia sets and associated objects, strange attractors and numerous other examples poured into both the scientific and popular literature as the idea of a dynamical systems approach took hold. Slowly it was becoming clear that the exotic behaviour encountered in theoretical studies could be reproduced in extremely simple systems on the computer – the challenge (and opportunity) this presented to theoretical researchers was irresistible and the new phase of theory/application in dynamics took hold.

During the next few decades, modern theoretical results due to Ornstein, Ratner, Bowen, Sinai, Furstenberg and Weiss to name just a few, were finding application in both pure mathematics (differential geometry, number theory, group theory and thermodynamics, for example) and in applied mathematics, (ODE, Kinetic theory, Billiards and other hard-sphere dynamics, population dynamics, mechanical models, finance and so-on) simultaneously.

This balance has continued into the current decade. It is hoped that even a cursory review of the presentations outlined in the next section will make this clear; in particular, that modern dynamical systems in general, and measurable dynamics in particular continues to be a productive mix of theoretical efforts linked with exciting applications both in mathematics and other sciences.

This in large part underlay the motivation for our choice to try to balance the participants between pure and applied mathematicians working in the field. It is apparent that there is a considerable spectrum in terms of paradigm and outlook amongst researchers in the field. We believe the meeting was highly successful and we look forward to having a chance to attend or organize another one soon.

Talks given during the workshop

We give a brief synopsis of the talks given at the workshop, in order of presentation. Additional information is contained in the speaker's abstract and/or through the cited web links.

James Yorke (Maryland) gave an entertaining presentation on the dynamics of a 'Taffy-pulling Machine' – a mechanical device with two overlapping arms which is used to stretch and fold a batch of taffy (candy). A mathematical model of this machine produces an interesting diffeomorphism of an open subset of the plane that contains a Plykin-like attractor. Various studies were presented to support the statement, and a 1-dimensional reduction of the model was described. This is joint work with J. Halbert. The talk was video-recorded and appears in the publications directory of the BIRS website. More details are available at
`http://www-chaos.umd.edu/~yorke/`

Oliver Jenkinson (Queen Mary, University of London) described an interesting and natural partial order on the set of (Borel) invariant measures $\mathcal{M}$ for the doubling map of the circle, $x \mapsto 2x \bmod 1$ (equivalently, for the 2-shift). Roughly speaking, a measure $\nu \succ \mu$ if ν is more spread out on $[0, 1]$; the precise definition may be found in [5]. The order is related to an ergodic optimization problem: for a given function f, find $\mu \in \mathcal{M}$ which maximizes $\int f d\mu$. An intriguing connection to classical *Sturmian* measures was noted: Sturmians are the unique maximizing measures for $f =$ characteristic function of a semicircle. Also, amongst periodic measures, Sturmians have the property that they are the only ones combinatorially conjugate to a rotation (either rational or irrational) and hence, not all periodic measures are Sturmian. Oliver's website is at
`http://www.maths.qmul.ac.uk/~omj/`

Erik Bollt (Clarkson University) investigated the notion of 'almost-conjugate' in the category of one-dimensional maps of the interval. Given two maps T and S, using a fixed point iteration scheme it is possible to construct an map f (which he calls a commutator) such that $f \circ T = S \circ f$ if no constraints such as surjectivity or continuity are enforced. Defects in the commutator, such as lack of injectivity, surjectivity, or continuity are used to give a measure of how different the two maps T and S are. Examples were presented that show how these measures can, in some simple cases, better match the heuristic notion of 'similar' than traditional approaches. This is joint work with J. Skufca. Erik's website is at
`http://people.clarkson.edu/~bolltem/`

Gerhard Keller (Erlangen-Nürnberg) The acronym GOPY is applied to a set of non-chaotic strange attractor examples due to Grebogi/Ott/Pelikan/Yorke from the mid-1980's. While not chaotic in the normal sense of the term, they necessarily exhibit chaotic-like behaviour and, in particular, have complex attractors and sensitive dependence to initial conditions. Many interesting questions remain open about these systems in general – the speaker gave a sample analysis of the attractor for a model problem developed by Grebogi *et al*. This is joint work with Glendinning and Jäger. The talk was video-recorded and appears in the publications directory of the BIRS website. More details are available at
`http://www.mi.uni-erlangen.de/~keller/english_index.html`

Judy Kennedy (University of Delaware) presented some problems from economics which, when posed in a dynamical systems language involve identification of the inverse limit (= natural extension) of a dynamical system. From this standard construction, one is able to compute expected utility for the process, and hence, to quantify monetary policy aimed at maximizing future utility. The main example discussed in the talk was the so-called 'cash-in-advance' model. This work is joint with two economists R. Raines and D. Stockman. Judy's website is at
`http://www.math.udel.edu/people/faculty/profile/kennedy.html`

Geon Ho Choe (Korean Advanced Institute for Science and Technology) presented a number of examples from the class of piecewise linear circle homeomorphisms where exact invariant densities could be determined using algebraic calculations. Maple was used extensively, as very few of these calculations are feasible by hand. Quantities of dynamical interest, such as rotation number, are then computed exactly with respect to this invariant density. Professor Choe is author of the book *Computational Ergodic Theory*, Springer Verlag, 2004.

Bryna Kra (Northwestern) gave an overview of the role of the so-called Gowers norms in the recent spectacular results in the application of dynamics to questions concerning arithmetic progressions and other patterns in positive density sets. Gowers norms (and their dynamical generalization by Kra and Host) are used to exploit parallelogram structures in a variety of abstract settings including abelian semigroups and suspensions of such groups to arbitrary sets. An abstract notion of parallelogram structure on a set was given, and characterized. Much more information can be found on Bryna's webpage:
`http://www.math.northwestern.edu/~kra/`

Wael Bahsoun (Victoria) Traditionally, dynamics has considered actions for closed systems, where the orbit of a point remains in the state space for all time. In some applications, a nonequilibrium model is required where the orbit of a point may eventually leave the system (and for convenience of description, never return). The *escape rate* gives the rate at which mass is lost to the system through this mechanism. A simple model for an open system was presented: an interval map with a hole (in the domain). The main question addressed in this talk was to produce a rigorous numerical scheme that can compute the escape rate for such a system. The algorithm is based on theoretical work of Keller and Liverani on spectral perturbation and Ulam method for discretization of the continous domain; the basic steps in the algorithm were reviewed and a simple example computation presented. From September 2006, Wael is with the Department of Economics at the University of Manchester. `http://www.socialsciences.man.ac.uk/economics`

Gary Froyland (University of New South Wales) Invariant sets and functions play a central role in the analysis of dynamical systems. In practice, almost invariant sets (or functions) also contain useful information and generically, one expects to have many such objects around. In certain cases, some of these almost invariant objects are also physically interesting and natural. The speaker showed how they can be found by spectral techniques applied to the associated transfer operator. Interesting properties of almost invariant objects include (relatively) slow mixing times and slow rates of correlation decay leading to interesting physical consequences. Examples were presented ranging from simple interval maps to a long-term project the speaker is working on to help model circulation patterns in the Great Southern Ocean. Gary's website is
`http://web.maths.unsw.edu.au/~froyland/`

Sinan Gunturk (Courant Institute) Gave us a useful introduction to the dynamical ideas underlying a functional approximation method called sigma-delta quantization. This method has applications in half-toning and analog-to-digital conversion. The talk also hinted at an intriguing sequencing problem where two competitors sequentially aim to hit a target which they have identical small unknown probability p of hitting. Sinan's webpage is at
`http://www.cims.nyu.edu/~gunturk/`

William Ott (Courant Institute) gave a very enjoyable talk on classical notions of recurrence and distality in topological dynamics. The basic definition is *product recurrence*: a point $x \in X$ is product recurrent if it is recurrent and, for every other topological system Y, for every other recurrent $y \in Y$, (x, y) is recurrent for the product system. A classical result identifies this concept with distality for $\mathbb{Z}-$ actions. The relation between these concepts for more general semigroup actions has been investigated by Auslander and Furstenberg. A related notion is *weak* product recurrence, where the test point $y \in Y$ is restricted to the class of uniformly recurrent points. This was shown by the speaker to be not equivalent to distality, even for $\mathbb{Z}$-actions. William's webpage is at
`http://www.cims.nyu.edu/~ott`

James Meiss (Colorado) brought a visually stunning display of recent computational experiments aimed at uncovering bifurcation of invariant sets in 3-D volume-preserving diffeomorphisms. The setting is a natural development from the area preserving 2-D diffeomorphisms that arise for example in Hamiltonian dynamics. On the other hand, compared to the 2-D situation, the scope for interesting and complicated behaviour is greatly increased. Using a few simple model maps the speaker was able to exhibit the appearance and destruction of invariant tori and to propose various mechanisms that could lead to these complex bifurcations. More stunning graphics and a lot of mathematics can be found at
`http://amath.colorado.edu/faculty/jdm/`

Peter Ashwin (Exeter) A classical example in ergodic theory is the *interval exchange transformation*. An interval (the state-space) is partitioned into finitely many subintervals and the dynamics rearranges these by a permutation. The dynamical properties of interval exchange transformations are well-studied. A multidimensional analogue of the interval exchange is a piecewise isometry from an open, connected domain onto itself. Very little is known in generality about such maps. The speaker discussed a class of such mappings (called pizza maps) on the plane which have an advantage in that the dynamics on $\mathbb{R}^2$ near infinity can be modelled by an interval exchange. Still, on the bounded component a very rich and complex behaviour of escape and attraction may be found. The speaker presented both results from numerical studies and theoretical work. This is a joint project with Arek Goetz. Peter's webpage is
`http://www.secam.ex.ac.uk/~PAshwin/`

Matt Nicol (Houston) The Young Tower construction (L.S. Young, ~1998) provides a convenient, abstract way to construct non-uniformly hyperbolic transformations, or, conversely, to analyze concrete systems with spatially contained non-hyperbolic features (such as indifferent fixed points). This talk discussed the derivation of large deviation estimates on Young Tower maps, that is, estimates on the decay rates of $m\{\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n) \geq \int \phi dm + \epsilon\}$. It was shown that structural features of the tower control the rate of decay, through both exponential and polynomial classes. A basic question arises from this work: can the exact results on the Tower be reproduced in a concrete intermittent map. Much more about these ideas may be found at
`http://math.uh.edu/~nicol/`

Vitaly Bergelson (Ohio State) One currently active area in ergodic theory is the study of subsequential limit theorems. The notion of an IP-subset of the integers (and correspondingly, IP convergence) plays a central role, both in establishing such theorems and in generalizing to other semigroup actions the types of results available for the integers. The speaker began with a self-contained introduction to the IP-notions, then moved on to a tour of some of the known results from the *multiple recurrence* literature. Here are a couple of striking results mentioned. Suppose (X, m, T) is weakly mixing. Then

1. Generically, there is IP-rigidity. That is to say there is an IP-subset $n_\alpha \subseteq \mathbb{N}$ and (nontrivial) $f \in L^2$ which is IP-mixing: $f \circ T^{n\alpha} \to f$.

2. (IP-Krengel partition independence) For every finite partition $\mathcal{P}$ and $\epsilon > 0$ there exists a finite partition $\mathcal{P}'$ such that $d(\mathcal{P}, \mathcal{P}') < \epsilon$ and an IP-subset n_α such that $\{T^{-n_\alpha}\mathcal{P}'\}$ is exactly independent.

Other results, surveys, and open questions may be found at
`http://www.math.ohio-state.edu/~vitaly/`

Ryszard Rudnicki (Silesian University) A Markov semigroup $\{P_t\}$ is a generalization of a dynamical system – sufficiently rich to contain, for example, random dynamical systems. For such systems one has the Foguel Alternative: either $\{P_t\}$ is asymptotically stable, or it is sweeping out (mass escapes to 'infinity'). Of particular interest in applications is the case where the semigroup is generated by a partial differential equation. The speaker reviewed two such applications, one in transport theory, the other in a biological model of a gene population which can be used to explain observed properties of maturity-distribution for age profiles. Ryszard's webpage is at
`http://www.impan.gov.pl/User/rudnicki/`

Rua Murray (University of Waikato) Various methods using finite computations to estimate unknown invariant measures have been proposed. Ulam's method (discussed numerous times during the workshop) is one of the most popular and easy to implement, but theoretical problems arise when one tries to validate the method and prove convergence. Rigorous results are known only for a much smaller class of dynamical systems than the class on which numerical experiments would suggest them to hold. Rua in joint work

with Chris Bose, described a completely different approach to the approximation problem, based on convex optimization (a.k.a. the maximum-entropy method). These allow widely valid approximation schemes (they converge in norm under weak assumptions) and for which finite computations, although more delicate than in the case with Ulam's method, are still feasible. There appears to be a great deal of scope for future improvement. Rua's website is
`http://www.math.waikato.ac.nz/~rua/`

Evelyn Sander (George Mason) discussed bifurcations of low-dimensional dynamical systems giving rise to crises and more specifically explosions: parameter values where chaotic behaviour appears in neighbourhoods that previously contained no recurrent points. A key question is the existence of unstable dimension variability: parameter values for which different points in the attractor have different dimensional unstable manifolds. Evelyn's talk outlined the construction of a three dimensional example exhibiting unstable dimension variability arising from a crisis. Evelyn's web page is at
`http://math.gmu.edu/~sander/`

Ian Melbourne (Surrey) An interesting model for dynamicists is the billiard flow on the plane outside finitely many convex bodies. This has been proposed as a deterministic model for Brownian motion. Such a map is (Sinai and others) uniformly hyperbolic with singularities and leads to a central limit theorem (CLT), a functional central limit theorem (FCLT) and more generally almost sure invariance principles (ASIP). In joint work with Matt Nicol, the speaker has investigated vector-valued ASIP's for non-Axiom A dynamics, once again using the formal structure of a Young Tower. Ian's website is
`http://www.maths.surrey.ac.uk/people/index.php?display=I.Melbourne`

Arno Berger (Canterbury). Arno discussed the use of shadowing to show that for certain classes of non-autonomous mappings, almost every orbit satisfies the 'first digit property' known as Benford's Law (where the frequency of different initial digits base b is given by a logarithmic distribution). Arno's webpage in New Zealand is
`http://www.math.canterbury.ac.nz/~abe34/`

Open Problems Session

On the evening of Monday, August 7 a problem session was convened and a number of participants presented interesting problems for consideration by workshop participants.

Vitaly Bergelson asked about Ergodic theorem along polynomials and the lazy physicist paradox.

Assume that T_v, $v \in \mathbb{R}$, is a continuous ergodic measure-preserving flow on a probability Lebesgue space. Note that due to the ergodic decomposition, the assumption of ergodicity does not limit the generality of our our discussion. It is not too hard to show that for all but countably many v the measure-preserving transformation $S = T_v$ is totally ergodic (meaning that all the non-zero powers of S are ergodic as well). Consider now the following situation. A physicist fixes first a time unit v (and we assume, without too much loss of generality, that the corresponding $S = T_v$ is totally ergodic) and then performs the sampling of the flow along "quadratic" instances of time, that is, considers the averages

$$A_N = 1/N \sum_{n=0}^{N-1} f(S^{n^2}(x)),$$

where f is, say, a bounded measurable function on X which describes an important physical parameter (so that $f(S^{n^2}x)$ describes the values of the parameter along the trajectory of the point x in X, measured at quadratic instances of time).

According to a theorem due to Bourgain, (which applies to any totally ergodic transformation and any non-trivial polynomial taking on integer values on integers) the physicist will see that despite the increasing gaps between time measurements, the averages A_N will converge (for almost every x in X) to the space average, $\int f$. Note also that if the flow T_v is weakly mixing, then $S = T_v$ is weakly mixing (and hence totally ergodic) for EVERY non-zero v.

Problem 1 (Philosophical). What is the physical meaning of this? Why does nature (in the case of totally ergodic transformations) work so well along the polynomials? Apropos, there are many more "good" sequences of times with similar properties but the sequences of exponential growth, such as 2^n are not "good".

Problem 2 (Mathematical). Assume that the flow T_v is comprised of smooth enough transformations and that the function f is also smooth enough. What can be said (in terms of smoothness of T_v and f) about the speed of convergence of A_N to $\int f$? Can one show that the convergence along the squares n^2 is (in some sense) faster than that along the cubes n^3 ?

Oliver Jenkinson asked for a continuous f with Lebesgue measure as the unique $\times 2$-invariant f-maximizing measure.

The following is Problem 3.9 in [5]

Problem 1 Let $T(x) = 2x \pmod 1$. Explicitly exhibit a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int f(x)\,dx > \int f\,d\mu$ for all T-invariant probability measures μ other than Lebesgue measure.

Remarks:

(a) The strict inequality is key; if the inequality were weak then a constant function would suffice.

(b) It is known that such functions f exist (see [6, Cor. 1]).

(c) By an "explicit" representation of f we have in mind some sort of series expansion, for example a Fourier expansion.

(d) It is known that any such f cannot be too "regular"; for example f cannot be Hölder (see e.g. the discussion in [5, 6]). There are heuristic reasons (see [6]) for expecting such an f to be highly oscillatory.

Since periodic orbit measures are weak-$*$ dense in the set of T-invariant measures, the following weaker version of the above problem is perhaps no easier to solve.

Problem 2 Let $T(x) = 2x \pmod 1$. Explicitly exhibit a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int f(x)\,dx > \frac{1}{n}\sum_{i=0}^{n-1} f\big(T^i(\frac{j}{2^n-1})\big)$ for all $n \geq 1$ and $0 \leq j \leq 2^n - 1$.

Gerhard Keller asked about (Non)minimality of transitive quasiperiodically forced Denjoy circle diffeomorphisms.

Let T be a quasiperiodically forced circle homeomorphism, i.e. a continuous map of the form

$$T : \mathbb{T}^2 \to \mathbb{T}^2\ ,\ (\theta, x) \mapsto (\theta + \omega, T_\theta(x))\ , \tag{15.1}$$

where $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and where the *fibre maps* T_θ are orientation-preserving circle diffeomorphisms with the derivative DT_θ depending continuously on (θ, x). To ensure all required lifting properties we additionally assume that T is homotopic to the identity on $\mathbb{T}^2$.

Let $\hat{T} : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a lift of T. Then the quantities

$$\rho_{\hat{T}} := \lim_{n\to\infty} \frac{1}{n}(\widehat{T}^n_\theta(\hat{x}) - \hat{x})\ , \quad \rho_T := \rho_{\hat{T}} \mod 1 \tag{15.2}$$

exist and are independent of θ, $\hat{x}$ and the choice of the lift $\hat{T} : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$. They are called the fibrewise rotation numbers of $\hat{T}$ and of T, respectively. (This result is due to Herman ([3]), an alternative proof can be found in [9].)

Suppose from now on that T satisfies the following Denjoy condition:

$$\int_{\mathbb{T}^1} \mathrm{var}(\log DT_\theta)\,d\theta < \infty\ .$$

The following is known [4, Theorem 4.4]:

Theorem: If ρ_T is irrational, then $T : \mathbb{T}^2 \to \mathbb{T}^2$ is topologically transitive.

Problem: In this situation, is it true that T is necessarily topologically minimal?

It is also known that, if such a T is non-minimal, each minimal invariant subset $M \subset \mathbb{T}^2$ is highly disconnected in the sense that each connected component of M is contained in a single fibre $\pi^{-1}(\theta)$ [4, Theorem 4.5].

Example: A concrete example where, to the best of my knowledge, the answer to the above question is not known is the *critical Harper map* where T_θ is given by

$$T_\theta(x) = \frac{-1}{x + 2\cos(2\pi\theta)}\ .$$

If this map has a nontrivial minimal subset, then it should like the figure below reproduced from [4].

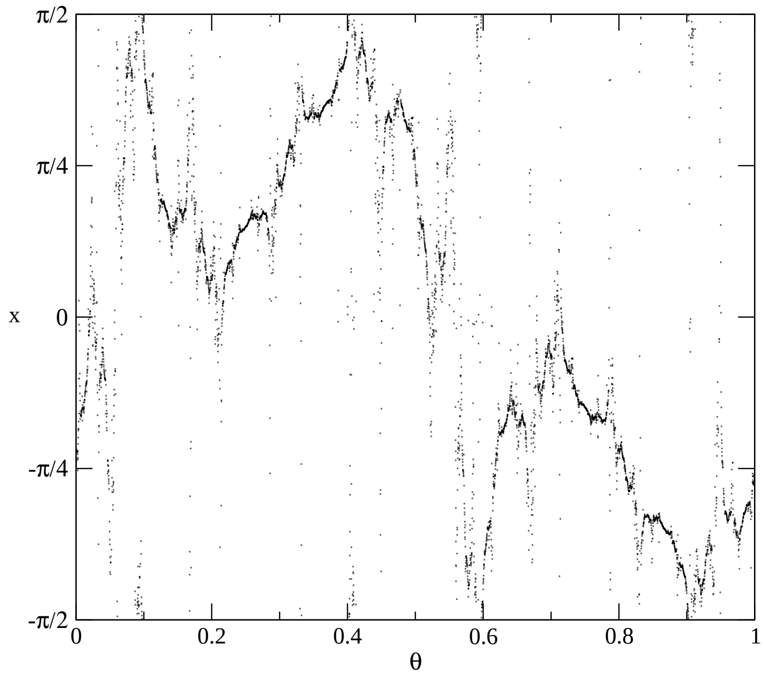

Figure 15.1: Numerical reconstruction of the invariant measure support for the critical Harper map.

Ian Melbourne asked a question about intermingled attractors.

Let $f : M \to M$ be a C^∞ diffeomorphism on a compact manifold M. We say that f has k intermingled attractors $A_1, \ldots, A_k$ if the A_j are closed f-invariant topologically transitive sets and the basins of attraction $B_j = \{x \in M : \omega(x) = A_j\}$ satisfy

(i) $\mathrm{Leb}\,(M - \{A_1 \cup \cdots \cup A_k\}) = 0$,

(ii) $\mathrm{Leb}\,(A_j \cap U) > 0$ for all nonempty open subsets $U \subset M$ and all $j = 1, \ldots k$.

Similarly, we can speak of countably many intermingled attractors.

For $k = 2$, there are three different constructions with $\dim M = 3$: Kan 1994 ($M = T^2 \times [0,1]$), Fayad 2003 ($M = T^3$), Melbourne & Windsor 2005, ($M = T^3$). For each $k = 3, 4, \ldots, \infty$, there is a 4-dimensional construction due to Melbourne & Windsor 2005 ($M = T^2 \times S^2$).

Problem Can the dimension of M in the above constructions be reduced?

Ian Melbourne and Vitaly Bergelson asked a question about weak mixing versus mixing: For measure-preserving transformations with the weak topology it follows from Halmos 1944 and Rokhlin 1948, that generically such transformations are weak mixing but not mixing. In the smooth category, it is possible to construct examples that are weak mixing but not mixing, but genericity is certainly false (mixing Axiom A diffeomorphisms form a nonempty open set of C^r diffeomorphisms for any $r \geq 1$).

In fact, the following anti-Halmos-Rokhlin situation is plausible: Consider C^r diffeomorphisms on a compact boundaryless manifold M. Perhaps there exists an r_0 (say $r_0 = 3$, or $r_0 = 2 + \epsilon$, etc) such that for any $r \geq r_0$, typical C^r diffeomorphisms $f : M \to M$ have the property that if A is a weakly mixing locally maximal ω-limit set for f then A is mixing. (Here, typical could be open-dense, generic, or prevalent.)

Problem: Prove or disprove.

Anthony Quas noted that for sofic $\mathbb{Z}$-shifts there is always a finite-to-one extension to a subshift of finite type. Once consequence is that the topological entropy of this extension is equal to that of the sofic. The finite-to-one ness fails for some $\mathbb{Z}^d$ actions but it is an open question as to whether every $\mathbb{Z}^2$ sofic admits an extension to a subshift of equal entropy.

Outcome of the Meeting

This workshop was designed to connect people and research areas across the sprawling, modern discipline of measurable dynamics. The extent to which we succeeded will only be evident some time in the future and even then may be difficult to quantify. However, the organizers are quite satisfied that the goal of creating such connections was bearing fruit already during the few days of the workshop. In addition to the positive impressions we received during our stay at BIRS we received many email comments from participants after the conference. We reproduce a few of these (both praise and constructive criticism) as representative.

- Let me thank the organizers for an excellent workshop. I think that part of the success is due to the cleverly executed implementation of the idea of bringing together representatives of different flavors of dynamics. I personally learned a lot and got plenty of new ideas which will be useful not only in my research but also in my advising activities. We should have more such workshops!
- The most interesting aspect of the meeting (for me) was bringing together people from applied and pure dynamics for the conference - this is something that should be happenning more. I would have liked to even see more applied people, to find out what they are interested in. The length of talks was optimal, and I found the problem session useful. Perhaps for future conferences it would be interesting to have someone take notes for the problem session and post them on the web.
- Thanks (also) for having me at the workshop which I found excellent and enjoyable indeed. As I said already last week, I shall be more than happy to help organizing future events.
- First of all, thanks to you and all the organizers. It was a fantastic conference. I would say as a "new guy" that the group meals and the scheduling of many breaks and social activities was great for me as far as meeting new people and getting conversations going.
- My only small complaint is food related: they served precious few green vegetables other than green peas, which I despise.
- I probably didn't explicitly mention it, but the conference was the most enjoyable I've been to for some time, so thanks for the invite!
- I found the format of the meeting highly conducive to scientific discussion and discovery. Each day included a good number of talks while providing ample time for informal discussion. Bryna, Anthony and Ronnie addressed some of the open problems that I stated at the conclusion of my talk. In general, discussion of both technical challenges and future directions permeated the meeting. I found the problem session highly useful. This idea should be implemented more generally for mathematics conferences.
- The meeting covered a fantastic breadth of subject matter; clearly a great deal of thought had gone into the organization. The BIRS facilities were great, with the natural informality guaranteeing plenty of constructive mathematical interaction. The scheduling was particularly good: the talks were a nice length, and having a few full days with six lectures, and a few days with 3 or 4 lectures but plenty of mingling time made for a very good pace.

List of Participants

Ashwin, Peter (University of Exeter)
Bahsoun, Wael (Univertiy of Victoria/PIMS)

Bergelson, Vitaly (Ohio State University)
Berger, Arno (University of Canterbury)
Bollt, Erik (Clarkson University)
Bose, Chris (University of Victoria)
Branton, Sheena (University of Houston)
Campbell, James (University of Memphis)
Choe, Geon-H (Korea Advanced Institute of Science and Technology)
Froyland, Gary (University of New South Wales)
Gora, Pawel (Concordia University)
Gunturk, Sinan (Courant Institute, NYU)
Hunt, Brian (University of Maryland)
Jenkinson, Oliver (Queen Mary - University of London)
Keller, Gerhard (Universitat Erlangen-Nuernberg)
Kennedy, Judy (University of Delaware)
Kra, Bryna (Northwestern University)
Marcus, Brian (University of British Columbia)
McClendon, David (University of Maryland)
Meiss, Jim (University of Colorado, Boulder)
Melbourne, Ian (University of Surrey)
Murray, Rua (Waikato University - New Zealand)
Nicol, Matthew (University of Houston)
Ott, William (Courant Institute of Mathematical Sciences)
Pavlov, Ronnie (University of British Columbia)
Quas, Anthony (University of Victoria)
Rudnicki, Ryszard (Institute of Mathematics PAS and Silesian University-Poland)
Sahin, Ayse (DePaul University)
Sander, Evelyn (George Mason University)
Santitissadeekorn, Naratip (Clarkson University)
Yorke, James (University of Maryland)

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Chapter 16

Geometric and Nonlinear Analysis (06w5090)

Aug 12 – Aug 17, 2006

Organizer(s): Matthew Gursky (University of Notre-Dame), Emmanuel Hebey (Universite de Cergy-Pontoise), Frederic Robert (Universite Nice-Sophia Antipolis)

Objectives of the workshop

The objective of this workshop was to cover a wide range of topics in nonlinear partial differential equations, both on the theoretical and on the applied point of view. As the title suggests, we insisted mainly on pde's arising in geometric analysis.

The idea was to make meet together people working in various topics of this very wide subject. We insisted mainly on conformal geometry and fully nonlinear equations, without neglicting other topics. As a matter of fact, the main objective of this workshop was to discuss recent developments in many branches of geometric analysis and to stress on possible new applications.

However, some new techniques, geometric or not, developed by teams concerned with these applications, do not make their way to others who may be using similar techniques but on different types of problems. One particularity, which is common to these very active areas, is the appearance of singularities. These singularities are of various type, often specific to each domain, and always need to be understood. Therefore, the other objective of this workshop was to mix senior experts and young researchers from either a more geometric flavour or a more "pure" pde flavour. In particular concerning young researchers, we were very happy to see that PhD students and post-docs were highly interested in this event and attended the workshop with pleasure.

The organizers were quite delighted to see that these objective were fullfilled. Many researchers interacted with each others and discovered new branches and new methods. Such an interaction provided an opportunity to exchange ideas from various sensibilities so that new applications could be developed. The balance between talks and free time, in the inspiring and majestuous environment of the Banff National Park, allowed the researchers in all these fields to interact together and spread their techniques one to each other.

The organizers thank the BIRS staff for their particular efficiency to manage the station and for always being happy to oblige.

Thema treated

Conformal Geometry

Conformal geometry in the large was clearly the most considered theme during the workshop. We focused on various conformal invariants. The most natural invariant is the scalar curvature, which is deeply related to the topology of two-dimensional surfaces: it also enjoys conformal invariance in dimension $n \geq 3$, and the prescription of the scalar curvature in a conformal class has been the target of investigations for decades, generalizing the Yamabe problem.

There has been some recent progresses towards the understanding of the structure of the Yamabe equation. Indeed, the difficulty in this question is due to the conformal invariance of the equation: this invariance allows bubbling to occure and singularities to appear. The difficulty is to tackle these singularities. F.Coda Marques presented his recent work concerning compactness of the Yamabe equation, a work that is a step towards the full analysis of the equation. O.Druet presented the asymptotics for a critical equation in dimension two which is a more general version of the Yamabe equation. The Yamabe equation is related to the study of nonlinear PDEs of second order with critical growth: therefore, the framework of the Yamabe equation is naturally applied to various nonlinear PDEs, geometric or not. In particular, despite the equation is not geometric, the bubbling appears in the MEMS equations studied by N.Ghoussoub: a control on the bubbling allows to prove multiplicity results and to use bifurcation methods.

The questions related to the conformal Yamabe equation can be generalized in two directions. If one considers the scalar curvature as attached to a second order problem, one will find interest in higher order conformal invariants, like the $Q-$curvature for fourth-order operators related to the conformal Paneitz-Branson operator. In this spirit, A.Malchiodi presented a recent result concerning the prescription of this curvature in dimension four in the compact setting, and H.-C.Grunau was interested in multiplicity of metrics in the noncompact setting of the hyperbolic space in higher dimension.

By the way, following Graham and al., it is possible to construct conformal invariants attached to arbitrary high order linear operators related to conformal infinity of noncompact manifolds. This theme was studied by C.Guillarmou who related the determinants of these operators to the Selberg's zeta function. In the context of the conformal infinity of hyperbolic spaces, R.Mazzeo presented existence and uniqueness of foliations with prescribed curvature, and P.Albin presented results for the renormalized index theorem. Finally, D.Raske considered compactness issues for an extension of the Yamabe equation at higher order.

The direction described above concerned linear operators. If one sees the scalar curvature as the sum of the eigenvalues of the Schouten tensor, one can consider other symmetric functions of the eigenvalues, which leads to fully nonlinear equations of second order, that is the σ_k equation. We had many talks in this directions. In particular, Z.-C.Han discussed the loss of compactness associated to this equation, non-existence results (in the spirit of Kazdan-Warner) and existence results for the prescription of σ_k. Y.Ge discussed the prescription of another invariant via the flow.

"'Non-conformal" Differential Geometry

Despite conformal geometry was dominating, it was not the only aspect of geometry considered in this workshop.

In her talk, S.-Y.A.Chang reported on regularity results concerning Bach-flat metrics on four manifolds with boundary: indeed, she provided an $\epsilon-$regularity theorem which is delicate for points on the boundary. S.Kim also considered Bach-flat four-manifolds, but when they are noncompact and complete: he presented a rigidity result by proving that when the curvature is small in the L^2-sense, then these manifolds are flat. Still concerning the structure (and also conformal geometry), J.Qing established a classification of degenerate Riemannian metrics. P.Yang discussed regularity issues for conformal minimal surfaces in pseudo-Hermitian geometry.

This concerned Riemannian geometry. We also had contribution in Spin geometry by E.Humbert. F.Pacard reported on existence of extremal Kähler metrics on some specifice manifolds, and concerning contact geometry, RCAM Van der Vorst presented the proof of the Weinstein's conjecture for closed characteristic in the noncompact setting.

At the intersection of hyperbolic equations and geometry, M.Dafermos presented in his talk some new decay estimates in the context of black hole exterior spacetime: an important question which is related to the stability of black holes in general relativity.

Fully nonlinear equations

The two main problems discussed in the conference were the fully-nonlinear version of the Yamabe problem (the σ_k equation) which connects this theme with the preceding conformal geometry theme, and the optimal transportation theory. The recent developments in transport, and especially its new applications, like in functional inequalities or evolutions pde's, are naturally of particular interest. This very active area had a natural place in this conference. We refer to the preceding contributions by Z.-C.Han and Y.Ge for issues related to the σ_k equation in conformal geometry. Concerning optimal transportation, we had a general survey by R.McCann: this survey explained really nicely the history, difficulties and issues related to this very old subject having its roots in the Monge-Kantorovich problem. Regularity issues being quite intricate for fully-nonlinear equations, R.McCann also detailed optimal results, examples and counterexamples.

Other subjects...

Some of the contributions were very general. Indeed, the techniques and ideas developed or them ahave really an important potential for other areas in Mathematics.
Y.Brenier presented a very elegant L^2 approach to hyperbolic conservation laws. A remarkable point in this formulation is that it permits to recover directly many existing results and that it can also be applied to many other problems.
M.Struwe revisited the regularity theory for harmonic maps. Indeed, in a joint work with T.Rivière, he proved the well-known regularity results with a completely new point of view and new techniques via Gauge theory. One advantage here is that this very elegant point of view applies to an important class of elliptic problems, and probably to many others.

Contents of the talks

Yann Brenier (Nice, France):
"L^2-formulation of some hyperbolic conservation laws."

Abstract: It is customary to address hyperbolic conservation laws (or Hamilton-jacobi equations) in functional spaces that are neither Hilbertian nor reflexive (typically L^1, BC, C^0, Lip, etc). We show that, in some simple but significative cases (multidimensional scalar conservation laws, Chaplygin gas etc...), a straightforward L^2 formulation can be introduced, leading to simple well posedness and stability results. This approach can be extended to some degenerate parabolic equations too.

Sun-Yung Alice Chang (Princeton, USA):
"'Regularity on a boundary value problem of generalized Yamabe type."

Abstract: In this talk, I will report on some recent joint work of Sophie Chen, Paul Yang and myself on a regularity problem of Bach-flat metrics on 4-manifolds with boundary. I will discuss both the set up and some $\epsilon-$regularity result of the problem.

Fernando Coda Marques (IMPA, Brasil):
"'Blow-up analysis for the Yamabe equation in high dimensions."

Abstract: In this talk we will discuss recent progress in understanding how solutions to the Yamabe equation can blow-up in high dimensions. We will describe how to use the Pohozaev identities, under a nondegeneracy condition, to get sufficient vanishing of the Weyl tensor at a blowup point and derive compactness results. This is joint work with Marcus Khuri and Richard Schoen.

Mihalis Dafermos (Cambridge, UK):
"The redshift effect and radiation decay for black hole spacetimes."

Abstract: I shall present new results on decay rates for the wave equation on black hole exterior spacetimes, like Schwarzschild and Schwarzschild-de Sitter, and discuss the relation of this to the problem of non-linear stability of black holes in general relativity. This is joint work with Igor Rodnianski.

Olivier Druet (ENS Lyon, France):
“’Quantification of blow-up levels for a 2d elliptic equation with critical exponential growth”

Abstract: We consider sequences of solutions of some 2-dimensional PDE with critical Trudinger-Moser growth and we show that they split into their weak solution plus a sum of standard bubbles.

Yuxin Ge (Paris 12, France):
“’On a conformal quotient equation.”

Abstract: In this talk, we consider a conformal quotient equation $\frac{\sigma_2(g)}{\sigma_1(g)} = 1$ in a given conformal class and prove the existence for $n > 4$ and prove a related Sobolev-type inequality.

Nassif Ghoussoub (UBC, Canada):
“’On PDEs arising from Electrostatic Micro-Electromechanical Systems”

Abstract: We analyze the nonlinear parabolic problem $u_t = \Delta u - \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary condition. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage -represented here by λ- is applied, the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value $\lambda^\star$ (pull-in voltage). This creates a so-called ”pull-in stability” which greatly affects the design of many devices. the challenge is to estimate $\lambda^\star$ in terms of material properties of the membrane, which can be fabricated with a spacially varying dielectric permittivity profile f. Applying analytical and numerical techniques, the existence of $\lambda^\star$ is established together with rigorous bounds. We show the existence of at least one steady-state when $\lambda = \lambda^\star$. More refined properties of steady states -such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results- are shown to depend on the dimension of the ambient space and on the permittivity profile. As to the dynamic case, the membrane globally converges to its unique maximal negative steady-state when $\lambda \leq \lambda^\star$, with a possibility of touchdown at infinite time when $\lambda = \lambda^\star$. On the other hand, if $\lambda > \lambda^\star$ the membrane must touchdown at finite time T, and touchdown cannot take place at the location where the permittivity profile vanishes. Both larger pull-in distance and larger pull-in voltage can be achieved by properly tailoring the permittivity profile. We analyze and compare finite touchdown times by applying various analytical and numerical techniques. This is joint work with Yujin Guo.

Hans-Christoph Grunau (Magdeburg, Germany):
“’The Paneitz equation in the hyperbolic ball.”

Abstract: Existence of a continuum of conformal radially symmetric complete metrics in the hyperbolic ball in $\mathbb{R}^n$, $n > 4$, is shown, all having the same constant Q-curvature. Moreover, similar results can be shown also for suitable non-constant prescribed Q-curvature functions.
(Joint work with M. Ould Ahmedou (Tuebingen), W. Reichel (Aachen))

Colin Guillarmou (Nice, France):
“’Generalized Krein formula for Poincaré-Einstein manifolds.”

Abstract: We prove a kind of Birman-Krein formula for the Laplacian on even dimensional asymptotically hyperbolic manifolds that have an asymptotic evenness for the metric expansion at infinity. This relates a renormalized trace of the spectral projector of the Laplacian to the phase of the Ray-Singer determinant of the Scattering operator. This is used for instance to obtain a functional equation for Selberg’s zeta function on convex co-compact hyperbolic manifolds and to compute the determinant of the GJMS conformal Laplacians in terms of this Selberg’s zeta function.

Zheng-Chao Han (Rutgers, USA):
“On the prescribing σ_k curvature equations.”

Abstract: Let A_g denote the Schouten-Weyl tensor of a Riemannian metric g on M^n, for $1 \le k \le n$, $\sigma_k(g^{-1} \circ A_g)$ denote the k^{th} elementary symmetric function of the eigenvalues of the $1-1$ tensor $g^{-1} \circ A_g$. For a given function $K(x)$ on M^n, we address several issues in the question of when there exists an admissible conformal metric $g_w = e^{2w(x)}g$ such that

$$\sigma_k(g^{-1} \circ A_g) = K(x) \quad \text{on} \quad M^n. \tag{16.1}$$

First we will give an elementary and unified discussion to the Kazdan-Warner type necessary conditions for the solvability of (16.1). Then we will discuss the potential loss of compactness to the solutions of (16.1), and show that under appropriate non-degeneracy conditions on $K(x)$, so such loss of compactness can happen. This latter part is part of joint work with S.-Y.A. Chang and P. Yang.

Emmanuel Humbert (Nancy, France):
"Surgery and harmonic spinors."

Abstract: Let M be a compact spin manifold with Riemannian metric g. Suppose that N is obtained from M by a surgery of codimension at least 2. We prove that N carries a Riemannian metric h such that the dimension of the kernel of the Dirac operator on (N, h) is not larger than the dimension of the kernel of the Dirac operator on (M, g). As an application, we show that for generic metrics on a spin manifold, the dimension of the kernel of the Dirac operator attains the lower bound given by the index theorem.

Seongtag Kim (Inha, Korea)
"Bach-flat manifolds with small L^2-curvature."

Abstarct: In this talk, I will sketch the rigidity of noncompact complete Bach-flat 4-manifolds with small L^2-curvature norm. let (M, g) be a Riemannina four-manifold with Weyl curvature W and Ricci curvature R_{ij}. The Bach tensor B_{ij} is defined by $B_{ij} = \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl}$. The Bach tensor is conformally invariant. A manifold (M, g) is called Bach-flat if $B_{ij} = 0$. Important examples of Bach-flat metrics are Eistein metrics, conformally Eistein metrics and self-dual Einstein metrics. We describe geometric conditions that ensure that noncompact complete Bach-flat manifolds with zero scalar curvature and small L^2-curvature norm are flat. For this, we use elliptic estimates on the equation arising from $B_{ij} = 0$.

Andrea Malchiodi (SISSA, Italy):
"'Existence of conformal metrics with constant Q-curvature."

Abstract: We discuss the problem of finding conformal metrics on a compact four-manifold which have constant Q-curvature. This consists in solving a fourth order elliptic PDE with exponential nonlinearity and with variational structure. When the total Q-curvature is large, the Euler functional is unbounded from below, and critical points have to be found via min-max methods. Using a new variational scheme based on concentration of conformal volume, we solve the problem for a large class of manifolds.

Rafe Mazzeo (Stanford, USA):
"'Geometric foliations near infinity in asymptotically hyperbolic manifolds."

Abstract: I will describe recent work with Frank Pacard concerning the existence and uniqueness of foliations near infinity in asymptotically hyperbolic spaces such that each leaf has constant mean (or σ_k) curvature, and the relationship of these foliations to the conformal infinity of these spaces.

Robert McCann (Toronto, Canada):
"'Regularity and counterexamples in optimal transportation."

Abstract: I shall give a rapid survey of new and old results in optimal transportation, including the regularity theory of Ma, Trudinger, Wang and Loeper, and counterexamples to regularity due to Gregoire Loeper.

Frank Pacard (Paris 12, France):
"'Elliptic aspects of extremal Kähler metrics."

Abstract: I will report some recent results with C. Arezzo and M. Singer about the existence of extremal Kähler metrics on the blow up at finitely many points of manifolds wich carry an extremal Kähler metric.

Jie Qing (Santa Cruz, USA):
"'Metrics degeneration and its applications in conformal geometry."

Abstract: In this talk, we develop a bubble tree structure for a degenerating class of Riemannian metrics satisfying some global conformal bounds on compact manifolds of dimension 4. Applying the bubble tree structure, we establish a gap theorem, a finiteness theorem for diffeomrphism type for this class, and raise a question concerning a diameter bound for a family of coformal metrics satisfyning a suitable curvature equation.

Michael Struwe (ETH Zürich, Switzerland):
"Partial regularity for harmonic maps, revisited."

Abstract: Via gauge theory, we give a new proof of partial regularity for harmonic maps in dimensions $m \geq 3$ into arbitrary targets. This proof avoids the use of adapted frames and permits to consider targets of "minimal" C^2 regularity. The proof we present moreover extends to a large class of elliptic systems of quadratic growth.

Robert Van der Vorst (Amsterdam, Holland):
"'Closed characteristics on non-compact hypersurfaces."

Abstract: C.Viterbo proved that any $(2n-1)$-dimensional compact hypersurface $M \subset (\mathbb{R}^{2n}, \omega)$ of contact type has at least one closed characteristic. This result proved the Weinstein conjecture for the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. Various extensions of this theorem have been proved since, all for compact hypersurfaces. In this paper we consider *non-compact* hypersurfaces $M \subset (\mathbb{R}^{2n}, \omega)$ coming from mechanical Hamiltonians, and prove an analogue of Viterbo's result. The main result provides a strong connection between the homology groups $H_k(M)$, $k = n, \cdots, 2n-1$, and the existence of closed characteristics in the non-compact case (including the compact case).

Paul Yang (Princeton, USA)
Title: *"'Minimal surfaces in pseudo-Hermitian geometry."*

Abstract: I will discuss the notion of minimal surfaces in a pseudo-Hermitian manifold, the existence and uniqueness questions as well as that of the regularity. It turns out that in dimension three, there are serious regularity questions. These are joint works with J.Cheng, J.Huang and A.Malchiodi. I will also discuss the Sobolev inequality on a pseudo-Hermitian manifold, and report on some joint work with S.Chanillo.

List of Participants

Albin, Pierre (Massachusetts Institute of Technology (MIT))
Brenier, Yann (University of Nice)
Cassani, Daniele (University of British Columbia)
Catino, Giovanni (Universita di Pisa)
Chang, Alice (Princeton University)
Coda Marques, Fernando (Instituto de Matematica Pura e Aplicada (IMPA))
Dafermos, Mihalis (University of Cambridge)
Druet, Olivier (Ecole Normale Supérieure de Lyon & CNRS)
Fan, Edward (Princeton University)
Ge, Yuxin (University of Paris 12)
Ghoussoub, Nassif (BIRS)
Grunau, Hans-Christoph (Otto-von-Guericke-Universität Magdeburg)
Guan, Pengfei (McGill University)
Guillarmou, Colin (Université de Nice)
Han, Zheng-Chao (Rutgers University)
Hebey, Emmanuel (Universite de Cergy-Pontoise)
Humbert, Emmanuel (Université Henri Poincaré Nancy)
Kim, Seongtag (Inha University)
Malchiodi, Andrea (International School for Advanced Studies (SISSA))
Mazzeo, Rafe (Stanford University)

McCann, Robert (University of Toronto)
Moller, Niels Martin (University of Aarhus)
Pacard, Frank (Université Paris 12-val de Marne)
Qing, Jie (University of California, Santa Cruz)
Raske, David (University of California, Riverside)
Robert, Frederic (Universite de Nice-Sophia Antipolis)
Schwartz, Fernando (Duke University)
Struwe, Michael (ETH Zentrum)
Van der Vorst, Robertus C.A.M. (Vrije Universiteit Amsterdam)
Williams, Catherine (University of Washington)
Yan, Yu (University of British Columbia)
Yang, Paul (Princeton University)

Chapter 17

Recent Advances in Computational Complexity (06w5031)

August 26 – August 31, 2006

Organizer(s): Stephen Cook (University of Toronto), Arvind Gupta (Simon Fraser University), Russell Impagliazzo (University of California, San Diego), Valentine Kabanets (Simon Fraser University), Madhu Sudan (Massachusetts Institute of Technology), Avi Wigderson (Institute for Advanced Study, Princeton)

Computational Complexity Theory is the field that studies the inherent costs of algorithms for solving mathematical problems. Its major goal is to identify the limits of what is efficiently computable in natural computational models. Computational complexity ranges from quantum computing to determining the minimum size of circuits that compute basic mathematical functions to the foundations of cryptography and security.

Computational complexity emerged from the combination of logic, combinatorics, information theory, and operations research. It coalesced around the central problem of ”P versus NP” (one of the seven open problems of the Clay Institute). While this problem remains open, the field has grown both in scope and sophistication. Currently, some of the most active research areas in computational complexity are the following:

- the study of hardness of approximation of various optimization problems (using probabilistically checkable proofs), and the connections to coding theory,
- the study of the role of randomness in efficient computation, and explicit constructions of ”random-like” combinatorial objects,
- the study of the power of various proof systems of logic, and the connections with circuit complexity and search heuristics,
- the study of the power of quantum computation.

Many new developments in these areas were presented by the participants of the workshop. These new results will be described in the following sections of this report, grouped by topic. For each topic, we give a brief summary of the presented results, followed by the abstracts of the talks.

Computational Randomness

Computational randomness, or pseudorandomness, is the area concerned with explicit constructions of various “random-like” combinatorial objects. New constructions of one type of such objects, *randomness extractors*, have been reported by Anup Rao, Ronen Shaltiel, and David Zuckerman. The work by Shaltiel and his co-authors also yields a new explicit construction of a (bipartite) Ramsey graph with better parameters than those of the construction by Frankl and Wilson. Chris Umans reported on a new construction of lossless condensers based on derandomized curve samplers.

ANUP RAO, **Extractors for a Constant Number of Polynomially Small Min-Entropy Independent Sources**

We consider the problem of randomness extraction from independent sources. We construct an extractor that can extract from a constant number of independent sources of length n, each of which have min-entropy n^γ for an arbitrarily small constant $\gamma > 0$. Our extractor is obtained by composing seeded extractors in simple ways. We introduce a new technique to *condense* independent somewhere-random sources which looks like a useful way to manipulate independent sources. Our techniques are different from those used in recent work [BIW04, BKS$^+$05, Raz05, Bou05] for this problem in the sense that they do not rely on any results from additive number theory.

Using Bourgain's extractor [Bou05] as a black box, we obtain a new extractor for 2 independent block-sources with few blocks, even when the min-entropy is as small as $\mathrm{polylog}(n)$. We also show how to modify the 2 source disperser for linear min-entropy of Barak et al. [BKS$^+$05] and the 3 source extractor of Raz [Raz05] to get dispersers/extractors with exponentially small error and linear output length where previously both were constant.

In terms of Ramsey Hypergraphs, for every constant $1 > \gamma > 0$ our construction gives a family of explicit $O(1/\gamma)$-uniform hypergraphs on N vertices that avoid cliques and independent sets of size $2^{(\log N)^\gamma}$.

RONEN SHALTIEL, **2-Source Dispersers for $n^{o(1)}$ Entropy, and Ramsey Graphs Beating the Frankl-Wilson Construction** (joint work with B. Barak, A. Rao, and A. Wigderson)

We present an explicit disperser for two independent sources on n bits, each of entropy $k = n^{o(1)}$. Put differently, setting $N = 2^n$ and $K = 2^k$, we construct explicit $N \times N$ Boolean matrices for which no $K \times K$ submatrix is monochromatic. Viewed as adjacency matrices of bipartite graphs, this gives an explicit construction of K-Ramsey *bipartite* graphs of size N.

This greatly improves the previous the previous bound of $k = o(n)$ of Barak, Kindler, Shaltiel, Sudakov and Wigderson [BKS$^+$05]. It also significantly improves the 25-year record of $k = \tilde{O}(\sqrt{n})$ on the very special case of Ramsey graphs, due to Frankl and Wilson [FW81].

The construction uses (besides "classical" extractor ideas) almost all of the machinery developed in the last couple of years for extraction from independent sources, including:

- Bourgain's extractor for 2 independent sources of some entropy rate $< 1/2$ [Bou05]
- Raz' extractor for 2 independent sources, one of which has any entropy rate $> 1/2$ [Raz05]
- Rao's extractor for 2 independent block-sources of entropy $n^{\Omega(1)}$ [Rao06]
- The "Challenge-Response" mechanism for detecting "entropy concentration" of [BKS$^+$05].

The main novelty comes in a bootstrap procedure which allows the Challenge-Response mechanism of [BKS$^+$05] to be used with sources of less and less entropy, using recursive calls to itself. Subtleties arise since the success of this mechanism depends on restricting the given sources, and so recursion constantly changes the original sources. These are resolved via a new construct, in between a disperser and an extractor, which behaves like an extractor on sufficiently large subsources of the given ones.

DAVID ZUCKERMAN, **Deterministic Extractors For Small Space Sources** (joint work with Jesse Kamp, Anup Rao, and Salil Vadhan)

We give explicit deterministic extractors for sources generated in small space, where we model space s sources on $\{0,1\}^n$ by width 2^s branching programs. We give extractors which extract almost all of the randomness from sources with constant entropy rate, when the space s is a small enough constant times n. We can extract from smaller min-entropies assuming efficient algorithms to find large primes. Previously, nothing was known for entropy rate less than $1/2$, even for space 0.

Our results are obtained by a reduction to a new class of sources that we call independent symbol sources, which generalize both the well-studied models of independent sources and symbol-fixing sources. These sources consist of a string of n independent symbols over a d symbol alphabet with min-entropy k. We give deterministic extractors for such sources when k is as small as $\mathrm{polylog}(n)$, for small enough d.

CHRIS UMANS, **Better lossless condensers through derandomized curve samplers** (joint work with Amnon Ta-Shma)

Lossless condensers are unbalanced expander graphs, with expansion close to optimal. Equivalently, they may be viewed as functions that use a short random seed to map a source on n bits to a source on many fewer bits while preserving all of the min-entropy. It is known how to build lossless condensers when the graphs are slightly unbalanced [CRVW02]. The highly unbalanced case is also important but the only known construction does not condense the source well. We give explicit constructions of lossless condensers with condensing close to optimal, and using near-optimal seed length.

Our main technical contribution is a randomness-efficient method for sampling F^D (where F is a field) with low-degree curves. This problem was addressed before [BSSVW03, MR06] but the solutions apply only to degree one curves, i.e., lines. Our technique is new and elegant. We use sub-sampling and obtain our curve samplers by composing a sequence of low-degree manifolds, starting with high-dimension, low-degree manifolds and proceeding through lower and lower dimension manifolds with (moderately) growing degrees, until we finish with dimension-one, low-degree manifolds, i.e., curves. The technique may be of independent interest.

Cryptography and Quantum Communication Complexity

Cryptography aims to develop protocols that will hide sensitive information from any unauthorized observer. One of the famous examples of such protocols is a “zero knowledge” protocol, which allows one one to convince an untrusting party of the truth of some statement without revealing any sensitive information about the statement. Salil Vadhan gave a survey and reported some new exciting results on zero-knowledge proofs. Adi Akavia presented the results showing that (essentially) one needs much more than P≠NP in order to build any cryptographic protocols. Scott Aaronson explained how one might be able to copy-protect quantum-computer software. Paul Valiant talked about a new notion of incrementally verifiable computation. Finally, Dmitry Gavinsky explained some differences between classical shared random string and quantum shared entanglements in the setting of communication complexity.

SALIL VADHAN, **The Complexity of Zero Knowledge**

I will survey our efforts in the complexity-theoretic study of zero-knowledge proofs, where we have characterized the classes of problems having various types of zero-knowledge proofs, established general theorems about these classes, and minimized (indeed, often eliminated) complexity assumptions in the study of zero knowledge. In particular, I will discuss our most recent result, showing that all of NP has ”statistical zero-knowledge arguments” under the (minimal) assumption that one-way functions exist, which resolves an open problem posed by Naor, Ostrovsky, Venkatesan, and Yung in 1992 [NOV$^+$].

The talk covers joint works with Minh Nguyen, Shien Jin Ong, and others, focusing on the papers [Vad04, NV06, NOV06].

ADI AKAVIA, **On Basing One-Way Function on NP-Hardness** (joint work with Oded Goldreich, Shafi Goldwasser and Dana Moshkovitz)

One-way functions are the cornerstone of modern cryptography. Informally speaking, one-way functions are functions that are easy to compute but are hard to invert (on the average case). There are several candidate functions, such as RSA or discrete-log, that are believed to be one-way, nonetheless, to date, no function was proved to be one-way. A puzzling question of fundamental nature is what are the minimal assumptions required for proving that a function is one-way. A necessary condition is that P does not equal NP (or more precisely, BPP does not equal NP, namely, that there is a problem in NP that cannot be solved by any probabilistic polynomial time algorithm). We ask whether this is also a sufficient condition. Namely, we ask whether there can be an efficient reduction from NP (that is, from the task of deciding an NP-complete language on the worst case) to a one-way function (that is, to the task of inverting a one-way function on the average case).

We proved two results on the impossibility of reducing NP to a one-way function; both results hold under the (widely believed) complexity assumption that coNP is not contained in AM. 1. There cannot be a reduction (not even an adaptive reduction) from NP to a ”size verifiable” one-way function; where we call f size-verifiable if, given y, the number of pre-image $|f^{-1}(y)|$ is efficiently computable, or, more generally, efficiently verifiable via an AM protocol. 2. There cannot be a non-adaptive reduction from NP to any one-way function (be it size-verifiable or not).

Our results improve on previously known negative results of [FF93, BT03] by (i) handling adaptive reductions (whereas previous works were essentially confined to non-adaptive reductions), and by (ii) relying on a seemingly weaker complexity assumption.

In the course of proving the above results, we designed a new constant round interactive protocol for proving upper bounds on the sizes of NP sets. We believe this protocol may be of independent interest.

SCOTT AARONSON, **Quantum Copy-Protection**

In the classical world, copy-protecting software is trivially impossible (not that that's stopped numerous companies from trying). But what if your computer program were a quantum state? In this talk, I'll present evidence that there exist quantum states that (1) can be used to evaluate some function f, but (2) can't be used to efficiently prepare more states with which to evaluate f. Indeed, in the black-box model, *any* function at all can be quantumly copy-protected, except in the degenerate case that one can efficiently learn the function by querying it. The proof of this result uses several new ideas that might be of interest on their own. These include an explicit construction of "d-wise independent quantum states," and a common generalization of the No-Cloning Theorem and the quantum search lower bound.

PAUL VALIANT, **Incrementally Verifiable Computation**

The probabilistically checkable proof (PCP) system enables proofs to be verified in time polylogarithmic in the length of a classical proof. Computationally sound proofs improve upon PCPs by additionally shortening the length of the transmitted proof to be polylogarithmic in the length of the classical proof. In this paper we explore the limits of such non-interactive proof systems. We present a proof system that in addition to the above properties allows proofs to be constructed in space polynomial in the space that it takes to classically accept the language, and time that is essentially linear in the time to classically accept. Our proof system is also *incremental*, a new notion that allows proofs of partial results to be composed together so that the length of the composition is no more than that of each part. Our construction relies on the hypothesized existence of a *proof of knowledge* system that reduces the length of classical proofs by a constant factor.

DMITRY GAVINSKY, **On the role of shared entanglement**

Despite the apparent similarity between shared randomness and shared entanglement in the context of Communication Complexity, our understanding of the latter is not as good as of the former. In particular, there is no known "entanglement analogue" for the famous theorem by Newman [New91, NS96], saying that the number of shared random bits required for solving any communication problem can be at most logarithmic in the input length (i.e., using more than $O(log(n))$ shared random bits would not reduce the complexity of an optimal solution).

We prove that the same is not true for entanglement. We establish a wide range of tight (up to a logarithmic factor) entanglement vs. communication tradeoffs for relational problems. The "low-end" is: for any $t > 2$, reducing shared entanglement from $log^t(n)$ to $o(log^{t-1}(n))$ qubits can increase the communication required for solving a problem almost exponentially, from $O(log^t(n))$ to $\omega(\sqrt{n})$. The "high-end" is: for any $\epsilon > 0$, reducing shared entanglement from $n^{1-\epsilon}\log(n)$ to $o(n^{1-\epsilon})$ can increase the required communication from $O(n^{1-\epsilon}\log(n))$ to $\omega(n^{1-\epsilon/2})$.

Circuit complexity

Classical complexity theory aims to understand the power and limitations of efficient computation. One way to understand the limitations is to prove circuit lower bounds. While no strong circuit lower bounds are known for the general circuit model, there are some results for weaker models as well as there are some connections between circuit lower bounds and other areas of complexity, e.g., pseudorandomness. Eric Allender reported on new connections among arithmetic circuit complexity, real computation, and derandomization. Toni Pitassi described new constructions of small monotone circuits for computing the Majority function. Pierre McKenzie discussed lower bounds for a special case of branching programs. Dieter van Melkebeek presented results on time hierarchy for probabilistic complexity classes. Rahul Santhanam showed a new circuit lower bound for the "promise" version of complexity class MA. Amnon Ta-Shma discussed limitations of "black-box" reductions. Finally, Josh Buresh-Oppenheim explained how one could construct computationally hard Boolean functions via "hardness condensing".

ERIC ALLENDER, **Arithmetic Circuits, Real Numbers, and the Counting Hierarchy**
Arithmetic circuit complexity is the object of intense study in three different subareas of theoretical computer science:

1. **Derandomization.** The problem of determining if two arithmetic circuits compute the same function is known as $ACIT$ (arithmetic circuit identity testing). $ACIT$ is the canonical example of a problem in BPP that is not known to have a deterministic polynomial-time algorithm. Kabanets and Impagliazzo showed that the question of whether or not $ACIT$ is in P very tightly linked to the question of proving circuit size lower bounds [KI03].

2. **Computation over the Reals.** The Blum-Shub-Smale model of computation over the reals is an algebraic model that has received wide attention [BCS$^+$98].

3. **Valiant's Classes VP and VNP.** Valiant characterized the complexity of the permanent in two different ways. Viewed as a function mapping n-bit strings to binary encodings of Natural numbers, the permanent is complete for the class CP [Val79b]. Viewed as an n-variate polynomial, the permanent is complete for the class VNP [Val79a].

The general thrust of these three subareas has been in three different directions, and the questions addressed seem quite different from those addressed by work in the numerical analysis community, such as that surveyed by Demmel and Koev [DK03].

This talk will survey some recent work that ties all of these areas together in surprising ways. Most of the results that will be discussed can be found in [ABK$^+$05, Bur06], but I will also discuss some more recent progress.

TONIANN PITASSI, **Monotone circuits for MAJORITY** (joint work with Shlomo Hoory and Avner Magen)
First I discuss what is currently known: the constructions by Ajtai, Komlos, and Szemeredy and by Valiant. Then I give our new results. We get smaller monotone circuits for MAJORITY; the size is roughly n^2 (rather than Valiant's n^5.3), while the depth is still $O(\log n)$. The circuit construction is also partially derandomized. The second phase which solves the promise problem uses belief propagation algorithm, and is derandomized and optimal; the first phase is still randomized.

PIERRE MCKENZIE, **Incremental branching programs** (joint work with Anna Gál and Michal Koucký)
We propose a new model of restricted branching programs which we call *incremental branching programs*. We show that *syntactic* incremental branching programs capture previously studied structured models of computation for the problem GEN, namely marking machines [Coo74] and Poon's extension [Poo93] of jumping automata on graphs [CR80]. We then prove exponential size lower bounds for our syntactic incremental model, and for some other restricted branching program models as well. We further show that nondeterministic syntactic incremental branching programs are provably stronger than their deterministic counterpart when solving a natural NL-complete GEN subproblem. It remains open if syntactic incremental branching programs are as powerful as unrestricted branching programs for GEN problems.

DIETER VAN MELKEBEEK, **Time Hierarchies for Semantic Models of Computation** (joint work with Konstantin Pervyshev)
A basic question in computational complexity asks whether somewhat more time allows us to solve strictly more decision problems on a given model of computation. Despite its fundamental nature, the question remains unanswered for many models of interest. Essentially, time hierarchies are known for every syntactic model of computation but open for everything else, where we call a model syntactic if there exists a computable enumeration consisting exactly of the machines in the model.

There has been significant progress in recent years, namely in establishing time hierarchies for nonsyntactic models with small advice. In this talk, we survey these results and present a generic theorem that captures and strengthens all of them. We show that for virtually any semantic model of computation and for any rationals $1 \leq c \leq d$, there exists a language computable in time n^d with one bit of advice but not in time n^c with one bit of advice, where we call a model semantic if there exists a computable enumeration that contains all machines in the model but may also contain others.

Our result implies the first such hierarchy theorem for randomized machines with zero-sided error, quantum machines with one- or zero-sided error, unambiguous machines, symmetric alternation, Arthur-Merlin games of any signature, etc. Our argument also yields considerably simpler proofs of earlier hierarchy theorems with one bit of advice for randomized or quantum machines with two-sided error.

RAHUL SANTHANAM, **Circuit Lower Bounds for Promise-MA**

We show that for each $k > 0$, $MA/1$ doesn't have circuits of size n^k. This implies the first superlinear circuit lower bounds for the promise versions of the classes MA, AM, $ZPP_{||}^{NP}$ and BPP_{path}.

We extend our lower bound to the average-case setting, i.e., we show that $MA/1$ is not approximable by circuits of size n^k. Earlier, it was not even known if there is a language computable in Σ_2 with sublinear advice which is inapproximable by linear-size circuits.

AMNON TA-SHMA, **New connections between derandomization, worst-case complexity and average-case complexity** (joint work with Dan Gutfreund)

There has been a long line of research trying to explain our failure in proving worst-case to average-case reductions within NP [FF93, Vio03, BT03, AGGM06]. The bottom line of this research is, essentially, that under plausible assumptions black-box techniques cannot prove such results. A simple generalization of [BT03] shows:

Theorem 1 *Suppose that there is a language $L \in NP$ and a distribution D sampleable in time $n^{\log n}$ such that there is a* black-box *and non-adaptive reduction from solving SAT on the worst-case to solving L on the average with respect to D. Then every language in $coNP$ can be computed by a family of nondeterministic Boolean circuits of size $n^{polylog(n)}$.*

In particular, assuming no unexpected collapse occurs for the polynomial time hierarchy, the above worst-case to average-case reduction cannot be obtained via a black-box, non-adaptive reduction.

On the other hand, we show that the reduction of Gutfreund, Shaltiel and Ta-Shma [GSTS05] breaks the above lower bound. Specifically,

Theorem 2 *There exists a distribution D sampleable in time $n^{\log n}$, such that there is a non-adaptive reduction from solving SAT on the worst-case to solving SAT on the average with respect to D.*

In particular, the [GSTS05] reduction bypasses the black-box limitation imposed by Theorem 1 (if the above collapse does not happen), and indeed the [GSTS05] reduction is non black-box.

As it turns out, the [GSTS05] reduction is black-box in the reduction function (mapping an algorithm good on average to a worst-case algorithm), and this reduction is simply the search to decision reduction. However, it is not black-box in the proof. Instead, the proof of correctness only shows that any *efficient* algorithm to the average-case problem, is mapped to an efficient algorithm for the worst-case problem. We call such reductions *class black-box*. We believe such reductions are often as useful as black-box reductions, and yet, our work demonstrates that they can break black-box limitations.

Finally, we are now in a position where there are no negative results to stop us. How far can we go? Given the techniques of [GSTS05] a natural goal is to answer the following Open Question: Does $NP \not\subseteq BPP$ imply the existence of a language in $QNP = NTIME(n^{O(\log n)})$ that is hard on average for BPP?

Using the [IL90] reduction we show such a result, but only using some weak, *unproven* derandomization assumption. Resolving this Open Question without any assumptions remains a challenge.

JOSHUA BURESH-OPPENHEIM, **Making Hard Problems Harder** (joint work with Rahul Santhanam)

Proving circuit lower bounds for explicit Boolean functions is one of the most fundamental and challenging questions in theoretical computer science. We consider an approach to this question which aims to improve hardness rather than give a direct proof of hardness. We define "hardness extractors," which are procedures taking in a Boolean function as input together with a relatively small advice string, and outputting a Boolean function on a smaller number number of bits which has greater hardness when measured in terms of its input length. We show a construction of a hardness extractor with linear advice extracting deterministic hardness from non-deterministic hardness. As a consequence, we obtain a "gap" theorem for E with linear advice: if E with linear advice requires exponential non-uniform space, then E with linear advice requires non-uniform space $2^n/n$.

We also define a natural class of "relativizing" hardness extractors and give lower bounds on the advice required by such extractors. This indicates that hardness extraction without advice and extraction of deterministic hardness from deterministic hardness in general will require novel techniques. On the other hand, we show two special cases where we can extract from deterministic hardness without advice: biased functions and functions that are hard on average.

Error-correcting codes and PCPs

Error-correcting codes play a major role in modern complexity theory. Many important results in complexity (e.g., the famous PCP theorem) are best viewed as constructions of special error-correcting codes. Once this connection between coding and complexity theory is realized, both areas enjoy mutual benefits by using ideas and insights from the other area. Venkatesan Guruswami gave an explicit construction of list-decodable codes with optimal rate. Ran Raz presented a construction of a very efficient low-degree test (useful for PCPs). Ragesh Jaiswal talked about some error-correcting codes directly inspired by complexity-theoretic questions. Eli Ben-Sasson discussed some limitations of list-decoding. Finally, Oded Regev showed an improved hardness of approximation result for the problem of finding a shortest vector in a lattice.

VENKATESAN GURUSWAMI, **List Decoding with Optimal Rate: Folded Reed-Solomon Codes** (joint work with Atri Rudra)

Suppose you want to communicate a message of k packets on a noisy communication channel. So you judiciously encode it as a redundant collection of $n = ck$ packets and transmit these. What is the fewest number of correct packets one needs to receive in order to have any hope of recovering the message?

Well, clearly one needs at least k correct packets. In this talk, I will describe an encoding scheme that attains this information-theoretic limit: for any desired $\epsilon > 0$, it enables recovery of the message as long as at least $k(1+\epsilon)$ packets are received intact. The location of the correct packets and the errors on the remaining packets can be picked adversarially by the channel.

This achieves the optimal trade-off (called "capacity") between redundancy and error-resilience for a malicious noise model where the channel can corrupt the transmitted symbols arbitrarily subject to a bound on the total number of errors. These results are obtained in an error-recovery model called list decoding. The talk will introduce and motivate the problem of list decoding, and then give a peek into the algebraic ideas and constructions that lead to the above result.

RAN RAZ, **Sub-Constant Error Low Degree Test of Almost Linear Size** (joint work with Dana Moshkovitz)

Given a function $f : F^m \to F$ over a finite field F, a *low degree tester* tests its agreement with an m-variate polynomial of total degree at most d over F. The tester is usually given access to an oracle A providing the *supposed* restrictions of f to affine subspaces of constant dimension (e.g., lines, planes, etc.). The tester makes very few (probabilistic) queries to f and to A (say, one query to f and one query to A), and decides whether to accept or reject based on the replies.

We wish to minimize two parameters of a tester: its *error* and its *size*. The *error* bounds the probability that the tester accepts although the function is far from a low degree polynomial. The *size* is the number of bits required to write the oracle replies on all possible tester's queries.

Low degree testing is a central ingredient in most constructions of probabilistically checkable proofs (PCPs) and locally testable codes (LTCs). The error of the low degree tester is related to the soundness of the PCP and its size is related to the size of the PCP (or the length of the LTC).

We design and analyze new low degree testers that have both *sub-constant error* $o(1)$ and *almost-linear size* $n^{1+o(1)}$ (where $n = |F|^m$). Previous constructions of *sub-constant error* testers had *polynomial size* [AS03, RS97]. These testers enabled the construction of PCPs with *sub-constant soundness*, but *polynomial size* [AS03, RS97, DFK$^+$99]. Previous constructions of *almost-linear size* testers obtained only *constant error* [GS02, BSSVW03]. These testers were used to construct *almost-linear size* LTCs and *almost-linear size* PCPs with *constant soundness* [GS02, BSSVW03, BSGH$^+$04, BSS05, Din06].

RAGESH JAISWAL, **Approximately list-decoding direct product codes and uniform hardness amplification** (joint work with Russell Impagliazzo and Valentine Kabanets)

We consider the problem of locally list-decoding *direct product* codes. For a parameter k, the k-wise direct product encoding of an N-bit message msg is an N^k-length string over the alphabet $\{0,1\}^k$ indexed by k-tuples $(i_1,\dots,i_k)\in\{1,\dots,N\}^k$ so that the symbol at position $(i_1,\dots,i_k)$ of the codeword is $msg(i_1)\dots msg(i_k)$. Such codes arise naturally in the context of hardness amplification of Boolean functions via Yao's Direct Product Lemma (and closely related Yao's XOR Lemma), where typically $k \ll N$ (e.g., $k = poly \log N$).

We describe an efficient randomized algorithm for approximate local list-decoding of direct product codes. Given oracle access to a word which agrees with a k-wise direct product encoding of some message msg in at least ϵ fraction of positions, our algorithm outputs a list of $poly(1/\epsilon)$ Boolean circuits computing N-bit strings (viewed as truth tables of $\log N$-variable Boolean functions) such that at least one of them agrees with msg in at least $1-\delta$ fraction of positions, for $\delta = O(\frac{\log(1/\epsilon)}{k}+k^{-0.1})$, provided that $\epsilon = \Omega(poly(1/k))$; the running time of the algorithm is polynomial in $\log N$ and $1/\epsilon$. When $\epsilon > e^{-k^\alpha}$ for a certain constant $\alpha > 0$, we get a randomized approximate list-decoding algorithm that runs in time quasipolynomial in $1/\epsilon$ (i.e., $(1/\epsilon)^{poly \log 1/\epsilon}$).

By concatenating the k-wise direct product codes with Hadamard codes, we obtain locally list-decodable codes over the binary alphabet, which can be efficiently approximately list-decoded from fewer than $1/2-\epsilon$ fraction of corruptions as long as $\epsilon = \Omega(poly(1/k))$. As an immediate application, we get ***uniform*** hardness amplification for $P^{NP_\parallel}$, the class of languages reducible to NP through one round of parallel oracle queries: If there is a language in $P^{NP_\parallel}$ that cannot be decided by any BPP algorithm on more that $1-1/n^{\Omega(1)}$ fraction of inputs, then there is another language in $P^{NP_\parallel}$ that cannot be decided by any BPP algorithm on more that $1/2+1/n^{\omega(1)}$ fraction of inputs.

ELI BEN-SASSON, **Subspace Polynomials and List Decoding of Reed-Solomon Codes** (joint work with Swastik Kopparty and Jaikumar Radhakrishnan)

We show combinatorial limitations on efficient list decoding of Reed-Solomon codes beyond the Johnson and Guruswami-Sudan bounds. In particular, we show that for arbitrarily large fields $F_N, |F_N| = N$, for any $\delta \in (0,1)$, and $K = N^\delta$:

- **Existence:** there exists a received word that agrees with a super-polynomial number of distinct degree K polynomials on approximately $N^{\sqrt{\delta}}$ points each;
- **Explicit:** there exists a polynomial time constructible received word that agrees with a super-polynomial number of distinct degree K polynomials, on approximately $2^{\sqrt{\log N}}K$ points each.

In both cases, our results improve upon the previous state of the art, which was about N^δ/δ for the existence case and about $2N^\delta$ for the explicit one. Furthermore, for δ close to 1 our bound approaches the Guruswami-Sudan bound (which is $\sqrt{NK}$) and implies limitations on extending their efficient RS list decoding algorithm to larger decoding radius.

Our proof method is surprisingly simple. We work with polynomials that vanish on subspaces of an extension field viewed as a vector space over the base field. These subspace polynomials are a subclass of linearized polynomials that were studied by Ore in the 1930s and by coding theorists. For us their main attraction is their sparsity and abundance of roots, virtues that recently won them pivotal roles in probabilistically checkable proofs of proximity and sub-linear proof verification.

ODED REGEV, **Tensor-based Hardness of the Shortest Vector Problem to within Almost Polynomial Factors** (joint work with Ishay Haviv)

We show that unless $NP \subseteq RTIME(2^{poly(\log n)})$, the Shortest Vector Problem (SVP) on n-dimensional lattices in the ℓ_p norm ($1 \le p < \infty$) is hard to approximate in polynomial-time to within a factor of $2^{(\log n)^{1-\epsilon}}$ for any $\epsilon > 0$. This improves the previous best factor of $2^{(\log n)^{1/2-\epsilon}}$ under the same complexity assumption due to Khot [Kho05]. Under the stronger assumption $NP \not\subseteq RSUBEXP$, we obtain a hardness factor of $n^{c/\log\log n}$ for some $c > 0$. Our proof starts with SVP instances from [Kho05] that are hard to approximate to within some constant. To boost the hardness factor we simply apply the standard tensor product of lattices. The main novel part is in the analysis, where we show that the lattices of [Kho05] behave nicely under tensorization. At the heart of the analysis is a certain matrix inequality which was first used in the context of lattices by de Shalit [deS06].

Computational Learning

The area of computational learning is concerned with the problems of learning a function, given a number of samples drawn according to some distribution on the inputs to the function. Ryan O'Donnell explained how to test, using very few samples, whether a given function is a Boolean halfspace. Adam Klivans presented results on learning halfspaces. Finally, Scott Aaronson showed how to learn quantum states.

RYAN O'DONNELL, **Testing Halfspaces** (joint work with Kevin Matulef (MIT), Ronitt Rubinfeld (MIT), and Rocco Servedio (Columbia))

In this talk we describe work showing that the class of Boolean halfspaces – -i.e., functions $f : \{-1, 1\}^n \to \{-1, 1\}$ representable as $f(x) = \text{sgn}(c_1x_1 + \cdots + c_nx_n - \theta)$ — has a property testing algorithm making only poly$(1/\epsilon)$ queries.

ADAM KLIVANS, **Agnostically Learning Halfspaces** (joint work with A. Kalai, Y. Mansour, and R. Servedio)

We give the first algorithm that efficiently learns halfspaces (under distributional assumptions) in the notoriously difficult agnostic framework of Kearns, Schapire, and Sellie. In this model, a learner is given arbitrarily labeled examples from a fixed distribution and must output a hypothesis competitive with the optimal halfspace hypothesis.

Our algorithm constructs a hypothesis whose error rate on future examples is within an additive ϵ of the optimal halfspace in time $poly(n)$ for any constant $\epsilon > 0$ under the uniform distribution over $\{0, 1\}^n$ or the unit sphere in R^n, as well as under any log-concave distribution over R^n. It also agnostically learns Boolean disjunctions in time $2^{\tilde{O}(\sqrt{n})}$ with respect to *any* distribution. The new algorithm, essentially L_1 polynomial regression, is a noise-tolerant arbitrary-distribution generalization of the "low-degree" Fourier algorithm of Linial, Mansour, and Nisan. Our Fourier-type algorithm over the unit sphere makes use of approximation properties of various classes of orthogonal polynomials.

SCOTT AARONSON, **The Learnability of Quantum States**

Traditional quantum state tomography requires a number of measurements that grows exponentially with the number of qubits n. But using ideas from computational learning theory, we show that "for most practical purposes" one can learn a state using a number of measurements that grows only linearly with n. Besides possible implications for experimental physics, our learning theorem has two applications to quantum computing: first, a new simulation of quantum one-way protocols, and second, the use of trusted classical advice to verify untrusted quantum advice.

Research Emerging from Workshop

The goal of the workshop was to bring some of the best researchers in the area of computational complexity to discuss the current state of the art in the area, and point out further directions of research. The workshop has been very successful from the point of view of many fruitful interactions among various groups of the workshop participants. Some ideas first discussed during the workshop already found their way into research papers. One example is the following paper *Extractors and Condensers from Univariate Polynomials* by Venkatesan Guruswami, Chris Umans, and Salil Vadhan, which was posted on *Electronic Colloquium on Computational Complexity*, October 2006 [GUV06]. Below is the description of this work provided by the authors.

Context and Genesis: There is a long body of work in theoretical computer science on constructions of both *randomness extractors* — functions that extract almost-uniform bits from sources of biased and correlated bits, and *expander graphs* — graphs that are sparse but highly connected. These two kinds of objects are closely related, and both have a wide variety of applications in theoretical computer science. The paper [GUV06] resulting from the BIRS workshop presents new constructions of both extractors and expanders (described as 'lossless condensers') that significantly improve previous work, while also being simpler and more direct.

The work began with a conversation between the participants Guruswami, Umans, and Vadhan in the Corbett Lounge after Guruswami had presented his new work [GR06] on capacity-achieving error-correcting codes. Indeed, for a few years, it has been known that randomness extractors can be viewed as a generalization of "list-decodable" error-correcting codes. Because of this connection and similarities between the Guruswami–Rudra codes and a previous extractor construction of Shaltiel and Umans [SU01], it seemed natural to explore whether the ideas underlying the Guruswami–Rudra codes and their predecessors [PV05] could be applied to construct better extractors and expander graphs. The participants pursued this idea via email after the workshop, and within a few weeks, the new results had emerged.

Abstract: We give new constructions of randomness extractors and lossless condensers that are optimal to within constant factors in both the seed length and the output length. For extractors, this matches the parameters of the current best known construction [LRV$^+$03]; for lossless condensers, the previous best constructions achieved optimality to within a constant factor in one parameter only at the expense of a polynomial loss in the other.

Our constructions are based on the Parvaresh-Vardy codes [PV05], and our proof technique is inspired by the list-decoding algorithm for those codes. The main object we construct is a condenser that loses *only* the entropy of its seed plus one bit, while condensing to entropy rate $1 - \alpha$ for any desired constant $\alpha > 0$. This construction is simple to describe, and has a short and completely self-contained analysis. Our other results only require, in addition, standard uses of randomness-efficient hash functions (to obtain a lossless condenser) or expander walks (to obtain an extractor).

Our techniques also show for the first time that a natural construction based on univariate polynomials (i.e., Reed-Solomon codes) yields a condenser that retains a $1 - \alpha$ fraction of the source min-entropy, for any desired constant $\alpha > 0$, while condensing to constant entropy rate and using a seed length that is optimal to within constant factors.

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Chapter 18

Algebraic groups, quadratic forms and related topics (06w5025)

Sep 02 – Sep 07, 2006

Organizer(s): Vladimir Chernousov (University of Alberta), Richard Elman (University of California, Los Angeles), Alexander Merkurjev (University of California, Los Angeles), Jan Minac (University of Western Ontario), Zinovy Reichstein (University of British Columbia)

A brief historical introduction

The origins of the theory of algebraic groups can be traced back to the work of the great French mathematician E. Picard in the mid-19th century. Picard assigned a "Galois group" to an ordinary differential equation of the form

$$\frac{d^n f}{dz^n} + p_1(z)\frac{d^{n-1} f}{dz^{n-1}} + \cdots + p_n(z)f(z) = 0\,,$$

where $p_1, \ldots, p_n$ are polynomials. This group naturally acts on the n-dimensional complex vector space V of holomorphic (in the entire complex plane) solutions to this equation and is, in modern language, an algebraic subgroup of $\mathrm{GL}(V)$.

This construction was developed into a theory (now known under the name of "differential Galois theory") by J. F. Ritt and E. R. Kolchin in the 1930s and 40s. Their work was a precursor to the modern theory of algebraic groups, founded by A. Borel, C. Chevalley and T. A. Springer, starting in the 1950s. From the modern point of view algebraic groups are algebraic varieties, with group operations given by algebraic morphisms. Linear algebraic groups can be embedded in GL_n for some n, but such an embedding is no longer a part of their intrinsic structure. Borel, Chevalley and Springer used algebraic geometry to establish basic structural results in the theory of algebraic groups, such as conjugacy of maximal tori and Borel subgroups, and the classification of simple linear algebraic groups over an algebraically closed field. (The latter used the classification of simple Lie algebras, developed earlier by Lie, Cartan, Killing and Weyl.) A more detailed historical account of these developments can be found in [19].

The main focus of the workshop was on linear algebraic groups over fields that are not necessarily algebraically closed. In this context the theory of linear algebraic groups turned out to be closely related to several areas in algebra which previously had an independent existence. Among these areas are Galois theory, the theory of central simple algebras (including Brauer groups and Brauer-Severi varieties), the algebraic theory of quadratic forms, and non-associative algebra. Informally speaking, these connections may be viewed as another manifestation of the idea, championed by F. Klein at the turn of the 20th century. Klein believed many mathematical objects (in particular, in geometry) are best understood and described in terms of their symmetry groups. A crucial role in implementing this idea in the algebraic context (where the objects to be studied are central simple algebras, quadratic forms, octonion algebras, etc.) is played by the theory of Galois cohomology pioneered by J.-P. Serre and J. Tate in the 1950s and 60s.

This approach has been particularly successful within the algebraic theory of quadratic forms. In the context of number theory the study of quadratic forms goes back to Gauss (and probably earlier). The algebraic theory of quadratic forms began with a seminal paper of Witt in 1937, in which what are now called ”Witt's Theorem” and the ”Witt ring” first appeared. But it was not until a remarkable series of papers by Pfister in 1965 - 1967 that the theory was transformed into a significant field in its own right. In these papers, Pfister generalized the well-known two, four, and eight square identities of Euler and Cayley, determined the minimum number of squares representing -1 in an arbitrary field, and developed the finer ring structure of the Witt ring of quadratic forms. This phase of the subject is well documented in the books of Lam [12] and Scharlau [17]. The connection with the theory of algebraic groups was introduced into he subject by T. A. Springer who, in 1959, recasted some of the classical invariants of quadratic forms in terms of the Galois cohomology of the orthogonal group.

Recent Developments and Open Problems

In the past 25 years there has been rapid progress in the theory of quadratic forms (and more generally, in the theory of algebraic groups) due to the introduction of powerful new methods from algebraic geometry and algebraic topology. This new phase began in 1981 with the first use of sophisticated techniques from algebraic geometry and K-theory by A. Merkurjev and A. Suslin who established a deep relationship between Milnor's K-groups and Brauer groups. The Merkurjev-Suslin theorem was a starting point of the theory of motivic cohomology constructed by V. Voevodsky. Voevodsky developed a homotopy theory in algebraic geometry similar to that in algebraic topology. He defined a (stable) motivic homotopy category and used it to define new cohomology theories such as motivic cohomology, K-theory and algebraic cobordism. Voevodsky's use of these techniques resulted in the solution of the Milnor conjecture (for which he was awarded a Fields Medal in 2002) and more recently of the Bloch-Kato conjecture (a detailed proof of the latter is yet to appear in print). For a discussion of the history of the Milnor conjecture and some applications, see [15].

These developments have, in turn, led to a virtual revolution in the theory of quadratic forms. Using motivic methods and Steenrod operations (defined by Voevodsky in motivic cohomology and independently by P. Brosnan on Chow groups), Merkurjev, Karpenko, Izhboldin, Rost and Vishik, and others have made dramatic progress on a number of long-standing open problems in the field. In particular, the possible values of the u-invariant of a field have been shown to include all positive even numbers (by A. Merkurjev, disproving a conjecture of Kaplansky), 9 by O. Izhboldin, and every number of the form $2^n + 1$ by A. Vishik. (Vishik's result is new; it was first announced at the workshop.) Another break-through was achieved by Karpenko, who described the possible dimensions of anisotropic forms in the nth power of the fundamental ideal I^n in the Witt ring, extending the classical theorem of Arason and Pfister.

An unrelated important development in the theory of central simple algebras is the recent proof, by A. J. de Jong, of the long standing period-index conjecture; see [8]. This conjecture asserts that the index of a central simple algebra defined over the function field of a complex surface coincides with its exponent. Previously this was only known in the case where the index of had the form $2^n \cdot 3^m$ (this earlier result is due to M. Artin and J. Tate). In a subsequent paper de Jong and J. Starr found a new striking solution of the period-index problem by constructing rational points on families of Grassmannians. Yet another geometric approach for index-period problem was developed by M. Lieblich. Lieblich's approach is based on constructing compactified moduli stacks of Azumaya algebras and studying their properties. These methods and their refinements are likely to play an important role in future research on currently open problems in the theory of algebraic groups; in particular, on Serre's Conjecture II, Albert's conjecture on cyclicity of central simple algebras of prime degree and Bogomolov's conjecture on the Galois group of a maximal pro-normal closure.

Many fundamental questions in algebra and number theory are related to the problem of classifying G-torsors and in particular of computing the Galois cohomology set $H^1(k, G)$ of an algebraic group defined over an arbitrary field k. In general the Galois cohomology set $H^1(k, G)$ does not have a group structure. For this reason it is often convenient to have a well-defined functorial map from this set to an abelian group. Such maps, called cohomological invariants have been introduced and studied by J-P. Serre, M. Rost and A.

Merkurjev. Among them, the Rost invariant plays a particularly important role. This invariant has been used by researchers in the field for over a decade but the details of its definition and basic properties have not appeared in print until the recent publication of the book [10] by S. Garibaldi, A. Merkurjev and J-P. Serre. This book is expected to give further impetus to this line of research.

The presentations at the workshop were loosely grouped into the following general categories:

- Galois theory
- K-theory
- Algebraic stacks
- Homogeneous spaces
- Arithmetic groups
- Brauer groups
- Quadratic forms in characteristic $\neq 2$
- Quadratic forms in characteristic 2

We will now briefly report on the contents of these presentations.

Galois theory

Lecture by Florian Pop. Let K be an arbitrary field containing an algebraically closed subfield. Let p be a prime number, $p \neq \operatorname{char} F$. Set $G_{K,p}$ to be a Sylow p subgroup of the absolute Galois group of K. Bogomolov's freeness conjecture asserts that the commutator group $[G_{K,p}, G_{K,p}]$ of $G_{K,p}$ is a free pro-p group; or equivalently, it has cohomological dimension one.

This conjecture was motivated by considerations about the cohomology ring $H^*(K)$ of K, with coefficients in, say, μ_p, and in particular by the Merkurjev-Suslin theorem. The evidence before Pop's work relied on generalizations of Tsen's theorem which asserts that if k is algebraically closed and the transcendence degree K/k is 1 then the cohomological dimension of the absolute Galois group is 1. Also if K satisfies Bogomolov's conjecture, then so do its fields of formal power series $K((t))$ in t over K.

This type of question was also investigated by Chernousov–Gille–Reichstein [9], who ask whether the maximal abelian extension K^{ab} of a field as above has cohomological dimension one; more concretely, whether the maximal abelian extension of the rational field in two variables $\mathbf{C}(t,u)^{ab}$ over the complex numbers has cohomological dimension one. If so, then this would have applications to tackling Serre's Conjecture II.

One could say that the power series $K((t))$ are "local" objects over K, thus one should rather speak here about an "obvious evidence" for the above conjectures. The work of Pop aims at giving less obvious evidence for the above two conjectures. In fact, the examples presented by Pop given evidence for an even stronger conjecture, namely that in the cases of interest (i.e., if one considers function fields $K|k$ over some base fields k which contain all the roots of unity) K^{ab} has a *free profinite absolute Galois group*.

The evidence given by Pop is the following:

1) Suppose that k is an algebraic extension of a local field such that k contains all roots of unity. If $K|k$ is a function field in one variable over k, then the absolute Galois group of K^{ab} is profinite free.

2) A more global version of (1): Let p be a prime number. Set $K = \bar{\mathbf{F}}_p(t,u)$, where $\bar{\mathbf{F}}_p$ is an algebraic closure of a field of order p and t, u are algebraically independent variables over $\mathbf{F}_p$. Let k_0 be a rational function field of one variable over $\mathbf{F}_p$ in K and let $\tilde{k}_0$ be a maximal algebraic extension of k_0 in $K_{\rm sep}$ which

is unramified over k_0 outside the infinite place. ($K_{\rm sep}$ is a separable closure of K.) Finally set K^{ab} to be a maximal abelian extension of K. Then the compositum $K^{ab}\tilde{k}_0$ has a free profinite Galois group.

This work is a promising step towards settling Bogomolov's conjecture for fields like $\bar{\mathbf{F}}_p(t,u), \mathbf{C}(t,u), \ldots$ The use of the field $\tilde{k}_0$ in Pop's construction is rather interesting. The advantage of using the field extension $\tilde{k}_0|k_0$ is that it has a well understood arithmetic description. Suppose that $k_0 = \mathbf{F}_p(X)$. Then the roots of Artin-Schreier equations $y^p - y = f(X)$ (where p does not divide $\deg f(X)$) are in $\tilde{k}_0$. Moreover by Abhyankar's conjecture (see [11] or [16]) it is known that all finite groups which are generated by p-subgroups (finite quasi p-groups) are quotients of $\mathrm{Gal}(\tilde{k}_0/k_0)$. More precise information about the solution of Galois embedding problems inside $\tilde{k}_0/k_0$ may be obtained from [16]. Hence $\tilde{k}_0$ is a rather large extension of k_0.

Lecture by John Swallow. Absolute Galois groups of fields are mysterious. Therefore one would like to identify quotients of Galois groups which are non-trivial, and yet possible to completely classify.

A pro-p group A is called a T-group if there exists some maximal closed abelian subgroup B of A (This means in particular that $[A:B]=p$) such that the exponent of B divides p.

Let Γ be a pro-p group and let Δ be a closed subgroup of index p in Γ. Set also $\Phi(\Delta)$ to be a Frattini subgroup of Δ. (This means $\Phi(\Delta) = \Delta^p[\Delta,\Delta]$, the closed subgroup of Δ generated by pth-powers and commutators.) Then we define $T(\Gamma/\Delta) := \Gamma/\Phi(\Delta)$ to be the T-group associated with the pair Γ, Δ. If Γ is an absolute group of a field F and Δ fixes a cyclic extension E/F, then we say that $T := T(\Gamma/\Delta) = T(E/F)$ is the T-group associated with the extension E/F. (In [BeLMS], all such groups are classified.) In fact, one does not need to require that the absolute Galois group of F is a pro-p group; we restrict our attention to this case to simplify the exposition.

Each $T(E/F)$ as above is a T-group. In order to classify all $T(E/F)$ among T-groups one defines certain invariants of T-groups. First recall that the central series $T_{(i)}$ of a group T is defined recursively as follows

$$T_{(1)} = T, T_{(i+1)} = [T, T_{(i)}], i = 1, 2, \ldots.$$

Further, $Z(T)$ is the center of T and $Z(T)[p]$ is the subgroup of elements of $Z(T)$ of order dividing p.

Swallow defined invariants $t_1, t_2, \ldots, t_p$ and u of T by

$$\begin{aligned} t_1 &= \dim_{\mathbf{F}_p} H^1\left(\frac{Z(T)[p]}{Z(T)\cap T_{(2)}}, \mathbf{F}_p\right), \\ t_i &= \dim_{\mathbf{F}_p} H^1\left(\frac{Z(T)\cap T_{(i)}}{Z(T)\cap T_{(i+1)}}, \mathbf{F}_p\right), 2 \le i \le p, \\ u &= \max\{i : 1 \le i \le p, T^p \subset T_{(i)}\}. \end{aligned}$$

and gave a complete description of which values of $t_1, \ldots, t_p$ can occur for T-groups. For an odd prime p he also explained for which values of $t_1, \ldots, t_p$ there exists a field extension E/F as above such that $T \cong T(E/F)$.

Thus if p is an odd prime, the possible quotients $T(E/F)$ of the absolute groups Γ_F are substantially restricted. These restrictions imply further restrictions on the presentation of Γ_F via generators and relations; for details see [5] and [6]. In contrast, if $p=2$, there is no restriction on $T(E/F)$, and all pro-2 T-groups occur for suitable quadratic extensions E/F.

Swallow also described $\mathbf{F}_p[\mathrm{Gal}(E/F)]$ module structure of $H^n(\Gamma_E, \mathbf{F}_p)$, where F/F is a cyclic extension of degree p, F contains a primitive pth-root of 1, and $\Gamma_E \subset \Gamma_F$ are absolute Galois groups of E and F respectively. For details see [13]

In recent joint work with F. Chemotti and J. Mináč, Swallow described the $\mathbf{F}_p[\mathrm{Gal}(E/F)]$-module structure of $H^1(\Gamma_E, \mathbf{F}_2)$ in the case where $\mathrm{Gal}(E/F)$ is $C_2 \times C_2$. An interesting byproduct of this description is that although the Klein-4 group has infinitely many indecomposable modules over $\mathbf{F}_2$, only finitely many of these modules can occur as a summand of $H^1(\Gamma_E, \mathbf{F}_2)$. This fact points out the possibility of obtaining the full structure of Galois modules for other Galois groups, even in cases where the classification of indecomposable modules is a hopeless task.

Lecture by Eva Bayer-Fluckiger. Let F be a field of characteristic different from two and G be a finite group. A G-form is a pair (M, φ) with M an $F[G]$ module of finite F-dimension and φ a quadratic form such that $\varphi(gx, gy) = \varphi(x,y)$ for all $x, y \in M$ and for all $g \in G$. The problem is to determine when are two G-forms $\varphi := (M, \varphi)$ and $\psi := (M, \psi)$, i.e., are G-isomorphic. This problem is a natural generalization of the classical problem of determining when (M, φ) admits a self-dual basis, i.e., when $(M, \varphi) \cong (F[G], q)$. Here $q(\sigma, \tau) = \delta_{\sigma\tau}$ for each $\sigma, \tau \in G$ and $\delta_{\sigma\tau} = 1$ or 0 depending upon whether $\sigma = \tau$ or $\sigma \neq \tau$. The problem of the existence of a self-dual basis is especially interesting when the module $F[G]$ is a Galois field extension L of F, $G = \mathrm{Gal}(L/F)$ and form q is a trace form $q_L : L \times L \to F, (x, y) \to \mathrm{Tr}_{L/F}(xy)$. In [1] it was proved that any Galois algebra has a self-dual basis. The situation is more complicated when $\mid G \mid$ is even; the paper [4] treated the cases where the 2-Sylow subgroup G_2 of the Galois group G is elementary abelian (i.e., $G_2 \simeq C_2 \times \cdots \times C_2$) or a quaternion group of order 8 ($G_2 \simeq Q_8$).

Let $W(F)$ be the Witt ring of non-degenerate quadratic forms over F and $I(F)$ its fundamental ideal of even dimensional forms and $I^n(F)$ its nth power. Suppose that $\mathrm{cd}_2 F$, the cohomological 2-dimension of F, is finite and equal to d. This means that the absolute Galois group Γ_F of F has cohomological dimension d. Let L and L' be two G-Galois algebras and let φ and ψ be their corresponding trace forms. Then in [7] it was proved that $\rho \otimes \varphi \cong_G \rho \otimes \psi$ for any $\rho \in I^d(F)$. Now let L be any Galois algebra over F with Galois group G. Let Γ be an absolute group of F. Then L can be viewed as an element of $H^1(\Gamma, G)$, where the action of Γ on G is trivial. In particular there is a corresponding 1-cocycle $\varphi : \Gamma \to G$, which is just a continuous homomorphism associated to L. Consider any homomorphism $x \in H^1(G, \mathbf{F}_2)$. Set $x_L = x_0\varphi \in H^1(\Gamma, \mathbf{F}_2)$. Keeping our assumption that L and L' are two Galois extensions of F having Galois group G, assume now that $\rho \in I^{d-1}(F)$. Then Bayer-Fluckiger showed that $\rho \otimes \varphi \cong_G \rho \otimes \psi$ if and only if $e_{d-1}(\rho) \cup x_L = e_{d-1}(\rho) \cup x_{L'}$ for all $x \in H^1(G, \mathbf{Z}/2\mathbf{Z})$. Here e_i are the isomorphisms given by the Milnor conjecture $I^i(F) \to H^i(\Gamma_F, \mathbf{Z}/2\mathbf{Z})$. An analogous result holds for (finite) ordered systems of quadratic forms (or hermitian forms) $\Sigma := (\varphi_1, \ldots, \varphi_m)$, with the obvious notion of isomorphism. If Σ and Σ' are two such ordered systems of quadratic forms (respectively hermitian forms) of size m such that they become isomorphic over the separable closure of F then $\rho\Sigma \cong \rho\Sigma'$ for all $\rho \in I^d(F)$ (respectively, $\rho \in I^{d-1}(F)$).

At the end of the lecture Bayer-Fluckiger showed that similar results hold if the field F is replaced by (D, σ), where D is an F-division algebra with involution σ. One can also look at ordered systems of such hermitian forms; for details see [2, 3]

K-theory

Lecture by Stefan Gille. Let X be noetherian scheme and $X^{(i)}$ the set of points in X of codimension i. If $x \in X^{(i)}$ let $F(x)$ be the residue field. We have a Gersten complex

$$(*) \quad 0 \to K'_n(X) \to \oplus_{X^{(0)}} K'_n(F(x)) \to \oplus_{X^{(1)}} K'_n(F(x)) \to \cdots$$

in coherent K-theory. The sequence $(*)$ is exact (Gersten Conjecture) for $X = \mathrm{Spec}\ R$ if R is a regular semi-local ring by work of Quillen and Panin. One wants a similar result for Hermitian Witt groups. Let $\mathcal{M}_c(X)$ be the category of coherent $\mathcal{O}_X$-modules. We have a filtration by Serre subcategories $\mathcal{M}_c(X) = \mathcal{M}^0 \supset \mathcal{M}^1 \supset \cdots$ with $\mathcal{M}^i := \{\mathcal{F} \in \mathcal{M}_c(X) \mid \text{codim supp } \mathcal{F} \geq i\}$. If $\dim X$ is finite, we get a spectral sequence $E_1^{p,q} := K_{p-q}(\mathcal{M}^p/\mathcal{M}^{p+1}) \Rightarrow$ cohomological K-theory of X and $K_{p-q}(\mathcal{M}^p/\mathcal{M}^{p+1}) = \oplus_{X^{(p)}} K_{-p-q}(F(x))$ by Gabriel's thesis and devissage. If A is an Azumaya algebra over X, we can look at $\mathcal{M}_c(A)$ the category of coherent (left) A-modules and filter it by $\mathcal{M}_A^p := \mathcal{M}_c(A) \cap \mathcal{M}^p$. We get another spectral sequence and can ask if the Gersten conjecture is true for it. Replacing X by A and $F(x)$ by $A \otimes F(x)$ in $(*)$, we get a complex introduced by Colliot Thélène and Ojanguran and showed to be exact by them and Panin-Suslin. Gille studies this problem if A has an involution which consists of an automorphism σ of X of order two and an $\mathcal{O}_X$-linear map $\tau : A \to \sigma_* A$ satisfying $\sigma_*(\tau) \circ \tau = 1_A$ and $\tau(ab) = \tau(b)\tau(a)$. The map τ is of the first kind of $\sigma = 1$ and the second kind otherwise. Assume that τ is of the first kind. Gille shows that if (A, τ) is an Azumaya algebra over a regular scheme X of finite dimension with τ of the first kind then there exist two complexes, the *hermitian* and *skew hermitian Gersten-Witt* complexes

$$(*) \quad 0 \to W^{\pm}(A, \tau) \to \oplus_{X^{(0)}} W^{\pm}(A \otimes F(x), \tau \otimes F(x)) \to \cdots$$

and this complex is exact if X is the spectrum of a semi-local ring of a smooth variety. Such a result could not be true if τ is of the second kind as, in general, it would not induce automorphisms of the residue fields. Gille then constructed these Gersten-Witt groups. To show exactness, one follows Quillen's proof but modifying

Quillen's last argument on the additivity of functors to Gille's result that given a Gorenstein ring R of finite Krull dimension, an Asumaya algebra A over R with an involution τ of the first kind, and $t \in R$ an element satisfying $\pi : R \to R/R/Rt$ has a flat splitting then the transfer $\pi_* : W^i(A/tA, \tau/t\tau) \to W^{i+1}(A, \tau)$ is zero.

Lecture by Alexander Nenashev. Balmer-Witt theory does not have Chern classes as it is not oriented and the Projective Bundle Theorem fails. Nenashev showed how to construct a twisted Thom isomorphism and deformation to the normal cone in the theory and used it to show the existence of a pushforward $f_* : W^n(Y, f^*L \otimes \omega_{Y/X}) \to W^{n+c}(X, L)$ for any projective morphism $f : Y \to X$ of pure codimension c where L is a line bundle on X and $\omega_{Y/X}$ is the relative dualizing sheaf. The difficult point is to show if $j : Z \to Y$ and $i : Y \to X$ are closed imbeddings then the pushforwards $(ij)_* = j_* i_*$ which uses the theory of "double" deformation spaces.

Lecture by Marco Schlichting. Balmer-Witt groups of a regular scheme X have long exact Mayer-Vietoris sequences. In general, this is no longer true if the scheme is singular. Schlichting lectured on a way to deal with this by defining new Witt groups called *stabilized Witt groups* generalizing certain L-groups defined by Ranichi and Witt rings with involution defined by Karoubi. Although this theory is not known to hold for triangulated categories with involution, it does hold for categories of rings with involution, exact categories with involution, dg categories, and exact categories with isomorphisms weak equivalences. In particular, in these cases, one can generalize the notion of suspensions and cones. Thes stabilized Witt groups have periodicity 4 and satisfy Mayer-Vietoris and homotopy invariance. They coincide with Witt-Balmer Witt rings if $K_n X = 0$ for all negative n, e.g., if X is regular. The case when the characteristic of the underlying field is zero was also discussed and the relationship with blowups, reflecting work done jointly with G. Cortinas, C. Haesemeyer, and C. Weibel.

Algebraic stacks

Lecture by Patrick Brosnan. Let $\mathcal{F} : \text{fields}/F \to \text{sets}$ be a functor. Merkurjev, generalizing the idea of Buhler-Reichstein defined the essential dimension of a set a to be ed $a := \min\{\text{tr deg}_F K \mid L/K/F$ with $a \in \text{im}(\mathcal{F}(K) \to \mathcal{F}(L))\}$ and the essential dimension of $\mathcal{F}$ to be ed $\mathcal{F} := \{\text{ed}\, a \mid a \in \mathcal{F}(L),\ L/F\}$. If G is a group let ed $G :=$ ed $H^1(-, G)$. Generalizing the definition of essential dimension to include stacks, Brosnan discussed his joint work with Z. Reichstein. An Artin stack can be viewed as a functor from rings to categories (usually groupoids) satisfying various properties. An interesting example of χ is the moduli stack of smooth curves of genus g; there are many other interesting examples, related, e.g., to various other families algebro-geometric objects, such as curves, hypersurfaces, abelian varieties, etc., possibly with additional structures, such as marked points. If χ is a stack then edχ is defined as the essential dimension of the functor $L \longrightarrow \{$isomorphism classes of objects in $\chi_L\}$. Assuming that an Artin stack has a filtration of closed stacks $\chi = \chi_n \supset \chi_{n-1} \supset \cdots \supset \chi_0 = \emptyset$ with $\chi_i \setminus \chi_{i-1} = [Y_i/G_i]$, the stack associated to the G_i-torsors of scheme Y_i where G_i is a linear algebraic group, then they show ed χ is finite. Brosnan also discussed his theorem that the essential dimension of a complex abelian variety A (i.e., of the Galois cohomology functor $H^1(-, A)$) is $2\dim(A)$.

Lecture by Angelo Vistoli. Vistoli lectured on the use of stacks to investigate the theory of hyperelliptic curves. (Cf. the summary of Brosnan's lecture for definitions.) If X is a scheme and G a group acting on X, let $[X/G]$ be the stack associated to G-torsors of X. For example, the stack of moduli spaces of genus g is $[X/PGL_N]$. Define the homology of a stack $\mathcal{F} \to \text{schemes}/F$ by $H^*(\mathcal{F}, \mathbf{Z}) := H^*_G(X, \mathbf{Z})$. Then $\text{Pic}([X/G]) = \text{Pic}_G(X)$, where $\text{Pic}_G(X)$ is the G-equivariant Picard group. The stack of elliptic curves is $[U/\mathbf{G}_m]$ where $U := \{(a, b) \in \mathbf{A}^2 \mid -4a^3 - 27b^2 \neq 0\}$ for elliptic curves (given in Weierstrass form): $y^2 = x^3 + ax + b$. This difficult theorem shows that the stack of elliptic curves is a quotient stack. The problem is to generalize this to find X_g so that the stack of curves of genus g is $[X_g/GL_g]$. For $g = 2$, it an be shown that X_2 is a subspace of $\mathbf{A}^7$. This generalizes to the stack of hyperelliptic curves of genus g, a closed substack of the stack of curves of genus g if $g \geq 2$. Together with A. Arsie, this stack was identified as $[X/G]$ with $X = \{f \mid f$ a homogeneous form of degree $2g+2$ with distinct zeros$\}$ an open subset of $\mathbf{A}^{2g+3}$ and $G = GL_2$ if g is even and $G = \mathbf{G}_m \times PGL_2$ if g is odd (with specified action). Moreover the Picard group of this stack is $\mathbf{Z}/(2g+1))$ if g is even and $\mathbf{Z}/(4(2g+1)))$ if g is odd. The case of trigonal curves was also discussed.

Homogeneous spaces

Lecture by Prakash Belkale. Let G be a simply connected simple algebraic group and P a maximal parabolic subgroup. Belkale lectured on his joint study with S. Kumar on the ring structure of $H^*(G/P, \mathbf{C})$ in terms of structure constants for multiplication of the Schubert basis. By introducing a new twisted product on this basis, they are able to apply to give additional information to the eigenvalue problems and its relation to the Horn Conjecture and the Klyachko, Knutson-Tao theorem on the sum of eigenvalues of hermitian matrices.

Lecture by Kirill Zainoulline. Let G be a linear algebraic group over F and X a projective homogeneous G-variety. One wishes to decompose the Chow motive $\mathcal{M}(X)$ of X into a sum of motives of varieties Y having "trivial splitting patterns". This has been done for some cases, e.g., if G is split, if X has a rational point (by V. Chernousov, S. Gille and A. Merkurjev), or if G is isotropic (by P. Brosnan). So assume that G is anisotropic. If X is an Pfister form then Rost showed that $\mathcal{M} = \oplus R(i)$ with $R(i)$ indecomposable, $R_{\bar{F}} = \mathbf{Z} \oplus \mathbf{Z}(2^{n-1}-1)$. But $R(i)$ is not the motive of a variety. N. Karpenko showed that the motive of the Severi-Brauer variety of a division algebra is indecomposable A. Vishik decomposed the motive of an anisotropic quadric. Zainoulline discussed other cases. A projective smooth variety over F is called *generically split* if $\mathcal{M}(X_{F(X)}) \cong \oplus_* \mathbf{Z}(*)$ and L/F is called a *splitting field* for X if X_L is generically split. Fix a prime p. Let $\bar{A} = \mathrm{CH}(X_L)/p$ where L is a splitting field of X and $\bar{A}_{rat} := \mathrm{im}(\mathrm{CH}(X)/p \to \mathrm{CH}(X_L)/p)$ (cf. the generically discrete invariant of Vishik). If p is prime and there exists a $\rho \in \bar{A}^r$ satisfying $\bar{A}^s = \bar{A}^s_{rat}$ for all $s < r$, $\bar{A}^r = \langle \rho, \bar{A}^r_{rat} \rangle$ and there exists finite subset $\mathcal{B}$ of $\bar{A}_{rat}$ such that $\mathcal{B} \times \{\rho^i\}_{i=0,p-1}$ is a basis for $\bar{A}$ then $\mathcal{M}(X) \otimes \mathbf{Z}/p = \oplus_* R(*)$ with R indecomposable if and only if R has no 0-cycles of degree 1. If X and Y both satisfy the conditions of this result for the same r, X splits over $F(Y)$, and Y splits over $F(X)$ then $R_X \cong R_Y$. This applies to the case of G split with $G = \xi G$, $\xi \in H^1(\Gamma_F, G)$ (an inner form) with Γ_F the absolute Galois group of F and $X = {}_\xi(G/P)$, P a parabolic subgroup, This applies when $X = SB(\mathbf{M}(D))$, where D is an F-division algebra of degree p, an n-fold Pfister form with $p = 2$, ${}_\xi(F_4/P_1)$ with $p = 2$ or 3, and ${}_\xi(E_8/P_8)$ with $p = 5$.

Arithmetic groups

Lecture by Philippe Gille. Let Γ_F be the absolute Galois group of a number field F. If v is a place of F, we will denote the completion of F at v by F_v and the algebraic closure of F_v by $\overline{F_v}$. We will also denote a finite set of primes by S, the ring of integers in F by A, the ring of S-integers by A_S, and the ring of integerst in $\overline{F_v}$ by $\overline{A_v}$. The Borel-Serre Theorem states that for a linear algebraic group G over F, the map $W'_S(F,G) := \ker(H^1(\Gamma_F, G) \to \prod_{v \notin S} H^1(\Gamma_{F_v}, G))$ is proper, i.e., has finite fibers. Gille discussed his joint work with L. Moret-Bailly on the integral version of this theorem.

Let X be variety over F having an action of a linear algebraic group G on it and $Z_0 \subset X$ a flat closed A_S-subscheme. Let $W'_s(x_0) := G(F)\backslash\{x \in X(F) \mid x \in G(F_v)x_0 \text{ for all } v \notin S\}$. This is a finite set. Suppose that G/A_S is a flat affine group scheme and X/A_S is a flat scheme with an algebraic action $G \times_{A_S} X \to X$ given by $g \cdot x \mapsto \rho(g) \cdot x$. Then $G(A_S)\backslash\mathrm{loc}(Z_0)$ is finite, where $\mathrm{loc}(Z_0) := \{Z \subset X \mid Z$ a flat closed A_S-subscheme with $\rho(g_v) : Z \times_{A_S} \overline{A_v} \xrightarrow{\sim} Z \times_{A_v} \overline{A_v}$ for some $g_v \in G(\overline{A_v})$ for all $v \notin S\}$. An example of this is $G = GL_m/\mathbf{Z}$ acting on G by conjugation. Suppose this is the case. Fix $g_0 \in G$. Then there are only finitely many $g \in GL_n\mathbf{Z}$ satisfying $g = g_p g_0 g_p^{-1}$ for $g_p \in GL_m\mathbf{Z}_p$ for all p. The theorem follows from a more general one, viz., if G/A_S is an affine group scheme (but not necessarily flat) then the cohomology set $H^1_{fppg}(A_S, G)$ is finite where $fppf$ is the faithfully flat of finite presentation topology. To prove this one makes various reductions. First one reduces to a flat group scheme over A_S. This can be done because over a number field as the normalization of G is still a group scheme. One shows that the result holds for a flat affine group scheme. Reducing to the case that G is also connected, the result for such G is proven.

Lecture by Uzi Vishne. Vishne discussed his joint work with M. Katz and M. Schaps on traces in congruence subgroups $\Gamma(I)$ of finite index in an arithmetic lattice Γ. Let K be a totally real number field lying in $\mathbf{R}$ via one of the real embeddings so $K \otimes \mathbf{R} = \mathbf{R} \times \mathbf{R}^{d-1}$. Let $\mathcal{O}_K$ be the ring of integers in K and D/K a quaternion algebra with $D \otimes \mathbf{R} = \mathbf{M}_2(\mathbf{R})$ but $D \otimes_\sigma \mathbf{R}$ a division algebra at the $(d-1)$ non-inclusion real embeddings. Let Q be an order in D. The lattice Γ is taken to be Q^1, the elements of norm one. Let $X := \Gamma\backslash\mathcal{H}$ where $\mathcal{H}$ is the upper half plane and $X_I := \Gamma(I)\backslash\mathcal{H}$ where I is an ideal in $\mathcal{O}_K$. Then $X_I \to X$ is a cover of Riemannian manifolds. Let $g(X)$ be the genus of X. The length of the shortest non-trivial closed loop in $\pi_1(X)$ is called the *girth* of X. Vishne and his collaborators showed that for any metric Riemannian surface Y of genus g, one has $(\text{girth}(Y))^2/\text{area}(Y) \leq (\log g)^2/\pi g$ and for $Z = X$, or $X(I)$ above that $(\text{girth}(Z))^2/\text{area}(Z) \geq 4(\log g(Z))^2/9\pi g(Z)$. So $\text{girth}(X_I) \geq (2 \cdot 2/1 \cdot 3)(\log(g(X_I) - c)$ for some constant $c = c(\Gamma)$. All the integers in the coefficient are constants that can be explained except for the second 2. For example, the first 2 is the trace of 1. More generally, if $\pm 1 \neq x \in \Gamma(I) := \ker(Q^1 \to (Q/IQ)^1$ then $|\text{tr}\, x| \geq (N(I)^2/2^d N(2\mathcal{O}_K + \gamma I)) - 2$ where $Q \subset (1/\gamma)Q_0$ with Q_0 the standard order $\mathcal{O}_K[i,j]$ in Q and γ minimal. Computation shows that there exists a constant $\lambda_{D,Q}$ satisfying $[\Gamma : \Gamma(I)] \leq \lambda_{D,Q} N(I)$. This is used to show $\text{girth}(X_I) \geq 4/3(\log(g(X_I)) - \log 2^{3d-5}\text{vol}(X)\lambda_{D,Q}/\pi$. For a Hurwitz surface, i.e., a compact Riemann surface X the order of whose automorphism group achieves the maximum possible size $84(g-1)$, this gives $\text{girth}(X) \geq (4/3)\log(X)$.

Brauer Groups

Lecture by Daniel Krashen. Krashen discussed joint work with M. Lieblich. Let F be a perfect field, D a central F-division algebra, and C a curve over F of genus 1. Krashen discussed the problem of determining the index of $D_{F(C)}$. They show that the index $\text{ind}\, D_{K(C)} := \min\{[E : F(C)] \mid D_E \text{ splits }\}$ is in fact equal to $\min\{[L : F] \mid D_{L(C)} \text{ splits}\}$. This solves the problem if F is a local field, viz., $\text{ind}\, D_{K(C)} = \min\{[L;F] \mid \text{ind}\, D/\text{gcd}(\text{ind}\, D, [L : F]) \text{ divides ind}\, C_L\}$ where $\text{ind}\,(C) := \min\{[E : F] \mid C(E) \neq \emptyset\}$. Krashen then discussed the theory of twisted sheaves and its relation to the index. Let X be a nice scheme over F, i.e., integral, noetherian, ... Let $\alpha \in H^2(X, \mathbf{G}_m)$ (the cohomological Brauer group). An α-*twisted sheaf* on X is a collection of $\mathcal{O}_{U_i}$-modules M_i where $\{U_i\}$ is an (étale) open cover of X with (glueing) isomorphisms $\varphi_{ij} : M_i|_{U_i \cap U_j} \to M_j|_{U_i \cap U_j}$ satisfying $\varphi_{ij}\tilde{\alpha} = \varphi_{jk}\varphi_{ij}$ with $\tilde{\alpha}$ a (Cech) cocycle in the class of α. (This can be shown to be independent of choices.) There exists an α-twisted locally free sheaf of rank r on X if and only if there exists an Azumaya algebra A on X of degree r such that the class of A is α, i.e., α lies in the Brauer group of X. Going to the generic point, this implies that there exists an α-twisted coherent sheaf of rank r on X if and only if $\text{ind}\, \alpha_{F(X)} \mid r$. Next Krashen discussed how this relates to the problem of determining when an element $l \in \text{Pic}\, C$ comes from $L \in \text{Pic}\, C(F)$, i.e., $l = [L]$; equivalently l arises from a line bundle $\mathcal{L}$ on $C_{\bar{F}}$, where $\bar{F}$ is the algebraic closure of F. Choosing isomorphisms $\varphi_{\sigma,\tau} : {}^\sigma\mathcal{L} \to {}^\tau\mathcal{L}$ for σ, τ in the absolute Galois group of F may not be compatible glueing data but does give a 2-cocycle $\alpha_{\sigma,\tau,\gamma}$ in $\mathbf{G}_m$ hence leads to an α_C-twisted line bundle defined over F hence in the Brauer group of F. If C is an elliptic curve and V an α_C-locally free sheaf of rank n then $\text{ind}\, \alpha_{F(C)} = n$ implies there exists E/F of degree n such that α_{C_E} splits.

Quadratic forms in characteristic $\neq 2$

Lecture by Alexandr Vishik. The u-invariant of a field F is defined to be the maximal dimension of anisotropic quadratic forms defined over F. For fields of characteristic different from two, it known that u cannot be $3, 5$, or 7. Merkurjev showed that any even integer could be the u-invariant of a field and Izhboldin showed the value of 9 was achievable. Vishik lectured on his construction of fields having u-invariant $2^n + 1$ for any $n \geq 3$. Let $G(Q,i)$ be the Grassmannian of i-dimensional projective planes in a smooth D-dimensional quadric Q over F for $0 \leq i \leq d := [D/2]$. The *generic discrete invariant* $GDI(Q)$ is defined to be the image of $\text{Ch}^*(G(Q,i) \to \text{Ch}^*(G(Q,i)/\bar{F})$ where $\text{Ch}^*(X)$ is the Chow group of X mod 2 and $\bar{F}$ is the algebraic closure of F. If $Fl(Q,0,i)$ is the flag variety of Q, there exists a correspondence $f : Q \longrightarrow G(Q,i)$. Let $z_j(i-d) = f_*(l_{D-i-j})$ for $D-d-i \leq j \leq D-i$ where $l_0, l_1, \ldots, l_d$ in $\text{CH}_i(Q/\bar{F})$, $\leq i \leq d$, are the classes of projective subspaces of $Q_{\bar{F}}$ of dimension i (choose one if d is even). The ith *elementary discrete invariant* $EDI(Q, i)$ of Q is set $\{j \mid z_j(i-j)$ is defined over F mod 2. To each Q one can draw a $d \times d$ square with the lattice point (x, y) colored if $z_x(y-d)$ is defined over F. Vishik proved the if the characteristic of F is zero and $D = 2^r - 1$ with $r \geq 3$ and the square for Q has only the (d, d) point possibly colored then for all quadrics Q' of dimension $> D$ the invariant $EDI(Q'_{F(Q)})$ will have the same property. In particular, $Q'_{F(Q)}$ is anisotropic. Using this result one can construct a field having u-invariant $2^r + 1$ for any $r \geq 3$ in the usual way. The proof of the theorem utilizes a more general result that Vishik proved, viz., if the characteristic of F is zero and $y \in \mathrm{Ch}^m(Y/\bar{k})$ with Y a smooth quasi-projective variety over F and Q as above then for any $m \leq [(1+D)/2]$, the element y is defined over F if and only if $y|_{F(Q)}$ is defined. The proofs use symmetric operations in cobordism theory. Because of interest in the result, Vishik gave a second lecture with more details of the proofs.

Quadratic forms in characteristic 2

Lecture by Ricardo Baeza. Let F be a field of characteristic two. Let $W(F)$ denote the Witt ring of non-singular symmetric bilinear forms and $I(F)$ the fundamental ideal of even dimensional forms. Let $I^n(F)$ be the nth power of $I(F)$. (Cf. the summary of Hoffman's lecture for definitions and notation.) Let $W_q(F)$ be the Witt group of (even dimensional) non-singular quadratic forms over F; it is a $W(F)$-module. If $a, b \in F$ let $[a, b]$ be the binary quadratic form $ax^2 + xy + by^2$. Every non-singular quadratic form is an orthogonal sum of such binary forms. The submodule $I_q^{n+1}(F) := I^n(F)W_q(F)$ is generated by $(n+1)$-fold quadratic Pfister forms $\varphi \otimes [1, a]$ with φ a bilinear n-fold Pfister form. J. Arason and R. Elman found a presentation for $I^n(K)$ when the field K was of characteristic different from two. Baeza with J. Arason found analogous presentations for $I^n(F)$ and $I_q^{n+1}(F)$ for all n. For $I^n(F)$ the generators are isometry classes $[\mathbf{b}]$ of bilinear n-fold Pfister forms $\mathbf{b}$ with generating relations given by

1. $[\mathbf{b}] = 0$ if $\mathbf{b}$ is metabolic.
2. $[\langle 1, a\rangle \otimes \mathbf{c}] + [\langle 1, b\rangle \otimes \mathbf{c}] = [\langle 1, a+b\rangle \otimes \mathbf{c}] + [\langle 1, ab(a+b)\rangle \otimes \mathbf{c}]$ with $\mathbf{c}$ an $(n-1)$-fold Pfister form and $a + b \neq 0$.
3. $[\langle 1, ab\rangle \otimes \langle 1, c\rangle \otimes \mathbf{d}] - [\langle 1, a\rangle \otimes \langle 1, c\rangle \otimes \mathbf{d}] = [\langle 1, ac\rangle \otimes \langle 1, b\rangle \otimes \mathbf{d}] - [\langle 1, a\rangle \otimes \langle 1, b\rangle \otimes \mathbf{d}]$ with $\mathbf{d}$ an $(n-2)$-fold Pfister form.

where the second relation is only needed if $n = 1$ and for $I_q^n(F)$ the generators are isometry classes of quadratic n-fold Pfister forms $[\varphi]$ with generating relations given by

1. $[\mathbf{c} \otimes [1, d_1 + d_2]] - [\mathbf{c} \otimes [1, d_1]] + [\mathbf{c} \otimes [1, d_2]]$ with $d_1, d_2 \in F$ and $\mathbf{c}$ a (bilinear) $(n-1)$-fold Pfister form.
2. $[\langle 1, a\rangle \otimes \varphi] + [\langle 1, b\rangle \otimes \varphi] = [\langle 1, a+b\rangle \otimes \varphi] + [\langle 1, ab(a+b)\rangle \otimes \varphi]$ with φ a quadratic $(n-1)$-fold Pfister form and $a + b \neq 0$.

where the second relation is only needed for $n = 1$ The proof uses the ideas to prove this result if the field is of characteristic different from two together with a result about forms $[[a_1, \ldots, a_n]]$ defined to be $\otimes_{i=1}^n \langle 1, a_i\rangle \otimes [1, a_1 \cdots a_{n+1}]$ if $a_1, \ldots a_n \in F^\times$ otherwise to be zero. These generate $I^{n+1}(F)$ with generating relations

1. $[[a_1, \ldots, a_{n+1}]] = 0$ if some $a_i = 1$.
2. $[[a_1, \ldots, r^2 a_i, \ldots a_j, \ldots, a_{n+1}]] = [[a1, \ldots, a_i, \ldots r^2 a_j, \ldots, a_{n+1}]]$.
3. $[[a_1, \ldots, a_{n+1}]] = 0$ if some $a_1, \ldots, a_{n+1} \in \wp(F)$.

Lecture by Detlev Hoffman. Let F be a field of characteristic two and $\mathbf{b}$ be a non-degenerate symmetric bilinear form over F. The form $\mathbf{b}$ decomposes as an orthogonal sum of an anisotropic part, unique up to isometry, and a metabolic part and each metabolic form is a sum of binary metabolic forms isometric to $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$. The form $\mathbf{b}$ is diagonalizable if it represents a non-zero element. In particular, the similarity classes of non-degenerate symmetric bilinear form the Witt ring $W(F)$. The even dimensional forms constitute the fundamental ideal $I(F)$ of this ring. We have the usual filtration by the powers $I^n(F)$ of $I(F)$ and $I^n(F)$ are generated by n-fold Pfister forms $\otimes_{i=1}^n \langle 1, a_i \rangle$ for some non-degenerate diagonal binary forms $\langle 1, a_i \rangle$. Let $\overline{I}^n(F) := I^n(F)/I^{n+1}(F)$. The Arason-Pfister Hauptsatz holds, i.e., the non-metabolic forms in $I^n(F)$ have dimension at least 2^n. To each $\mathbf{b}$, we can associate the corresponding quadratic form $\varphi_{\mathbf{b}}, v \mapsto \mathbf{b}(v,v)$. This form is totally singular, i.e., its polar form is trivial. Let $F(\mathbf{b}) := F(\varphi_{\mathbf{b}})$ be the function field of the projective quadric determined by $\varphi_{\mathbf{b}}$. Laghribi showed that $\mathbf{b}_{F(\mathbf{b})}$ is metabolic if and only if $\mathbf{b}$ is a scalar multiple of a Pfister form just as in the case that the field is of characteristic not two. Moreover, we can construct a splitting tower by inductively defining $F_0 = F$ and $F_i = F(\mathbf{b}_i)$ where $\mathbf{b}_i$ to be the anisotropic part of $\mathbf{b}_{F(\mathbf{b}_{i-1})}$. If h is the smallest integer such that $\dim \mathbf{b}_h \leq 1$ then $\mathbf{b}_{h-1}$ is a scalar multiple of an n-fold Pfister form for some n called the degree of $\mathbf{b}$. Let $J_n(F) := \{\mathbf{b}_i \mid \deg \mathbf{b} \geq n\}$ (with the zero form having infinite degree). Then Laghribi showed $J_n(F) = I^n(F)$. If φ is a quadratic form over F then it is an orthogonal sum of a non-degenerate (non-singular) part φ_{ns} and a totally singular part. If φ is a quadratic form over F let $\overline{I}^n(F(\varphi)/F) := \ker(I^n(F) \to \overline{I}^n(F(\varphi)$. Hoffman showed that the following (which proves the second Laghribi result): Let φ be a quadratic form over F. If the non-degenerate part of φ is of dimension at least two then $\overline{I}^n(F(\varphi)/F) = 0$ for all $n \geq 0$ and if $\varphi := \langle 1, a_1, \dots a_l \rangle$, so totally singular, and $2^m = [F^2(a_1, \dots a_l) : F^2]$ then $\overline{I}^n(F(\varphi)/F) = 0$ for $m > n$ and $\overline{I}^n(F(\varphi)/F)$ is generated by the forms $\psi \otimes (\otimes_{i=1}^m \langle 1, b_i \rangle) + I^{n+1}(F)$ with $\psi \in I^{n-m}(F)$ and $b_1, \dots, b_m$ satisfying $F^2(b_1, \dots b_m) = F^2(a_1, \dots a_l)$. This uses the analogue of the Milnor conjecture for quadratic forms in characteristic not two proven by Kato using differential forms.

Lecture by A. Laghribi. Let F be a field of characteristic two. We use the notation and definitions in the talks by R. Baeza and D. Hoffmann. If K/F is a field extension, let $i_K : W(F) \to W(K)$ and $j_K : W_q(F) \to W_q(K)$ be the maps induced by the inclusion $F \subset K$. In the case of fields of characteristic not two, kernels of these maps for various field extensions were studied by R. Elman, A. Wadsworth, T.-Y-. Lam, J.-P. Tignol, and R. Fitzgerald. In characteristic two, the multiquadratic case was studied by D. Hoffmann and Laghribi. Let p be an irreducible monic polynomial in the polynomial ring $F[T] := F[t_1, \dots, t_n]$ (monic relative to a fixed lexicographic ordering) and $F(p)$ the quotient field of $F[T]/(p)$. M. Knebusch proved the Norm Theorem: If $\mathbf{b}$ is an anisotropic symmetric bilinear form then $\mathbf{b}_{F(p)}$ is metabolic if and only if $\mathbf{b}_{F[T]} \cong p\mathbf{b}_{F[T]}$ (without a characteristic assumption) using the theory of specializations and induction, where the case $n = 1$ is handled by the Milnor exact sequence for $W(F(t))$. Aravire-Jacob used the analogue of this sequence for $W_q(F)$ if F is perfect and another if F is not perfect to prove the analogue of the Norm Theorem for non-singular quadratic forms with hyperbolic replacing metabolic. Let φ be a quadratic form then $\varphi \cong \varphi_{ns} \perp \varphi_{ts}$ with φ_{ns} non-singular and φ_{ts} totally singular. (The form φ_{ns} is not unique but φ_{ts} is.). Call a form *semi-singular* if neither summand is trivial. We can study three cases: the form is non-singular, totally singular, or semi-singular. The Norm Theorem for totally singular forms was proven by Hoffmann-Laghribi. This leaves the case of semi-singular quadratic forms. We can also write $\varphi \cong \varphi_{\mathbf{H}} \perp \varphi_0 \perp \varphi_{an}$ where $\varphi_{\mathbf{H}}$ is hyperbolic, φ_0 is the trivial form of some dimension, and φ_{an} is the anisotropic part. Let $i_W(\varphi) = (1/2) \dim \varphi_{\mathbf{H}}$, the *Witt index* of φ and $j_d(\varphi) := \dim \varphi_0$, the *defect index* of φ. Call $i_t(\varphi) = i_W((\varphi) + j_d(\varphi)$ the *total index* of φ. The form φ is called *quasi-hyperbolic* if $\dim \varphi$ is even and $i_t(\varphi) \geq \dim \varphi/2$. The Norm Theorem holds for semi-singular quadratic forms. Its proof depends on this notion of quasi-hyperbolicity replacing hyperbolicity (as it does in the totally singular case). Laghribi-Mammone prove the following Norm Theorem: If φ is anisotropic semi-singular then $\varphi_L = p\varphi_L$ implies that p is inseparable and $\varphi_{F(p)}$ is quasi-hyperbolic and if p is a totally singular quadratic form representing 1 then the converse is true. They also prove a Subform Theorem: If φ is even dimensional and anisotropic and p is a quadratic form such that $\varphi_{F(p)}$ is quasi-hyperbolic then p is totally singular and for all values a of φ_{ns}, non-zero values b of φ_{ts}, and non-zero values c of p , there exists a non-singular form ψ such that $\varphi \cong \psi \perp \varphi_{ts}$ with abp a subform of ψ and acp a subform of $ab\varphi_{ts}$. The proof also uses there theorem that if $p - t_1^{2^m} + d$ (with $m \geq 1$) and $i_w(\varphi_{F(p)}) = \dim \varphi_{ns}/2$ then there exists a non-singular ψ over F such that $\psi_{F(p)}$ is hyperbolic and $\varphi \cong \psi \perp varphi_{ts}$. This also leads to the generalization of when a quadratic form splits over its function field.

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Chapter 19

Topics on von Neumann algebras (06w5086)

September 16 – 21, 2006

Organizer(s): Juliana Erlijman (University of Regina), Hans Wenzl (University of California, San Diego)

The emphasis of the workshop was given to active areas in the theory of von Neumann algebras with connections to other fields as well as to these areas themselves. Among the participants were almost all the leaders in von Neumann algebras, but also representatives from group theory, quantum computing and conformal field theory. As well, in attendance were one Fields medalist (V. Jones), two speakers who gave plenary talks at an ICM (U. Haagerup, 2002, and S. Popa, 2006), and five additional mathematicians who have given invited addresses at ICMs, with two of them at the recent one in 2006 (N. Ozawa and N. Monod). Besides these and other well-established experts in von Neumann algebras and related fields, there were also many younger participants (including recent Ph.D.'s, graduate students, and a good representation of women) who had the opportunity to interact with these leaders. Participants expressed in numerous occasions that the workshop was very stimulating and allowed for fruitful discussions of joint projects. We are glad that the objectives for this workshop were fulfilled.

Short overview of the field and topics targeted in the workshop

Von Neumann algebras are algebras of bounded linear operators on a Hilbert space which are closed under the topology of pointwise convergence. If their center only consists of multiples of the identity, they are called factors. Von Neumann algebras were first studied in a series of papers by Murray and von Neumann in the 1930's, such as in [MvN]. Their motivation was to have a tool for studying quantum mechanics and representations of infinite groups. As it will be seen below, these are still some of the major driving forces in research related to von Neumann algebras with exciting recent developments.

In order to better describe the structure of the workshop scientific content –in which the allotted time was separated into thematic groupings– we will roughly divide the recent activities in von Neumann algebras into the study of subfactors and the study of factors.

1. *Subfactors.* The study of subfactors was initiated by V. Jones in the 1980's by introducing an important invariant for them called the index, [J1]. Moreover, he proved a surprising and fundamental theorem on the set of possible index values and he produced an important class of examples called the Jones subfactors. This class of examples carried a representation of braid groups, and was later used to define link invariants, [J2]. This in turn led to invariants of 3-manifolds and to connections to conformal field theories, representations of loop groups and quantum groups and fusion categories (see e.g. [Wi], [RT], [Wa] and [We]). Some of these

connections will be discussed below.

An important classification result for amenable subfactors of the hyperfinite II_1 factor was proved by S. Popa, [P1]. He showed that they can be reconstructed by what he calls the standard invariant; it is, however, still a wide open problem what values this standard invariant can take in general. The following topics are still very active areas in connection with subfactors, which were addressed in the workshop.

(a) *Fusion*. A. Wassermann's construction [Wa] was in part inspired by results in algebraic quantum field theory, where von Neumann algebras have played an important role for a long time. His crucial result in this context was the definition and explicit computation of a highly nontrivial tensor product between two representations of loop groups of the same level, which is usually called the fusion tensor product. A. Wassermann's fusion can be considered a limiting case of a fusion proposed by G. Segal in connection with conformal field theory. The latter fusion still has not been mathematically rigorously established. Wassermann is currently working on these problems, which he has discussed in his talk in the workshop. In a related development, relative tensor products of bimodules are also investigated by Jones, who gave a talk on his current progress at this workshop as well.

(b) *Conformal Field Theory and Subfactors*. As already mentioned, Wassermann's work established a useful link between conformal field theory and von Neumann algebras: As an application, he obtained examples of subfactors, which could also be constructed on the type II_1 level via quantum groups [We],[Xu1]. However, the conformal field theory approach also yields additional constructions which, at least so far, can not be done via technically less demanding approaches. More progress in the construction of new examples has been achieved in recent works by F. Xu by his own, [Xu2], and also in joint work with V. Kac and R. Longo, [KLX]. Additional constructions are to be expected from the machinery of conformal field theory. Moreover, various aspects of the connections between the theory of von Neumann algebras and algebraic as well as conformal field theory also appeared in talks given by D. Evans, T. Gannon, Y. Kawahigashi, and R. Longo (see 'presentation highlights').

(c) *Other constructions.* There are also constructions of subfactors via other methods. One of these yields the famous Haagerup subfactor, the irreducible subfactor of the hyperfinite II_1 factor of smallest known index > 4, [AH]. This subfactor was obtained from a list of possible standard invariants provided by Haagerup, [H]. There has been recent progress in showing that these other candidates for possible standard invariants probably do not give subfactors, by M. Asaeda [A], about which she gave a talk in this Banff workshop. Her proof uses results by T. Gannon, [CG], coming from his algebraic/combinatorial studies of conformal field theory related topics, and by P. Etingof, D. Nikshych and V. Ostrik, [ENO].

(d) *Further examples/counterexamples.* Popa's recent results (see part 2) have also had consequences for the study of subfactors. In particular, D. Bisch, R. Nicoara, and Popa, [BNP], have recently constructed a continuous family of mutually non-isomorphic irreducible finite-index subfactors of the hyperfinite type II_1 factor with the same standard invariant, about which Bisch gave a talk at this workshop. Also S. Vaes recently showed the existence of type II_1 factors whose only finite-index subfactors are the trivial ones, [Va], using techniques by Popa, A. Ioana and J. Peterson (of the type described in 2.(b)); a talk about these techniques was given at this workshop as well.

(e) Certain tensor categories connected to subfactors and conformal field also play an important role in Freedman's approach towards building a quantum computer. A survey on that was given by E. Rowell.

2. *Factors and free probability.* One of the big problems in von Neumann algebras is the classification of II_1 factors. One can define an important class of examples of such factors from the group von Neumann algebras of infinite discrete groups for which all nontrivial conjugate classes are infinite. However, it is very difficult to decide when these factors are isomorphic. Various invariants for II_1 factors have been introduced by A. Connes, e.g. [Co], and several deep results have been proved by him. He showed all the factors obtained from amenable groups are isomorphic to the hyperfinite II_1 factor. It is known that this factor is not isomorphic to the one obtained from a free group with n generators.

(a) It has been a longstanding unsolved problem to decide whether the factors obtained from the free groups with n and m generators respectively are isomorphic if n is not equal to m with both $n, m > 1$. This problem was one of the inspirations for Voiculescu's theory of free probability. While this has not led to a solution of the original problem yet, it produced many interesting results in its own right as well as surprising applications to the theory of von Neumann algebras such as e.g. Voiculescu's proof of the absence of Cartan subalgebras for free group factors, [Vo]. Voiculescu and his school have found amazing analogs in free probability of well-known phenomena in classical probability, such as e.g. free Fisher information and free entropy. Questions involving these concepts have been discussed in talks in this Banff workshop (see the talks by D. Shlyakhtenko, K. Dykema and K. Jung).

(b) The most exciting developments in the theory of von Neumann algebras in the last few years undoubtedly took place in connection with group theory. D. Gaboriau defined a notion of ℓ^2 Betti numbers for countable measure preserving equivalence relations in a Borel space, [Ga]. This proved a crucial tool in Popa's proof of a long-standing problem in von Neumann algebras, the construction of a II_1 factor with trivial fundamental group, [P2]. In addition Popa has continued proving exciting (super)rigidity results concerning group actions on probability spaces. More precisely, he shows for certain groups acting on probability spaces that an equivalence between their orbits already induces an equivalence between the groups themselves, e.g. [P3]. These results, among others, strongly contributed to the fact that Popa was an invited plenary speaker at the ICM 2006.

There have been similar superrigidity results within geometric group theory by Y. Shalom, N. Monod and A. Furman, with the first two researchers also honored at the recent ICM. From our 2006 workshop, N. Monod and N. Ozawa expect to collaborate on the study of von Neumann algebras associated to arithmetic groups. This is a very active area, with interesting new results also coming from young people such as A. Ioana and J. Peterson who also spoke at our workshop.

Additional interesting results were recently obtained by N. Ozawa: Based on his notion of solid von Neumann algebras, [O1], he obtained many examples of prime factors (i.e. II_1 factors which are not the tensor product of two II_1 factors) [O2]; moreover, in collaboration with Popa, [OP], they prove unique prime factorization results for tensor products of factors coming from subgroups of hyperbolic groups. New results in this direction have also been presented in this workshop by Peterson.

(c) In another interesting development Haagerup talked about his recent work with M. Musat on classification of hyperfinite factors up to completely bounded isomorphism of their preduals. It follows from their work that they can distinguish between various hyperfinite III_0 factors. This is an important class of factors which still are not very well understood. So this is an important result whose details have not appeared yet. The Banff workshop turned to be useful for both Haagerup and Musat to continue their collaboration.

Presentation Highlights (following same thematic order as in the workshop schedule)

Day 1.

Sorin Popa (University of California, Los Angeles) *On the Superrigidity of Malleable Actions* (Abstract): Let $\Gamma \curvearrowright X$ be a measure preserving action of a countable discrete group on a probability space. It is well understood by now that some weak form of property (T) for Γ combined with a *malleability* assumption on the way it acts on X entails sharp rigidity phenomena for the associated II_1 factor and orbit equivalence relation. I will present a new set of rigidity results for malleable actions, in which the property (T) assumption on Γ is no longer needed. Instead, the group needs to have a non-amenable subgroup H with infinite centralizer.

Jesse Peterson (University of California, Berkeley) *L^2-rigidity in von Neumann algebras* (Abstract): I will present a new approach for showing primeness in von Neumann algebras. Specifically I will apply Popa's deformation/rigidity techniques in the context of Sauvageot's deformations arising from closable derivations to conclude that all free product II_1 factors, as well as all group factors arising from groups with positive first L^2-Betti number are prime. These techniques also give a new approach to Ozawa's result that all nonamenable subfactors of a free group factor are prime.

Narutaka Ozawa (University of Tokyo) *A comment on the free group factors* (Abstract): For a finite von Neumann algebra M, there are natural inclusions $M \subset L^2 \subset L^1$. I will talk about the space of those operators in $B(L^2)$ that are compact when viewed as operators from M into L^p ($p = 2, 1$). I will particularly discuss the free group factors.

Nicolas Monod (University of Geneva) *Splitting and rigidity* (Abstract): The study of Orbit Equivalence for type II_1 relations is a classical topic in ergodic theory. It is known to be equivalent to the von Neumann algebra setting of Cartan subalgebras of factors of type II_1. We start by presenting results from joint work with Shalom in which strong restrictions, and indeed superrigidity, are established for relations produced by suitable product groups. These results hinge upon bounded cohomology, in particular upon a splitting theorem in the latter formalism. We then briefly present a set of results bearing a formal analogy with the preceding: A general version of Margulis' superrigidity for latices in products, which relies on a geometric splitting theorem generalising the classical Lawson-Yau/Gromoll-Wolf theorem for Hadamard manifolds.

Mikaël Pichot (Institut des Hautes Etudes Scientifiques (IHES)) *The space of triangle buildings* (Abstract): I will present the notion of measured equivalence relation from a geometric point of view and discuss the concentration of measure property in that framework. Examples of measured equivalence relations arise, for instance, from nonsingular actions of groups on probability spaces, foliation theory, questions of classification,.... A part of the talk will be devoted to the problem of classification of triangle buildings (joint work with Sylvain Barre).

Adrian Ioana (University of California, Los Angeles) *Orbit inequivalent actions for groups containing a copy of F_2.*

Day 2.

Vaughan Jones (University of California, Berkeley) *Connes tensor product in quantum physics?* (Abstract): We present evidence for treating highly constrained physical systems using the Connes tensor product of correspondences. The idea is that the constraints should force some observables on one system to be identified with observables on the other system.

Pinhas Grossman (Vanderbilt University) *Forked Temperley-Lieb Algebras and Intermediate Subfactors* (Abstract): We consider noncommuting pairs P, Q of intermediate subfactors of an irreducible, finite-index inclusion N in M of II_1 factors such that P and Q are supertransitive with Jones index less than 4 over N. We show that in the hyperfinite case, there is a unique such pair corresponding to each even value $[P : N] = 4\cos^2(\pi/2n)$ but none for the odd values $[P : N] = 4\cos^2(\pi/(2n + 1))$.

Antony Wassermann (CNRS, Institut de Mathématiques, Luminy) *Segal and Connes' fusion*: In this talk an important technical problem for a rigorous definition of Segal fusion is dicussed, fusion with corners. This leads to classical questions in connection with uniformization.

V.S. Sunder (IMSc, Chennai) *Kac algebras, doubles and planar algebras* (Abstract): We wish to describe the planar algebra of the 'double' (=asymptotic inclusion) of the fixed subfactor $R^H \subset R$ of an outer action of a finite-dimensional Kac algebra on the hyperfinite factor. It turns out that this happens to be identifiable as a sub-planar algebra of the related subfactor $R^{H^{*op}} \subset R$. We describe this result, some corollaries and an idea of the ingredients of the proof.

Dietmar Bisch (Vanderbilt University) *A continuous family of hyperfinite subfactors* (Abstract): I will present a construction of continuous families of non-isomorphic, irreducible, finite index subfactors of the hyperfinite II_1 factor with the same standard invariant. This is joint work with Remus Nicoara and Sorin Popa.

Masaki Izumi (Kyoto University) *Type III factors distinguish (some) type III E_0-semigroups* (Abstract): I'll give an account of new examples of uncountably many type III E_0-semigroups, which are distinguished by the type of analogues of the local observable algebras. Joint work with R. Srinivasan.

Eric Rowell (Purdue University) *Algebraic Problems in Topological Quantum Computing*: A brief overview was given about the Freedman approach towards building quantum computers. This motivated various algebraic and combinatorial questions related to unitary tensor categories, some of which were discussed in more detail.

Stefaan Vaes (Institut de Mathématiques de Jussieu) *Type II_1 factors without non-trivial finite index subfactors* (Abstract): We call a subfactor trivial if it is isomorphic with the diagonal inclusion of N into matrices over N. We prove the existence of type II_1 factors M such that every finite index subfactor is trivial. Also, every M-M-bimodule with finite coupling constant, is a multiple of $L^2(M)$. In particular, these II_1 factors do not have outer automorphisms: such factors were shown to exist by Ioana, Peterson, Popa and our methods are a generalization of theirs.

Day 3.

Uffe Haagerup (University of Southern Denmark) *Classification of hyperfinite factors up to completely bounded isomorphism of their preduals (joint work with Magdalena Musat)* (Abstract): By a result of Christensen and Sinclair, all infinite dimensional hyperfinite factors are cb-isomorphic (i.e. isomorphic as operator spaces), but if one looks at the preduals instead, the story is totally different: It turns out, that one can for instance separate Type II from Type III this way, and there are uncountably many non-cb-isomorphic preduals of hyperfinite Type III_0 factors, while the preduals of hyperfinite Type III_λ factors are all isomorphic when $0 < \lambda \leq 1$. The proof uses Connes classification of injective (=hyperfinite) factors and the Connes-Takesaki "flow of weights" for Type III-factors.

Kenley Jung (University of California, Los Angeles) *Microstate Spaces and Geometric Measure Theory in Free Probability* (Abstract): I will discuss the microstate theory in free probability and its operator algebra applications. Emphasis will be placed on the geometric measure theory approach to microstates and on how this alternative interpretation leads to nonisomorphism results for von Neumann algebras.

Dmitri Shlyakhtenko (University of California, Los Angeles) *Estimates for free entropy dimension* (Abstract): We discuss some new and old estimates on free entropy dimension and connections with a free analog of an inequality of Otto and Villani (previously considered in the free case by Biane and Voiculescu) that occurs in their work on the Talagrand inequality.

Ken Dykema (Texas A&M University) *Free Entropy Dimension in Amalgamated Free Products* (Abstract): We calculate the free entropy dimension of natural generators in an amalgamated free product of the hyperfinite II_1-factor with itself, with amalgamation over an atomic, type I subalgebra. In particular, some 'exotic' Popa algebra generators of free group factors are shown to have the expected free entropy dimension. (Joint work with Nate Brown and Kenley Jung.)

Day 4.

Terry Gannon (University of Alberta) *The braid group and modular forms (among other things)*: In this talk it is argued that for various problems in conformal field theory the group $SL(2, Z)$ should be replaced by the 3-strand braid group.

Marta Asaeda (University of California, Riverside) *Galois group obstruction to principal graphs* (Quoted from math.OA/0605318): The Galois group of the minimal polymonal of a Jones index value gives a new type of obstruction to a principal graph, thanks to a recent result of P. Etingof, D. Nikshych, and V.Ostrik. We show that the sequence of the graphs given by Haagerup as candidates of principal graphs of subfactors, are not realized as principal graphs for $7 < n \leq 27$ using GAP program. We further utilize Mathematica to extend the statement to $27 < n \leq 55$. We conjecture that none of the graphs are principal graphs for all $n > 7$, and give evidence using Mathematica for smaller graphs among them for $n > 55$. The problem for the case $n = 7$ remains open, however, it is highly likely that it would be realized as a principal graph, thanks to numerical computation by Ikeda.

David Evans (Cardiff University) *Modular invariants, Subfactors and Twisted K-theory*.

Shamindra Ghosh (Vanderbilt University) *Planar algebras (Abstract): A category theoretic point of view*: We define Jones's planar algebra as a map of multicategories and construct a planar algebra starting from a 1-cell in a pivotal strict 2-category. We introduce the concept of an affine representations of a planar algebra and prove some finiteness results for the affine representations of finite depth planar algebras. We also show that the radius of convergence of the dimension of an affine representation of the planar algebra associated to a finite depth subfactor is at least as big as the inverse-square of the modulus.

Yasuyuki Kawahigashi (University of Tokyo) *Superconformal nets of factors and their classification* (Abstract): Super Virasoro nets with central charge less than $3/2$ are constructed as Fermionic extensions of certain coset nets arising from the $SU(2)$-nets, as studied by A. Wassermann and F. Xu. This construction is an operator algebraic counterpart for the Goodard-Kent-Olive coset construction for the discrete series for the $N = 1$ superconformal algebras. We study their extensions, and give a complete classification, using the work on modular invariants by Cappelli and Gannon-Walton. This is a "super" counterpart of our previous complete classification of local conformal nets with central charge less than 1. This is a joint work with Roberto Longo.

Roberto Longo (University of Rome Tor Vergata) *Nuclearity for inclusions of real Hilbert spaces, representations of $SL(2, R)$ and CFT* (Abstract): (Based on a joint work with C. D'Antoni and D. Buchholz) We introduce a new type of spectral density condition, that we call L^2-nuclearity. One formulation concerns lowest weight unitary representations of $\mathrm{SL}(2, R)$ and turns out to be equivalent to the existence of characters. A second formulation concerns inclusions standard real Hilbert subspaces of a complex Hilbert spaces. We consider Moebius covariant nets of real Hilbert subspaces associated with interval on the circle and set up a relation with the above nuclearity conditions. We show the corresponding nuclearity conditions to agree for a local conformal net of von Neumann algebras on the circle (chiral conformal Quantum Field Theory) and, starting from the trace class condition for the semigroup generated by the conformal Hamiltonian L_0, we infer and naturally estimate the Buchholz-Wichmann nuclearity condition and the (distal) split property. As a corollary, if L_0 is log-elliptic, the Buchholz-Junglas set up is realized and so there exists a beta-KMS state for the translation dynamics on the net of C^*-algebras for every inverse temperature beta> 0. We further mention a formulation on higher dimensional spacetimes. In particular, L^2-nuclearity is satisfied for the scalar, massless Klein-Gordon field.

Feng Xu (University of California, Riverside) *Mirror extensions of local nets* (Abstract): In this talk we will discuss a general theorem which under certain conditions constructs extensions of local nets from given ones. Such extensions are called mirror extensions since the corresponding link invariants are related to their mirror images in the given nets. When applying the theorem to conformal inclusions and diagonal cosets, we obtain infinite series of new examples of completely rational chiral conformal field theories. The talk is based on math.QA/0505367.

Final comments

As mentioned at the beginning, the workshop was very successful in bringing together almost all the leading experts in von Neumann algebras as well as researchers from related areas. We think it provided an excellent reflection of the current exciting developments in this subject and its influences on/from other areas. This should be particularly helpful for the many younger researchers which attended our workshop.

We received positive comments and feedback about the meeting from many participants. So we believe that it was indeed stimulating and did contribute to further progress in our field.

List of Participants

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Dykema, Ken (Texas A&M University)
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Erlijman, Juliana (University of Regina)
Evans, David (Cardiff University)
Gannon, Terry (University of Alberta)
Ghosh, Shamindra (University of New Hampshire)
Goodman, Frederick (University of Iowa)
Grossman, Pinhas (University of California, Berkeley)
Haagerup, Uffe (University of Southern Denmark)
Hauschild Mosley, Holly (Grinnell College)
Ioana, Adrian (Caltech)
Izumi, Masaki (Kyoto University)
Jones, Vaughan (University of California, Berkeley)
Jung, Kenley (University of California, Los Angeles)
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Rowell, Eric (Texas A&M University)
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Vaes, Stefaan (Institut de Mathématiques de Jussieu)
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Chapter 20

Positive Polynomials and Optimization (06w5060)

October 7 – October 12, 2006

Organizer(s): Salma Kuhlmann (University of Saskatchewan), Sanjay Lall (Stanford University), Victoria Powers (Emory University), Frank Sottile (Texas A&M University)

The following is the scientific report on our 5-days workshop "Positive Polynomials and Optimization". The aim of holding this workshop at this time was to continue a recent tradition of bringing researchers in optimization to gain access to the new mathematical tools related to positive polynomials. The conference was to give an opportunity for pure mathematicians to interact and exchange ideas/results with applied researchers in optimization. In a sense, the distinguishing property of the real numbers is positivity. There has been a recent explosion of interest in positive polynomials. This is due to the many interesting applications, the introduction of numerical algorithms for computing with sums of squares, as well as new theoretical results about sums of squares and representations of positive polynomials. Positive polynomials can be used to formulate problems in control theory, optimization, and other areas, and then these problems can be solved using the theory of positive polynomials coupled with numerical techniques from semidefinite programming.

This tradition goes back to a February 2002 meeting, "Positivität von Polynomen" held at Oberwolfach. This was the first time that people working on theoretical and algebraic geometrical aspects of positive polynomials had gotten together with people in optimization. Interest in this subject has grown through subsequent meetings; this was one of several topics at the MSRI Semester on Topological Aspects of Real Algebraic Geometry in Spring 2004, a meeting on optimization was held in Amsterdam in June 2004, a Luminy workshop on Positive Poynomials was held in March 2005, and Kuhlmann, Lall and Sottile have taught several short courses on real algebra and optimization at the "Trimestre Géométrie Algébrique Réelle" held at Institut Henri Poincaré, Paris September to December, 2005. These activities have featured particular aspects of these developments and have neither been comprehensive nor brought people together again for a period of sustained interaction. We felt that it was time for a comprehensive meeting on this subject to give the people working in different mathematical areas an overview of the developments since February 2002. The goal of this BIRS workshop was to deepen the connections between people working in different communities, as well as to disseminate these ideas and developments to students, post-doctoral fellows, and researchers working in nearby areas of mathematics. Special effort was made to invite "young researchers" to participate; several graduate students and post-doctoral fellows attended the meeting, and contributed actively to its success.

Overview of the Field

A multivariate polynomial is positive if it takes only non-negative values on its domain. For example, if a polynomial is a square, or a non-negative linear combination of squares (sos), then it is positive and this representation is an algebraic certificate of its positivity. Unfortunately, Hilbert showed that a polynomial which is positive on all of real affine n-space, where $n > 1$, is not necessarily sos. One may similarly ask about polynomials that are positive on some domain defined by polynomial inequalities (a semi-algebraic set). The solution of the moment problem by Schmüdgen, Putinar, and Jacobi and Prestel shows that polynomials positive on compact semi-algebraic sets have a similar algebraic certificates of positivity involving squares and the polynomials defining the semi-algebraic set.

Finding such a certificate having a particular form and involving squares of bounded degree d may be formulated and thus solved as a semidefinite program (SDP), for which there exist very efficient numerical tools. For example, Parrilo has developed software, SOStools, using SDPs to numerically solving problems involving sums of squares.

Lassere used this methodology to devise an algorithm to compute the minimum value of a polynomial f(x) on a compact semialgebraic set (an NP-complete problem, so it is hopeless to find an efficient algorithm). This algorithm finds the largest number m_d such that $f(x) - m_d$ has an algebraic certificate of positivity on the set, involving squares of degree at most d. This sequence of approximate minima m_d converges to the actual minimum, and thus we obtain a sequence of computationally feasible relaxations to this problem in class NP.

At the same time as these mathematical developments, there has been a new emphasis on control problems that are intrinsically computationally hard. There is a growing need for control techniques for problems where there is an explicit combinatorial structure, such as power control in wireless sensor networks with interference, combined task assignment and path planning for multiple vehicle systems, and multi-limbed robotic systems. There is also renewed interest and applications for control systems where the source of apparent intractability is not due to a combinatorial growth of discrete possibilities; instead the need for decentralization introduces complexity due to the structure of the desired control design. Examples of this kind of system include formation control of vehicles, and congestion control in wireline networks.

A certificate provides a mathematical proof which is easily verifiable automatically. In control theory, it is used to show that desirable outcomes will always occur, such as in stability analysis, where one shows that for all initial conditions states converge to the equilibrium. Equivalently, it is used to show that undesirable outcomes never occur, for example in reachability analysis, where one shows that two vehicles will never collide. The idea of certificates is a striking example of the common threads that interconnect the fields of optimization, algebraic geometry, control, and computational complexity. Another example of the applicability of these methods is found in recent work by K. Gatermann and P. Parrilo which looks at the global optimization problem for polynomials invariant under certain small finite groups. Using invariant theory, they show that the SDPs involved decompose into a series of much smaller, hence more computationally tractable, SDPs. They have used this idea to solve problems arising in chemistry, where the polynomials involved much symmetry.

Recent Developments and Open Problems

Recent developments, presented at the workshop, include the following: Lasserre's SDP relaxations exploiting some structural properties of the positive polynomials under consideration. Lasserre considers "sparse polynomials", that is, polynomials in which the variables occuring in the monomials satisfy specific partition and overlap conditions. On the same topic, and exploiting these ideas, Nie lectured on optimization of such structured polynomials. Both talks were very well recieved by the audience, and produced many intense discussions among the participants. Several open problems emerged naturally: e.g. to provide other, algebraic or analytic proofs of Lasserre's Theorem on the representation of sparse positive polynomials by sparse sums of squares. See the section "outcome of the workshop"below.

Curto's talk highlighted new aspects of the moment problem, by presenting his ideas about truncated moment problems; that is, representation by linear functionals of truncated moment sequences. This approach calls for many natural questions, for example, to investigate the various approximation methods available to date (such as saturation and closure of the preorderings, Schmúdgen's positivstellensatz) to polynomials of bounded degrees only.

Laurent's talk presented very interesting ideas to develop an algorithmic approach to compute real radical ideals, in the spirit of Gröbner basis in algebraic geometry. Many participants were inspired by these new ideas. A Ph.D student from Regensburg (Doris Augustin) have been studying the membership problem for a preorder in the polynomial ring (that is, searching for conditions which are necessary and sufficient for a polynomial f to lie in the preorder generated by finitely many other polynomials). She found a lot of inspiration in Laurent's talk and is now analyzing the relationship between the two approaches.

Netzer (Ph.D student from Konstanz, Germany) presented a new, elementary proof of Schmüdgen's celebrated result that a finitely preordering satisfies the strong moment property if the preorderings associated to the fibers do (a finitely generated preordering in the real polynomial ring is said to have the Strong Moment Property, if its closure with respect to the finest locally convex topology equals its saturation). Netzer solved another open problem: In [8] and [9], a property which implies the Strong Moment Property, known as the "Double-Dagger Property", was found. Netzer shows that this property is strictly stronger than the Strong Moment Property.

Scheiderer's talk presented recent progress made in [2] concerning representation of invariant (under the action of a locally compact group) positive polynomials by sums of squares. Several exciting open problems emerge from this work, for example, to characterize invariant semi-algebraic sets for which the invariant moment problem is solvable (that is, invariant linear functionals are represented by an invariant measure with support in the given invariant semi-algebraic set) whereas the usual moment problem fails (that is, not every linear functional is representable by a measure). His talk was related in many aspects to Theobald's talk about exploiting invariance in SDP-relaxations for polynomial optimization.

Presentation Highlights

List of Scientific Presentations:

Sunday

Mihai Putinar, *The trigonometric moment problem in several variables and related SOS decompositions*
Pablo Parillo, *Exact semidefinite representations for genus zero curves*
Konrad Schmüdgen, *Positivity and Positivstellensätze for matrices of polynomials*
Bill Helton, *Real algebraic geometry in a free *-algebra*
16:30–17:00 Second Chances

Monday

Raul Curto, *Algebraic geometric techniques for the truncated moment problem*
Claus Scheiderer, *Sums of squares and moment problems with symmetries*
Tim Netzer, *An elementary proof of Schmüdgen's Theorem*
Luis Zuluaga, *Closed-form solutions to certain moment problems with applications to business*
17:00–17:30 Second Chances

Tuesday

M.-F. Roy, *Certificate of positiviy in the Bernstein basis*
11:30–12:00 Second Chances

Wednesday

Monique Laurent, *A numerical algorithm for the real radical ideal*
Thorsten Theobald, *Symmetries in SDP-based relaxations for constrained polynomial optimization*
Markus Schweighofer, *Global optimization of polynomials using gradient tentacles and sums of squares*
Jiawang Nie, *Sparse SOS Relaxation and Applications*
Murray Marshall, *Representations of non-negative polynomials and applications to optimization*
17:00–17:30 Second Chances

Thursday

Bruce Reznick, *Hilbert's construction of psd quartics and sextics that are not sos*
J.-B. Lasserre, *A Positivstellensatz which preserves the coupling of variables*
11:00–11:30 Last Chances

ABSTRACTS

Speaker: **Raúl Curto** (Iowa)
Title: *Algebraic Geometric techniques for the truncated moment problem*
Abstract: For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \le 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy rank$\mathcal{M}(n) \le$ card$\mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \le 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $p(\beta) := \sum_{|i| \le 2n} a_i \beta_i = 0$. In joint work with Lawrence Fialkow and Michael Möller, we employ tools and techniques from algebraic geometry (e.g., Hilbert polynomials, Gröbner and H-bases, representation of positive polynomials) to prove that for the *extremal* case (rank$\mathcal{M}(n) =$ card$\mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, rank$\mathcal{M}(n)$-atomic) representing measure.

Truncated moment problems (TMP) as above for which the support of a representing measure is required to lie inside a closed set K are called truncated K-moment problems (TKMP). In case K is a semi-algebraic set determined by polynomials $q_1, ..., q_m$, the study of TKMP is dual to determining whether a polynomial nonnegative on K belongs to the positive cone consisting of polynomials of degree at most $2n$ which can be expressed as sums of squares, and of squares multiplied by one or more distinct q_i's.

The extremal case, which we have now solved, is inherent in the TMP. A recent result of C. Bayer and J. Teichmann (extending a classical theorem of V. Tchakaloff and its successive generalizations given by I.P. Mysovskikh, M. Putinar, and L. Fialkow and the speaker) implies that if $\beta^{(2n)}$ has a representing measure, then it has a finitely atomic representing measure. Fialkow and the speaker had previously shown that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ admits an extension to a positive moment matrix $\mathcal{M}(n+k)$ (for some $k \ge 0$), which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $\mathcal{M}(n+k+1)$. Further, we proved that any flat extension $\mathcal{M}(n+k+1)$ is an extremal moment matrix for which there is a computable rank $\mathcal{M}(n+k)$-atomic representing measure μ. In this sense, the existence of a representing measure for $\beta^{(2n)}$ is intimately related to the solution of an extremal TMP.

Speaker: **Bill Helton** (UC San Diego)
Title: *Real Algebraic Geometry in a Free $*$-Algebra*
Abstract: The talk will describe recent results and focus on new directions.

Speaker: **J.-B. Lasserre** (Toulouse)
Title: *A Positivstellensatz which preserves the coupling of variables*
Abstract: We specialize Schmüdgen's Positivstellensatz and its Putinar and Jacobi–Prestel refinement, to the case of a polynomial $f^2\mathbb{R}[X,Y] + \mathbb{R}[Y,Z]$, positive on a compact basic semi-algebraic set K described by polynomials in $\mathbb{R}[X,Y]$ and $\mathbb{R}[Y,Z]$ only, or in $\mathbb{R}[X]$ and $\mathbb{R}[Y,Z]$ only (i.e. K is cartesian product). In particular, we show that the preordering $P(g,h)$ (resp. quadratic module $Q(g,h)$) generated by the polynomials $\{g_j\} \subset \mathbb{R}[X,Y]$ and and $\{h_k\} \subset \mathbb{R}[Y,Z]$ that describe K, is replaced with $P(g) + P(h)$ (resp. $Q(g) + Q(h)$), so that the absence of coupling between X and Z is also preserved in the representation. A similar result applies with Krivine's Positivstellensatz involving the cone generated by $\{g_j, h_k\}$.

Speaker: **Monique Laurent** (CWI, Amsterdam)
Title: *Semidefinite characterization and computation of real radical ideals*
Abstract: For an ideal $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ given by a set of generators $h_1, \ldots, h_m$, we propose a semidefinite characterization and a numerical method for finding the real radical ideal $\sqrt[\mathbb{R}]{I} = I(V_\mathbb{R}(I))$, provided it is zero-dimensional (even if I is not). Our method relies on expressing $I(V_\mathbb{R}(I))$ as the kernel of a suitable positive semidefinite moment matrix and uses semidefinite optimization for finding such a matrix.

One of our results can be sketched as follows. Let $M_t(y)$ be a maximum rank feasible solution to the system:

$$M_t(y) \succeq 0,\ M_{t-d_j}(h_j y) = 0\ (j = 1, \ldots, m),$$

where $d_j := \lceil \deg(h_j)/2 \rceil$ and $t \ge \max_j d_j$. Then, $I(V_\mathbb{R}(I)) = \langle \mathrm{Ker} M_t(y) \rangle$ if the rank condition: $\mathrm{rank} M_t(y) = \mathrm{rank} M_{t-d}(y)$ holds. A maximum rank solution $M_t(y)$ can be found with a semidefinite programming solver; if the rank condition holds we have found $I(V_\mathbb{R}(I))$, otherwise iterate replacing t by $t+1$. The algorithm is +guaranteed to terminate when $V_\mathbb{R}(I)$ is finite. With our method we can compute directly from the optimal matrix $M_t(y)$ the following objects: the set $V_{\mathbb{R}}(I)$ of real roots, a linear basis of the quotient vector space $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$, a border basis of $I(V_{\mathbb{R}}(I))$ as well as a Gröbner basis for a total-degree monomial ordering.

A feature of our method is that it exploits right from the beginning the real algebraic nature of the problem. In particular, it does not need the determination of a Gröbner basis of the ideal I and we do not compute (implicitly or +explicitly) the complex variety $V(I)$. The method also applies to finding $I(V_{\mathbb{R}}(I) \cap S)$ where S is basic closed semialgebraic set.

This is joint work with J.-B. Lasserre and P. Rostalski

Speaker: **Murray Marshall** (Saskatchewan)
Title: *Representations of non-negative polynomials, degree bounds and applications to optimization*
Abstract: Natural sufficient conditions for a polynomial to have a local minimum at a point are considered. These conditions tend to hold with probability 1. It is shown that polynomials satisfying these conditions at each minimum point have nice presentations in terms of sums of squares. Applications are given to optimization on a compact set and also to global optimization. In many cases, there are degree bounds for such presentations. These bounds are of theoretical interest, but they appear to be too large to be of much practical use at present. In the final section, other more concrete degree bounds are obtained which ensure at least that the feasible set of solutions is not empty.

Speaker: **Tim Netzer** (Konstanz)
Title: *An Elementary Proof of Schmüdgen's Theorem on the Moment Problem of Closed Semi-Algebraic Sets*
Abstract: We discuss a more elementary proof of the main result from Schmüdgen's 2003 article "On the moment problem of closed semi-algebraic sets". The result states, that the question whether a finitely generated preordering has the so called Strong Moment Property can be reduced to the same question for preorderings corresponding to fiber sets of bounded polynomials.

Speaker: **Jiawang Nie** (IMA, Minnesota)
Title: *Sparse SOS Relaxation and Applications*
Abstract: SOS relaxation provides very good approximation for finding global minimum and minimzer of polynomial functions. However, the size of the resulting SDP is often very large and makes it difficult to solve large scale problems. This talk will discuss the global optimisation of large polynomial functions that are given as the summation of small polynomials. The sparse SOS relaxations are proposed. We analyze the computational complexity and the quality of lower bounds. Some numerical implementations of randomly generated problems shows that this sparse SOS relaxation is very successful. This sparse SOS relaxation is very useful in solving large scale sparse polynomial systems, like the polynomial systems derived from nonlinear PDEs and distance geometry problems (e.g., sensor network localization).

Speaker: **Bruce Reznick** (Illinois)
Title: *Hilbert's construction of psd quartics and sextics that are not sos*
Abstract: We will discuss both of these constructions, which become almost intuitive when one "counts constants". New and simple examples will be derived. Speaker: **Pablo Parrilo** (MIT)

Title: *Exact semidefinite representations for genus zero curves*
Abstract: The characterization of sets that admit an exact representation in terms of semidefinite programming constraints (perhaps with additional variables) is one of great interest in optimization. There have been a few recents results in this direction, based mainly on the work of Helton and Vinnikov and the related Lax conjecture, that point out to the existence of specific obstructions for (the interior of) a plane curve to be semidefinite representable. In this talk we discuss a procedure to explicitly construct exact representations for convex hulls of arbitrary segments of genus zero plane curves. In particular, it is shown that the new method enables the computation of representation for particular curves, for which a generic SOS-based construction fails.

Speaker: **Mihai Putinar** (UC Santa Barbara)
Title: *The multivariate trigonometric moment problem and related sums of squares decompositions*

Abstract: In the case of the one dimensional torus, Riesz-Fejer factorization of a non-negative trigonometric polynomial as the modulus square of another polynomial provides the basis of all positivity results related to the unit disk: Riesz-Herglotz parametrization of all non-negative harmonic functions, the solution to the trigonometric moment problem, as proposed by Schur, and separately by Caratheodory-Fejer, the spectral theorem for unitary operators.

In several variables, on the torus in $\mathbb{C}^n$, we reverse the flow, and start with Bochner's characterization of Fourier transforms of positive measures. This provides a sum of squares decomposition for positive trigonometric polynomials. And the result can easily be adapted to compact, semi-algebraic supports on the torus. The most intriguing case is however the unit sphere in $\mathbb{R}^n$, where a decopmpsition into squares of spherical harmonics is available. For odd dimensional sphere, the complex structure induced from $\mathbb{C}^n$ provides a decomposition (of a positive polynomial) into squares of pluriharmonic polynomials. As a consequence I will indicate a novel proof of an old theorem of Quillen.

Finally, the non-commutative sphere, or torus, associated to the free-$*$ algebra reveals stronger SOS deecompositions.

Speaker: **M.-F. Roy** (Rennes)
Title: *Certificate of positiviy in the Bernstein basis*
Abstract: We prove the existence of a polynomial size (in the degree d and bitsize t of coefficients) certificate of positivity for a positive univariate polynomial on $[-1, 1]$ using Berstein basis of degree d on well-chosen subintervals. This improves by an exponential factor previsouly known results by Powers and Reznick.

Speaker: **Claus Scheiderer** (Konstanz)
Title: *Sums of squares and moment problems with symmetries*
Abstract: Let G be a real algebraic subgroup of $GL(V)$, the general linear group of a finite-dimensional real vector space V, and assume that G is (semi-algebraically) compact. We study the cones of sums of squares in $R[V]$ and in $R[V]^G$, the ring of G-invariants, and relate them through the operations of contraction and extension. More generally, we do the same for arbitrary quadratic modules. In doing this, we use (and partially re-prove, partially generalize) results of Procesi-Schwarz, Bröcker and Gatermann-Parrilo. We prove that the Reynolds operator maps the cone $\Sigma R[V]^2$ into itself, and that this property is characteristic of the case where G is compact.

Given a basic closed set K in V which is G-invariant, we ask for (finite) characterizations of the G-invariant K-moment functionals. We isolate two conditions under which such characterizations exist, and show by examples that this may happen at the same time when the usual (full) K-moment problem is not finitely solvable.

The talk will contain (plenty of) explicit examples and (a few) open problems. (Joint work with Salma Kuhlmann and Jaka Cimpric.)

Speaker: **Konrad Schmüdgen** (Leipzig)
Title: *Positivity and Positivstellensätze for Matrices of Polynomials*
Abstract: The notion of k-positivity ,$0 \leq k \leq n$, for (n, n)-matrices of polynomials is introduced and discussed. Generalizations of Stengle's Positivstellensatz to matrices are given.

Speaker: **Markus Schweighofer** (Konstanz)
Title: *Global optimization of polynomials using gradient tentacles and sums of squares*
Abstract: We combine the theory of generalized critical values with the theory of iterated rings of bounded elements (real holomorphy rings). We consider the problem of computing the global infimum of a real polynomial in several variables. Every global minimizer lies on the gradient variety. If the polynomial attains minimum, it is therefore equivalent to look for the greatest lower bound on its gradient variety. Nie, Demmel and Sturmfels proved recently a theorem about the existence of sums of squares certificatesfor such lower bounds. Based on these certificates, they find arbitrarily tight relaxations of the original problem that can be formulated as semidefinite programs and thus be solved efficiently. We deal here with the more general case when the polynomial is bounded from below but does not necessarily attain a minimum. In this case, the method of Nie, Demmel and Sturmfels might yield completely wrong results. In order to overcome this problem, we replace the gradient variety by larger semialgebraic sets which we call gradient tentacles. It now gets substantially harder to prove the existence of the necessary sums of squares certificates.

Speaker: **Thorsten Theobald** (Berlin)
Title: *Symmetries in SDP-based relaxations for constrained polynomial optimization*
Abstract: (joint work with L. Jansson, J.B. Lasserre and C. Riener) We study methods for exploiting symmetries within semidefinite programming-based relaxation schemes for constrained polynomial optimization. Our main focus is on problems where the symmetric group or the cyclic group is acting on the variables. From the exact point of view, we extend the representation-theoretical methods of Gatermann and Parrilo for the unconstrained case to the constrained case (i.e., to Lasserre's relaxation scheme). In contrast to the viewpoint merely from the resulting semidefinite programs, the symmetries on the original variables induce much additional symmetry structure on the moment matrices of the relaxation scheme. We characterize the combinatorics of the resulting block decompositions in terms of Kostka numbers. Moreover, we present methods to efficiently compute lower and upper bounds for a subclass of problems where the objective function and the constraints are given by power sums.

Speaker: **Luis Zuluaga** (New Brunswick)
Title: *Closed-form solutions to certain moment problems with Applications to Business*
Co-authors: Donglei Du, Javier Pena, and Juan Vera.

Abstract: We present new closed-form solutions to certain moment problems with applications in mathematical finance, inventory theory, supply chain management, and Actuarial Science. In particular, we extend Lo's classical semiparametric closed-form bound for European call options by considering third-order moment information, and by finding a related semiparametric bound on the option's risk. Furthermore, we present a closed-form solution for a class of arbitrage bounds. We show how the latter result can be used to obtain a novel model for portfolio allocation with desirable properties for today's investors.

Scientific Progress Made and Outcome of the Meeting

Intensive discussions amongst the participants resulted in several new collaborations. The " second chances" offered a wonderful opportunity for lively question-answer sessions. The feedback from the participants was in general very positive. Here is a list of some of the collaborations initiated during this meeting. Many of these have already resulted in preprints or submitted papers (see the preprints in the bibliography below dated October or November 2006).

Putinar is collaborating with Kuhlmann on: the decision problem for systems of polynomial inequalities in a free *-algebra. We also collaborated on sparse polynomials [10]. Putinar, Lasserre and Helton started a collaboration on: sparsity pattern of inverses of truncated moment matrices associated to the multivariate distribution of independent random variables. Putinar and Schmüdgen started a collaboration on: an extension of Marcel Riesz ρ function for moment data in two variables; this function being the classical and most refined object to study moment sequences in one dimension.

An old paper of Gondard and Ribenboim [4] - in french - was quoted in Schmüdgen's talk, moreover this finally resulted into a new, constructive proof found by Hillar and Nie [6] , of the main theorem of [4].

Laurent and Schweighofer started collaborating on the project of writing an extensive survey on the recent developments on the use of moment matrices and sums of squares of polynomials in optimization. Schweighofer remarked that this very nice workshop is perhaps the nicest he has ever attended.

Reznick was inspired by the company and the interest to discover sufficiently simple proofs of Hilbert's construction [16].

Theobald used the breaks and evenings to exchange further ideas with Lasserre continuing work on symmetries, and also had interesting discussions with other participants on this.

Cimpric, Marshall and Kuhlmann discussed some further planned joint work on Invariant saturated preorders.

Cimpric, Scheiderer and Kuhlmann used this opportunity to discuss final details of [2] before submitting it.

Concluding Remarks

This meeting in the Banff Centre has been very interesting and enjoyable. Talks of one hour give a much better understanding than shorter talks as in some others meetings. The combination of optimization and real algebraic geometry is currently extremely fruitful. The workshop was an excellent opportunity for researchers in optimization to gain access to the new mathematical tools related to positive polynomials. This was one of the few conferences where pure mathematicians were able to interact and exchange ideas/results with applied researchers in optimization.

It is of great benefit to mathematical research in Canada to hold such workshops in this country. The Banff centre attracts visitors from abroad, and this provides an opportunity for Canadian researchers to invite these visitors further to their own institutions. A mini-conference " Topics in Real Algebraic Geometry" (http://math.usask.ca/ skuhlman/confgond.htm) was held at the University of Saskatchewan immediately before the Banff Workshop , to give an opportunity to the "young researchers" among the BIRS workshop participants to give talks in a less formal context.

D. Augustin lectured on The membership problem for preorders. She showed that the membership problem for a preorder in the polynomial ring over a real closed field is solvable if the set of coefficients of the polynomials which belong to the preorder is weakly semialgebraic; this means: the intersection of this set of coefficients with every finite dimensional subspace of the polynomial ring is semialgebraic. In the talk she concentrated on finitely generated preorders in the polynomial ring over the field of real numbers in one variable. By describing the structure of preorders generated by one polynomial in the local power series ring at a point she derived conditions which are necessary and sufficient for a polynomial f to lie in the preorder generated by another polynomial g if the basic closed semialgebraic set generated by g is compact. These conditions imply that the membership problem is solvable for preorders generated by one single polynomial.

Netzer lectured on The doubledagger condition: Recall that a finitely generated preordering in the real polynomial ring is said to have the Strong Moment Property, if its closure with respect to the finest locally convex topology equals its saturation. In the literature, there has been introduced a property which implies the Strong Moment Property, known as the "Double-Dagger Property". He presented an example which shows that this property is strictly stronger than the Strong Moment Property. And unlike the Strong Moment Property, a preordering does not necessarily have it if all preorderings corresponding to the fibers of some bounded polynomials have it.

Plaumann (Ph.D in Konstanz, Germany) lectured on "Sums of squares on reducible real curves": Scheiderer has classified all irreducible real affine curves for which every non-negative regular function is a sum of squares in the coordinate ring. Plaumann showedhow to extend some of these results to reducible curves. He also discussed the moment problem for reducible curves and applications to the moment problem in dimension 2.

List of Participants

Augustin, Doris (University of Regensburg)
Castle, Mari (Emory University)
Cimpric, Jaka (University of Ljubljana)
Curto, Raul (University of Iowa)
Gondard, Danielle (Université Pierre et Marie Curie)
Helton, Bill (University of California, San Diego)
Hillar, Christopher (Texas A&M University)
Kuhlmann, Salma (University of Saskatchewan)
Lall, Sanjay (Stanford University)
Lasserre, Jean-Bernard (LAAS-CNRS)
Laurent, Monique (CWI Amsterdam)
Marshall, Murray (University of Saskatchewan)
Netzer, Tim (University of Konstanz)
Nie, Jiawang (University of California, Berkeley)
Papachristodoulou, Antonis (University of Oxford)

Parrilo, Pablo (Massachusetts Institute of Technology)
Pasechnik, Dmitrii (Dima) (Nanyang Technological University)
Pena, Javier (Carnegie Mellon University)
Plaumann, Daniel (University of Konstanz)
Powers, Victoria (Emory University)
Prestel, Alexander (Universitaet Konstanz)
Putinar, Mihai (University of California at Santa Barbara)
Renegar, James (Cornell University)
Reznick, Bruce (University of Illinois)
Roy, Marie-Francoise (Université Rennes 1)
Scheiderer, Claus (Universität Konstanz)
Schmuedgen, Konrad (University of Leipzig)
Schwartz, Niels (Universität Passau, Germany)
Schweighofer, Markus (Universität Konstanz)
Sottile, Frank (Texas A&M University)
Theobald, Thorsten (Technische Universitaet Berlin)
Tuncel, Levent (University of Waterloo)
Wolkowicz, Henry (University of Waterloo)
Zinchenko, Yuriy (McMaster University)
Zuluaga, Luis (University of New Brunswick)

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Chapter 21

Topological graph theory and crossing numbers (06w5067)

Oct 21 - Oct 26, 2006

Organizer(s): Bojan Mohar (Simon Fraser University), Janos Pach (Courant Institute and City College), Bruce Richter (University of Waterloo), Robin Thomas (Georgia Institute of Technology), Carsten Thomassen (Technical University of Denmark)

Objectives

The main objective of this workshop is to bring together two groups of researchers, those working in topological graph theory and graph minors, and those working with crossing numbers. Both areas have developed methods, mathematical tools and powerful results that have great potential for being used in the other area. For instance, the most basic open problem about crossing numbers is the Turan's Brickyard problem. Would it be possible to use results about the genus of graphs and graph minors to get some new insight into this problem? On the other hand the study of crossing numbers of graphs on nonsimply connected surfaces may yield new results of interest for the topological graph theory.

We plan to organize some survey lectures where best mathematicians from both areas will present the current state of the art of the theory. Additionally, there will be corresponding problem sessions with intention to motivate the participants to apply their knowledge towards problems in the other area.

Overview of the Field

Roots of the Topological Graph Theory lie in the Heawood problem, one of very early discovered generalizations of the Four Color Problem. Heawood [4] proved in 1890 that every graph embedded in a closed surface of Euler characteristic $c \neq 2$ can be colored with $H(c) = \lfloor \frac{7+\sqrt{49-24c}}{2} \rfloor$ colors. However, it was left open for another 78 years if that many colors are really needed. G.A. Dirac proved in the 1950's that the answer to this question is equivalent to the fact that the genus (and the nonorientable genus) of the complete graph of order n is equal to $\lceil \frac{1}{12}(n-3)(n-4) \rceil$ (and $\lceil \frac{1}{6}(n-3)(n-4) \rceil$, respectively). Ringel and Youngs solved this problem completely in 1968, cf. [6]. Their solution and related works by other authors motivated further extensive research on embeddings of graphs in surfaces. The books by White [13] and later by Gross and Tucker [3] show the state of the art of the theory at the end of the 1980's.

In the late 1980's, two new directions of research brought additional insight and boosted the topological graph theory into even higher levels. The most important results came from Robertson and Seymour's theory of graph minors. Very influential was also Thomassen's work in which he first presented stimulating new proofs of the fundamentals of the theory, and later produced a number of deep results, most notably related to colorings of graphs on surfaces. These new developments are now accessible in a monograph by Mohar and Thomassen [5].

Today, topological graph theory and the related theory of graph minors are battling its way into the area of computer science. Applications have been discovered in areas like computational complexity, theory of algorithms, graph drawing, computer graphics, computer vision, etc. This is the direction for which we believe that important advances will be made in the future.

One particular branch of topological graph theory, the crossing number problems, has received particular attention in the last decade. Discoveries of F.T. Leighton in the early 1980's made this area of high importance in the theoretical computer science. Stunning discoveries made by Pach, Szekely, and many others in the 1990's have advanced the theory of crossing numbers into an independent subject with many applications in discrete geometry, combinatorics and computer science.

This workshop has brought together mathematicians working in these two areas with intention to meet together, exchange ideas, and use recent advances in both subjects to create new results on the extended grounds.

Scientific program

The program has been composed of long lectures (mostly surveys), short lectures (important new results), open problems sessions, and five-minute lectures. Each participant (except some of those giving another talk) have been asked to present a short summary of their work of no more than five minutes in duration. Although this is hard to obey, the time limit was strictly enforced (with relatively good success). The purpose was to get to know each other's research programs and interests so that like-minded individuals can pursue informal interactions. This applied to all participants from senior researchers to graduate students.

Saturday, October 21, 2006

17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Sunday, October 22, 2006

09:00–09:15 Introduction and Welcome by the BIRS Station Manager
09:15–10:00 Carsten Thomassen, Planar representations of finite and infinite graphs
10:00–10:30 Coffee
10:30–12:15 Five-minute lectures
12:15–14:00 Lunch
14:00–14:40 Gelasio Salazar, A biased survey on crossing numbers, plus two doable important open problems
14:40–14:45 Short break
14:45–15:30 Five-minute lectures
15:30–16:00 Coffee
16:00–17:30 Five-minute lectures
17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Monday, October 23, 2006

09:00–10:00 MohammadTaghi Hajiaghayi, Algorithmic graph minor theory
10:00–10:30 Coffee
10:30–11:30 Kenichi Kawarabayashi, Linear-time algorithm for computing crossing number
11:30–11:40 Short break
11:40–12:00 Matt DeVos, Describing Fullerenes
12:00–14:00 Lunch
14:00–17:30 Free

17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Tuesday, October 24, 2006

09:00–10:00 Janos Pach, Extremal Graph Theory and Geometric Graphs
10:00–10:30 Coffee
10:30–11:15 Marcus Schaefer, Graphs with rotation
11:20–11:40 Laszlo Szekely, On lower bounds for the minor crossing number
11:45–12:05 Michael Albertson, Distinguishing labelings of geometric graphs
12:00–14:00 Lunch
14:00–15:00 Gabor Tardos, Extremal Theory of Topological Graphs
15:00–15:30 Coffee
15:30–16:15 Jacob Fox, Ramsey-type results for intersection graphs
16:20–16:40 Petr Hlineny, Crossing number of almost planar graphs
16:50–17:10 Drago Bokal, Crossing-critical graphs with presecribed average degree and crossing number
17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Wednesday, October 25, 2006

09:00–09:45 Neil Robertson, My favorite open problems in Topological Graph theory
09:50–10:20 Henning Bruhn, MacLane's criterion for higher surfaces
10:20–10:50 Coffee
10:50–11:10 Bruce Richter, Cycle spaces of infinite graphs
11:15–11:35 Mark Ellingham, The orientable genus of some joins of complete graphs with large edgeless graphs
Afternoon: Work in groups

Thursday, October 26, 2006

No scheduled talks.

Abstracts of talks

On the Crossing Number of Almost Planar Graphs

Petr Hlineny and Gelasio Salazar

Crossing minimization is one of the most challenging algorithmic problems in topological graph theory, with strong ties to graph drawing applications. Despite a long history of intensive research, no practical good algorithm for crossing minimization is known (that is hardly surprising, since the problem itself is NP-complete). Even more surprising is how little we know about a seemingly simple particular problem: to minimize the number of crossings in an almost planar graph, that is, a graph with an edge whose removal leaves a planar graph. This problem is in turn a building block in an "edge insertion" heuristic for crossing minimization. In this paper we prove a constant factor approximation algorithm for the crossing number of almost planar graphs with bounded degree. On the other hand, we demonstrate nontriviality of the crossing minimization problem on almost planar graphs by exhibiting several examples, among them new families of crossing critical graphs which are almost planar and projective.

MacLane's criterion for higher surfaces

Henning Bruhn and Reinhard Diestel

MacLane's planarity criterion can be seen as listing a number of properties of the facial cycles of a plane graph which, together, are strong enough to imply the following: that whenever we have any collection of cycles with these properties and attach a 2-cell to each of them, the 2-complex obtained is homeomorphic to the sphere. We shall provide such a list for higher surfaces. The difficulty here does not lie in the proof but rather in finding properties that are as simple and natural as possible.

Ramsey-type results for intersection graphs

Jacob Fox

The intersection graph of a collection C of sets has vertex set C and two elements of C are adjacent if and only if they have nonempty intersection. J. Pach, Cs. Toth, and I recently proved several Ramsey-type results for intersection graphs of geometric objects that are outlined below.

(1) There is a positive constant c such that for every intersection graph G of $n > 1$ convex bodies in the plane, G or its complement contains a complete bipartite graph with at least cn vertices in each of its vertex classes.

(2) An arrangement of pseudosegments is a collection of continuous arcs in the plane such that no pair cross more than once. There is a positive constant c such that the intersection graph of any arrangement of n pseudosegments in the plane contains a clique or independent of size at least n^c.

(3) An x-monotone curve is a continuous arc in the plane such that no vertical line intersects it in more than one point. There is a positive constant c such that for every intersection graph G of $n > 1$ x-monotone curves in the plane, G contains a complete bipartite graph on at least $cn/\log n$ vertices in each of its vertex classes or the complement of G contains a complete bipartite graph with at least cn vertices in each of its vertex classes. In the other direction, Pach and G. Tóth showed that for each $\epsilon > 0$ and n sufficiently large, there is an intersection graph G of a collection of n x-monotone curves in the plane that does not contain a complete bipartite graph with at least $\frac{14n}{\epsilon \log n}$ vertices in each of its vertex classes and every vertex of G is adjacent to all but at most n^ϵ other vertices.

(4) For each positive integer k, there is a positive constant c_k such that for every intersection graph G of $n > 1$ x-monotone curves in the plane with no pair intersecting in more than k points, G or its complement contains a complete bipartite graph with at least $c_k n$ vertices in each of its vertex classes.

The above results are proved using structural theorems that demonstrate close relationships between certain families of intersection graphs of geometric objects and cocomparability graphs.

Turán-Type Results for Arrangements of Curves

J. Pach and M. Sharir

Let C be a family of n compact connected sets in the plane, whose intersection graph $G(C)$ has no complete bipartite subgraph with k vertices in each of its classes. Then $G(C)$ has at most n times a polylogarithmic number of edges, where the exponent of the logarithmic factor depends on k. In the case where C consists of convex sets, we improve this bound to $O(n \log n)$. If in addition $k = 2$, the bound can be further improved to $O(n)$.

Linear time algorithm for computing crossing number

Ken-ichi Kawarabayashi and Bruce Reed

We show that for every fixed k, there is a linear time algorithm that decides whether or not a given graph has crossing number at most k, and if this is the case, then the algorithm computes a drawing of the graph into the plane with at most k crossings. This answers the question posed by Grohe (STOC01 and JCSS 2004). Our algorithm can be viewed as a generalization of the seminal result by Hopcroft and Tarjan, which says that planarity of graphs can be decided in linear time. Our algorithm can also be compared with the algorithm by Mohar (STOC96 and Siam J. Disrete Math 2001), which says that there is a linear time algorithm for a given graph G to give either an embedding of G into a fixed surface S, i.e, Euler genus k for fixed k, or a minimal forbidden subgraph for embeddability in S. Our algorithm has several appealing features. First, unlike the algorithm by Grohe, our algorithm does not involve any huge hidden constant. In fact, the time complexity can be written as $O(2^{O(k^4)}n)$. This is because our proof does not depend on the excluded grid minors theorem, which is the case in Grohe. Second, our algorithm consists of several interesting ingredients. It uses a deep result of Mohar. It also uses techique by Reed, Robertson, Schrijver and Seymour, which improves the time complexity of the seminal result of Robertson and Seymour for planar graphs and graph on a fixed graph. The algorithm also needs a deep result in discrete geometry. Finally, it uses the algorithm by Bodlaender which shows how to give linear time algorithms for many NP-hard problems in graphs of bounded tree-width. Third, we can apply our algorithm to other problem. For instance, given a graph G and fixed k. Can we make G planar after deleting at most k edges? Our algorithm does give a linear time algorithm for this problem.

Describing Fullerenes

Matt DeVos and Bojan Mohar

With a notion of a polyhedral surface, one can use a fundamental result of Alexandrov to represent various families of planar graphs in a particularly simple geometric way. In particular, we show how this can be done for all triangulations of the sphere with maximum vertex degree 6. Dually, this applies to cubic planar graphs of maximum face size 6. In particular, this representation can be used for the class of fullerenes, cubic planar graphs whose face sizes are only 5 or 6.

An independent discovery of essentially the same result was obtained earlier by Thurston.

My interests in Topological graph theory

Serguei Norine

I am interested in topological description of Pfaffian and k-Pfaffian graphs along the lines of the following conjecture of mine. Conjecture 1. For a graph G and a non-negative integer g the following are equivalent

(1) There exists a drawing of G on an orientable surface of genus g such that the number of pairwise crossings of edges of M is even for every perfect matching M of G.

(2) G is $4g$-Pfaffian.

(3) G is $(4g + 1 - 1)$-Pfaffian.

I have been able to prove that the above conjecture holds for $g = 0$, that (1) implies (2) in general, and that (1) and (2) are equivalent for $g = 1$. However, for other implications the method I used breaks down for $g > 1$ and new ideas are required to settle the conjecture.

I am also interested in applying methods used in proofs of the subcases of the above conjecture to the theory of crossing numbers and in unifying various existing approaches to studying the parity of crossing numbers. In particular, I want to study the hypergraph problem described below, which is related to Turans brickyard problem.

Cycle spaces of infinite graphs

Bruce Richter

Recent work by several authors have developed the theory of cycle spaces of infinite graphs to usefully allow infinite circuits. A single locally finite graph can have many different compactifications. Each of these has its own, typically different, cycle space. If compactification C_1 is the continuous image of compactification of C_2, preserving the graph itself, then the C_2-cycle space contains the C_1-cycle space. What is a basis for the quotient space? There are many other questions one could ask.

2-crossing-critical graphs

Bruce Richter

Bogdan Oporowski has made substantial progress on trying to determine all 2-crossing-critical graphs. I have gotten sucked into this project. But I am also interested in determining all large 3-crossing-critical graphs. (A graph is k-crossing-critical if its crossing number is at least k and all proper subgraphs have crossing number ¡ k. The graph $C_3 \times C_3$ has crossing number 3, but all its proper subgraphs have crossing number at most 1, so it is both 2-crossing-critical and 3-crossing-critical.)

Some recent work on Topological Graph Theory

Robin Thomas

With M. DeVos, R. Hegde, K. Kawarabayashi, S. Norine and P. Wollan we have shown:

THEOREM. There exists an integer N such that every 6-connected graph G with no K_6 minor has a vertex v such that $G - v$ is planar.

Jorgensen conjectured that the above statement holds for all 6-connected graphs regardless of size. That remains open. The theorem suggests the following conjecture.

CONJECTURE. For every integer t there exists an integer N such that every t-connected graph G with no K_t minor has a set X of at most $t - 5$ vertices such that $G - X$ is planar.

Here the assumption that G be big is necessary. Notice that if G has a set X as above, then it has no K_t minor. The excluded clique theorem of Robertson and Seymour gives structural information about graphs with no K_t minor, but not enough to easily deduce the above conjecture.

With Daniel Kral we have obtained a polynomial-time algorithm to compute the chromatic number of a graph G embedded on a fixed surface S with all faces even. This is a warm-up for the more general and still open problem whether the same can be done for triangle-free graphs, regardless of face sizes. An exact statement of our theorem requires a number of definitions, and so let me state it informally. We need a definition and an observation. Let C be a cycle with vertex-set $\{v_1, \ldots, v_k\}$ in order, let $v_0 = v_k$, and let $c : V(C) \to \{1, 2, 3\}$ be a (proper) 3-coloring of C. We define $w(C)$, the winding number of C, as the number of indices $i \in \{1, \ldots, k\}$ such that $c(v_i) = 1$ and $c(v_{i-1}) = 2$ minus the number of indices i such that $c(v_i) = 2$ and $c(v_{i-1}) = 1$. Now let G be a graph embedded in an orientable surface S, let $C_1, \ldots, C_k$ be facial cycles, and assume that every other face is bounded by a cycle of length four. Then if c is a 3-coloring of G, then $\sum_{i=1}^{k} w(C_i) = 0$. In other words, if some faces are precolored and all other faces are bounded by cycles of length four, then a necessary condition for the precoloring to extend to a 3-coloring of G is that the sum of the winding numbers of the precolored faces be zero. We will call this the winding number condition. Our theorem says that for every surface S (orientable or not) there exists an integer N such that for every graph G embedded in S with all faces even there exists a subgraph H of G on at most N vertices such that for every face f of H:

(i) if f includes a face of G with boundary length exceeding four, then every 3-coloring of the boundary of f extends to a 3-coloring of the subgraph G_f of G contained in f, and

(ii) otherwise a 3-coloring of the boundary of f extends to a 3-coloring of G_f if and only if it satisfies the winding number condition.

A 3-coloring algorithm follows: for every 3-coloring of H test if it extends into every face of H.

Summary of my recent research activities related to the topic of the workshop

Daniel Kral'

I am working in the area of graph colorings and I have recently started being interested a lot in colorings of graphs embedded in surfaces. As examples of my recent work, let me name the following results:

Theorem 1 (K., Mohar, Nakamoto, Pangrac, Suzuki) An Eulerian triangulation of the Klein bottle is 5-colorable unless it contains a complete graph of order six as a subgraph.

Theorem 2 (K., Stehlik) Every triangle-free graph on the double-torus is 4-colorable.

Theorem 3 (K., Thomas) A triangle-free quadrangulation of the torus is 3-colorable if and only if it does not contain a Cayley graph for the group Z_{13} with generators 1 and 5 as a subgraph.

Theorem 4 (K., Thomas) For every fixed $g \geq 0$, the chromatic number of an even-faced graph embedded in the orientable surface of genus g can be determined in polynomial time.

Planar representations of finite and infinite graphs

Carsten Thomassen

Results on representations of finite planar graphs may be non-trivial to extend to the infinite case. We have recently found an extension method which applies to rectangular representations, bar representations and visibility graphs.

A biased survey on crossing numbers, plus two doable important open problems

Gelasio Salazar

We will try to cover most of the mainstream avenues of research in Crossing Numbers, including the work that has steadily led to the study of crossing numbers in nonplanar surfaces.

Regarding open problems, I'll take Donald Knuth's position when a Computer Science student asked him, during a lecture at Munchen: What are the 5 most important open problems in Computer Science? Knuth replied: "I don't like this "top ten" business. It's the bottom ten that I like. You've got to go for the little things, the stones that make up the wall". I have two serious candidates for the "bottom ten".

Graphs with Rotation

Michael Pelsmajer, Marcus Schaefer, Daniel Štefankovič

A *rotation system* for a graph specifies a clockwise ordering of incident edges at each vertex. Rotation systems have traditionally been used to describe embeddings of graphs in surfaces. We have recently applied them in various graph drawing problems concerned with *imbeddings* of graphs: odd crossing number, minor-monotone crossing number, the Hanani-Tutte theorem, and generalized thrackles.

The crossing number, $\texttt{cr}(G)$, of a graph G is the smallest number of crossings in any drawing of the graph.[1] The *odd crossing number*, $\texttt{cr}_{odd}(G)$, is the smallest number of pairs of edges that cross an odd number of times in any drawing. Obviously, $\texttt{cr}_{odd}(G) \leq \texttt{cr}(G)$. We were able to show that for every $\epsilon > 0$, there is a graph G such that $\texttt{cr}_{odd}(G) < (\sqrt{3}/2 + \epsilon)\texttt{cr}(G)$ (2005).

The proof uses a multigraph on two vertices with rotation. Combining contractions with rotations, we can also show that

$$\texttt{cr}(G) \leq \texttt{cr}_{odd}(G)\binom{n+4}{4}/5$$

for a multigraph on n vertices.

Call an edge in a drawing *even* if it intersects every other edge an even number of times.

Theorem 1 (Hanani-Tutte) *If a non-planar graph is drawn in the plane, then the drawing contains two non-adjacent edges that intersect an odd number of times.*

We give a new and direct geometric proof of this result which, in turn, is based on a strengthening of a result by Pach and Tóth:

Theorem 2 (Pelsmajer, Schaefer, Štefankovič) *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

[1] No more than two edges are allowed to cross in a point, and edges cannot pass through vertices.

The proof of this result relies on a geometric redrawing idea, using rotations in an essential way. This result leads to an easy proof of another result of Pach and Tóth: $\mathtt{cr}(G) \leq 2\mathtt{cr}_{odd}(G)^2$. It also allows us to prove that odd crossing number and crossing number are the same for small values: $\mathtt{cr}(G) = \mathtt{cr}_{odd}(G)$ for $\mathtt{cr}_{odd}(G) \leq 3$. The case analysis in the proof again uses rotation systems.

A graph is a *thrackle* if it can be drawn such that any pair of edges intersects exactly once, where a common endpoint of two edges counts as an intersection of these two edges. A *generalized thrackle* is a graph that can be drawn such that any pair of edges intersects an odd number of times (again counting endpoints).

As it turns out, our proof techniques allow us another simple proof of the following well-known result.

Theorem 3 (Cairns, Nikolayevsky (2000)) *If G is a bipartite, generalized thrackle on a surface of genus g, then G can be embedded on that surface.*

The special case $g = 0$ of the theorem was first proved by Lovász, Pach, and Szegedy, 1997: if a bipartite graph is a generalized thrackle, then it is planar.

We can naturally ask about the complexity of problems—such as the crossing number—for graphs with rotation system. We can show that computing the crossing number (or odd-crossing number or pair-crossing number) of a graph with rotation system is **NP**-hard. As a corollary we obtain Hliněný's result (2004) that computing the crossing number of a cubic graph (without rotation system) and computing the minor-monotone crossing number is **NP**-complete.

If we restrict the number of vertices to 1 or 2, the crossing number problem (with rotation) lies in **P**. The case of three vertices is open. For pair crossing number even the case $k = 2$ is open.

Crossing-critical graphs with prescribed average degree and crossing number

Drago Bokal

Crossing-critical graphs were introduced by Širáň, who proved existence of infinite families of 3-connected k-crossing-critical graphs for every $k \geq 3$. Kochol proved existence of infinite families of simple 3-connected k-crossing-critical graphs, $k \geq 2$. Richter and Thomassen started the research on degrees in crossing-critical graphs by proving that there are only finitely many simple k-crossing-critical graphs with minimum degree r for every two integers $r \geq 6$ and $k \geq 1$. Salazar observed that their argument implies the same conclusion for every rational $r > 6$, integer $k \geq 1$, and simple k-crossing-critical graphs with average degree r. For every rational $r \in [4, 6)$ he proved existence of an infinite sequence $\{k_{r,i}\}_{i=0}^{\infty}$ such that for every $i \in \mathbf{N}$ there exists an infinite family of simple 4-connected $k_{r,i}$-crossing-critical graphs with average degree r and asked about existence of such families for rational $r \in (3, 4)$. The question was partially resolved by Pinontoan and Richter, who answered it positively for $r \in (3\frac{1}{2}, 4)$.

In the talk, we extend the theory of tiles, developed by Pinontoan and Richter, to encompass a generalization of the crossing-critical graphs constructed by Kochol. Combining tiles with a new graph operation, the zip product, which preserves the crossing number of the involved graphs, we settle the question of Salazar and combine the answer with the results of Širáň and Kochol into the following theorem: there exists a convex continuous function $f : (3, 6) \to \mathbf{R}^+$, such that, for every rational number $r \in (3, 6)$ and every integer $k \geq f(r)$, there exists an infinite family of simple 3-connected crossing-critical graphs with average degree r and crossing number k.

Recent and current work on geometric and topological graphs

Michael O. Albertson

My most recent work (with Debra Boutin) concerns distinguishing labelings of geometric graphs. A labeling of a graph $f : V(G) \to \{1, 2, \ldots, d\}$ is said to be d-*distinguishing* if no nontrivial automorphism of G preserves the labels. The *distinguishing number* of a graph G, denoted by $\mathrm{Dist}(G)$, is the minimum d such that G has a d-distinguishing labeling.

An automorphism of a geometric graph that preserves both crossings and noncrossings of edges is called a *geometric automorphism*. We prove two theorems constraining the action of a geometric automorphism on the boundary of the convex hull of a geometric clique. First, any geometric automorphism that fixes the boundary of the convex hull fixes the entire clique. Second, if the boundary of the convex hull contains at least four vertices, then it is invariant under every geometric automorphism.

We use the above results, and the theory of determining sets, to prove that certain geometric cliques are 2-distinguishable. A subset of vertices is a *determining set* for a graph if every automorphism is uniquely determined by its action on this subset. The main theorem connecting these concepts says that a graph is d-distinguishable if and only if it has a determining set that can be $(d-1)$-distinguished. These ideas readily extend to geometric graphs. Using these results, we prove that if $(n \geq 7)$ and $\overline{K}_n$ is a geometric clique in which the boundary of the convex hull contains at least 4 vertices, then $\mathrm{Dist}(\overline{K}_n) \leq 2$. In prior work we had shown that for $n \geq 6$ there does exist a rigid $\overline{K}_n$. We have conjectured that $\mathrm{Dist}(\overline{K}_n) \leq 2$ when $n \geq 7$ and the boundary of the convex hull of $\overline{K}_n$ is a triangle. We know from prior work that $\mathrm{Dist}(\overline{K}_n) \leq 3$ in these circumstances.

Current projects include

1. Geometric automorphisms of other families of geometric graphs;
2. Graphs (geometric graphs) with large distinguishing number;
3. Edge coloring embedded graphs; and
4. Induced acyclic subgraphs of embedded graphs.

Distances in embedded graphs

Sergio Cabello (joint work with Erin W. Chambers)

We give an $O(g^2 n \log n)$ algorithm to represent the shortest path tree from all the vertices on a single specified face f in a genus g graph. ¿From this representation, any query distance from a vertex in f can be obtained in $O(\log n)$ time. This generalizes a result of Klein (SODA'05) for plane graphs. We also show how to use these shortest path trees to find a shortest non-contractible cycle and a shortest non-separating cycle in a graph embedded in an orientable surface in $O(g^3 n \log n)$ time.

The orientable genus of some joins of complete graphs with large edgeless graphs

Mark Ellingham and Chris Stephens

In an earlier paper the authors showed that with one exception the nonorientable genus of the graph $\overline{K_m} + K_n$ with $m \geq n-1$, the join of a complete graph with a large edgeless graph, is the same as the nonorientable genus of the spanning subgraph $\overline{K_m} + \overline{K_n} = K_{m,n}$. The orientable genus problem for $\overline{K_m} + K_n$ with $m \geq n-1$ seems to be more difficult, but here we find the orientable genus of some of these graphs. In particular, we determine the genus of $\overline{K_m} + K_n$ when n is even and $m \geq n$, the genus of $\overline{K_m} + K_n$ when $n = 2^p + 2$ for $p \geq 3$ and $m \geq n-1$, and the genus of $\overline{K_m} + K_n$ when $n = 2^p + 1$ for $p \geq 3$ and $m \geq n+1$. In all of these cases the genus is the same as the genus of $K_{m,n}$, namely $\lceil (m-2)(n-2)/4 \rceil$.

Embedding metrics into constant-dimensional geometric spaces

MohammadTaghi Hajiaghayi

Embedding metrics into constant-dimensional geometric spaces, such as the Euclidean plane, is relatively poorly understood. Motivated by applications in visualization, ad-hoc networks, and molecular reconstruction, we consider the natural problem of embedding shortest-path metrics of unweighted planar graphs (planar graph metrics) into the Euclidean plane. It is known that, in the special case of shortest-path metrics of trees, embedding into the plane requires $T(vn)$ distortion in the worst case, and surprisingly, this worst-case upper bound provides the best known approximation algorithm for minimizing distortion. We answer an open question posed in this work and highlighted by Matousek by proving that some planar graph metrics require $O(n^{2/3})$ distortion in any embedding into the plane, proving the first separation between these two types of graph metrics. We also prove that some planar graph metrics require $O(n)$ distortion in any crossing-free straight-line embedding into the plane, suggesting a separation between low-distortion plane embedding and the well-studied notion of crossing-free straight-line planar drawings. Finally, on the upper-bound side, we prove that all outerplanar graph metrics can be embedded into the plane with $O(vn)$ distortion, generalizing the previous results on trees (both the worst-case bound and the approximation algorithm) and building techniques for handling cycles in plane embeddings of graph metrics.

List-coloring classes of planar graphs when the lists vary in size

Joan P. Hutchinson

We prove the following theorem. If a graph is a 2-connected outerplanar near-triangulation and a list assignment L satisfies $|L(v)| \geq \min\{\deg(v), 5\}$ for every vertex v, then the graph is L-list-colorable except for K_3 with three identical 2-lists.

Connectivity cannot be reduced and the result does not hold for 2-connected K_4-minor-free graphs. The bound of five is best possible due to an example of A. Kostochka.

Another theorem is: If a 2-connected bipartite outerplanar graph has $|L(v)| \geq \min\{\deg(v), 4\}$ for every vertex v, then the graph is L-list-colorable.

This result does not hold for non-bipartite graphs, for 1-connected graphs, nor for K_4-minor-free graphs, and the bound of four cannot be decreased to three. We have conjectures about other variations in which this sort of list-coloring might be possible.

Pivot-vertex-minors of graphs

Sang-il Oum

Our main research interest is on graph structure theory related to pivot-vertex-minors of graphs. Local complementation at a vertex v of a graph is an operation to replace the subgraph of G induced by neighbor of v by its complementary graph. A vertex-minor of a graph is a graph obtained by local complementation and vertex deletion. Pivoting an edge uv is an operation to toggle adjacency of two vertices in different sets among three sets of vertices defined by adjacency to u and v (common neighbors of u, v, neighbor of u but nonneighbor of v, or neighbor of v but nonneighbor of u). A pivot-minor is a graph obtained by pivoting and vertex deletion.

The Kuratowski-Wagner theorem states that a graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$. A circle graph is an intersection graph of chords of a circle. Bouchet proved in 1994 that a graph is a circle graph if and only if it has no “vertex-minor” isomorphic to one of three graphs.

With Jim Geelen, we proved that there are finitely many excluded “pivot-minors” for circle graphs, and obtained 15 excluded pivot-minors by computer search. In particular, this implies Kuratowski-Wagner theorem.

Circle graphs have a quite topological feature. De Fraysseix (1984) showed that a bipartite graph is a circle graph if and only if it is a fundamental graph of a cycle matroid of a planar graph. In this sense, planar graphs are strongly related to circle graphs.

We would be very interested to know any other graph classes that are pivot-minor-closed or vertex-minor-closed. Are there any examples arising from surfaces other than sphere? I know that bipartite graphs, circle graphs, distance-hereditary graphs, PU-orientable graphs, graphs of rank-width at most k are closed under pivot-minors.

On lower bounds for the minor crossing number

Drago Bokal, Éva Czabarka, László A. Székely, and Imrich Vrťo

The *minor crossing number* of a graph G is defined as the minimum crossing number of all graphs that contain G as a minor:

$$mcr(G) := \min\{cr(H) \ : \ G \leq_m H\}.$$

This concept was introduced by D. Bokal, G. Fijavž, and B. Mohar in "The minor crossing number", to appear in *SIAM J. Discrete Mathematics*. The point of this definition is that the family of graphs with minor crossing number at most k is minor closed, unlike the family of graphs with ordinary crossing number at most k. Bokal et al. showed for the complete graph $(1+o(1))n^2/4 \leq mcr(K_n) \leq (1+o(1))n^2/2$, and for the hypercube $4^n \frac{1+o(1)}{5n^2} \leq mcr(Q_n) \leq 2(1+o(1)) \cdot 4^{n-2}$.

We improve on the bounds for the hypercube by showing $4^n \frac{1+o(1)}{3263n} \leq mcr(Q_n) \leq 4^n \frac{1+o(1)}{\sqrt{\pi n}}$. The key tool for establishing the lower bound is an adaptation of the bisection width lower bound to the minor crossing number.

As the cited result for K_n shows, one cannot expect the Leighton Lemma lower bound for $mcr(G)$, as $mcr(K_n) = O(n^2)$. We present an adaptation of the Leighton Lemma that still might be useful, and also discuss the adaptation of the embedding method to set lower bound for $mcr(G)$.

Ramsey-Type Results for Arrangements of Curves

Janos Pach (joint with J. Fox and Cs. Tóth)

An arrangement of *pseudosegments* is a family of continuous curves such that any pair of them cross at most once. We prove that

1. there is a positive constant c such that the intersection graph of any arrangement of n pseudosegments contains a clique or independent set of size at least n^c.

2. there is a positive constant c such that if G is the intersection graph of $n > 1$ convex compact sets in the plane, then G or its complement contains a complete bipartite graph with cn vertices in each of its vertex classes.

An x*-monotone* curve is a continuous curve in the plane that is intersected by any vertical line in at most one point. We prove that there is a positive constant c such that for every intersection graph G of n x-monotone curves in the plane, G contains a complete bipartite graph with at least $\frac{cn}{\log n}$ vertices in each of its vertex classes or the complement of G contains a complete bipartite graph with at least cn vertices in each of its vertex classes.

Degenerate Crossing Numbers

J. Pach and G. Tóth

Let G be a graph with n vertices and $e \geq 4n$ edges, drawn in the plane in such a way that if two or more edges (arcs) share an interior point p, then they must properly cross one another at p. It is shown that the number of crossing points, counted without multiplicity, is at least constant times e and that the order of magnitude of this bound cannot be improved. If, in addition, two edges are allowed to cross only at most once, then the number of crossing points must exceed constant times $(e/n)^4$.

Carsten Thomassen's exposition about Planar representations of finite and infinite graphs

Many results and problems on finite graphs have natural counterparts for infinite graphs. One of the most useful tools for going from finite graphs to infinite graphs is the following which is called König's infinity lemma.

Theorem 4 *Let v be a vertex in an infinite connected graph K, and let $D_1, D_2, \ldots$ be the distance classes from v. Then K has a one-way infinite path $vv_1v_2\ldots$ such that v_i belongs to D_1 for $i = 1, 2, \ldots$.*

One of the first applications (if not the first) is the extension of Kuratowski's theorem which characterizes the finite planar graphs in terms of forbidden subgraphs. This was generalized to infinite graphs by P. Erdős, see [2], as follows.

Theorem 5 *Let G be a countably infinite graph such that every finite subgraph is planar. Then G can be drawn in the plane such that no two edges intersect except at a common end.*

To prove this, Erdos constructed an auxiliary graph K as follows. First the vertices of G are enumerated $x_1, x_2, \ldots$. Then G_n denotes the subgraph induced by $v_1, v_2, \ldots, v_n$. Two planar embeddings of G_n are said to be *equivalent* if there is a homeomorphism of the plane taking one to the other. Then there are only finitely many non-equivalent embeddings. Each of them will be a vertex in a set D_n. The vertex set of the graph K is the union of all the sets D_n, $n = 1, 2, \ldots$. If we delete x_n from an embedding of G_n, then we obtain an embedding of G_{n-1}, and we add an edge between these embeddings in K. Then we apply König's infinity lemma to K, and we use the resulting path in K to draw successively $x_1, x_2 \ldots$ such that each drawing is equivalent with the corresponding vertex in the path of K.

Numerous results on finite graphs can be extended to infinite graphs by the same argument. However, the limitations of the method are perhaps more interesting. For example, it is a well-known result on finite graphs (also attributed to Erdős) that every finite graph has a so-called *unfriendly partition*, that is, a vertex partition such that each vertex has at least as many neighbors in the opposite part as in its own part. Using König's infinity lemma, it is easy to prove that the same holds for locally finite graphs. However, it is not known whether every countable graph has an unfriendly partition.

The notation and terminology below are the same as in [5]. In addition, we say that an infinite graph is *locally finite* if every vertex has finite degree. Following [8] we say that an embedding of a graph in the plane is *rectangular* if every edge is a vertical or horizontal straight line segment. The planar graphs having rectangular embeddings have not been characterized, but Ungar [12] proved that every finite planar, cubic, cyclically 4-edge-connected graph has a rectangular embedding after four edges on the outer cycle are subdivided. In such an embedding every face is bounded by a rectangle. Over 20 years ago I conjectured in [9] that Ungar's theorem extends to infinite graphs. One may try to repeat the above argument by Erdős. But, the infinite path in the auxiliary graph K cannot always be used to find the embedding because it is not always possible to extend a rectangular representation to a bigger one even if the homeomorphism properties are satisfied. In [11] we overcome this obstacle by introducing what we call a *grid representation* which is defined as follows.

Let $L_1, L_2, \ldots, L_p$ be a collection of pairwise parallel horizontal lines in the plane, and let $Q_1, Q_2, \ldots, Q_q$ be a collection of pairwise parallel vertical lines. Let G be a finite graph such that each vertex of G can be represented by an intersection point of some L_i and Q_j, and such that each edge of G is contained in one of $L_1, L_2, \ldots, L_p, Q_1, Q_2, \ldots, Q_q$, and such that no two edges cross. If each of the lines $L_1, L_2, \ldots, L_p, Q_1, Q_2, \ldots, Q_q$ contains a vertex of G, then these lines together with the representation of G is called a *grid representation* of G. We say that the intersection point of L_i and Q_j has the *coordinates* i, j. Two grid representations of G are *equivalent* if they have the same number of vertical lines and the same number of horizontal lines, and if every vertex of G has the same coordinates in the two representations. Clearly, a finite graph has only finitely many non-equivalent grid representations. The following was proved in [11].

Theorem 6 *Let G be a countably infinite graph. Let E be a collection of edges of G. (E may equal the edge set of G. E may be empty.) Let E be colored with the two colors $0, 1$. Assume that every finite subgraph of G has a rectangular representation such that each edge of E in the subgraph is vertical if it has color 1 and horizontal if it has color 0. Then G has a rectangular representation such that each edge of E is vertical if it has color 1 and horizontal if it has color 0.*

The proof ensures that all vertices and edges can be kept inside a prescribed square of the plane if we wish so. In the proof of the above theorem we get accumulation points. (A *vertex accumulation point* is a point each neighborhood of which contains infinitely many vertices of the graph. An *edge accumulation point* is a point each neighborhood of which intersects infinitely many edges of the graph.) The representation can be chosen such that no point on an edge of G is an accumulation point. It is easy to give examples of a cubic plane graph which has a grid representation and also has a planar drawing without accumulation points but with the property that every grid representation has accumulation points if we allow E to be the whole edge set. It is also easy to give such examples where E is empty and G is 4-regular. In fact, the only 4-regular graph which has a grid representation without accumulation points is the 2-dimensional grid. (For, if some facial walk is not a 4-cycle, then it is a two-way infinite path forming a spiral, and one of the one-way infinite subspirals is bounded.)

A cubic graph G is *cyclically* 4-*edge connected* if it is 3-connected and any edge-cut of G consists of the three edges incident with a vertex. Ungar [12] proved that every finite, planar, cubic, cyclically 4-edge-connected graph has a rectangular embedding provided some four edges on the outer cycle are subdivided. It follows from [9] that any facial cycle can play the role of the outer cycle

One noteworthy feature of Ungar's theorem is that it emphasizes that the difficulties in the 4-color problem (and many other problems on planar graphs) are of purely combinatorial nature and not of geometric or topological nature, as the countries can be chosen to be rectangles.

In [9] it was conjectured that Ungar's theorem extends to the infinite case. In [11] we prove that conjecture by combining Ungar's result with Theorem 6.

Theorem 7 *Every infinite, planar, cubic, cyclically* 4*-edge-connected graph* G *has a rectangular representation in the plane.*

This proof of this result applies in other contexts as well. A *bar representation* of a graph G is a representation such that the vertices of G are pairwise disjoint horizontal straight line segments and each edge is a vertical straight line segments joining its two ends and intersecting no other vertex. If all possible edges between vertices are present, then G is a *visibility graph*. If G is a graph, then we define the graph G^* as follows: If each component of G is 2-connected, then $G^* = G$. If some component of G is not 2-connected, then G^* is obtained from G by adding a new vertex and joining it to all cutvertices of G.

Tamassia and Tollis [7] proved that a finite graph G is a visibility graph if and only if G^* is planar. In other words, G is planar and has a plane representation such that all cutvertices are on the outer face boundary. In particular, every 2-connected finite planar graph is a visibility graph, and every finite planar graph has a bar representation. In [11] the following infinite counterpart is proved.

Theorem 8 *A countably infinite graph has a bar representation if and only if it is planar.*

Consider now a countably infinite graph G with the property that each finite subgraph is planar. The proof of Theorem 8 shows that G can be representated such that every vertex is a rectangle and every edge is a vertical straight line segment joining its ends and intersecting no other rectangle. (Instead of successively adding one horizontal line in the proof of Theorem 8 we add two horizontal lines close to one another.) By representing a vertex by a point inside a rectangle it is now easy to obtain a planar drawing of the graph. This argument may not be as natural as the one by Erdős but it avoids the geometric details in the transition from the finite case to the infinite case.

Perhaps the rectangles in the previous paragraph can even be chosen to be squares, see Conjecture 4 below.

Theorem 9 *If* G *is a countably infinite, locally finite graph* G *such that* G^* *is planar, then* G *is a visibility graph. In particular, every countably infinite, locally finite,* 2*-connected graph is a visibility graph.*

The converse of Theorem 9 is not true. To see this, consider any finite or countably infinite visibility graph. Every edge xy can be represented by two vertical straight line segments which, together with parts of the line segments representing x, y form a rectangle whose interior intersects no vertex or edge. We divide any such rectangle into two rectangles by adding a horizontal straight line segment inside the rectangle, and then we add in each of the two smaller rectangles a one-way infinite path starting at x or y. The resulting graph is a visibility graph, and every vertex is a cutvertex. More elaborate examples are possible so a complete characterization of the infinite visibility graphs seems complicated, even in the locally finite case.

An additional complication occurs if we allow vertices of infinite degree. The proof of Theorem 9 shows that any graph G such that G^* is planar has a bar representation such that the only edges that can be added to the representation are some which join two vertices of infinite degree.

All previous results are 2-dimensional. However the method can be extended to higher dimensions although there may not be many examples demonstrating that. We shall here mention one nontrivial example. A *box graph* is a graph such that every vertex is a box in 3-space, that is, the cartesian product of three closed bounded intervals in the real line. No two boxes have an interior point in common. Two vertices are neighbors if they have a rectangle (with positive area) in common. The box graphs have not been characterized in terms of forbidden subgraphs. In [10] I proved that every finite, planar graph is a box graph. In [11] this is extended to the infinite case as follows.

Theorem 10 *A countably infinite graph is a box graph if and only if every finite subgraph is a box graph. In particular, every countably infinite, planar graph is a box graph.*

The proof of Theorem 10 is analogous to that of Theorem 6.

We conclude with some open problems which are related to the results above but seem to require more elaborate methods.

In [9] I conjectured the following:

Conjecture 1 *Every infinite, cubic, cyclically 4-edge-connected graph which has a planar representation with no vertex-accumulation point and no edge-accumulation point has a rectangular representation in the plane with no vertex-accumulation point and no edge-accumulation point.*

There is an analogous problem for bar representations:

Conjecture 2 *Every infinite, locally finite graph which has a planar representation with no vertex-accumulation point and no edge-accumulation point has a bar representation with no vertex-accumulation point and no edge-accumulation point.*

In Theorem 9 the graphs are locally finite. Perhaps this condition can be omitted.

Conjecture 3 *Every countably infinite, 2-connected graph is a visibility graph.*

Conjecture 4 *Every countably infinite, planar graph has a representation such that every vertex is a square such that no two squares intersect and every edge is a vertical straight line segment joining the squares representing its ends.*

Conjecture 5 *Every countably infinite, planar graph has a representation such that every vertex is a closed disc such that no two discs intersect and every edge is a vertical straight line segment joining the discs representing its ends.*

Note that Conjecture 4 implies Conjecture 5 by taking the largest discs inside the squares. So the remark following Conjecture 4 shows that Conjecture 5 is true for finite graphs. This also follows from the theorem of Koebe (see e.g. [5], page 51) that every finite planar graph is a *coin graph*, that is, a graph whose vertices are closed discs no two of which have an interior point in common and such that two discs are neighbors if and only they have a point in common. To obtain Conjecture 5 in the finite case, just shrink the discs a little (and possibly rotate the collection of discs slightly).

Conjecture 6 *Every countably infinite, locally finite, planar graph is a coin graph.*

Conjecture 6 cannot be extended to all countable graphs. The graph obtained from a two-way infinite path by adding two new vertices each of which is joined to all other vertices in the graph is a countable, planar graph, but it cannot be represented as a coin graph.

Some open problems presented at the problem sessions

In problem sessions, many old and new open problems have been presented. Some of them are recorded below.

Problem 1 *Let G be a planar graph and $x, y \in V(G)$. How difficult is to compute the crossing number of $G + xy$?*

Note: Salazar and Hlineny have conjectured that this problem is solvable in polynomial time. On the other hand, Cabello and Mohar believe that this problem is NP-hard at least when the planar graph has weighted edges and a crossing of edges with weights a and b counts as ab.

Problem 2 (Bruce Richter) *A graph G is said to be k-*crossing-critical *if $\mathtt{cr}(G) \geq k$ and $\mathtt{cr}(G - e) < k$ for every edge e of G. It is known that 2-crossing-critical graphs have "cyclic" structure apart from a finite number of sporadic cases. Is there a similar result for large 3-crossing-critical graphs?*

Problem 3 (Petr Hlineny) *Is it true that for every positive integer k there exists an integer D such that every k-crossing-critical graph has maximum degree less than D?*

Petr Hlineny thinks that this may not be true.

Problem 4 *Is it true that for every positive integer k there exists an integer B such that every k-crossing-critical graph has bandwidth at most B?*

Problem 5 (Robin Thomas) *Let $\mathcal{I}$ be a lower ideal for minor ordering, i.e., a minor-closed class of graphs which does not contain all graphs. Is there a polynomial time algorithm which for every $G \in \mathcal{I}$ computes an integer k such that $|\chi(G) - k| \leq 10$?*

The constant 10 is fictious. The problem makes sense if it is replaced by any positive integer. Robin Thomas conjectured that the answer to the above problem is affirmative.

Problem 6 (Drago Bokal) *Is there an integer k for which there exist infinitely many k-crossing-critical (simple) graphs of average degree 6.*

It is known that infinitely many k-crossing-critical graphs of average degree d exist if d is any rational number in the interval $(3, 6)$ and that this is no longer the case if $d > 6$.

Problem 7 (Sergio Cabello) *Find a polynomial fixed-parameter-tractable (FPT) algorithm for deciding isomorphism of graphs of genus at most g, i.e.,the time complexity should be $O(f(g)p(n)$, where p is a polynomial and n is the size of the input.*

Note: Known algorithms have polynomial complexity $O(n^{O(g)})$, which is not FPT.

Problem 8 *Is there a graph which is a minimal forbidden minor for two distinct surfaces?*

If yes, then the two surfaces would both be nonorientable and their nonorientable genera would differ by one.

Problem 9 (Gelasio Salazar) *For a graph G, let $\mathtt{cr}_g(G)$ be the crossing number of G with respect to the drawings of G in the orientable surface of genus g. Is it possible that a graph satisfies $\mathtt{cr}_0(G) = N$, $\mathtt{cr}_1(G) = N - 1$, and $\mathtt{cr}_2(G) = 0$, where N is a very large integer?*

This problem has been recently solved by Matt DeVos, Bojan Mohar, and Robert Šamal [1].

Let G and H be geometric graphs, represented by straight-line segments in the plane. We say that G and H are *isomorphic* if there is an isomorphism f of (abstract) graphs G and H such that for every pair of edges ab and cd of G, the line segments corresponding to these two edges intersect if and only if the line segments of edges $f(a)f(b)$ and $f(c)f(d) \in E(H)$ intersect in the representation of H. The isomorphism f is *strong* if each line segment representing an edge e of G intersects other edges of G in the same order as the line segment of $f(e)$ intersects their f-images in the representation of H. The following problem was motivated by the lecture of Mike Albertson on distinguishing geometric graphs.

Problem 10 (Bojan Mohar) *How hard it is to decide if two geometric graphs are (strongly) isomorphic.*

Further open problems have been presented by Matt DeVos, Mark Ellingham, Luis Goddyn, and others.

Scientific progress and some outcomes of the meeting

Some of the spirit of the workshop can be seen from the exposition of Carsten Thomassen in Section 21. Old results and open problems presented in a new light, many open problems and conjectures.

The exchange between the various research groups at the workshop proved to be very fruitful. From the point of view of crossing numbers, there were two main outcomes.

The presentation by Salazar on the problem of determining the crossing number of a graph G such that G has an edge e so that $G - e$ is planar led to much discussion. The question is:

Is there an efficient (that is, polynomial time) algorithm to determine the crossing number of such a graph G?

This is such a “simple” problem and yet there is no known solution. Examples show that the crossing number of G can be arbitrarily large. Earlier work by Mutzel et al. showed that there is an efficient algorithm to find the drawing of G having the fewest crossings subject to the additional requirement that all the crossings involve e. These are quite natural drawings of such a graph, but Hlineny and Salazar show by example that they are not necessarily optimal in the sense of giving a drawing of G having fewest crossings. (The same result was discovered independently by Mutzel et al. and published in the journal version of their original presentation.) They further show that they can efficiently find a drawing that is within a constant factor of being optimal.

Petr Hlineny gave a short presentation about the important class of k-crossing-critical graphs. These are the graphs G having crossing number at least k, but every proper subgraph has crossing number less than k. There are several standard questions about such graphs. Two standard ones have the same form:

Is there a number $f(k)$ so that any k-crossing-critical graph has maximum degree or bandwidth at most $f(k)$?

The best result along these lines is Hlineny’s theorem that they have bounded path-width. The result is known for $k = 2$ but open for $k \geq 3$. The discussion during the workshop led to further discoveries of k-crossing-critical graphs with larger maximum degrees than were previously known, but still did not resolve the question.

The presentation by Richter on a problem, deriving from the Graph Minors Project, concerning how often a set of disjoint arcs, joining specified pairs of points in the boundary of a bordered surface must meet the different homeomorphism types of non-separating curves led quite directly to its complete solution shortly after the conference.

János Pach reported of a new joint paper that he has just finished with Jacob Fox, that grew out of work started in Banff. This improves on the results of his earlier joint work with Micha Sharir. The paper is entitled “Separator theorems and Turán-type results for planar intersection graphs” (by Jacob Fox and János Pach) and its contents can be summarized as follows:

> “We establish several geometric extensions of the Lipton-Tarjan separator theorem for planar graphs. For instance, we show that any collection C of Jordan curves in the plane with a total of m crossings has a partition into three parts $C = S \cup C_1 \cup C_2$ such that $|S| = O(\sqrt{m})$, $\max\{|C_1|, |C_2|\} \leq \frac{2}{3}|C|$, and no element of C_1 has a point in common with any element of C_2. These results are used to obtain various properties of intersection patterns of geometric objects in the plane. In particular, we prove that if a graph G can be obtained as the intersection graph of n convex sets in the plane and it contains no complete bipartite graph $K_{k,k}$ as a subgraph, then the number of edges of G cannot exceed $c_k n$, for a suitable constant c_k.”

Bojan Mohar also reported of a couple of new results about crossing numbers that have resulted from discussions at this meeting. First, he has extended results related to János Pach’s exposition on Degenerate Crossing Numbers. He proved that this version of a crossing number is almost equivalent to the notion of the non-orientable genus of graphs – an unexpected relation whose importance lies in the fact that the non-orientable genus is monotone under graph minors relation. Another achievement is a recent proof by Matt DeVos, Bojan Mohar, and Robert Šamal [1] which, in particular, completely solves Problem 9.

Drago Bokal reports on an ongoing research jointly with Éva Czabaraka, Lszl A. Szkely, and Imrich Vrťo on General lower bounds for minor crossing number, on which they also made important progress during the workshop:

> "There are three general techniques for bounding crossing numbers of graphs: the Crossing lemma, the Bisection method, and the Embedding method. Recently we established similar results in the context of the minor crossing number. As a result, we were able to tighten the bounds for the minor crossing number of hypercubes. We are looking for further applications of the new tools."

Additionally, Petr Hlineny informed us that he has finished and submitted a paper with a crossing-critical construction with high even degrees, and he acknowledged great influence of this workshop in obtaining this result.

List of Participants

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Chapter 22

Optimization and Engineering Applications (06w5081)

November 11 – November 16, 2006

Organizer(s): Jiming Peng (University of Illinois at Urbana-Champaign), Tamás Terlaky (McMaster University), Robert Vanderbei (Princeton University), Henry Wolkowicz (University of Waterloo), Yinyu Ye (Stanford University)

Objectives

This workshop had several objectives:

- to bring together outstanding researchers from the fields of optimization and engineering;
- to establish new collaborations; and
- to expose both sides to the challenge and opportunities in the respective fields, through identifying new problems and either matching these with potential solution approaches or developing new models and methodologies.

Among the participants were optimization researchers that work on engineering optimization problems and vice versa. The cross fertilization of experts in both fields is crucial for the development of new methods in both communities. Although there is an increasing awareness of common interests, until recently there have been only very few opportunities for the two sides to meet at conferences. In the past two years there have been a few occasions where optimizers, e.g., Z.Q. Luo and S. Boyd have been invited to give tutorials in signal processing and electrical engineering conferences. The MOPTA conference series at McMaster University, University of Waterloo, and University of Windsor, organized by T. Terlaky, H. Wolkowicz and their colleagues, has featured several engineers who presented their work on the application of optimization in their fields. However, there has not been a workshop specifically targeting the exchange of cutting edge expertise in mathematical optimization and engineering optimization, where an elite club of expertise from both fields was formed to pave the road for future directions.

A particular fact that should be mentioned here is the MITACS project "High Performance Optimization: Theory, Algorithm Design and Engineering Applications" led by T. Terlaky, which will have been in operation for five years by 2007. The workshop provided a good opportunity to review the achievements of the entire project, discuss future research plans for the project, and identify the areas in engineering for which optimization research has a major impact in the future and new challenges of engineering optimization problems offer to the mathematical, algorithmic optimization community.

Overview of the Field

Optimization is a subject that deals with the problem of minimizing or maximizing a certain function in a finite dimensional Euclidean space over a subset of that space, which is usually determined by functional inequalities. During the past century, optimization has been developed into a mature field that includes many branches, such as linear conic optimization, convex optimization, global optimization, discrete optimization, etc. Each of such branch has a sound theoretical foundation and is featured by an extensive collection of sophisticated algorithms and software. Optimization, as a powerful modelling and problem solving methodology, has a broad range of applications in management science, industry and engineering.

The application of optimization in engineering has a very long history. It is well known that two special classes of optimization problems, linear least squares and linear optimization problems, have been widely used in a tremendous number of application areas, such as transportation, production planning, design and data fitting. Since 1990, the appearance of highly efficient interior point method (IPM) based algorithms and software for convex optimization has motivated people to apply convex optimization models in several new areas, such as automatic control systems, signal processing, communications and networks, product and shape design, truss topology design, electronic circuit design, data analysis and modelling, statistics and financial engineering. A celebrated example of successful application of optimization in electrical engineering is the company, Barcelona Design, established by Prof. S. Boyd of Stanford University, which sells custom designed electronic circuits and uses convex optimization as the core technology to optimize its circuit designs. The application of convex optimization in engineering enables engineers to design and solve problems more efficiently. In recognition of the contribution of optimization to engineering, S.S. Rao wrote: "Optimization is now viewed as an indispensable tool of the trade for engineers working in many different industries, especially the aerospace, automotive, chemical, and manufacturing industries. "

Recent Developments and Open Problems

Starting in the middle 1980's, the field of optimization has experienced a revolutionary development, in particular in the area of convex optimization. This was sparked by Karmarkar's ground-breaking work on IPMs for linear optimization, and later accomplished by many excellent optimization experts. A remarkable work in the IPM stream is the self-concordant theory by Nesterov and Nemirovski that shows several important classes of optimization problems, such as second-order conic and semidefinite optimization (SDO) problems, can be solved efficiently by IPMs. Several high quality optimization packages based on IPMs, such as LOQO, MOSEK, SeDuMi and McIPM, have been developed and they are widely used problem solving tools today. The IPM revolution has brought new theoretical and practical powerful tools for solving classes of convex optimization problems and led to new research areas such as SDO and robust optimization (RO). These novel optimization models and methodologies provide powerful approaches for attacking problems in size and complexity that could not be handled before.

The IPM revolution has brought not only the celebrated theory and powerful tools, but more challenges to the optimization community. This is particularly true in the new areas of SDO and RO. In many scenarios such as signal processing and data analysis, the original physical model is a highly nonlinear and non-convex optimization problem, which is typically NP-hard. The SDO approach can be used only as a relaxed approximation to the original problem. It is very important to investigate under what conditions, the approximate solution obtained from the SDO relaxation can match the exact solution of the original problem, or to what degree it can approximate the exact solution. When we use RO approach to attack optimization problems under uncertainty, it is crucial to investigate under what distribution of the data, the resulting RO model is robust and can be solved efficiently. There are also challenges from an algorithmic perspective. For example, all the current SDO solvers can solve only SDO problems in size up to a couple of thousands variables unless the underlying problem has a special data structure, while there is a huge number of problems with larger size in real applications that cannot be efficiently solved by the existing solvers. Some optimization problems, such as SDO problem with nonnegative constraints on its elements, are theoretically polynomially solvable. However, current SDO solvers can only solve this kind of problems up to sizes of 100 variables. It is important to develop new efficient methodologies for these theoretically solvable but practically extremely hard problems and analyze the properties of these new methods.

The application of optimization in engineering not only helps engineers in their design and analysis of systems, but also leads to significant advances and new discoveries in optimization theory and techniques. For example, numerous engineering design problems need to deal with noisy data, manufacturing error or uncertainty of the environment during the design process. Such engineering optimization problems, like the antenna synthesis problem, lead to the birth of robust optimization, a new emerging research area in the context of convex optimization. Further, the application of optimization in chemical engineering has resulted in several powerful optimization packages in that field and these packages have also been proved to be useful tools for solving general optimization problems.

The interaction between engineering and optimization communities has led to numerous significant achievements in both fields. Besides convex optimization, other optimization techniques, such as integer programming, dynamic programming, global optimization and general nonlinear optimization, have also been successfully applied in engineering. On the other hand, the broad application of optimization methodology in engineering yields a strong stimulus to develop new optimization models and algorithms to meet the increasing demand from engineering practice.

In spite of the broad application of optimization in engineering and occasional close collaboration between individual optimizers and engineers, there have been few chances for experts from both sides to meet to discuss new challenges in the subject area, and explore potential opportunities for fruitful collaboration from both sides.

The Talks

Speaker: M.F. Anjos
Title: Finding Nash Equilibria in Electricity Markets: An AC-Network Approach
Coauthors: G. Bautista and A. Vannelli.
Abstract: Using an AC transmission network, oligopolistic competition in power markets is formulated as a Nonlinear Programming (NLP) problem, and characterized by a multileader single-follower game. The follower is composed of a set of competitive suppliers, demands and the system operator, while the leaders are the dominant suppliers. The transmission network is modeled with a detailed nonlinear system. This approach allows one to capture the strategic behavior of suppliers regarding not only active but also reactive power. With this setting, the impact of voltage and apparent power flow constraints can be easily explored. Based on a three-node system, an illustrative study case is used to highlight the features of the formulation. A larger system is also used to describe computational issues.

Speaker: A. d'Aspermont
Title: Smooth Semidefinite Optimization and Applications
Abstract: We describe an application of Nesterov's optimal first-order method on large scale semidefinite optimization problems arising in multivariate statistics. The method's key numerical step involves computing a matrix exponential, and we show that this can be done very efficiently for favorable matrix spectrum structures.

Speaker: John T. Betts
Title: Planning a Trip to the Moon? ... And Back?
Abstract: Designing an optimal trajectory to the moon and back leads to an extremely challenging optimal control problem. This presentation will describe the practical and computational issues associated with the solution of these problems, including treatment of the underlying nonlinear boundary value problem, and nonlinear programming algorithm.

Speaker: A.R. Conn
Title: An Initial Algorithmic Framework for Convex Mixed Integer Nonlinear Programs
Coauthors: P. Bonami, L.T. Biegler, A. Wächter, G. Cornuéjols, I.E. Grossmann, C.D. Laird, J. Lee, A. Lodi, F. Margot, and N. Sawaya
Abstract: In my opinion mixed integer nonlinear programming is an area of ever increasing importance and applications that is significantly under researched – no doubt because it presents many difficult challenges. I

will present a basic hybrid framework for convex mixed-integer nonlinear programming. In one extreme case, the method becomes the branch-and-bound approach, where a nonlinear optimization problem is solved in each node of the enumeration tree, and in the other extreme it reduces to the polyhedral outer approximation algorithm, which alternates between the solution of a nonlinear optimization problem and a mixed-integer linear program. Numerical results are presented, using an open source software implementation available on `http://www.coin-or.org`. This work results from an on-going research collaboration between IBM and CMU.

Speaker: M. Duer
Title: Solving Copositive Programs through Linear Approximations
Coauthor: S. Bundfuss
Abstract: Optimization over convex cones has become increasingly popular in recent years, the most prominent cones being the positive semidefinite cone and the second order cone. Semidefinite programming provides good and efficiently computable relaxations for several hard combinatorial and quadratic problems. However, it is known that these bounds may be improved by solving optimization problems over the copositive cone. The price of this gain in quality is an jump in complexity, as copositive programs are NP-hard. In this talk, we propose new polyhedral approximations of the cone of copositive matrices which we show to be exact in the limit. This gives rise to necessary as well as sufficient criteria for copositivity, and it can also be used to approximate copositive programs. We present an algorithm resulting from this approach, and conclude by presenting preliminary numerical results.

Speaker: M.C. Ferris
Title: Optimization of Noisy Functions: Application to Simulations
Coauthor: G. Deng
Abstract: In many real-world optimization problems, the objective function may come from a simulation evaluation so that it is (a) subject to various levels of noise, (b) not neccesarily differentiable, and (c) computationally hard to evaluate.

We propose a two-phase approach for optimization of such functions. Phase I uses classification tools to facilitate the global search process. By learning a surrogate from existing data the approach identifies promising regions for optimization. Additional features of the method are: (a) more reliable predictions obtained using a voting scheme combining the options of multiple classifiers, (b) a data pre-processing step that copes with imbalanced training data and (c) a nonparametric statistical method to determine regions for multistart optimizations

Phase II is a collection of local trust region derivative free optimizations. Our methods apply Bayesian techniques to guide appropriate sampling strategies, while simultaneously enhancing algorithmic efficiency to obtain solutions of a desired accuracy. The statistically accurate scheme determines the number of simulation runs, and guarantees the global convergence of the algorithm.

We present results on two practical simulations: a Wisconsin breast cancer simulation and the robust design for a floating sleeve coaxial antenna for hepatic microwave ablation. The use of resampling of particular organ structures in this context will be outlined. Particular emphasis will be on general principles that are applicable to large classes of treatment planning problems. Specific examples will also be detailed showing enormous increase in speed of planning, without detriment to the quality of solutions found.

Speaker: R.M. Freund
Title: On Efficient Randomized Methods for Convex Optimization
Abstract: Randomized methods for convex optimization rely on stochastic processes and random number/vector generation as part of the algorithm and/or its analysis. In this talk we will discuss some recent developments in randomization-based algorithms for convex optimization from both a theoretical and practical point of view. We will show some interesting parallels between one randomization-based method and interior-point methods, and will forecast some possible trends both in theory and practice and pose some pertinent research questions.

Speaker: D.Y. Gao

Title: Canonical Duality Theory and Applications in Global Optimization
Abstract: This paper presents a canonical (i.e., strong) duality theory for solving nonconvex programming problems subjected to box constraints. It is proved that the dual problems are either concave maximization, or convex minimization problems. Both global and local extrema of the constrained nonconvex problems can be identified by triality theory proposed by the author. Applications to nonconvex integer programming and Boolean least squares problem are discussed. Examples are illustrated. A conjecture on NP-hard problems is proposed.

Speaker: D. Goldfarb
Title: Total Variation Based Image Restoration by Second-Order Cone Programming and Min-Cuts
Coauthor: W. Yin
Abstract: The traditional approach for solving total variation based image restoration problems is based on solving partial differential equations. We describe here how to formulate and solve these problems either by interior-point algorithms for second-order cone programs or by parametric max flow algorithms.

Speaker: F. Jarre
Title: An Augmented Primal-Dual Method for Linear Conic Minimization
Coauthors: F. Rendl
Abstract: We present a new iterative method for solving linear minimization problems over convex cones. The problem is reformulated as an unconstrained problem of minimizing a differentiable convex function. The method does not use any homotopy parameter but solves the primal-dual problem in one step using a nonlinear conjugate gradient type approach. Some approaches for preconditioning of the algorithm will be illustrated with numerical examples.

Speaker: E. Krislock
Title: The Nonsymmetric Semidefinite Least Squares Problem and Compliance Matrix Estimation
Abstract: An important step in the process of making an interactive computer model of a deformable object is to estimate the compliance matrix at various points on the object. This estimation is accomplished by taking experimental measurements of the object and computing the least squares solution of a linear matrix equation of the form $AX = B$. For such a compliance matrix X, it is required that $\frac{1}{2}(X + X^T)$ be positive semidefinite, otherwise the computer model may respond to the user touching some contact point by pulling the user's hand further in the direction of the touch. Adding this constraint to the least squares problem we get the nonsymmetric semidefinite least squares NS-SDLS

$$\begin{array}{rl} \min & \|AX - B\|_F \\ \text{subject to} & 0.5(X + X^T) \succeq 0 \end{array}$$

When the matrix A has linearly independent columns, the solution of the NS-SDLS problem exists and is unique. We will provide the Karush-Kuhn-Tucker equations which characterize this solution and show how these equations can be stated as a semidefinite linear complementarity problem. Finally, we will discuss how interior-point methods can be used for the numerical solution of the NS-SDLS problem.

Speaker: K. Kostina
Title: Model Based Design of Optimal Experiments for Parameter Estimation and Applications
Coauthor: H. G. Bock, S. Koerkel, and J. P. Schloeder
Abstract: The development and quantitative validation of complex nonlinear differential equation models is a difficult task that requires the support by numerical methods for sensitivity analysis, parameter estimation, and the optimal design of experiments. We first present particularly efficient "simultaneous" boundary value problems methods for parameter estimation in nonlinear differential algebraic equations, which are based on constrained Gauss-Newton-type methods and a time domain decomposition by multiple shooting. The method include a numerical analysis of the well-posedness of the problem and an assessment of the error of the resulting parameter estimates. Based on these approaches, efficient optimal control methods for the determination of one, or several complementary, optimal experiments are developed, which maximize the

information gain subject to constraints such as experimental costs and feasibility, the range of model validity, or further technical constraints.

Special emphasis is placed on issues of robustness, i.e. how to reduce the sensitivity of the problem solutions with respect to uncertainties - such as outliers in the measurements for parameter estimation, and in particular the dependence of optimum experimental designs on the largely unknown values of the model parameters. New numerical methods will be presented, and applications will be discussed that arise in satellite orbit determination, enzyme kinetics and robotics. They indicate a wide scope of applicability of the methods, and an enormous potential for reducing the experimental effort and improving the statistical quality of the models.

Speaker: Z.Q. Luo and S. Zhang

Title: Optimization in Resource Management: Complexity, Lyapunov Theorem and Approximation

Abstract: We consider a class of nonconvex optimization problems arising from resource (e.g., spectrum) management in multiuser communication. For the discretized version of this problem, we characterize its computational complexity under various practical settings and study the structure of its global optimal solutions. It is shown that this discretized nonconvex optimization problem is NP-hard in general and has a positive duality gap. Surprisingly this duality gap disappears asymptotically as the size of discretization step decreases to zero, thanks to a hidden convexity that can be uncovered by the Lyapunov Theorem in functional analysis. Based on this asymptotic zero duality result and a Lagrangian dual relaxation, we present, for any positive ϵ, a polynomial time approximation scheme to compute an ϵ-optimal solution for the continuous version of the resource management problem. Finally, we also establish a general min-max theorem for a game theoretic formulation under the continuous framework.

Speaker: J. Moré

Title: Derivative-Free Methods for Simulation-Based Optimization Problems

Coauthor: S. Wild

Abstract: We give a brief overview of the current state of the art of derivative-free methods that emphasizes the viewpoint that the performance of derivative-free methods should be measured when there is a constraint on the computational budget, that is, when there is a (small) limit on the number of function evaluations. We discuss how this viewpoint is appropriate for simulation-based optimization problems, and outline current research on new algorithms for this class of optimization problems.

Speaker: J. Martins

Title: Multidisciplinary Optimization: Current Status and Future Directions

Abstract: The objective of this talk is to present an overview multidisciplinary design optimization (MDO), with emphasis on the most significant challenges currently faced by academia and industry in its utilization. All current MDO architectures will be described and a unified mathematical framework for describing these architecture will be proposed. On the more applied side, a new software package that uses these ideas to automatically implement the various MDO architectures will be presented, together with results obtained in the solution of a suite of test problems. The suite itself represents another important focus of the current research and includes scalable problems in order to investigate how the relative merits of each MDO architecture vary. Finally, future research directions will be identified and discussed.

Speaker: J. Nie

Title: SOS Methods for Sensor Network Localization

Abstract: We formulate the sensor network localization problem as finding the global minimizer of a quartic polynomial. Then sum of squares (SOS) relaxations can be applied to solve it. However, the general SOS relaxations are too expensive for practical problems. Exploiting special features of this polynomial, we propose a new Structured SOS relaxation. It works well for large scale problems. At each step of interior-point methods solving the resulting SOS relaxation, the computational cost is $O(n^3)$. When distances have errors and localization is unique, we show that the sensor location given by this SOS relaxation is accurate within a constant factor of the distance error under some technical assumptions.

Speaker: A. Ozgadlar
Title: Differential Topology for the Uniqueness of Equilibrium in Network Control Models
Coauthors: A. Simsek and D. Acemoglu
Abstract: In this talk, we first present an extension of the Poincare-Hopf Theorem of index theory to generalized critical points of a function defined on a compact region with nonsmooth boundary, defined by a finite number of smooth inequality constraints. We use the generalized Poincare-Hopf Theorem to present sufficient (local) conditions for the global uniqueness of solutions to finite-dimensional variational inequalities and the uniqueness of stationary points of optimization problems. We finally use our results to establish uniqueness of equilibria in two recent models of communication networks.

Speaker: J.D. Pintér
Title: Global Optimization in Practice: State-of-the-Art and Perspectives
Abstract: Global optimization (GO) – the theory and methods of finding the best solution in multi-extremal models – has become a subject of significant interest in recent decades. The key theoretical results have been followed by software implementations, and a growing range of real-world applications. We present a concise review of these developments, with an emphasis on practical aspects, including modeling environments, software and applications.

Speaker: G. Savard
Title: Pricing a Segmented Market Subject to Congestion
Coauthors: M. Fortin, L. Brothers, and P. Marcotte
Abstract: The optimal setting of prices, taxes or subsidies on goods and services can be naturally modeled as a bilevel program. Indeed, bilevel programming is an adequate framework for modeling optimization situations where a subset of decision variables is not controlled by the main optimizer (the leader), but rather by a second agent (the follower) who optimizes its own objective function with respect to this subset of variables. In this presentation we address the problem of setting profit-maximizing tolls on a congested transportation network involving several user classes. At the upper level, the firm (leader) sets tolls on a subset of arcs and strives to maximize its revenue. At the lower level, each user minimizes its generalized travel cost, expressed as a linear combination of travel time and out-of-pocket travel cost. We assume the existence of a probability density function that describes the repartition of the value of time (VOT) parameter throughout the population. This yields a bilevel optimization problem involving a bilinear objective at the upper level and a convex objective at the lower level. Since, in this formulation, lower level variables are flow densities, it follows that the lower level problem is infinite-dimensional. We devise a two-phase algorithm to solve this nonconvex problem. The first phase aims at finding a good initial solution by solving for its global optimum a discretized version of the model. The second phase implements a gradient method, starting from the initial point obtained in the initial phase.

Speaker: P. Tseng
Title: p-Order Cone Relaxation for Sensor Network Localization
Abstract: Building on recent work on 2nd-order cone relaxation for sensor network localization, we discuss extensions to p-order cone relaxation when measured distances are based on p-norm instead of Euclidean norm.

Speaker: C. Visweswariah
Title: Challenges in Statistical Timing and Optimization of Integrated Circuits
Abstract: As transistors and wires on a chip get smaller, they are exhibiting proportionately increasing variability. This variability is changing the design methodology, and the tools and techniques used to analyze and optimize chip designs. The first part of this presentation will give an overview of our research work in statistical timing and optimization. In the second part, some mathematical problems will be formulated that, if solved, would be of tremendous utility to the design automation community.

Speaker: H. Wolkowicz
Title: Semidefinite Relaxations for Anchored Graph Realization and Sensor Localization

Abstract: Many applications use ad hoc wireless sensor networks for monitoring information. Typical networks include a large number of sensor nodes which gather data and communicate among themselves. The location of a subset of the sensors is known; these sensors are called anchors. From the intercommunication, we are able to establish distances between a subset of the sensors and anchors. The sensor localization problem is to find/estimate the location of all the sensors. We study several semidefinite programming relaxations for this numerically hard problem.

Speaker: H. Zhang
Title: Approximation Algorithms for Routing in VLSI Design and Multicast Networks
Coauthors: A. Deza, C. Dickson, T. Terlaky, and A. Vannelli
Abstract: Given a computer chip represented as a grid graph and groups of pins as vertices to be connected, the goal of the global routing problem in VLSI design is to find one tree along the channels for each group of pins such that the number of trees crossing a channel is bounded by its capacity and the total cost (a combination of the overall tree length and the total number of bends) is minimized. Global routing is regarded as one of the hardest problems in discrete optimization due to the scales of real instances. We present a concurrent routing algorithm by mathematical programming methods in order to approach the global optimum. Our algorithm runs in a polynomial time and delivers a near-optimal solution with a provably good approximation bound for any instance. Promising numerical results for challenging benchmarks are also reported. We also show that this algorithm can be applied to a multicast routing problem in communication networks.

Highlights of the Panel Discussions

Title: Where we need breakthroughs? What is the next breakthrough?
Panel: D. Goldfarb, T. Marlin, J. Pintér, R. Vanderbei
Moderator: T. Terlaky

- The dramatic progress in LP technology was highlighted, from the 1947 diet problem (9 constraints, 77 variables, 120 “soldier-days”. The software was 9 people!) through the 1980s (up to few thousands variables and constraints in 6 seconds-300 seconds) and up to 2004 (many large problems can be solved in a few seconds on any decent computer). These improvements were due roughly equally to hardware and software. Preprocessing probably brought the most dramatic improvement! The breakthrough in IPMs for LP came from an obscure, theoretical Soviet mathematician. The question is of course: Where will the next breakthrough come from?

  We *really* don’t know where the breakthrough will come from! IPMs were very exciting for many years. Now we have symmetric cone optimization, robust optimization... could this have been predicted?

- Dealing with uncertainty is very important and very difficult. The effect of uncertainty on the objective was pointed out: monitor, diagnose, reduce (e.g., control in a chemical plant to maximize the profit). Reduce uncertainty by systematic, focused experimentation to improve the performance of model-based optimization.

- A related point was that for engineers to pay attention, mathematicians must show a genuine interest in the problems. Also, proper modelling is really important, and therefore so is understanding and caring about the problem.

- The really interesting stuff happens in inter-disciplinary research, where the most significant contributions are made (e.g., cancer therapy research needed global optimization).

  The only way to succeed is for optimizers and engineers to get together and talk, as is taking place this week at BIRS.

Title: What do engineers need? What do optimizers need to know about engineering?
Panel: Z.Q. Luo, T. Marlin, J. Martins, J. Moré, C. Viswesvariah, L. Behjat
Moderator: R. Vanderbei

- There is tremendous power in optimization algorithms. The challenge is to harness their power.
- What can optimization offer?
  - a suite of good algorithms and software;
  - a theory that helps to characterize the structure of optimal solutions (can serve as a guide to designing heuristics).
- What do engineers need?
  - Education: good optimization, complexity courses, books; course projects (missing some basic ideas and techniques);

    Many engineering students just use the Matlab optimization toolbox blindly. Some engineers develop in-house software without exploiting what is already available. In short, there is a need to train a large number of engineers with a solid understanding of optimization.

    Engineers want to understand the solutions using post-optimality analysis. They often do this visually, e.g., using plots of the result, not analytically.
- Software: robust, efficient, and able to solve large-scale problems.
- Some theory, such as: distributed optimization; game theory over energy-constrained networks with unreliable links.

  Need specialized codes to compete with engineering heuristics. General codes usually not efficient, and high accuracy is typically not needed, it is even useless!
- Key: For optimizers,
  - find a good engineering partner.
  - work closely with engineers to:
    - formulate the problem properly;
    - suggest good software to use (or develop specialized ones);
    - identify interesting theoretical issues for further investigation;
    - interpret computational and theoretical results in engineering context.
  - have a lot of patience!
- The real value is in the *interaction* when dealing with really hard problems. Engineers need to appreciate what optimizers can deliver; and the optimizers need to understand the needs of engineers.
- Challenges for the future:
  - Encourage communication between engineers and mathematicians: tutorials? specialized workshops?
  - Need to understand each other's challenges;
  - Start at the undergraduate level, and try to bring optimization into the engineering courses;
  - Better education is really key!

Title: Optimization Software
Panel: A. Conn, J. Moré, J. Pintér, I. Pólik, K. Toh, R. Vanderbei
Moderator: T. Terlaky

- Ease of use more important than quality of algorithm!
- Commercial software often more attractive than open source (the fact they pay for it makes a difference);

- Nonlinear optimization cannot be used like a black-box;
- We publish our theory and build on it. We do not publish our software so others can build on it;
- Software needs good documentation; including simple examples so that users can copy code;
- Where's the reward in writing good software?
- Commercial software does not keep pace with academic breakthroughs;
- Almost all academics and students don't know how to write extensible, maintainable software;
- Lacks: we lack *standards*, we lack *component libraries*, we lack *support*;
- There has been a serious effort in implementing some semi-general purpose SDP solver for large-scale problems;
- Testing is a very important part of developing software; be imaginative on how your code will perform badly!

Scientific Progress Made

Participants with similar research interests formed several special interest groups. There was thorough discussion about their research work and future plan in each group. Sketch of the discussed topics in each group are as follows:

- **VLSI design** (M. Anjos, L. Behjat, A. Conn, T. Terlaky, C. Visweswariah, H. Zhang): Participants in this group are from both industry and academia.
  - Mathematical models for placement, floorplanning and global routing in VLSI design;
  - Current methods: IPMs, integer nonlinear programming, approximation algorithms;
  - More realistic models fitting the engineering demands needed;
  - Efficient algorithms and fast heuristics desired;
  - Implementation and software packages for industrial benchmark.
- **Digital communication** (Z.Q. Luo, J. Nie, P. Tseng, W. Yu, H. Wolkowicz, S. Zhang):
  - Hidden convexity in resource management in multicast communication;
  - Relaxation for sensor network localization: SDP, Sum of Square, or cone;
  - More application of Lyapunov theorem expected;
  - Alternative proof of Lyapunov theorem?
  - Better approaches for sensor network localization?
  - Link between resource management and fractional packing problems.
- **Bilevel and equilibrium problems** (M. Anjos, M. Ferris, A. Ozdaglar, G. Savard, P. Tseng):
  - Mathematical formulations of equilibria;
  - Computational methods and software for equilibrium problems;
  - Role and importance of congestion in markets;
  - Comparison of different methodologies for bilevel problems.
- **Engineering design and optimization** (J. Betts, E. Kostina, T. Marlin, J. Martins, R.Vanderbei): Participants in this group are from both industry and academia.
  - Challenges for optimization in industry;
  - Importance of MDO;
  - Reduction of uncertainty by systematic, focused experimentation;
  - The challenges of industry-academia interactions.

Outcomes of the Meeting

There will be a special issue of the journal Optimization and Engineering, edited by E. Kostina and J. Martins, dedicated to this scientific meeting.

This meeting clearly generated a significant amount of interaction between members of both communities, and we hope that these discussions have seeded future exciting developments at the interface between optimization and engineering.

List of Participants

Adler, Ilan (University of California, Berkeley)
Anjos, Miguel F. (University of Waterloo)
Behjat, Laleh (University of Calgary)
Betts, John (Boeing)
Brown, David (Duke University)
Chen, Lifeng (Columbia University)
Chiang, Mung (Princeton University)
Conn, Andrew R. (IBM)
d'Aspremont, Alexandre (Princeton University)
Ding, Yichuan (University of Waterloo)
Duer, Mirjam (T.U. Darmstadt)
Ferris, Michael C. (University of Wisconsin)
Freund, Robert M. (MIT)
Gao, David Yang (Virginia Tech)
Goldfarb, Donald (Columbia University, New York)
Grodzevich, Oleg (University of Waterloo)
Jarre, Florian (University of Duesseldorf)
Kostina, Ekaterina (University of Heidelberg)
Krislock, Nathan (University of Waterloo)
Luo, Zhi-Quan (University of Minnesota)
Marlin, Tom (McMaster University)
Martins, Joaquim R. R. A. (University of Toronto)
Moré, Jorge (Argonne)
Nematollahi, Eissa (McMaster University)
Nie, Jiawang (University of California, Berkeley)
Ozdaglar, Asu (Massachusetts Institute of Technology (MIT))
Peng, Jiming (University of Illinois at Urbana-Champaign)
Pinter, Janos (Pinter Consulting Services, Inc.)
Polik, Imre (McMaster University)
Richtarik, Peter (Cornell University)
Savard, Gilles (Ecole Politechnic Montréal)
Terlaky, Tamás (McMaster University)
Toh, Kim-Chuan (National University of Singapore (NUS))
Tseng, Paul (University of Washington)
Vanderbei, Robert (Princeton University)
Visweswariah, Chandu (IBM Thomas J. Watson Research Center)
Wild, Stefan (Cornell University)
Wolkowicz, Henry (University of Waterloo)
Yu, Wei (University Toronto)
Zhang, Shuzhong (Chinese University of HongKong)
Zhang, Hu (McMaster University)
Zhang, Yin (Rice University)

Chapter 23

Polynomials over Finite Fields and Applications (06w5021)

November 18 - -23, 2006

Organizer(s): Ian Blake (University of Toronto), Stephen Cohen (University of Glasgow), Gary Mullen (Pennsylvania State University), Daniel Panario (Carleton University)

Introduction

Finite fields are discrete mathematical objects satisfying all the axioms of a field, much as for the real or complex numbers, except for their finiteness. They are also referred to as Galois fields after the French mathematician Evariste Galois who was one of the first to show interest in them. It is easy to show that such objects exist only when the number of their elements is a power of a prime number and that any two finite fields with the same number of elements are isomorphic. Fundamental to the study of finite fields is the study of polynomials over finite fields which is the focus of this workshop.

The properties of finite fields and polynomials over them are of interest in their own right as they play a central role in many areas of pure mathematics. However it is perhaps the fundamental role they play in applications that propels them to such prominence and is the central interest of this workshop. These applications include such areas as:

Error correcting codes: Such codes are fundamental to many digital communication and storage systems, to improve the error performance over noisy channels. First proposed in the seminal work of Claude Shannon, they are now ubiquitous and included even in consumer electronic systems such as compact disc players and many others. Virtually all such error correcting codes and their decoding algorithms depend on the structure and properties of finite fields and polynomials over finite fields.

Cryptography: The advent of public key cryptography in the 1970's has generated innumerable security protocols which find widespread application in securing digital communications, electronic funds transfer, email, internet transactions and the like. Most of these schemes use either the integers or finite fields as the domain of computation to achieve their goals. More recent systems use elliptic curves, hyperelliptic curves and Abelian varieties over finite fields and these are assuming ever greater importance in applications and depend crucially on properties of finite fields for their study and implementation.

Sequences: Pseudorandom sequences i.e. deterministic sequences with random-like properties, are needed in a great many applications, including such areas as cryptography where random-like seeds are required. In addition, the design of sequences, mainly binary, for low correlation are central to such cell phone technology as CDMA. Most such systems require application of various finite field properties, including polynomials over finite fields.

While many of the talks of this workshop found their motivation in the above and related areas, such as combinatorics and design theory, many other talks addressed questions of fundamental importance to the theory of finite fields and their relationship to other areas of mathematics such as algebraic geometry and number theory.

Press Release

World's top experts on finite field theory meet at BIRS: Finite fields are finite sets of objects which have an arithmetic that allows the usual operations of addition, subtraction, multiplication, and division, except that contrary to the real numbers, the set contains only a finite number of distinct elements. Some of the best finite field researchers from around the world will converge on The Banff Centre during the period Nov. 18-23, 2006, where the Banff International Research Station will be hosting a workshop on recent developments in the theory of polynomials over finite fields. This event is co-organized by Professors Ian Blake of the University of Toronto, Stephen Cohen from the University of Glasgow, Gary Mullen from The Pennsylvania State University and Daniel Panario of Carleton University.

The workshop will focus on new results and methods in the study of various kinds of polynomials over finite fields. Finite fields are not only of deep mathematical interest in their own right but also play a critical role in modern information theory including algebraic coding theory for the error-free transmission of information and cryptology for the secure transmission of information. Polynomials over finite fields play an essential role in these and other very practical and important technologies; thus the emphasis of the workshop on various aspects related to polynomials with coefficients in finite fields.

Overview of the Field and Objectives of the Workshop

Polynomials over finite fields have been studied since the time of Gauss and Galois. The determination of special types of polynomials such as irreducible, primitive, and permutation polynomials, is a long standing and well studied problem in the theory and application of finite fields.

On the other hand, in recent years there has been intensive use of special polynomials in many areas including algebraic coding theory for the error-free transmission of information, cryptography for the secure transmission of information, and combinatorics, especially design theory. Polynomials over finite fields are the key ingredient in the construction of error-correcting codes such as BCH, Goppa, Reed-Solomon, and Reed-Muller codes, among others. Moreover, polynomials also play a key role in other areas of coding theory such as the determination of weight enumerators, the study of distance distributions, and decoding algorithms.

Large extensions of finite fields (especially over the two-element field) are important in cryptography. Elements in these extension fields can be represented by polynomials over the prime subfield. Thus, constructions of extension fields and fast arithmetic of polynomials are important practical questions. Moreover, many central problems in cryptography such as the discrete logarithm problem can be immediately translated into problems involving polynomials over finite fields.

Polynomials over finite fields appear very naturally in several areas of combinatorics. First, due to the finite number of elements, the enumeration of various special kinds of polynomials over finite fields is an interesting and extremely important research area in combinatorics. In design theory, polynomials are used to construct and describe cyclic difference sets and special types of designs such as group divisible designs. Divisibility conditions on trinomials over finite fields have been shown to produce orthogonal arrays with certain strengths, and bivariate and multivariate polynomials can be used to represent and study latin squares and sets of orthogonal latin squares and hypercubes of prime power orders.

In addition, polynomials over finite fields are important in engineering applications. Linear recurrence relations over finite fields produce sequences of field elements. Linear feedback shift registers are used to implement these recurrences. Characteristic polynomials over finite fields are one of the main tools when dealing with shift registers. In particular, primitive characteristic polynomials produce sequences with large periods, and thus have found many applications in areas such as random number generation.

As an example of an important theoretical problem involving polynomials over a finite field F_q is the study of the value set $\{f(a) | a \in F_q\}$ of a polynomial f. Polynomials with maximal value sets are permutation polynomials which have applications in various areas of combinatorics such as the study of sets of orthogonal latin squares, as well as in cryptography. Can one characterize the polynomials of degree n which have maximal value sets; minimal value sets, value sets of cardinality k with $1 \leq k \leq q$? What is the number of such polynomials? These are fundamental theoretical problems which are not only worthy of study in their own right, but also because of their applications.

This workshop addressed new theoretical results about polynomials over finite fields as well as the ways in which polynomials are used in algebraic coding theory, cryptography, and design theory. Given the increasing number of relevant research papers, as can be seen by checking recent issues of journals such as *Finite Fields and Their Applications, Designs, Codes and Cryptography*, and the *IEEE Transactions on Information Theory*, along with other journals such as the *J. Combinatorial Theory Series A*, *J. Number Theory*, and *Discrete Mathematics* which regularly publish papers dealing with finite fields and their applications, we believe the time has come to summarize these achievements and formulate new challenges in this very important area.

The purpose of the workshop was to establish links between the community of researchers working on polynomials over finite fields and researchers and practitioners working in various areas of application. The workshop brought together researchers from finite fields, coding theory, cryptography, combinatorics, number theory and engineering. As a result of this interaction researchers became acquainted with recent techniques and results dealing with various theoretical properties of polynomials over finite fields. The diversity of the attendees stimulated joint works and fostered work across fields, leading to new mathematics and to the solution of interesting applied problems.

A balanced group of talks in each of the areas of focus of the workshop was presented. These areas included theoretical properties of polynomials over finite fields, connections to algebraic coding theory, combinatorial design theory, and cryptography.

Recent Developments

We enumerate some important recent developments discussed at the meeting.

In 1992 Hansen and Mullen conjectured that with only a few necessary exceptions for very small values of q and small degrees n, there is always at least one primitive polynomial of degree n over $\mathbb{F}_q$ with the coefficient of any power of x specified in advance. More precisely, given $a \in \mathbb{F}_q$ and j with $1 \leq j \leq n-1$, there is always at least one primitive polynomial of degree n over $\mathbb{F}_q$ where a is the coefficient of x^j. This conjecture led to a lot of research establishing various related results, but the Banff Workshop talk by M. Presern completed the proof that the Hansen/Mullen Conjecture was indeed true.

The computation of modular polynomials is crucial for many applications, including the problem of counting points on elliptic curves over finite fields. Enge's presentation centered on recent algorithmic developments that allow modular polynomial computation in essentially linear time in the output size. This algorithm has been used to establish new records for elliptic curve point counting.

Lenstra's talk focused on a very recent new algorithm (due to himself and Carl Pomerance) for the construction of irreducible polynomials over finite fields. Irreducible polynomials are normally used for constructing elements in finite field extensions. This algorithm comes as a by-product of their version of a new polynomial time primality test.

In recent years, there has been considerable progress in the constructions of algebraic curves with many rational points. The main leaders of these developments are Arnaldo Garcia and Henning Stichtenoth. This area has applications to coding theory (which for years was its main driving motivation). Garcia's talk focused on new results for certain type of towers (Artin-Schreier) recently studied in collaboration with Stichtenoth.

An important class of polynomials over finite fields are the polynomials that permute the elements in the field, the so-called *permutation polynomials*. Permutation polynomials have been largely studied since Hermite but even the simple case of characterizing permutation binomials remains vastly open. Masuda's talk concentrated on characterizing and counting permutation binomials over certain special finite fields.

Counting special type of polynomials is a classical problem in finite fields. Precise formulas exist for the number of univariate irreducible polynomials over a finite field and several related quantities such as squarefree (and in general, k-free) polynomials, and so on. The equivalent problem of counting polynomials in several variables has been less studied, although some results from the 1960's exist mostly due to Carlitz and Cohen. Von zur Gathen's talk presented new counting estimates for some classes of special bivariate polynomials such as reducible, the so-called "exceptional" and "singular" polynomials.

Many properties of integers have been considered for polynomials over finite fields. For example, questions about the factorization of integers can be easily translated into questions about the decomposition of polynomials in irreducibles; famous number theoretic problems such as the twin primes and the Goldbach conjectures have been studied over finite fields. Mullen's talk presented results dealing with a polynomial analogue of the famous unsolved 3n+1 problem.

On Monday evening there was a very active two hour problem session moderated by Gary Mullen during which numerous open problems relating to polynomials over finite fields were discussed. While we will not try to describe these problems here, it was clear that the problems generated considerable interest. Moreover, later in the week many participants continued to discuss some of these problems. We are confident that further discussion will continue long after the conference is over and that some related results will be published.

Titles and Abstracts

Speaker: **Omran Ahmadi** (University of Toronto)
Title: *Quadratic transformation of irreducible polynomials over finite fields*

Abstract: Self-reciprocal irreducible monic (srim) polynomials over finite fields have been studied in the past. These polynomials can be studied in the context of quadratic transformation of irreducible polynomials over finite fields. In this talk we present the generalization of some of the results known about srim polynomials to polynomials obtained by quadratic transformation of irreducible polynomials over finite fields.

Speaker: **Dan Bernstein** (University of Illinois at Chicago)
Title: *Faster factorization into coprimes*
Abstract: How quickly can we factor a set of univariate polynomials into coprimes? See `http://cr.yp.to/coprimes.html` for examples and applications. Bach, Driscoll, and Shallit achieved time $n^{(2+o(1))}$ in 1990, where n is the number of input coefficients; I achieved time $n(\lg n)^{(O(1))}$ in 1995; much more recently I achieved time $n(\lg n)^{(4+o(1))}$.

Speaker: **Antonia Bluher** (National Security Agency)
Title: *Hyperquadratic elements of degree 4*
Abstract: I will describe joint work with Alain Lasjaunias about the construction of degree-4 polynomials over fields K of char p whose roots α have a Frobenius property:

$$\alpha^q = \frac{A\alpha + B}{C\alpha + D},$$

where $A, B, C, D \in K$, $AD - BC$ is nonzero, and q is a power of p.

The case of interest is when K is a function field and α has a Laurent series expansion. It is conjectured that in such a case, α might have interesting patterns in its continued fraction expansion, such as those found by Buck and Robbins for a root of the polynomial $X^4 + X^2 - TX + 1 \in \mathbb{F}_{13}(T)[X]$.

Speaker: **Mireille Car** (Universite Paul Cezanne (Aix-Marseille III))
Title: *Ternary Quadratic forms that represent 0, the function field case*

Abstract: Let K be a global function field with field of constants a finite field k with q elements and odd characteristic. Let S be a finite set of $s > 0$ places of K and let R_S denote the set of S-integers of K. For s-tuples of rational integers $\mathbf{m} = (m_v)_{v\in S}$ and $\mathbf{n} = (n_v)_{v\in S}$, let $Q_S(\mathbf{m}, \mathbf{n})$ denote the number of pairs (a, b) of integers of R_S such that $v(a) = m_v, v(b) = n_v$ for all $v \in S$, and such that the quadratic form

$$(f_{a,b}) \qquad X^2 - aY^2 - bZ^2$$

represents 0 over the field K. We give an asymptotic estimate for the number $Q_S(\mathbf{m}, \mathbf{n})$ for s-tuples $\mathbf{m}$ and $\mathbf{n}$ such that the numbers

$$\|\mathbf{m}\| = -\sum_{v\in S} f_v m_v, \qquad \|\mathbf{n}\| = -\sum_{v\in S} f_v n_v$$

tend to $+\infty$, f_v denoting the degree of the place v.

In a previous work, we dealt with these questions in the case of a rational function-field. (Indeed, if $K = k(T)$, the rational function field, the polynomial ring $k[T]$ is the ring $R_{\{\infty\}}$ with ∞ the $\frac{1}{T}$-place, and if m and n are positive integers, if $(\mathbf{m}, \mathbf{n}) = ((-m), (-n))$, the number $Q_{\{\infty\}}(\mathbf{m}, \mathbf{n})$ is equal to the number $H(m, n)$ of polynomials a and b in $k[T]$ of degree m and n respectively, such that the quadratic form $(f_{a,b})$ represents 0 over the field $k(T)$.) The case of the rational function-field was a polynomial analogue of questions asked by Serre and solved by Hooley and Guo about the size of the number $H(x)$ of pairs $(a, b) \in \mathbb{Z}^2$, such that $|a| \le x$, $|b| \le x$ and such that the ternary quadratic form

$$X^2 + aY^2 + bZ^2$$

represents 0 over the field $\mathbb{Q}$. Presently now, no number field analogue of the theorems proved in what follows is known.

Speaker: **Michael Dewar** (University of Illinois, Urbana-Champaign)
Title: *When do pentanomials divide trinomials over $\mathbb{F}_2$?*
Abstract: Over $\mathbb{F}_2$, up to reciprocals, no pentanomial of degree m divides a trinomial of degree at most $2m$ except for 25 specific exceptions, all with degree $m < 14$, and one infinite family of pentanomials. A careful case analysis reveals that for large degree the coefficients cancel in a "staircase"-like manner. This divisibility property allows the construction of orthogonal arrays of strength 3.

[This is a joint work with L. Moura, D. Panario, B. Stevens and Q. Wang.]

Speaker: **John Dillon** (National Security Agency)
Title: *APN polynomials and related codes*
Abstract: A map $f : \mathrm{GF}(2^m) \to \mathrm{GF}(2^m)$ is APN (*almost perfect nonlinear*) if $x \mapsto f(x+a) - f(x)$ is 2-to-1 for all nonzero a in GF(2^m). Equivalently, the binary code of length $2^m - 1$ with parity-check matrix

$$H := \begin{bmatrix} \cdots & \omega^j & \cdots \\ \cdots & f(\omega^j) & \cdots \end{bmatrix}$$

is double-error-correcting, where ω is primitive in GF(2^m) and we may, without loss of generality, assume that $f(0) = 0$.

We give a brief review of these maps and their polynomials; and we present some new examples along with some related codes and designs which serve as invariants for their equivalence classes.

Speaker: **Andreas Enge** (Ecole polytechnique, Paris)
Title: *Asymptotically optimal computation of modular polynomials*
Abstract: Modular polynomials play an essential role in the Schoof-Elkies-Atkin algorithm for point counting on elliptic curves over finite fields, and they occur in several algorithms for constructing elliptic curves with prescribed complex multiplication. I present an algorithm based on floating point evaluation and interpolation that computes several flavours of modular polynomials in essentially linear time in the output size, and that has enabled us to set the recent records for elliptic curve point counting.

Speaker: **Luis H. Gallardo** (L'Universite de Bretagne Occidentale)
Title: *Sums of biquadrates in $\mathbb{F}_q[t]$*
Abstract: For most q's, it is known that every polynomial $P \in \mathbb{F}_q[t]$ that is a sum of biquadrates in $\mathbb{F}_q[t]$ is a strict sum, (i.e., if $P = A^4 + \ldots$ then $\deg(A^4) < \deg(P) + 4$), of 16 biquadrates.

Here we explain how we may reduce this to 11 biquadrates. Essentially this is done by using a new formula (that was discovered recently) to write t as a sum of 4 biquadrates when -1 is not a biquadrate in $\mathbb{F}_q$. (Its proof uses Jacobi sums). [This is joint work with Mireille Car.]

Speaker: **Shuhong Gao** (Clemson University)
Title: *Primary decomposition of zero-dimensional ideals over finite fields*
Abstract: A new algorithm is presented for computing primary decomposition of zero-dimensional ideals over finite fields. Like Berlekamp's algorithm for univariate polynomials, the new method is based on the

kernel of the Frobenius map acting on the quotient algebra. The dimension of the kernel equals the number of primary components, and a basis of the kernel yields a complete decomposition. Unlike previous approaches for multivariate polynomial systems, the new method needs no generic projections but reduces the problem directly to root finding of univariate polynomials over the ground field. If time permits, we shall show how Gröbner basis structure can be used to get partial primary components without root finding. [Joint work with Daqing Wan and Mingsheng Wang.]

Speaker: **Arnaldo Garcia** (IMPA)
Title: *Some Artin-Schreier towers are easy*
Abstract: Towers of function fields (resp. of algebraic curves) over finite fields with positive limit, for the ratios of numbers of rational places over the genera, provide examples of curves with large genus having many rational points over a finite field. It is in general a difficult task to calculate the limit of a tower. In this talk we present a simple method how to calculate the genus of certain Artin-Schreier towers. As an illustration of our method we obtain a very simple and unified proof for the limits of some towers which attain the Drinfeld-Vladut bound or the Zink bound.The limits of their Galois closures can also be obtained similarly. The method computes the limit of certain towers avoiding the hard computations involved in the determination of the individual genus of each function field in the tower.

Speaker: **Joachim von zur Gathen** (University of Bonn)
Title: *Counting bivariate polynomials: reducible, exceptional, and singular ones*
Abstract: Among the bivariate polynomials over a finite field, most are irreducible. We count some classes of special polynomials, namely the reducible ones, those with a square factor, the "exceptional" ones which are irreducible but factor over an extension field, and the singular ones, which have a root at which both partial derivatives vanish.

Speaker: **Guang Gong** (University of Waterloo)
Title: *Two-level Autocorrelation Sequences, Polynomials, and Exponential Sum Equalities*
Abstract: In this presentation, first, I will provide a survey of all the known constructions of (ideal) 2-level autocorrelation sequences with period $p^n - 1$ over a finite field $GF(p)$ where p is a prime and n is a positive integer. These sequences have important applications in code division multiple access (CDMA) communications. Any sequence over $GF(p)$ with period dividing $p^n - 1$ can be represented by a sum of multiple trace terms, which corresponds to a polynomial function from $GF(p^n)$ to $GF(p)$. Then I will present some conjectures on ternary sequences with 2-level autocorrelation, their trace representations, and some extremely surprising exponential sum equalities obtaining by iteratively applying the two operations of decimation and Hadamard transform.

Speaker: **James Hirschfeld** (University of Sussex)
Title: *Non-isomorphic maximal curves over a finite field*
Abstract: A *maximal curve* $\mathcal{F}$ over the finite field $\mathbf{F}_q$ is an algebraic curve attaining the Hasse–Weil upper bound,

$$q + 1 + 2g\sqrt{q},$$

where g is the genus of $\mathcal{F}$ and q is necessarily a square.

The genus of $\mathcal{F}$ satisfies the inequality,

$$g \leq \tfrac{1}{2}(q - \sqrt{q}),$$

where equality is achieved if and only if $\mathcal{F}$ is isomorphic to the Hermitian curve $\mathcal{H}_q$, given by the form,

$$X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1}.$$

If a curve is a quotient of $\mathcal{H}_q$, then it is maximal. It is conjectured that every maximal curve is a quotient of $\mathcal{H}_q$.

A family of quotient curves of $\mathcal{H}_q$ with genus $\sqrt{q} - 1$ is considered. The members of the family have many similar properties but provide many non-isomorphic maximal curves.

[Joint work with M. Giulietti, G. Korchmáros and F. Torres.]

Speaker: **Sophie Huczynska** (University of St Andrews)
Title: *The Strong Primitive Normal Basis Theorem*
Abstract: An element α of the extension E of degree n over the finite field $F = GF(q)$ is called free over F if $\{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$ is a (normal) basis for E/F. The Primitive Normal Basis, first established in full by Lenstra and Schoof (1987), asserts that for any such extension E/F, there exists an element $\alpha \in E$ which is simultaneously primitive (generates the multiplicative group of E) and free over F (equivalently, there exists a primitive free polynomial). in this talk, I will discuss the following strengthening of this theorem: aside from four specific extensions E/F, there exists an element $\alpha \in E$ such that both α and α^{-1} are simultaneously primitive and free over F (equivalent to the existence of a pair of reciprocal primitive free polynomials).

[This is joint work with S.D.Cohen (Glasgow).]

Speaker: **Hendrik W. Lenstra** (University of Leiden)
Title: *Constructing finite fields*
Abstract: We shall describe an algorithm that, given a prime number p and an integer D with $D > (\log p)^{46/25}$, produces an irreducible polynomial f over $\mathbf{Z}/p\mathbf{Z}$ with $D \leq \deg f < 4D$. It is a particularly attractive feature of the algorithm that its run time is essentially linear in terms of the length of the output; that is, it runs in time at most $(d \log p) \cdot (2 + \log d + \log\log p)^c$, where c is some universal constant. The algorithm was developed jointly with Carl Pomerance, as a byproduct of a new primality test.

Speaker: **Winnie Li** (Pennsylvania State University)
Title: *Characterizations of pseudo-codewords of a parity-check code*
Abstract: In this survey talk we shall explain how pseudo-codewords of a parity-check code arise, the role they play in the fast decoding algorithms, and give two ways to characterize them.

Speaker: **Petr Lisonek** (Simon Fraser University)
Title: *Caps and highly nonlinear functions on finite fields*
Abstract: We consider functions on binary vector spaces which are far from linear functions in different senses. Three well studied classes of such functions are crooked (CR) functions, almost bent (AB) functions and almost perfect nonlinear (APN) functions. In the binary case all known constructions of such functions arise from certain monomial functions on GF(2^n). In 2003 van Dam and Fon-Der-Flaass obtained a combinatorial characterization of all AB functions in terms of the number of solutions to a certain system of equations. We study a similar characterization for certain classes of APN functions. We discuss an application of this characterization in the study of caps in the finite projective spaces over GF(2).

Speaker: **Ariane Masuda** (Carleton University)
Title: *Permutation binomials over $\mathbb{F}_{2^k p+1}$ where p and $2^k p + 1$ are primes*
Abstract: In this talk we present results on the characterization of permutation binomials over $\mathbb{F}_{2^k p+1}$ where $k \geq 1$, p and $2^k p + 1$ are primes. We also give a formula for the number of permutation binomials of degree smaller than $q - 1$ when $k = 1$ and 2.

[This is joint work with Daniel Panario and Steven Wang.]

Speaker: **Marko Moisio** (University of Vaasa)
Title: *Kloosterman curves, their fibre products, and explicit enumeration of irreducible polynomials with two coefficients prescribed, I*
Abstract: Let $\mathbb{F}_q$ be a finite field with $q = p^r$ and let $c \in \mathbb{F}_q$. In this talk some preliminaries for an explicit enumeration of the irreducible polynomials

$$x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x]$$

with $a_{m-1} = 0$ and $a_1 = c$, are given.

First, for $a, b \in \mathbb{F}_q^*$ ($a \neq b$), the number of rational places of the function field $\mathbb{F}_{q^m}(x, y, z)$ with $y^q - y = x + ax^{-1}$, $z^q - z = x + bx^{-1}$, is given in terms of moments of Kloosterman sums over $\mathbb{F}_q$. Secondly, evaluation of moments in cases $p = 2, 3$ is considered. More precisely, the moments are connected by Pless's power moment identity to the weight distribution of Melas codes, which opens up a possibility to evaluate moments by using explicit weight formulae obtained by Schoof, van der Vlugt, and van der Geer.

We illustrate the method by giving explicitly the number of rational places of $\mathbb{F}_{q^m}(x, y, z)$ for $m = 1, \ldots, 10$.

Speaker: **Gary Mullen** (Pennsylvania State University)
Title: *A polynomial analogue of the $3n + 1$ problem*
Abstract: The integer $3n + 1$ problem is a well studied but still open problem. In particular, if n is an even positive integer, then one divides by 2 while if n is odd, one calculates $3n + 1$. The process is repeated and the $3n + 1$ problem conjectures that after a finite number of iterations, one always ends at the value 1.

I will discuss a polynomial analogue of this problem which was first motivated by considering polynomials over the binary field $\mathbb{F}_2$. We will consider an even more general version over any field. The interesting point is that after a finite number of iterations, our algorithm for monic polynomials over a field always ends at 1. We will also discuss several related open problems which arise in the case of the binary field $\mathbb{F}_2$.

Speaker: **Jang-Woo Park** (Clemson University)
Title: *Monomial dynamics over finite fields*
Abstract: Let k be a finite field and $f : k^n \to k^n$ a map defined by polynomials. It is an important problem to study the dynamics of f, particularly its fixed points and cycles of various lengths. This problem is well understood for linear functions, but wide open for nonlinear functions. In this talk, we present our recent work on dynamics defined by monomials.

Speaker: **Mateja Presern** (University of Glasgow)
Title: *Completing the Hansen-Mullen primitivity conjecture*
Abstract: The Hansen-Mullen Primitivity Conjecture (1992) is that, generally, there exists a primitive polynomial of degree n over a finite field $\mathbb{F}_q$ with any coefficient arbitrarily prescribed. This was proved by S. D. Cohen for $n \leq 3$ and $n \geq 9$ and S. D. Cohen and M. Prešern for $n = 4$. We present refinements of these ideas which yield the facts that the 3rd or 4th (or $(n-3)$rd or $(n-4)$th) coefficient can be prescribed, thus completing the proof of the Hansen-Mullen Primitivity Conjecture. We particularly focus on primitive polynomials of degree 8. A very small amount of computation is needed.

[This is joint work with S. D. Cohen.]

Speaker: **Kalle Ranto** (University of Turku)
Title: *Kloosterman curves, their fibre products, and explicit enumeration of irreducible polynomials with two coefficients prescribed, II*
Abstract: Let $\mathbb{F}_q$ be a finite field with $q = p^r$ and let $c \in \mathbb{F}_q$. We show how the number of irreducible polynomials $x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x]$ with $a_{m-1} = 0$ and $a_1 = c$ is connected to the number of rational places of the function field $\mathbb{F}_{q^m}(x, y, z)$ with $y^q - y = x + ax^{-1}$, $z^q - z = x + bx^{-1}$, $a, b \in \mathbb{F}_q^*$, and $a \neq b$. The number of these rational places in cases $p = 2, 3$ were recently obtained by Marko Moisio, and this enables us to give the number of irreducible polynomials in question when $m = 1, ..., 10$.

Speaker: **Frank Ruskey** (University of Victoria)
Title: *Exhaustive generation of irreducible polynomials over small finite fields*
Abstract: We have exhaustively generated all irreducible polynomials over GF(2), GF(3), GF(4), GF(5), GF(7), and GF(8) for "reasonable" values of n. Reasonable means that they will fit on a single CD after compression. For example, over GF(3) we generate up to degree $n = 20$, and there are 174,342,216 such polynomials. We also generate one million primitive polynomials for each degree $n \leq 64$. The basic technique is to generate Lyndon strings and convert each string to a polynomial using a shift and add technique on computer words. The optimization of the crucial add routine leads to interesting questions in circuit optimization that are addressed in the latest drafts of Knuth Volume IV.

Using the data we have produced tables computing various statistics of the polynomials and will present several conjectures based on those statistics.

[This research done together with my Ph.D. student Gilbert Lee.]

Speaker: **Igor Semaev** (University of Bergen)
Title: *On solving sparse algebraic equations over finite fields*
Abstract: A system of algebraic equations over a finite field is called sparse if each equation depends on a small number of variables. In this talk new deterministic algorithms for solving such equations are presented. The mathematical expectation of their running time is derived. These estimates are at present the best theoretical bounds on the complexity of solving average instances of the above problem.

Speaker: **Igor Shparlinski** (Macquarie University)
Title: *On the Sato-Tate conjecture on average*
Abstract: We obtain asymptotic formulae for the number elliptic curves $E_{a,b} : Y^2 = X^3 + aX + b$ over a field $\mathbb{F}_p$ where p is prime, satisfying certain "natural" properties. We consider the cases when:

- a and b are fixed but p is chosen at random with $p \le x$,
- p is fixed but a and b are chosen at random with $|a| \le A$ and $|b| \le B$,
- a, b and p are chosen at random with $|a| \le A$, $|b| \le B$ and $p \le x$.

Specifically, we investigate the behavior of such curves with respect to the Sato-Tate conjecture, cyclicity and divisibility of the number of points by a fixed integer m.

[Joint work with Bill Banks.]

Speaker: **Horacio Tapia-Recillas** (Universidad Autonoma Metropolitana-Iztapalapa)
Title: *The simplex code over finite chain rings*
Abstract: Codes over finite rings have been studied since the early seventies, particularly over the ring $\mathbb{Z}_m$ of integer modulo m. Recently the ring $\mathbb{Z}_4$ has been of particular interest after the work of Nechaev and later, Hammond et al. These authors show that certain non-linear binary codes with good parameters, including the Kerdock and Preparata codes, are the image of $\mathbb{Z}_4$ linear codes under the Gray isometry between $(\mathbb{Z}_4^n, d_L)$ and $(\mathbb{Z}_2^{2n}, d_H)$, (here d_L and d_H are the Lee and Hamming distance respectively). This result has motivated the study of several types of codes over finite rings and their image under the (generalized) Gray isometry. These rings include $\mathbb{Z}_{p^s}$ where p is a prime and s a positive integer, Galois rings, finite chain rings and Frobenious rings, to mention some of them. Recently the simplex code over the ring $\mathbb{Z}_{2^s}$ has been introduced and some of its properties studied. Since this ring, and, more generally, the ring $\mathbb{Z}_{p^s}$ where p is a prime and s a positive integer, is a particular case of a Galois ring and the latter is an example of a finite chain ring, it is natural to ask if it is possible to extend some of the results previously given for the (linear) simplex code over the ring $\mathbb{Z}_{2^s}$ to the case of finite chain rings. Also, the Gray isometry for codes over a finite chain ring has been introduced; thus, one can ask if the image under the Gray isometry of the (linear) simplex code defined over a finite chain ring is still linear. In this talk the simplex code over a finite chain ring is defined and its homogeneous weight distribution is determined. Furthermore, it is shown with an example that the image of this simplex code under the Gray isometry is not linear in general.

Speaker: **Felipe Voloch** (University of Texas at Austin)
Title: *Symmetric Cryptography and Algebraic Curves*
Abstract: The S-boxes of symmetric cryptography can be viewed as polynomials over finite fields (of characteristic two). Their non-linearity properties, which are important for their use in cryptography, translate into properties of certain algebraic curves. I will explain these facts and present some results obtained along these lines.

Speaker: **Daqing Wan** (University of Calirfornia, Irvine)
Title: *Counting rational points on varieties over finite fields*
Abstract: This is an expository lecture on both complexity and algorithms for counting the number of rational points on a hypersurface over a finite field, with an emphasis on modular reduction via p-adic methods.

Speaker: **Qiang (Steven) Wang** (Carleton University)
Title: *Permutation polynomials and sequences over finite fields*
Abstract: A polynomial over a finite field is called a permutation polynomial if it induces a bijective map from

the finite field to itself. Permutation polynomials were first investigated by Hermite, and since then, many studies concerning them have been devoted. Recently there has been a revival in the interest for permutation polynomials, in part due to their applications in coding theory, combinatorics, and cryptography. In this talk I will describe some new classes of permutation polynomials of finite fields in terms of sequences over finite fields. I will also explain the tight connections between permutation behaviors of binomials and the periodicity of certain class of sequences.

Speaker: **Joseph L. Yucas** (Southern Illinois University)
Title: *Generalized reciprocals and factors of Dickson polynomials*
Abstract: We discuss recent results on Dickson polynomials. In particular we give new descriptions of the factors of Dickson polynomials over finite fields in terms of cyclotomic factors. To do this generalized reciprocal polynomials are introduced and characterized.

[This is joint work with Robert W. Fitzgerald.]

Speaker: **Michael Zieve** (IDA Center for Communications Research)
Title: *Polynomial decomposition*
Abstract: Consider the operation of composition on polynomials over a field K, namely $(f \text{ o } g)(x) = f(g(x))$. A polynomial of degree at least 2 is called indecomposable if it cannot be written as the composition of polynomials of strictly lower degree. Every polynomial f of degree at least 2 can be written as the composition of indecomposable polynomials, but this decomposition need not be unique. However, if K has characteristic zero, then results of Ritt, Levi, Engstrom, and Schinzel provide a complete theory of polynomial decomposition – for instance, any two decompositions of f must have the same length, and it is known how to produce all decompositions of f from any single decomposition.

I will present several results and examples about the analogous problem in fields of positive characteristic. Along the way I will present new examples of indecomposable polynomials which decompose over an extension field; new types of reducible 'variables separated' polynomials $f(x) - g(y)$; and various results on computing the intersection of two subfields of $K(x)$.

Outcome of the Meeting

A great deal of collaboration between participants at the workshop was in evidence, either on previously discussed problems or initial work on new problems. Many of the participants commented favorably to the organizers how the setting and the format of the workshop was conducive to such cooperation. Indeed the only negative comments received, from two prominent researchers, was that there should have been fewer talks and that some of the talks should have been shorter in order to allow more time for such collaboration.

We briefly comment on some new results obtained during the workshop. It is very likely that more results will come up in the future from discussions that happened in Banff. Indeed, we are aware that other papers by participants are in various stages of preparation.

Boolean functions with small Fourier spectrum have been widely studied in recent years. They are interesting not only from a pure standpoint but also they are used nowadays in applications in areas such as communications, cryptography and information theory. The study of bent functions, that is boolean functions where the Fourier spectrum contains only the two values $\{\pm 2^{n/2}\}$, has been at the center of the research in this area. The classification of all bent functions seems to be a very hard problem. Some boolean functions of interest are the well-known Kasami-Welch functions: $\mathrm{Tr}(x^d)$, for certain exponent d. John Dillon and Gary McGuire's collaboration at the Banff meeting focus on finite fields of the form $\mathbb{F}_{2^n}$ where n is not divisible by 3. They show that in this case and when the Kasami-Welch exponent is $d = 4^k - 2^k + 1$, where $n = 3k \pm 1$, then $\mathrm{Tr}(x^d)$ is bent when restricted to the hyperplane formed by the trace 0 elements in $\mathbb{F}_{2^n}$.

Gao and Lenstra have characterized, in a fundamental paper, when optimal normal elements exist. Normal elements are vastly used in practice, specially for exponentiation in finite fields, a basic operation in cryptography. However, as Gao and Lenstra have shown, optimal normal elements do not exist for every finite field extension of a finite field. The question of finding low complexity normal elements when optimal elements do not exist remains wide open. At the BIRS meeting, Theo Garefalakis, Daniel Panario and David Thomson (later joined by Maria Christopoulou), teamed up to study whether one could obtain normal elements with guaranteed low complexity. They were able to construct such elements by studying traces of optimal normal elements.

The talk by Masuda on permutation binomials over certain finite fields triggered the collaboration, during the workshop, between her and Michael Zieve. As a consequence of this collaboration, they have now submitted a paper where some conjectures on Masuda's previous work are solved. In particular, this new work establishes the nonexistence of binomials for a larger class of finite fields than previously known, or even conjectured.

In recent years many results towards the enumeration of classes of univariate irreducible polynomials with certain characteristics have appeared in the literature. Marko Moisio and Kalle Ranto's collaboration in Banff centered on the study of irreducible polynomials with two prescribed coefficients. This problem is quite hard and only very basic results are known. For example, Carlitz has studied the case when the two fixed coefficients are the trace and the independent term of the polynomial. Using properties of Kloosterman sums and enumeration of the number of rational points on some super-singular curves, Moisio and Ranto obtain explicit counting formulas for the number of irreducible polynomials with two prescribed coefficients, for some new special cases.

Related to the Hansen/Mullen conjecture about primitive polynomials with prescribed coefficients, Gary Mullen and Frank Ruskey are considering various possible refinements of the conjecture for primitive and irreducible polynomials over finite fields.

(**Please note:** the references in the section below are for papers which either previously existed and were used in the talks or for work which was initiated or contributed to during the workshop.)

List of Participants

Ahmadi, Omran (University of Toronto)
Arikushi, Karin (Carleton University)
Bernstein, Daniel (University of Illinois, Chicago)
Blake, Ian (University of Toronto)
Bluher, Antonia (National Security Agency)
Car, Mireille (Universite Paul Cezanne (Aix-Marseille III))
Coulter, Robert (University of Delaware)
Dewar, Michael (University of Illinois, Urbana-Champaign)
Dillon, John (National Security Agency)
Enge, Andreas (Ecole polytechnique, Paris)
Gallardo, Louis (L'Universite de Bretagne Occidentale)
Gao, Shuhong (Clemson University)
Garcia, Arnaldo (IMPA)
Garefalakis, Theo (University of Crete)
Gathen, Joachim von zur (B-IT, University of Bonn, Germany)
Gong, Guang (University of Waterloo)
Hirschfeld, James (University of Sussex)
Huczynska, Sophie (University of St Andrews)
Lange, Tanja (Technische Universiteit Eindhoven)
Lenstra, H.W. (University of Leiden)
Li, Winnie (Pennsylvania State University)
Lisonek, Petr (Simon Fraser University)
Masuda, Ariane (Carleton University)
McGuire, Gary (University College Dublin)
Mills, Don (Rose-Hulman Institute of Technology)
Moisio, Marko (University of Vaasa, Finland)
Mullen, Gary (Pennsylvania State University)
Panario, Daniel (Carleton University)
Park, Jang-Woo (Clemson University)

Presern, Mateja (Univeristy of Glasgow)
Ranto, Kalle (University of Turku)
Ruskey, Frank (University of Victoria)
Semaev, Igor (University of Bergen)
Shparlinski, Igor (Macquarie University)
Tapia-Recillas, Horacio (Universidad Autonoma Metropolitana-Iztapalapa)
Thomson, David (Carleton University)
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Wang, Qiang (Carleton University)
Yucas, Joe (Southern Illinois University)
Zieve, Michael (IDA Center for Communications Research)

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[15] Ariane Masuda and Michael Zieve, Permutation binomials over finite fields, preprint.

[16] Marko Moisio and Kalle Ranto, Elliptic curves and explicit enumeration of irreducible polynomials with two coefficients prescribed, preprint, contributed to during workshop.

Chapter 24

Operator Methods in Fractal Analysis, Wavelets and Dynamical Systems (06w5027)

December 2 – December 7, 2006

Organizer(s): Ola Bratteli (University of Oslo), Palle Jorgensen (The University of Iowa), David Kribs (University of Guelph), Gestur Olafsson (Louisiana State University), Sergei Silvestrov (Lund University)

Introduction about the meeting

Recently our understanding of some of the most exciting new scientific discoveries has proved to rely on fractal features. They are understood by the coming together of mathematics, communication theory, computer graphics, signal/image processing, medical imaging, and quantum theory.

Leading researchers on fractals, wavelets, dynamics and their analysis and uses as well as talented younger researchers have converged on the Banff Centre in the week of December 2–7, 2006 for a hosted by Banff International Research Station international workshop on new developments in Operator methods in fractal analysis, wavelets and dynamical systems.

More then 40 participants from around the World: Europe, Scandinavian countries, Asia, Australia, the US and Canada took part.

The event consisted of three elements: (1) more then 25 scheduled research presentations covering the themes of the workshop; (2) discussions helping to bridge the separate topics; and (3) joint research interaction, including work on joint research papers and projects.

The conference activities aimed at creation of new international cooperation patterns and research advances as well as research training of young promising researchers.

This event was co-organized by Professors Ola Bratteli (University of Oslo, Norway), Palle Jorgensen (The University of Iowa, USA), David Kribs (University of Guelph), Gestur Olafsson (Louisiana State University, USA) and Sergei Silvestrov (Lund University, Sweden). The names Bratteli and Jorgensen are synonymous with significant advances in these fields. Professor David Kribs is conducting pioneering work in quantum computation, and Professors Gestur Olafsson and Sergei Silvestrov are leading mathematicians in areas covered by the workshop and their applications to neighboring fields such as engineering, physics, and medicine. The expert group that has been assembled focused on the hottest new results in fractal theory and related topics. The workshop brought together experts in areas of pure and applied mathematics who have made independent advances and it has provided a unique opportunity for advancing the field through teamwork and collaboration. Among participants there were a substantial number of distinguished senior researchers as well as junior researchers both with theoretical and applied backgrounds. This mix ensured a long-term impact of the program on the area, and on the mathematics and its applications as a whole. The research agenda was full of interesting problems of common interest to most of the participants.

Overview of the Field

Fractals are everywhere in nature and in technology. When you look at them in a microscope or in a telescope, you see hidden patterns as similar repeated structures and features, repeated at different scales. On occasion they are well hidden, for example in huge data sets from the internet. Fractal analysis and data mining are the tools that reveal these features, repeated at varying scales of resolution; and making up fundamental constituents in a yet new and relatively uncharted domain of science.

This workshop aimed at developing new approaches and mathematical foundations for wavelet analysis, dynamical and iterated function systems, spectral and tiling duality, fractal iteration processes and non-commutative dynamical systems. The basic methods involved in this work derive from a combination of operator theory, harmonic analysis and representation theory.

The program is in mathematics. However, the issues represented in the program originate both from within mathematics and from observed natural phenomena and the engineering practice. The interplay and unified approaches to these significantly interrelated areas of mathematics is of great significance both for mathematics itself and its connections to other subjects and applications.

Wavelet theory stands on the interface between signal processing, operator theory, and harmonic analysis. It is concerned with the mathematical tools involved in digitizing continuous data with a view to storage and compression, and with the synthesis process, recreating the desired picture (or time signal) from the stored data. The algorithms involved go under the name of filter banks, and their spectacular efficiency derives in part from the use of hidden self-similarity in the data or images that are analyzed. Self-similarity is built into wavelets too as they are intrinsically defined using dynamical and iterated function systems. This makes wavelets also closely related to fractals and fractal processes. Investigation of this relation has huge theoretical and practical potential and thus it is becoming a subject of growing interest both in and outside mathematics. It has been recently shown (by Palle Jorgensen, Ola Bratteli, David Larson, X. Dai and others) that a unifying approach to wavelets, dynamical systems, iterated function systems, self-similarity and fractals may be based on the systematic use of operator analysis and representation theory.

Recent Developments and Open Problems

Connections of operator methods to the applications are manifold.

Operator analysis. Observations or time signals are functions, and classes of functions make up vector spaces. The most useful spaces are Hilbert spaces of square integrable functions on domains in n-dimensional Euclidean space, but several applications require more general spaces, like Sobolev spaces or even spaces of distributions and also functional spaces on fractal sets and sets with fractal boundaries. The basic idea in wavelet theory is to study the operators of dilations and translation, and subspaces invariant under those operators. The computational aspect is represented in sampling and approximation. Given measured data one finds the elements in the subspace closest to those data. The interplay between those two aspects was the focus of the workshop.

Representation theory and operator algebras. Representation theory associates to abstract algebraic objects operators on Hilbert spaces or more general locally convex topological vector spaces satisfying the algebraic relations. We are interested in representations defined by filters, wavelets and similar objects. The continuous wavelet transform gives rise to square integrable representations of topological groups; the scaling identity gives rise to representations of the Cuntz algebra defined by the high pass and low pass filters. The high pass filters alone define transfer operators. A main point is the study of intertwining operators between, on one side, the “discrete world” of high-pass/low-pass filters of signal processing, and on the other side, the “continuous world” of wavelets. There are significant operator-algebraic and representation-theoretic issues on both sides of the ‘divide’, and the intertwining operators throw light on central issues for wavelets in higher dimensions.

Dynamical systems, operator algebras and operator analysis. Operators of multiplication and dilations and more general weighted composition operators with more general may be non-linear and non-invertible transformations of variables are central for wavelet analysis and analysis on fractals. The properties of dynamical and iterated function systems defined by these transformations govern the spectral properties and corresponding subspace decompositions for these operators. Operator algebras generated by such operators are concrete representations of generalized crossed product type algebras and C^*-algebras defined via various kinds and generalizations of covariance commutation relations. Invariant sets and measures for the dynamical and for the iterated function systems such as for example orbits or attractors and invariant or quasi-invariant measures on them are directly linked to such operator algebras. The interplay between dynamical and iterated function systems and actions of groups and semigroups on one side and operator algebras on the other side bring fruitful results and new methods for both areas. This yields new powerful results and methods for wavelets and fractal analysis and geometry.

There are many applications of the above mentioned methods and analysis to engineering and physics problems, as well as reach possibilities to gain insight into numerical analysis of corresponding applications.

Tools from diverse areas of analysis, as well as from dynamical systems and operator theory, merge into the research on wavelet analysis. The diversity of techniques is also a charm of the subject, which continues to generate new graduate students and postdoctoral activity.

Scientific Progress Made

The operator methods enter in that wavelets, signals and information may be realized as vectors in a real or complex Hilbert space, or in symbolic graph or path spaces. For the purpose of transmission, these vectors are encoded in for example a set of linear coefficients. In the case of images, including fractal images, this was worked out using wavelet and filter functions, for example corresponding to ordinary Cantor fractal subsets of $\mathbb{R}$, as well as for fractal measure spaces corresponding to Sierpinski Gasket fractals.

Several fractals, like a finitely summable infinite tree, and the Sierpinski gasket fit naturally within this framework. In these cases, we show that our spectral triples do describe the geodesic distance and the Minkowski dimension as well as. Furthermore, in the case of the Sierpinski gasket, the associated Dixmier-type trace coincides with the normalized Hausdorff measure.

More generally, fractals are related to more traditional wavelets, those of $L_2(\mathbb{R}^d)$. Two computational features were addressed: (a) Approximation of the father or mother functions by subdivision schemes, and (b) matrix formulas for the wavelet coefficients. A variety of data were considered; typically for fractals, L_2-convergence is more restrictive than is the case for $L_2(\mathbb{R}^d)$-wavelets.

A variety of wavelet applications were considered, involving a construction of certain groups of measure preserving transformations, and groups and algebras of operators, with special algebraic properties. Other results include applications of a theory of projection decompositions of positive operators, and a theory of operator-valued frames.

Operator algebra constructions of covariant representations are used in the analysis of orthogonality in wavelet theory, in the construction of super-wavelets, and orthogonal Fourier bases for affine fractal measures.

K-theoretic tools have been further developed. Smale spaces are abstract topological dynamical systems characterized by canonical coordinates of contracting and expanding directions. These include basic sets from Smale's Axiom A systems as well as shifts of finite type. In general, they are chaotic and the underlying geometry is fractal. There are C^*-algebras associated with such objects and the aim is to compute their K-theory providing invariants. Other C^*-algebras were associated to shift spaces. They can be viewed as generalizing the universal Cuntz-Krieger algebra, leading to simple and purely infinite C^*-algebras, understood via their K-theory, K_0 and K_1. Further understanding of the structure of these algebras leads to new approaches to classifications and better understanding of the dynamics of arbitrary shift spaces. One version of dynamics was explained as information is stored in certain ideal structure, referring to a recent classification result for certain non-simple Matsumoto C^*-algebras [6]. New insights also have been gained into the interplay between orbit space structure for dynamical systems or more general group and semigroup actions, and structure of ideals, subalgebras and representations for crossed product algebras constructed via twisted by dynamics generalized convolution products [18].

A different class of Banach algebras enter applied analysis as follows: (a) Wiener's Lemma for twisted convolution and for the rotation algebras; (b) algebras of infinite matrices with off-diagonal decay are inverse-closed in the algebra of all bounded operators; (c) inverse-closedness plays an essential role in quantitative studies of the finite section method to solve operator equations; (d) a new construction of inverse-closed matrix algebras by approximation properties.

We further worked on complex B-splines, i.e., a generalization of ordinary B-splines to complex degrees. This results in an infinite uniform knot sequence for complex B-splines. Result is that generalized fractional divided differences can be defined via the fractional Weyl-integral with complex B-splines as densities.

In the area of quantum information, the operator theoretic view begins with a von Neumann measurement. It is replaced by a more general concept called a positive operator valued measure (POVM), which is essentially a partition of unity in terms of positive semi-definite operators. POVM's formed from equally weighted rank one operators define a tight frame, and any frame defines a POVM.

Several themes were covered in the workshop, and we have grouped them loosely under the headers, fractals, wavelets and dynamics. But we are stressing an underlying unity based on operator theoretic tools. Covariant representations represent one such. This has been a central theme in operator algebras since the 1950s, and has played a key role in numerous applications since. One of these more recent applications is to a class of wavelets called "frequency localized" wavelets [2, 10] as well as to signal and image processing constructions. And more generally to symbolic dynamics! Another is the "transfer operator." This refers to a construction with origins in probabilistic path models from physics and ergodic theory. In our workshop we drew up some connections between the two areas, showed how operator algebraic ideas and representations throw new light on applications. Since several ideas are involved, participants from one area learned from the others. The workshop offered a unique opportunity for leading researchers in diverse but related areas to meet and learn from one another. One use of operator algebras (specifically, C^*-algebras) is to construction of representations, to states, and to dynamics. Initially [14], the focus was on groups and on harmonic analysis, but the notion of covariance from physics (see e.g., [4, 16]) suggested crossed products of groups with act by automorphisms on C^*-algebras ([9, 20]). Since the pioneering paper by Stinespring [17], a preferred approach (e.g., [1]) to constructing representations begins with a positive operator valued mapping, and Stinespring identified complete positivity; now widely used. At the same time, related notions of positivity are central in a variety of probabilistic path models, beginning with Doeblin [12], see also [8]. It is now a key tool in ergodic theory, [19]. As a result, over the years, Doeblin's operator has taken on many other incarnations, and it is currently known as "the transfer" operator, the Ruelle operator, or the Perron-Frobenius-Ruelle operator [3]. The name Ruelle is from its use in statistical mechanics as pioneered by David Ruelle, see [16] and [3]. Of a more recent vintage are applications to wavelets [11], and to analysis of fractals, i.e., special and computational bases in Hilbert space constructed from a class of unitary representation of certain discrete groups of affine transformations. It was realized (e.g., [5]) that there are transfer operators T_W for wavelets, that the solution to a spectral problem for T_W yields wavelet representations; and moreover that these representations come with a useful covariance. The workshop further explored an intriguing variant of the operators T_W used in quantum error correction codes, see [7, 13]. All versions of transfer operators involve hieratical processes with branching, and probabilities assigned by a weight function; touching again on the central feature of fractals.

Outcome of the Meeting

Parallel efforts are under way in the United States, Canada, and Europe and merging the activities add tremendously both to generating research advances, and to advance education. Bringing together groups of participants with common interests and prominent researchers in the fields of operator algebras, representation theory, fractal analysis, dynamical and iterated function systems, wavelets and harmonic analysis have lead to substantial new advances in both the theory and application of all these fields. Finally, there have been discovered enticing connections between the focus areas of this workshop and certain aspects of quantum computing and quantum information theory. The workshop had also important educational dimension since a number of junior researchers, post-docs and PhD students, have taken part and were given opportunity to present their results and take part in discussions with senior researchers and colleagues from other groups and countries.

An important outcome of the workshop is the appearance of a number of joint publications and the creation of several new collaborative research projects and initiatives involving different groups of participants. A special volume on the subject is in preparation.

Dissemination.

We have agreement with Springer Verlag about the publication of a special volume on the topic of the conference.

The aim of the volume is to broaden and deepen interplay between:

– wavelets and fractals and operator algebras, operator theory and representation theory;

– dynamics and operator algebras and operator theory;

– quantum computing and information;

– applications of the above in engineering, physics and beyond.

Contributions to the volume will address one or several of the above directions.

Acknowledgements

All the participants thoroughly enjoyed their stay at the BIRS facility. The venue is superb and the scenery, the hospitality and the food were fantastic. This brought together a number of important participants in the field. The scientific discussions continued long after the talks were over, in the meeting rooms, the hiking trails, and by the donation fridge. It was a meeting that will shape the future of our field for years to come. Our thanks to the staff and directorship of BIRS for this opportunity.

Participation from other countries nearby and far away from Canada have become possible thanks to crucial travel support from many national and international research foundations and networks, such as European networks (Liegrits and others), Research councils and foundations in Norway, Denmark, United states of America, Canada and other countries, STINT (The Swedish Foundation for International Cooperation in Research and Higher Education) and Crafoord foundation, private research foundations and participant's home universities. The support of all these funding institutions is gratefully acknowledged.

Presentation Highlights (in alphabetical order)

Florin Boca (University of Illinois)
Title: C^*-algebras and continued fractions
An AF algebra associated with the Farey tessellation and capturing the properties of the continued fraction algorithm has been considered. The Effros-Shen rotation AF algebras arise as quotients of this algebra.

Bernhard Bodmann (University of Waterloo)
Title: Optimal Redundant Packet Encoding for Loss-Insensitive Linear Transmissions
The objective of this talk was to characterize the optimal use of redundancy in transmitting a signal that is encoded in packets of linear coefficients. The signals considered here are vectors in a finite-dimensional real or complex Hilbert space. For the purpose of transmission, these vectors are encoded in a set of linear coefficients that is partitioned in packets of equal size. It was investigated how the encoding performance depends on the degree of redundancy it incorporates and on the amount of data-loss when packets are either transmitted perfectly or lost in their entirety. The encoding performance is evaluated in terms of the maximal Euclidean norm of the reconstruction error occurring for the transmission of unit vectors. The main result of this talk is the derivation of error bounds as well as the characterization of optimal encoding when up to three packets are lost.

Berndt Brenken (University of Calgary)
Title: Topological quivers as multiplicity free relations
For a discrete directed graph a certain graph C^*-algebra is invariant under a procedure that yields a multiplicity free graph. It was shown the analogue of this holds for topological quivers; a certain C^*-algebra, namely the unaugemented Cuntz-Pimsner C^*-algebra, of a topological quiver remains Morita equivalent to the C^*-algebra of an associated multiplicity free topological quiver.

Jonas D'Andrea (University of Colorado)
Fractal wavelets of Dutkay-Jorgensen Type for the Sierpinski gasket space
Several years ago, D. Dutkay and P. Jorgensen developed the concept of wavelets defined on a sigma-finite fractal measure space, developed from an iterated affine system. They worked out in detail the wavelet and filter functions corresponding to the ordinary Cantor fractal subset of $\mathbb{R}$. In this talk the construction of Dutkay and Jorgensen was examined as applied to the fractal measure space corresponding to the Sierpinski Gasket fractal. A variety of high-pass filters was developed, and as an application, the various families of wavelets were used to analyze digital images.

Kenneth R Davidson (University of Waterloo)
Operator algebras for multivariable dynamics I
To a locally compact Hausdorff space X with n proper continuous maps of X into itself, there were associated various topological conjugacy algebras; with two emerging as the natural candidates for the universal algebra of the system, the tensor algebra and the semicrossed product. The reasons for this were discussed, including dilation theory, representations and C^*-envelopes. Generalized notions of wandering sets and recursion were used to characterize when these algebras are semisimple.

Dorin Dutkay (Rutgers University)
bf Covariant representations, scaling functions and affine fractals
(a joint work with Palle Jorgensen)
In this talk it was demonstrated how some operator algebra constructions of covariant representations can be used to analyze orthogonality in wavelet theory, to construct super-wavelets, and to obtain orthogonal Fourier bases for affine fractal measures.

Soren Eilers (University of Copenhagen)
Classification of C^*-algebras associated to irreducible shift spaces
A construction by Matsumoto allows an invariant association of C^*-algebras to any shift space. Somewhat exceptionally, these C^*-algebras are not always simple when the shift space is irreducible, and in previous work, mainly with Carlsen, one has endeavored to explain what dynamical information is stored in the ideal structure in those cases. In this talk this problem was reviewed and discussed in light of a recent classification result for certain non-simple Matsumoto C^*-algebras obtained in joint work with Restorff and Ruiz.

Karlheinz Groechenig (University of Vienna)
Inverse-closed Banach algebras in applied analysis
A Banach algebra A is called inverse-closed in a larger Banach algebra B, if every element in A that is invertible in B is already invertible in the smaller algebra A. For instance, the algebra of absolutely convergent Fourier series is inverse-closed in the algebra of continuous functions on the torus. This is the classical Wiener Lemma. In the talk there were presented several results about inverse-closed Banach algebras in applied analysis:
(a) Wiener's Lemma for twisted convolution and for the rotation algebras;
(b) algebras of infinite matrices with off-diagonal decay are inverse-closed in the algebra of all bounded operators;
(c) inverse-closedness playing an essential role in quantitative studies of the finite section method to solve operator equations;
(d) A new construction of inverse-closed matrix algebras by approximation properties.

Cristina Ivan (University of Hannover, Germany)
Spectral triples for fractals
The purpose of this talk was to present two possible ways of associating a spectral triple to a fractal such that it encodes geometric data of the fractal. The spectral triple is obtained in both constructions as a countable sum of unbounded Fredholm modules. In the first construction (joint work with Erik Christensen) each summand is a spectral triple for a set consisting of just two points (a "two-point" spectral triple). It was Alain Connes who first constructed in this way a spectral triple for the middle Cantor set in the unit interval. Connes showed how the metric, the Hausdordff measure and dimension are encoded by this spectral triple

and its associated Dixmier trace. Connes' construction has been studied in details and extended to certain classes of fractals by Guido and Isola. Together with Erik Christensen, Cristina Ivan has investigated to which extend such a spectral triple may encode geometric data of a general compact metric space. They showed that for any compact metric space it is possible to associate a spectral triple which is a countable sum of "two-point" spectral triples and which reflects the Minkowski dimension of the space, and the metric induced by the spectral triple is equivalent to the given one. Explicit computations were performed for the unit interval, Cantor set and Sierpinski gasket and each time it was obtained that the spectral triple and its Dixmier trace recovers metric, dimension and volume measure of the compact metric space under discussion. In the second construction (joint work with Erik Christensen and Michel Lapidus) each summand is based on a curve in the space. Several fractals, like a finitely summable infinite tree, and the Sierpinski gasket fit naturally within this framework. In these cases, we show that our spectral triples do describe the geodesic distance and the Minkowski dimension as well. Furthermore, in the case of the Sierpinski gasket, the associated Dixmier-type trace coincides with the normalized Hausdorff measure. It is important to mention the advantage of each proposal for constructing spectral triples. The advantage of the first construction is that it is modeled for a general compact metric space. The advantage of the second construction is that, when it is possible to be done, it brings more information about the topological structure of the fractal (the spectral triple will induce a nontrivial element in the K-homology of the fractal).

Elias G. Katsoulis (East Carolina University)
Operator algebras for multivariable dynamics II
(This is a joint work with Ken Davidson.)
To a locally compact Hausdorff space X with n proper continuous maps of X into itself there were associated various topological conjugacy algebras; and two emerged as the natural candidates for the universal algebra of the system, the tensor algebra and the semicrossed product. A new concept of topological conjugacy for multidimensional systems, called piecewise conjugacy, was introduced and discussed. It was proved that the piecewise conjugacy class of the system can be recovered from the algebraic structure of either the tensor algebra or the semicrossed product. Various classification results follow as a consequence. For example, for n=2,3, the tensor algebras are (algebraically or even completely isometrically) isomorphic if and only if the systems are piecewise topologically conjugate.

Palle Jorgensen (The University of Iowa)
Wavelets on Fractals
(This is a joint work with Dorin Dutkay)
Joint work between Palle Jorgensen and Dorin Dutkay, recently led to wavelet constructions, and wavelet algorithms in Hilbert spaces built on fractals. The talk covered some highpoints, and provided a comparative study: the case of fractals was contrasted with the more traditional wavelets, those of $L^2(R^d)$. As a conclusion there were noted several instances of dichotomies; e.g., measure classes, regions of convergence, stability to mention three. Two computational features were addressed:
(a) Approximation of the father/mother functions by subdivision schemes;
(b) matrix formulas for the wavelet coefficients.
For (a) it was demonstrated that the variety of data when L^2-convergence holds is much smaller in the case of fractals than is the case for $L^2(R^d)$-wavelets.

Takeshi Katsura (Hokkaido University)
Cuntz-Krieger algebras and factor maps between topological graphs
A (one-sided) Markov chain is a topological dynamical system defined from a 0,1-matrix. Cuntz and Krieger introduced a C^*-algebra to examine a Markov chain. Although a Cuntz-Krieger algebra is defined from a 0,1-matrix, it only depends on the associated Markov chain. Later, a construction of Cuntz-Krieger algebras from Markov chains using groupoids were provided. In this talk, there were introduced topological graphs which contain 0,1-matrices and Markov chains as special cases. Also there was introduced a way to construct C^*-algebras from topological graphs, which generalizes the construction of Cuntz-Krieger algebras. Using the notion of "factor maps" between topological graphs, it was clarified the relation of the two constructions of Cuntz-Krieger algebras from $\{0,1\}$-matrices and from Markov chains.

David R. Larson (Texas A & M University)
Frames and Operator Theory
A few years ago David R. Larson and his collaborators developed an operator-interpolation approach to wavelets and frames using the local commutant (i.e. commutant at a point) of a unitary system. This is really an abstract application of the theory of operator algebras to wavelet and frame theory. The concrete applications of operator-interpolation to wavelet theory include results obtained using specially constructed families of wavelet sets. The methods include the construction of certain groups of measure preserving transformations, and groups and algebras of operators, with special algebraic properties. Other results include applications of a theory of projection decompositions of positive operators, and a theory of operator-valued frames. In the talk there have been discussed unpublished and partially published results, and some brand new results, that are due to David R. Larson and his former and current students, and other collaborators.

Nadia S. Larsen (University of Oslo)
Projective multi-resolution analyses arising from direct limits of Hilbert modules
(This is a joint work with I. Raeburn.)
In joint work with I. Raeburn it has been shown how direct limits of Hilbert spaces can be used to construct multi-resolution analysis and wavelets in $L^2(R)$. This talk was devoted to enlargement of the framework of this construction and use a direct limit of Hilbert modules over a fixed C^*-algebra to produce projective multi-resolution analysis in the limit module. In certain cases, existence of standard module frames for the limit module was proved. For modules over the algebra of continuous functions on the product of n-copies of the circle, these methods shed light on work of Packer and Rieffel on projective multi-resolution analysis for specific Hilbert modules of functions on R^n. New applications arise in the context of modules over the algebra of continuous functions on the compact infinite path space of a finite directed graph.

Peter Massopust (Technische Universitat Munchen, Germany)
Dirichlet-Poisson Processes meet Complex B-Splines
(This is a joint work with Brigitte Forster.)
Complex B-splines are a generalization of ordinary B-splines to complex degrees. This results in an infinite uniform knot sequence for complex B-splines. It was shown that generalized fractional divided differences can be defined via the fractional Weyl-integral with complex B-splines as densities. This representation leads to a generalized Hermite-Genocchi formula over infinite dimensional simplices. The generalized Hermite-Genocchi formula then allows the extension of complex B-splines to non-uniform knots and their interpretation as probability densities for a class of stochastic processes, namely the Dirichlet-Poisson processes.

Sergey Neshveyev (University of Oslo)
KMS-states on Hecke algebras crossed products
(This is a joint work with M. Laca and N. S. Larsen)
It was shown that KMS-states on crossed products of abelian C^*-algebras by Hecke algebras correspond to measures scaled by Hecke operators. In this work there was considered the $GL(2)$-system of Connes and Marcolli and their analysis of the system was completed by showing that for each value of the temperature in the critical region there exists a unique KMS-state.

Johan Oinert,
Commutativity and ideals in generalized crossed products
(This is a joint work with Sergei Silvestrov)
A short review was given of G-crossed product systems and the construction of generalized algebraic crossed products following C. Nastasescu, F. Van Oystaeyen, Methods of graded rings, LNM 1836, Springer-Verlag, 2004. Thereafter, inspired by the theory of C^*-dynamical systems, some results were presented relating commutativity, ideals, group actions and zero-devisors in algebraic crossed product algebras.

N Christopher Phillips (University of Oregon)
Crossed products of the irrational rotation algebras by the "standard" actions of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$ are AF
Let F be a finite subgroup of $SL_2(\mathbb{Z})$ (necessarily isomorphic to one of $Z/2Z$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$), and

let F act on the irrational rotational algebra A_θ via the restriction of the canonical action of $SL_2(\mathbb{Z})$. Then the crossed product of A_θ by F, and the fixed point algebra for the action of F on A_θ, are AF algebras. The same is true for the crossed product and fixed point algebra of the flip action of $\mathbb{Z}/2\mathbb{Z}$ on any simple d-dimensional noncommutative torus A_θ. Along the way, there were proved a number of general results which should have useful applications in other situations. (The paper is available at arXiv:math.OA/0609784)

Ian Putnam (University of Victoria)
K-theory for Smale spaces
Smale spaces are abstract topological dynamical systems characterized by canonical coordinates of contracting and expanding directions. These include basic sets from Smale's Axiom A systems as well as shifts of finite type. In general, they are chaotic and the underlying geometry is fractal. There are C^*-algebras associated with such objects and the aim is to compute their K-theory. For shifts of finite type, this is the usual dimension group invariant. More generally, there is a spectral sequence for this, but the answer can be given in purely dynamical terms as a kind of homology theory for chaotic systems, similar in spirit to Cech homology.

Iain Raeburn (University of Newcastle, Australia)
Direct limits, the Cuntz relations and wavelets
(This is a joint work with Nadia Larsen)
A famous theorem of Mallat shows how to build a wavelet basis for the Hilbert space of square-integrable functions on R starting from a quadrature mirror filter, which is a function on the unit circle satisfying an algebraic relation. From such a filter, Bratteli and Jorgensen constructed a pair of isometries satisfying the Cuntz relations well-known to operator algebraists. This talk was devoted to an approach to Mallat's theorem which uses a direct limit construction and exploits the geometric information inherent in the Cuntz relations.

Kjetil Roysland (University of Oslo)
Transition operators on bundle maps
In a joint work Kjetil Roysland and Dorin Dutkay have studied transition operators that act on the bundle maps of a vector bundle. This talk was about the fix points of such an operator. In some situations this turn out to be a finite-dimensional non-commutative C^*-algebra.

Christian Skau (Norwegian University of Science and Technology, Trondheim) Title: AF-equivalence relations and group actions
It was shown that a group acting freely as homeomorphisms on a zero-dimensional space gives rise to an AF-equivalence relation if and only if the group is locally finite. Furthermore, it was shown that the AF-equivalence relations that occur are exactly the ones that are associated to Bratteli diagrams that have the equal path number property. It was also shown that the ”super” order of the locally finite group is completely determined by the rational subdimension group of the AF-relation.

Sergei Silvestrov (Lund University, Sweden)
C^***-crossed Products and Shift Spaces**
(This is a joint work with Toke Meier Carlsen, arXiv:math.OA/0512488)
In this talk Exel's C^*-crossed product of non-invertible dynamical systems was used to associate a C^*-algebra to every shift space. It was shown that this C^*-algebra is canonically isomorphic to the C^*-algebra associated to a shift space in arXiv:math.OA/0505503, has the C^*-algebra defined by Toke Meier Carlsen and Kengo Matsumoto (Math. Scand. 95, 2 (2004), 145-160) as a quotient, and possesses properties indicating that it can be thought of as the universal C^*-algebra associated to a shift space. Also there were considered its representations, relationship to other C^*-algebras associated to shift spaces, shown that it can be viewed as a generalization of the universal Cuntz-Krieger algebra, discussed uniqueness and a faithful representation, shown that it is nuclear and satisfies the Universal Coefficient Theorem, were provided conditions for it being simple and purely infinite, shown that the constructed C^*-algebras and thus their K-theory, K_0 and K_1, are conjugacy invariants of one-sided shift spaces, presented formulas for those invariants, and also presented a description of the structure of gauge invariant ideals.

Christian Svensson (Lund University, Sweden and Leiden Universty, The Netherlands)
Dynamical systems and commutants in crossed products
(This is a joint work with Marcel de Jeu and Sergei Silvestrov, arXiv:math.DS/0604581)
Given a discrete dynamical system, one may construct an associative (non-commutative) complex algebra with multiplication determined via the action defining the system - a crossed product algebra. It turns out that for large classes of systems, one obtains striking equivalences between, in particular, dynamical properties of the system and algebraic properties of the crossed product. Quite a lot has been done in this direction for C^*-crossed products. It is satisfactory to see that many of the results obtained in purely algebraic setup are analogous to those well-known for C^*-case, and at the same time further progress on interplay between structure of crossed product and dynamics can be obtained outside the C^*-context. In this work, in particular, there was described the commutant of an arbitrary subalgebra A of the algebra of functions on a set X in a crossed product of A with the integers by a composition automorphism defined via a bijection of X. The conditions on A and on the dynamics, extending topological freeness, which are necessary and sufficient for A to be maximal abelian in the crossed product are subsequently applied to situations where these conditions can be shown to be equivalent to a condition in topological dynamics. As a further step, using the Gelfand transform, for a commutative completely regular semi-simple Banach algebra, there was obtained a topological dynamical condition on its character space which is equivalent to the algebra being maximal abelian in a crossed product with the integers.

Jun Tomiyama (Tokyo Metropolitan University, Japan Women's University)
Hulls and kernels with actions of topological dynamical systems and C^*-algebras
Let $\Sigma = (X, \sigma)$ be a topological dynamical system in a compact space X with a homeomorphism σ, and let $A(\Sigma)$ be the associated C^*-crossed product. In this context there were defined hulls and kernels with the action σ, and discussed the following problems.
1. What is the meaning in C^*-theory of the kernels of the elementary sets for the dynamical system Σ?
2. What is the meaning of the Hulls of those structural ideals of the C^*-algebra $A(\Sigma)$?

List of Participants

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Bodmann, Bernhard (University of Waterloo)
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D'Andrea, Jonas (University of Colorado at Boulder)
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de Jeu, Marcel (Leiden University - Holland)
Dutkay, Dorin (Rutgers University)
Eilers, Soren (University of Copenhagen)
Giordano, Thierry (University of Ottawa)
Groechenig, Karl-Heinz (University of Vienna)
Ivan, Cristina (University of Hannover, Faculty of Mathematics and Physics)
Johansen, Rune (University of Copenhagen)
Jorgensen, Palle (The University of Iowa)
Katsoulis, Elias (East Carolina University)
Katsura, Takeshi (University of Toronto)
Kribs, David (University of Guelph)
Laca, Marcelo (University of Victoria)
Lamoureux, Michael (University of Calgary)
Larsen, Nadia S. (University of Oslo)
Larson, David (Texas A&M University)
Lesosky, Maia (University of Guelph)
Massopust, Peter (GSF - Institute for Biomathematics and Biometry and Technical University Munich)
Mohari, Anilesh (N. Bose Centre for Basic Sciences - India)
Neshveyev, Sergey (University of Oslo)

Nikolaev, Igor (University of Calgary)
Niu, Zhuang (Fields Institute)
Oinert, Johan (Lund University - Sweden)
Olafsson, Gestur (Louisiana State University)
Packer, Judith (University of Colorado)
Phillips, N. Christopher (University of Oregon)
Putnam, Ian (University of Victoria)
Raeburn, Iain (University of Wollongong - Australia)
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Two-day Workshop Reports

Chapter 25

Math Fair Workshop (06w2319)

Apr 20 – Apr 22, 2006

Organizer(s): Ted Lewis (University of Alberta),, Tiina Hohn (Grant MacEwan College)

Overview of the Field

This was the fourth BIRS math fair workshop, which is becoming a popular annual event. The participants came from elementary schools, junior-high and high schools, from independent organizations, and from universities and colleges. The thirty-six participants at this year's workshop were educators of all types, from teachers to grad students to expert puzzle and game creators.

The purpose of the workshop was to bring together educators who are interested in using our particular type of math fair, called a SNAP math fair, to enhance the mathematics curriculum. (The name SNAP is an acronym for the guiding principles of this unconventional type of math fair: It is student-centered, non-competitive, all- inclusive, and problem-based.)The projects at a SNAP math fair are problems that the students present to the visitors. In preparation, the students will have solved chosen problems, rewritten them in their own words, and created hands-on models for the visitors. At a SNAP math fair, all the students participate, and the students are the facilitators who help the visitors solve the problems. This process of involving students in fun, rich mathematics is the underlying vision that makes the SNAP program so unique and effective. No first prize! No arguments about judging! Everyone is a winner!".

At the BIRS workshop, the participants learn about and try math-based puzzles and games that they can use in the classroom. They have a chance to see how other teachers have organized math fairs at their schools, how the SNAP math fair fits the curriculum, and what some schools have done for follow-ups. And then they go back to their schools and change the culture of mathematics in their class-room.

Two of the presenters gave an interesting contrast that displayed exactly how versatile the SNAP math fair is. One of them had a math fair that was spread over four days, and which was integrated with many other cultural activities. Another teacher explained how she overcame the problems of no budget and no space in which to hold the math fair.She figured out a way to make small inexpensive backboards to display the problems and utilized existing manipulatives. The whole affair cost her about $ 60.00 and was held in her own classroom.

Outcome of the Meeting

The concept of the SNAP math fair originated in Edmonton with Andy Liu and Mike Dumanski, and it has proved so successful that it led to the formation of a non- profit organization, the SNAP mathematics foundation, which has helped promote mathematics in schools around the world. As well as the SNAP foundation, the Calgary-based Galileo Education Network Association (GENA) helps schools organize math fairs, and provides valuable lesson-study follow-ups.

The BIRS math fair workshops have contributed greatly to the proliferation and popularization of the SNAP math fair. In some places, the use of a SNAP math fair to change children's attitudes about mathematics has almost become a ”grass-roots” movement, and so it is difficult to pin down exactly how many schools are now doing them. We have a fair idea about the numbers in Edmonton and Calgary - for example over 60 percent of the elementary schools in the Edmonton catholic system now hold regular math fairs, and as far as we can gauge, the numbers are high in the public system as well. GENA reports similar figures for the Calgary area.

SNAP and CMS are also providing some support for the launch of a similar math fair workshop in the Fields institute in Toronto, and PIMs is providing math fair booklets for the participants. The Fields workshop is being organized by Tanya Thompson who has been a valuable participant at past BIRS workshops. Altogether, the BIRS math fair workshops are having a noticeable impact on mathematics education.

List of Participants

Beisiegel, Mary (University of Alberta)
Bosscha, Angela (Edmonton Public Schools)
Burke, Joan (Calgary Schools)
Charters, Julie (Edmonton Public Schools)
Chasse, Danielle (Our Lady of the Prairies)
Chernishenko, Kelly (Edmonton Public Schools)
Christensen, Derek (Edmonton Public Schools)
Craven, Jodie (Talmud Torah School)
Desaulniers, Shawn (University of Alberta)
Francis-Poscente, Krista (University of Calgary)
Godwaldt, Terry (Edmonton Public Schools)
Gordon, Christie (Olympic Heights Elementary)
Graves, Sean (University of Alberta)
Hamilton, Gordon (Masters Academy and College)
Herwig, Brooke (Foundations for the Future Charter Academy)
Hohn, Tiina (Grant MacEwan College)
Hudani, Salima (Foundations for the Future Charter School)
Isaac, Vince (Annunciation)
Leblanc, Isabelle (Edmonton Catholic School District)
Lee, Jennifer (Edmonton Catholic School District)
Lewis, Ted (University of Alberta)
Livingstone, Kate (Calgary Schools)
MacKay, Lorraine (BC schools)
Marinova, Rossitza (Concordia University College of Alberta)
Mineva, Victoria (University of Alberta)
Pederson, Claudia (Edmonton Catholic Schools)
Serate, Hope (University of Calgary)
Shaw, Dolph (Edmonton Public Schools)
Sorenson, Wendy (Edmonton Public Schools)
Teixeira, Nadene (Edmonton Schools)
Wang, Qian (University of Alberta)
Watz, Tammy (Edmonton Catholic School District)

Chapter 26

CanQueue 2006: 8th Annual Conference for Canadian Queueing Theorists and Practitioners (06w2128)

Sep 14 – Sep 16, 2006

Organizer(s): Diane Bischak (University of Calgary), Van Enns (University of Calgary), Armann Ingolfsson (University of Alberta)

Overview of the Field

Queueing theory is concerned with developing and investigating mathematical models of systems where "customers" wait for "service." The terms "customers" and "servers" are generic. Customers could, for example, be humans waiting in a physical line or waiting on hold on the telephone, jobs waiting to be processed in a factory, or tasks waiting for processing in a computer or communication system. Examples of "service" include a medical procedure, a phone call, or a commercial transaction. Queueing theory started with the work of Danish mathematician A. K. Erlang in 1905, which was motivated by the problem of designing telephone exchanges. The field has grown to include the application of a variety of mathematical methods to the study of waiting lines in many different contexts. The mathematical methods include Markov processes, linear algebra, transform theory, and asymptotic methods, to name a few. The areas of application include computer and communication systems, manufacturing systems, and health care systems. Introductory treatments of queueing theory can be found, for example, in [1] and [2].

Recent Developments and Open Problems

Many recent developments in queueing theory have been driven in large part by a greater interest in applications that involve human customers, for example in the rapidly growing call centre sector (see [3]). Humans behave in less predictable ways than, say, jobs in a factory or tasks in a computer system. For example, they may *renege* (abandon the queue), and *retry* later. The needs of human customers are likely to be heteregeneous (motivating the use of *skills-based routing* to connect different customers to different servers) and to vary with time (sometimes requiring transient rather than steady-state solutions). All of these complications lead to interesting mathematical challenges. The interest in modeling reneging has led to a substantial literature by now, for example see [5]. Asymptotic analysis, which in the past typically considered situations where the arrival rate approached the capacity of a system with fixed number of servers, has been rejuvenated by a focus on situations where the arrival rate and the number of servers approach infinity simultaneously (see [4] for the first such analysis). Such many-server asymptotic analysis has resulted in a collection of simple-to-use formulas for recommended staffing, consisting of a linear term (minimum staffing for stability) and a square root term ("safety staffing," to protect against random fluctuations).

In addition to the focus on the call centre sector, applications in health care are becoming increasingly important, for example see [6]. Successful health care applications are likely to require further extensions to the queueing theory toolkit to accomodate "customers" that are given different priorities and have different needs and "servers" that may group together to work on one "service" and then move on to other tasks. Typical queue performance measures, such as average wait or average cost, will also need to be re-examined, and a greater focus on measuring equity as well as quantification of the medical consequences of waiting may be neccessary.

Presentation Highlights

- Carey Williamson (University of Calgary) gave an intriguing keynote lecture on performance modeling of stochastic networks, with a focus on modeling stochastic variation in the number of servers; an issue that is often overlooked in queueing theory.
- Sunil Kumar (Stanford University) discussed some of the economic implications of customer waiting, and models that include such implications.
- David Stanford (University of Western Ontario) described preliminary work on how queueing theory might inform the difficult choice involved in managing organ transplant waiting lists.
- Marvin Mandelbaum (York University) gave an after-dinner lecture about the history of queueing theory in Canada, highlighting important contributions from Canadian researchers in the last few decades and the role of the CanQueue workshops in continuing that trend.
- The workshop included a poster session with several poster presentations; something that has not been tried at past CanQueue workshops. This was a useful addition that facilitated conversations about research during coffee breaks and allowed all participants to present their work while scheduling sufficient time for discussion after each lecture.
- An unplanned highlight was an informal get-together to celebrate the career of Winfried Grassmann, one of Canada's most distinguished queueing theorists, who retired recently. Despite his retirement, Dr. Grassmann shows no signs of slowing down, as evidenced by the two lectures that he presented or co-authored during the workshop.

Scientific Progress Made

As at past CanQueue meetings, the unique feature that facilitiates progress on research is that these workshops attract both queuing theorists, who focus on developing new methodology, and researchers who apply queueing theory in various settings. Thus, it provides opportunities for theorists to learn about new application areas and the types of models that are needed for these areas and opportunities for researhers with a more applied bent to get suggestions from queueing theory specialists on potentially useful methodologies or approaches. As well, the 2006 workshop brought together theorists with different foci, for example those that focus on asymptotic analysis versus those that focus on matrix analytic methods, and the interchanges between these groups brought valuable insights on what each of these fields can add to the other. The workshop also provided a forum for graduate students to present their work, even when it is in the beginning stages. Funding from the School of Business at University of Alberta and the Schools of Business and Engineering at the University of Calgary allowed us to provide travel funding to graduate students to attend the workshop.

Although it is too early to mention specific instances, we know that conversations during the workshop have led to collaboration between researchers and changes in research direction. The organizers know from personal experiences that such interchanges during past CanQueue workshops have helped them advance their research.

A Note of Appreciation

The organizers would like to thank BIRS staff for their very competent assistance in organizing this workshop. Many participants commented favourably on the superb BIRS facility. The location and the amenities made it considerably easier for us to attract the group of distinguished researchers that attended the workshop.

List of Participants

Abadpour, Arash (University of Manitoba)
Alanis, Ramon (University of Alberta)
Alfa, Attahiru (University of Manitoba)
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Down, Douglas (McMaster University)
Enns, Van (University of Calgary)
Grassmann, Winfried (University of Saskatchewan)
Huang, Mei Ling (Brock University)
Ingolfsson, Armann (University of Alberta)
Jayaswal, Sachin (University of Waterloo)
Jewkes, Elizabeth (Waterloo University)
Kumar, Sunil (Stanford University)
Li, Song (University of Saskatchewan)
Luo, Jingxiang (University of Manitoba)
MacGregor, Mike (University of Alberta)
Mandelbaum, Marvin (York University)
Margolius, Barbara (Cleveland State University)
Oh, Sherry (University of Calgary)
Rogers, Paul (University of Calgary)
Srinivasan, Raj (University of Saskatchewan)
Stanford, David (University of Western Ontario)
Tavakoli, Javad (University of British Columbia, Okanagan)
Williamson, Carey (University of Calgary)
Wu, Rong (McMaster University)
Zhang, Zhidong (University of Saskatchewan)
Zhao, Yiqiang (Carleton University)

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Focused Research Group Reports

Chapter 27

Random Sorting Processes (06frg501)

April 22 – May 6, 2006

Organizer(s): Alexander E. Holroyd (University of British Columbia), Omer Angel (University of British Columbia), Dan Romik (University of California, Berkeley), Bálint Virág (University of Toronto)

Overview of the Field

The purpose of the focussed research group was to study the Uniform Sorting Network and related random models. These random models bring together ideas from several fields, including probability, interacting particle systems, algebraic combinatorics, Young tableaux and group theory.

Our main object of study was the Uniform Sorting Network; that is, a uniformly random reduced word factorization of the permutation $(n, \ldots, 2, 1)$ in the symmetric group S_n using the adjacent transpositions $(i\ i+1)$ as generators. The study of the combinatorial properties of sorting networks has been a popular area of research among algebraic combinatorialists; see the papers [8], [4], [6], [5]. The introduction of a probabilistic element in [1] has brought to light very interesting connections with the theory of interacting particle systems and with the geometry of polytopes, and a wealth of striking conjectures. It was with these developments in mind that we decided to organize a meeting to bring together experts from several different related fields and try to attack some of the open problems mentioned in [1].

Recent Developments and Open Problems

The central open problem in the field is the conjecture that, when viewed as a path on the permutohedron (a natural embedding into R^n of the Cayley graph of S_n generated by adjacent transpositions), the Uniform Sorting Network resembles half of a great circle in the $(n-2)$-sphere, asymptotically as $n \to \infty$. This conjecture is known to imply several remarkable statements about the behavior of random sorting networks, notably that the particle trajectories are approximately sine curves, that the density of particles in the halfway permutation is given by "Archimedes measure" on the disk, and that the swap locations satisfy a law of large numbers with intensity given by a semicircle law. We have proved this last statement in [1] using the recent work of Pittel and Romik [7] on random Young tableaux, thus offering some circumstantial evidence in support of the sphere geodesic conjecture. Other circumstantial and heuristic evidence, as well as overwhelming numerical data, also support the conjecture as a natural limiting law for the behavior of random sorting networks, and perhaps other models.

In addition to the sphere geodesic conjecture, which at the moment seems to be a difficult problem, the problem of deriving other rigorous results on sorting networks appears challenging but not impossible. Some important steps forward in this direction were made during the workshop – see below.

Other directions of research include analyzing other random sorting processes. In [1] the Uniform Transposition Sorting Network, a natural directed random walk on the permutohedron, is analyzed successfully. Some open questions remain.

Scientific Progress Made

BIRS provided the perfect setting for the focussed research group, and we made some important progress in the study of random sorting networks. On a broad level, it was extremely useful to bring together the experts in the field and discuss the main issues in a focussed way. The central problem of the sphere geodesic conjecture "proved its worth by fighting back", and we were not able to solve it. However, a number of major advances were made, as detailed below. In addition, the meeting enabled us to hugely increase our understanding of the problem, and a number of very promising approaches have been identified. It seems very likely that further progress will follow.

Specifically, the following advances were made:

- **"Octagon bounds".** We have proved a family of bounds on the location of the particles in a random sorting network as a function of time. Specifically, with high probability as $n \to \infty$, there are no particles outside a certain polygonal region. The region is illustrated below in the case of the half-way permutation (along with the conjectural sharp "circle bound" which would follow from the sphere geodesic conjecture mentioned above). Our argument also gives bounds on the particle trajectories, and shows that when suitably scaled, the trajectories are continuous, and in fact Hölder$_{1/2}$. The results are proved by analyzing the Edelman-Greene [4] algorithm for constructing a random staircase Young tableau from a random sorting network. We are hopeful that the same ideas may yield further bounds on the permutations and trajectories.

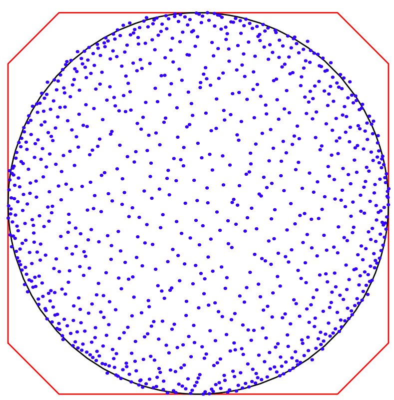

Figure 27.1: The half-way permutation of a random sorting network, the conjectural circle limit shape and the octagon bound.

- **Local structure.** A crucial step towards proving the sphere geodesic conjecture seems to be gaining a good understanding of the "local" structure of a random sorting network; that is the behavior of the wiring diagram over small regions of space-time. This local behavior is known to be stationary in time, but it would seem *a priori* that it might be different in different places along the spatial axis. A major step forward in our understanding was the realization that in fact the behavior should be the same in all places along the spatial axis, up to a scaling of the rate of swaps. This realization comes from rephrasing the problem in terms of staircase shape random Young tableaux. One may rigorously formulate this by saying that the swaps in a local region converge in distribution to a limiting process of swaps, and that this limit is the same for all spatial locations. We are currently working on a proof of this statement, which is likely to be completed in the near future.

Other progress made during the meeting includes the following. Several specific families of permutations have been ruled out as half-way permutations, in the sense that their probability is very small. A new and more efficient algorithm for exact simulation of random sorting networks was developed, resulting in simulations up to $n = 10000$, and even more compelling evidence for the main conjecture. We also explored, and gained some understanding into, a certain group of matrices defined using the Jacobi orthogonal polynomials that seems to exhibit the same limiting behavior as the random sorting networks.

Outcome of the Meeting

Several papers are in preparation: [1] was already in preparation, and has been improved as a result of the meeting; [2] and [3] are based on advances made during the meeting. It seems likely that more progress will follow soon.

List of Participants

Angel, Omer (University of British Columbia)
Berestycki, Nathanal (University of British Columbia)
Gamburd, Alex (Institute of Advanced Studies)
Hammond, Alan (University of British Columbia)
Holroyd, Alexander (University of British Columbia)
Kassabov, Martin (Cornell University)
Romik, Dan (University of California, Berkeley)
Virag, Balint (University of Toronto)
Wilson, David (Microsoft Research)

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Chapter 28

Infinite dimensional Lie algebras and local von Neumann algebras in CFT (06frg103)

May 6 – May 20, 2006

Organizer(s): Victor G. Kac (MIT), Roberto Longo (University of Rome"Tor Vergata")

The workshop set up interesting relations between Infinite dimensional Lie algebras and local conformal nets of von Neumann algebras. Each participant has given one of more talks and stimulated discussions on certain topics. A detailed list of the talks is the following.

Bojko Bakalov

Representations of Vertex Algebras

Bakalov gave a series of three lectures on representations of vertex algebras. After giving the definition of a vertex algebra, several consequences of the Borcherds identity were discussed, which led to several equivalent definitions of the notion of a module. Next, the introduction of various gradings gave three different variants of modules. The second lecture was devoted to Zhus C2-finiteness condition and its consequences; in particular its relationship to the notions of rationality and regularity. The third lecture discussed the correspondence between the representations of a vertex algebra and of its associative Zhu algebra. A recent induced-module construction of Dong and Jiang was reviewed.

Vertex Algebras in Higher Dimensions

Bakalov gave a talk on his recent joint works with Nikolov and with Nikolov-Rehren-Todorov on vertex algebras in higher-dimensional spacetime, which provide an algebraic framework for investigating axiomatic quantum field theory with global conformal invariance. The talk presented an introduction to these algebras, as well as new results. The latter included generalizations of the Borcherds identity and of the n-th product identity. Two examples were discussed: the complex massless scalar free field and the complex scalar bilocal field.

Sebastiano Carpi

From Vertex Algebras To Conformal Nets I

The unitary structure on a (local) Vertex Operator Algebra (VOA) is considered and its uniqueness up to automorphisms explained. The possible relations between the automorphism group of a unitary VOA and the correseponding unitary subgroup are discussed. In particular it is shown that the automorphism group is finite if and only if the unitary subgroup is and that in this case they coincide. It is explained how to associate a covariant net on the circle starting from a unitary VOA with (polynomial) energy bounds. A unitary VOA with energy bounds is sayd to be strongly local if the associated net is local. (Joint work with Y. Kawahigashi, R. Longo and M. Weiner)

From Vertex Algebras To Conformal Nets II

The one-to-one correspondece between vertex subalgebra of a strongly local VOA and covariant subnets of the corresponding local net of von Neumann algebra is explained. As a result sub VOA of a strongly local VOA are strongly local. It is shown that a unitary VOA with energy bounds is strongly local if is generated by a family of strongly local fields. As a consequence a VOA generated by currents and Virasoro elements is is strongly local. The coincidence of the unitary automorphism group of a strongly local VOA and the one of the corresponding net is explained. Some applications are discussed. (Joint work with Y. Kawahigashi, R. Longo and M. Weiner)

Short talk in the final joint informal discussion

Some open problems concerning the relation of VOA modules and representations of conformal nets of von Neumann algebras are outlined.

Alberto De Sole

Quantum and classical W algebras

We first reviewed the notions of vertex algebra and Poisson vertex algebra, and the relation between them via the so called quasi-classical limit. We also reviewd the definition of the Zhu algebra of a vertex algebra, which controls its representations, both in the quantum and in the classical case. We then introduced the definition of the affine and finite W algebras, obtained by the method of quantum Hamiltonian reduction, and we discussed the relaion among them.

Victor Kac

Basics on vertex algebras and principal W algebras

In my talks I explained the basics of the theory of vertex algebras in relation to common grounds with the theory of local conformal nets. Also, I discussed in some detail the principal W-algebras, their minimal models and unitary minimal models, and stated some related open problems.

Yasuyuki Kawahigashi

Representation theory of local conformal nets and complete rationality

I presented representation theory of local conformal nets based on the Doplicher-Haag-Roberts theory and introduced braiding structure. Next I presented the definition of complete rationality and explained various related conditions. Then I brielfy explained modular invariants and how to classifiy extensions of local conformal nets.

α-induction and modular invariants

As a continuation of the above talk, I explained more detailed propertoes of alpha-induction, which is an induction procedure for representations of a local conformal net to those of its extension using a braiding. Graphical method to prove modular invariance property was presented in detail. A comparison to representation theory of a vertex operator algebra was also given.

Realization of a modular tensor category as a representation category of a local conformal net

An open problem to reaized a given modular tensor category as a representation category of a local conformal net was explained. Then I explained in more detail the special case of modular tensor categories arising as quantum doubles.

Roberto Longo

Real Hilbert subspaces and conformal nets

This introductory talks explains the part of the structure associated with a local Möbius covariant net that depends only on the Hilbert space structure or, equivalently, on the Möbius group representation. In a sense, this is the universal structure associated with a net and can be descibed by the Tomita-takesaki modular theory (joint work with R. Brunetti and D. Guido). Borchers theorem and the Wisbrock-Araki-Zsido theorem on half-sided modular inclusions have a version in the real Hilbert subspace setting.

Jones index and Doplicher-Haag-Roberts superselection sectors

This talk was aimed to give an introduction to the fundamental relation between Jones index theory of subfactors and the DHR sector theory in algebraic QFT.

An analogue of the Kac-Wakimoto formula, black hole entropy, and boundary conformal QFT

We have explained a local version of a formula by Kac and Wakimoto tha can be proved in full generality in the framework of local conformal nets. This formula gives an expression for the statistical dimension that is related to the incremental black hole entropy. Finally we have explined the basic structure of boundary conformal QFT from the operator algebraic viewpoint (this last topic is taken by a joint work with H. Rehren).

Mihály Weiner

Examples of unitary VOAs that integrate to local conformal nets

This introductory talk aimed to establish the necessary background for the new results achieved in a joint work and presented later in the workshop by S. Carpi.

The passage from unitary VOAs to local conformal nets involves several technical difficulties. This is mainly due to the fact the vertex operators, in general, are unbounded.

For a set of densely defined, closed operators $\{A_\alpha\}$ there is a minimal von Neumann algebra $\mathcal{M}$ such that A_α is affiliated to $\mathcal{M}$ for every α. We may call $\mathcal{M}$ the "generated" von Neumann algebra, and denote it by $\{A_\alpha\}''$.

Let $\{A_\alpha\}$ and $\{B_\beta\}$ are two sets of densely defined, closed operators with a common invariant core $\mathcal{D}$. Suppose that $[A_\alpha, B_\beta] = 0$ (on $\mathcal{D}$) for every α and β. Does it follow that $\{A_\alpha\}''$ and $\{B_\beta\}''$ are commuting von Neumann algebras?

It is well known, that the answer, in general, is "no". Yet this is exactly what we should check (in our particular case), in order to ensure the locality of the generated net of von Neumann algebras.

In my talk I gave an overview of existing methods of dealing with such questions using certain estimates both in the general setting — e.g. by using Nelson's commutator theorem — and in the particular case of vertex operators, where it is implemented through the use of certain "energy bounds". Moreover, I explained how to find some energy bounds in the affine and in the Virasoro case.

In the mentioned examples, these energy bounds, for a generating set of fields, turn out to be linear (or better, than linear). Thus in these fortunate examples — at least for this generating set of fields — the mentioned commutator theorem of Nelson can be employed.

Finally, I gave the following conjecture on energy bounds: a (quasi) primary field of conformal dimension $d > 1$ should admit an energy bound of degree $d - 1$. This lead to several discussions after the talk. In particular, following an idea of B. Bakalov, we have managed to give such bounds for an infinite set of fields in the $W_{1+\infty}$ model. This infinite set contains a quasi primary field for *every* conformal dimension $d > 1$. (Before this, there was no known examples for such bound for a field of conformal dimension $d \geq 3$.) Hence this example seems to give some support to this conjecture.

Feng Xu

Some arithmetic properties of chiral quantities

We sketch a proof of some arithmetic properties of chiral quantities such as congruence subgroup properties in completely rational nets.

Mirror extensions of local nets

In this talk we give examples of new rational nets which are obtained as "mirrors" of exotic extensions such as conformal inclusions.

List of Participants

Bakalov, Bojko (North Carolina State University)
Carpi, Sebastiano (University of Chieti-Pescara)
De Sole, Alberto (Harvard University)
Kac, Victor (MIT)
Kawahigashi, Yasuyuki (University of Tokyo)
Longo, Roberto (University of Rome Tor Vergata)
Weiner, Mihaly (Universita' di Roma Tor Vergata)
Xu, Feng (University of California, Riverside)

Chapter 29

Complex Arrangements: Algebra, Geometry, Topology (06frg309)

Jun 10 – Jun 17, 2006

Organizer(s): Hal Schenck (Texas A&M University), Sergey Yuzvinsky (University of Oregon)

Overview of the Field

A hyperplane arrangement A is a finite collection of hyperplanes in some fixed (typically real or complex) vector space V. For simplicity, in this overview we work over the complex numbers $\mathbb{C}$. There is a host of beautiful mathematics associated to the complement $X = V\ A$. Perhaps the first interesting result in the area was Arnol'd's computation [2] of the cohomology ring of the complement of the pure braid arrangement; this was shortly followed by work of Brieskorn [3] computing the cohomology ring $H^*(X)$ of X in terms of differential forms. Subsequently, Orlik and Solomon [OS] gave a presentation of the ring $H^*(X)$ as the quotient of an exterior algebra by an ideal determined by the intersection lattice $L(A)$ of the arrangement; the rank one elements of $L(A)$ are the hyperplanes, and a rank i element is a set of hyperplanes meeting in codimension i.

A far more delicate invariant of X is the fundamental group; unlike the cohomology ring, $\pi_1(X)$ is not determined by $L(A)$. In [15], Hirzebruch wrote "The topology of the complement of an arrangement of lines in P^2 is very interesting, the investigation of the fundamental group very difficult." For any group G, the lower central series is a chain of normal subgroups defined by $G_1 = G$, $G_2 = [G, G]$, and $G_{k+1} = [G, G_k]$). Because X is formal (that is, its rational homotopy type is determined by the rational cohomology ring), the ranks of the successive quotients $\phi_k = G_k/G_{k+1}$ (the LCS ranks) are combinatorially determined for $\pi_1(X)$; so some information about the fundamental group does depend on the combinatorics of $L(A)$. Combining results of Priddy, Sullivan, Kohno, and the Poincaré-Birkhoff-Witt theorem, one can show that the LCS ranks are determined by the graded pieces of the diagonal Yoneda Ext-algebra $Ext^(H^*(X))_i(\mathbb{C},\mathbb{C})_i$; this leads to beautiful connections to Koszul algebras and duality, first observed by Shelton-Yuzvinsky [28].

Recent Developments and Open Problems

A good deal of the current work in the field centers around some very striking relationships relating the LCS quotients of $\pi_1(X)$ to a projective variety associated to $H^*(X)$. This variety, known as the resonance variety $R^1(A)$, is defined as follows: For each element $a \in H^*_1$, the Orlik-Solomon algebra can be turned into a cochain complex (H^*, a). The i^{th} term of this complex is simply the degree i graded piece of H^*, and the differential is given by multiplication by a:

$$(H^*, a): \quad 0 \longrightarrow H_0^* \xrightarrow{\cdot a} H_1^* \xrightarrow{\cdot a} H_2^* \xrightarrow{\cdot a} \cdots \xrightarrow{\cdot a} H_\ell^* \longrightarrow 0. \qquad (29.1)$$

This complex arose in the work of Aomoto [1] on hypergeometric functions, and in the work of Esnault, Schechtman and Viehweg [10] on cohomology with coefficients in local systems. In [11], Falk gave necessary combinatorial conditions that the components of $R^1(A)$ satisfy, and conjectured that the components are linear projective spaces; this was proved simultaneously by Cohen-Suciu [7] and Libgober-Yuzvinsky [17].

In [29], Suciu conjectured a relationship between $R^1(A)$ and the LCS ranks; in particular, under certain conditions the LCS ranks are solely by the dimensions of the components of $R^1(A)$. There are several means of attacking this conjecture, which has been proved in a number of cases (Falk–Randell [13], Papadima–Suciu [24], Jambu–Papadima [16], and Lima-Filho–Schenck[18]). Nevertheless, the conjecture remains open in general. Connections between higher homotopy and resonance provide a wealth of new and exciting research avenues that are just beginning to be explored; in short, this is a field with a wonderful array of open problems which connect algebra, geometry, topology and combinatorics.

Scientific Progress Made

In Fall of 2004 MSRI held a semester-long program on arrangements. The semester was wonderfully stimulating and served both to advance existing projects and foster new collaborations. Interaction among participants, postdocs, and graduate students at MSRI made clear the need for a good, up-to-date central reference for the field. In 1990 Orlik and Terao published ”Arrangements of Hyperplanes” [22], but the field has advanced dramatically since that time.

The aim of this workshop was twofold; the first objective was to work on a ”state of the art” book on hyperplane arrangements, and the second objective was to work on various joint research projects. Both of these aims were achieved; indeed, the meeting turned out to be crucial to the development of the book. We ended up restructuring the first third of the chapters, and changing the flow of the book quite dramatically. It is impossible to emphasize enough how important it was for us to all be in the same place, debating and challenging each other; we departed with a clear concensus on the selection of topics, flow, and notation. With seven authors involved, this is crucial. We plan to finish the first draft and hand it over to the publisher by the end of the summer. In addition to work on the book, the meeting also served to further a number of ongoing research collaborations:

- Cohen and Suciu made significant progress in their ongoing study of the boundary manifolds of hyperplane arrangements. Shortly after the Banff meeting, they completed a 29-page paper “The boundary manifold of a complex line arrangement”, available at `http://arxiv.org/math.GT/0607274`. This paper has now been submitted to Geometry & Topology Monographs.

- Denham and Suciu discussed their on-going work on moment-angle complexes, monomial ideals, and coordinate subspace arrangements. Shortly after the Banff meeting, they produced a substantial, expanded revision of their first paper on the subject, following referee’s comments. The paper, available at `math.AT/0512497`, was submitted to the Pure and Applied Mathematics Quarterly, for the Bob MacPherson Festschrift. They also discussed several follow-up projects, to be pursued in-depth this Fall at MSRI and Oberwolfach.

- Discussions between Cohen, Denham, and Falk facilitated progress on a project on resonance and critical points of arrangements. For a H of A, let α_H be a linear form with kernel H. A collection of weights $(\ldots \lambda_H \ldots)$ gives rise to a “master function” $\Phi_\lambda = \prod_{H \in A} \alpha_H^{\lambda_H}$, and a corresponding element $a = d \log \Phi_\lambda$ in H_1^*. Recent work on “discriminantal” arrangements suggests a relationship between resonance, $a \in R^k$, and the critical set of the master function, $\mathrm{crit}(\Phi_\lambda) = \{x \in V \mid a(x) = 0\}$. In the case where A is a free arrangement, the above participants have shown that resonance informs on the codimension of the critical set: if $a \in R^k$, then $\operatorname{codim} \mathrm{crit}(\Phi_\lambda) \leq k$. Discussions at the workshop focused on the potential reverse implication: does knowledge of the (codimension of the) critical set of Φ_λ inform on the resonance of $a = d \log \Phi_\lambda$?

- Falk and Yuzvinsky worked to put the finishing touches on their paper 'Cohomology of Orlik-Solomon algebras, multinets, and pencils of curves' and how to incorporate the very interesting comments they received after the first draft of the paper had appeared on the arXiv.

- Denham and Schenck made large strides on understanding the connection of resonance varieties, the support varieties of various Ext modules, and Fitting ideals. In particular, they Recent work shows that for arrangements, the first resonance variety $R^1(A)$ is equal to $V(annExt^{\ell-1}(F(A), S))$, where S is a symmetric algebra and $F(A)$ is a finitely generated, graded S–module depending only on the cohomology ring of $X(\mathcal{A})$.

  Denham and Schenck were able to generalize this result on the cohomology ring of an arrangement complement in two directions; showing that $F(A)$ may be replaced with an arbitrary S–module M, and that
  $$R^k(M) = \bigcup_{k' \leq k} V(annExt^{\ell-k'}(M, S)),$$
  where $R^k(M)$ is defined in terms of Koszul cohomology. By way of the Cartan-Eilenberg spectral sequence, this yields a stratification of $R^k(M)$ related to the filtration of M.

Outcome of the Meeting

The meeting yielded a firm outline and division of labor for the book on arrangements, as well as advances on a large number of research projects, described above. Overall, it was an extremely fruitful visit for all participants!

List of Participants

Cohen, Daniel (Louisiana State University)
Denham, Graham (University of Western Ontario)
Falk, Michael (Northern Arizona University)
Schenck, Hal (Texas A&M University)
Suciu, Alexander (Northeastern University)
Terao, Hiroaki (Hokkaido University)
Yuzvinsky, Sergey (University of Oregon)

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Chapter 30

Inverse Protein Folding (06frg321)

September 5 – September 9, 2006

Organizer(s): Ken Dill (University of California, San Francisco) Arvind Gupta (Simon Fraser University) Ladislav Stacho (Simon Fraser University

Overview of the Field

It has long been known that protein interactions depend on their native three-dimensional fold and understanding the processes and determining these folds is a long standing problem in molecular biology. Naturally occurring proteins fold so as to minimize total free energy. However, it is not known how a protein can choose the minimum energy fold amongst all possible folds [6].

Many forces act on the protein which contribute to changes in free energy including hydrogen bonding, van der Waals interactions, intrinsic propensities, ion pairing, and hydrophobic interaction. Of these, the most significant is hydrophobic interaction (see [8] for details). This led Dill to introduce the *Hydrophobic-Polar Model* [7]. Here the 20 amino acids from which proteins are formed are replaced by two monomers: hydrophobic (H) or polar (P) depending on their affinity to water. To simply the problem, the protein is laid out on a 2D spatial lattice with each monomer occupying exactly one square and neighboring monomers occupy neighboring squares. The free energy is minimized when the maximum number of non-neighbor hydrophobic monomers are adjacent in the lattice. Therefore, the "native" conformations are those with the maximum number of such HH contacts, also called *bonds*. Even though the hydrophobic-polar model is the simplest model of the protein folding process, computationally it is an NP-hard problem, cf. [5] for two- and [4] for three-dimensional square lattices.

In many applications such as drug design, we are actually interested in the complement problem to protein folding: *protein design*. Current protein designs often focus on local interactions such as intrinsic propensities of amino acids to form helices and turns [13, 2]. However, major forces of folding are due to hydrophobic and other *non-local* interactions [8]. To compensate for this unbalance, the existing designs work on selected small group of very stable protein motifs altering only some parts of the sequence appearing at the surface of the fold. For instance, due to its simplicity and regularity, the most extensively studied protein motif is the "coiled coil": alpha-helices wrapping around each other [18]. Another challenge in designing proteins that attain a specific native fold is to avoid proteins that have multiple native folds. We say that a protein is *stable* if the minimum free energy fold is unique. It is generally believed that all naturally occurring proteins are stable, however this is usually not true for arbitrary protein sequences.

Presentation Highlights

Ron Elber: *The space of sequences and structures of proteins*

There are numerous examples for a large number of sequences with diverse biological functions that share a similar fold. What is the number of sequences that fold into a particular protein structure? Is it possible by a point mutation to migrate from one stable protein fold to another? How probable are such transitions? Ron Elber proposed a global view of sequence — fold relationships and compute a directed graph of sequence migration between structures. He focused on the impact of protein structural stability on sequence selection and shifts between folds. A node in the graph is an experimental structure from the protein databank, and a directed and weighted edge between A and B is the fraction of sequences of A that migrate to B. An edge is a function of the energy of A. Two thousands and sixty experimental structures from the Protein Data Bank were considered, providing a good coverage of the known fold families. The directed graph is highly connected at the native energies with ”sinks” that attract substantial number of sequences. The number of incoming edges of a particular protein shape correlates significantly with the number of sequences that matches this shape in empirically determined genomes. The contact density (the average number of contacts per structural site) is a useful order parameter for sequence retention of a fold.

Ján Maňuch, Ladislav Stacho, Arvind Gupta: *Inverse protein folding: design of stable proteins*

The inverse protein folding problem is that of designing an amino acid sequence which has a particular native protein fold. This problem arises in drug design where a particular structure is necessary to ensure proper protein-protein interactions. Ján Maňuch showed that in the 2D HP model of Dill, it is possible to solve this problem for a broad class of structures. These structures can be used to closely approximate any given structure. One of the most important properties of a ”good” fold is its stability - the aptitude not to fold simultaneously into other structures. This is a necessary condition in for drug design, for example. We show that for a number of basic structures, our sequences have a unique fold.

David Bremner: *Stable and unstable foldings of proteins in the H-P model*

The ”H-P” model of protein folding considers only two classes of monomers (amino acids), namely hydrophobic (H) or hydrophilic (P). In this model hydrophobic monomers tend to cluster together to avoid water, but hydrophilic monomers are considered neutral. A folding in the H-P model consists of a self-avoiding embedding of the chain of monomers in a lattice. The ”goodness” of an embedding is measured by how many pairs of H monomers are adjacent (i.e. at unit distance) in the lattice. Despite the obvious shortcomings of this model in terms of realism, it has several attractive features. Its simple nature means that we have some chance of developing a theory predicting the folding of certain proteins (in the H-P model), rather than just waiting for the outcome of a simulation. Furthermore, it has the potential to provide insight into e.g. symmetry of optimal foldings. David Bremner presented work studying how well the H-P model captures the phenomenon of stability in protein folding, i.e. the fact that proteins almost always fold to a unique configuration. An optimal (or ”minimum energy”) embedding is one that maximizes the number of bonds. David started with a discussion of the strengths and weaknesses of the model, followed by a menagerie of (infinite families of) examples demonstrating stability in a globular (i.e. like real proteins) or non-globular form. We also looked at some examples that are extremely unstable, even though they contain a linear fraction of H nodes.

Anne Condon: *Computational challenges in prediction and design of nucleic acid structure*

RNA molecules are increasingly in the spotlight, in recognition of the important roles they are now known to play in our cells and their promise in therapeutics. Function follows form in the molecular world, and so our ability to understand RNA function is enhanced by reliable means for predicting RNA structure. Outside of the cell, exotic DNA structures are now finding use in the construction of biosensors, nanotubes, lattices, and much more, motivating the need for DNA structure prediction, as well as design of DNA sequences that fold to specific structures. Free energy minimization algorithms are widely used to predict the secondary structure of a DNA or RNA molecule from its base sequence. Such algorithms minimize over a restricted class of structures, since the general problem is NP-complete. Thus, the quality of an algorithm depends on its generality (the range of structures over which it performs energy minimization) and its accuracy (the degree to which the structure is correctly predicted). Anne Condon presented a comparison the generality and accuracy of the best free energy minimization algorithms for DNA and RNA secondary structure prediction, and discussed how such algorithms might be improved.

Ken Dill: *Molecular self-assembly in stages*

Molecular self-assembly gives rise to a great diversity of complex forms, from crystals and DNA helices to microtubules and holoenzymes. The formal study of pseudo-crystalline self-assembly, called the Tile Assembly Model (TA), started with a paper of Paul Rothemund and Erik Winfree [19] in 2000. In this model, we put together a collection of square tiles (each type of tile occurring in a large number of copies) and observe what kinds of structures are assembled from the tiles. Each type of tile is characterized by the types of glues on its four sides. For a given shape, the goal is to find a set of tiles with the minimum number of types of tiles which would uniquely assembly to the given shape. Interestingly, in a recent paper [20] of D. Soloveichik and E. Winfree, a strong connection between Kolmogorov (descriptional) complexity and the minimum number of types of tiles was shown.

Ken Dill suggested the following variation of the TA model. Instead of combining all tiles together at once, we will put tiles together in stages. For example, in stage 1, we combine tiles of type A and B; in stage 2, types C and D and in stage 3, we combine products from stage 1 and stage 2 with tiles of type E to obtain the final product. Of course, if we would mix all 5 types of tiles together, we would still be able to obtain the final product, but interactions between A and C tiles, for instance, could result in unwanted assembled shapes. Thus, under Dill's variation we might need a smaller number of distinct tiles to assembly a given shape compared to the original TA model.

Open Problems

1. Use SS bridges to fortify designed structures and to help prove stability.
2. Design of stable structures in 3D lattices (specifically, recently we have designed some simple stable structures in the hexagonal prism lattice)
3. Design of robust structures (design of structures with big energy gap between the native fold and any other fold).
4. Design of proteins with flexible parts (it is possible that flexibility of natural proteins is a key factor in protein-protein interactions).
5. Do real proteins fold uniquely in the HP model?
6. Asymptotically, what fraction of n-node HP sequences folds uniquely?
7. Is HP sequence folding still NP-complete when restricted to "nice" sequences?
8. RNA tertiary structure prediction from the base sequences?
9. Design of RNA molecules that have desired structural or functional properties? Is it hard?
10. Formalize Ken Dill's variation of self-assembly model and study its properties.

Outcome of the Meeting

The main outcome of the meeting is the proposal for 5-day workshop in BIRS 2008: "*The Biology-Combinatorics Interface: Addressing new challenges in computational biology*".

A short overview

Math and biology have a long history together [21], beginning with the work of Mendel and Darwin. Today, combinatorics and discrete mathematics are key tools for genome sequencing alignment and assembly, Markov modeling of RNA sequences, gene expression haplotyping, phylogeny construction, and biomolecular statistical mechanics and protein structure modeling.

Now, on the horizon, are the new and bigger challenges posed by systems biology. Systems biologists are interested in models at various levels, from the microscopic (genes, protein structures, and signaling pathways) to the macroscopic (metabolic and genetic circuits, cells, organs, organisms, and populations). Mathematics is poised to help to understand emergent large-scale properties from properties of smaller subsystems, in ways that less quantitative modeling cannot. A key goal is ultimately a quantitative theory of biological cells that begins with the folding and structures and binding of proteins and the regulation of genes, and scales up to the properties of full metabolic and genetic networks, and to the evolutionary dynamical processes that lead to them.

We propose a workshop that will bring biologists together with discrete mathematicians, aimed at understanding the emerging challenges in systems biology. We envision that this workshop could play a new and interesting nucleating role in biology. Biologists' workshops are typically framed around individual talks on work that has already been accomplished. Instead, here, we propose a more engaged problem-solving approach: we will define the problems, consider the variables, frame the possible models, work through toy problems, and envision the possible outcomes via the give and take of problem-based discussions. We think that this sort of mathematicians' approach could provide fresh new insights for biologists into some their most challenging current problems.

List of Participants

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Gupta, Arvind (MITACS)
Manuch, Jan (Simon Fraser University)
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Chapter 31

Quantum algorithms for algebraic problems (06frg323)

Sep 16 – Sep 23, 2006

Organizer(s): Ashwin Nayak (University of Waterloo and Perimeter Institute), Leonard Schulman (California Institute of Technology (Caltech)), John Watrous (University of Calgary)

Overview

Quantum computers are computational devices that are based on principles of quantum mechanics. Phenomena such as superposition—the ability to exist in several states simultaneously, and interference—the ability of different computation paths to combine constructively as well as destructively, allow quantum computers a much broader range of operations than possible with current computers, which are based on the laws of classical physics. The potential of quantum devices to outperform current computers was rigorously demonstrated by Bernstein and Vazirani in 1993. Since then, efficient quantum algorithms have been discovered for a number of important problems, including integer factorization and the discrete logarithms. No efficient (polynomial-time) classical algorithms are known for these problems. In fact, cryptosystems such as RSA, whose security rests upon the computational intractability of factorization and discrete logs, are in widespread use.

Quantum computers seem to be especially effective in solving problems with a group-theoretic flavour. The aim of the proposed meeting is to develop new techniques that will allow us to tackle some outstanding questions concerning algebraic problems such as Graph Isomorphism, and Closest Lattice Vector. These problems are of complexity in between P and NP, and are considered candidates for efficient quantum algorithms. Some of these problems are also the basis of classical cryptosystems believed to be more secure than RSA. The possibility of cryptosystems secure against quantum computers hinges crucially on the quantum complexity of the underlying problems.

Objectives

The meeting planned to focus on quantum algorithms and complexity theory issues surrounding algebraic problems such as Graph Isomorphism, and Closest Lattice Vector.

Graph Isomorphism

Originally motivated by problems in chemistry, the Graph Isomorphism (GI) has become a central problem in classical computation. Given two undirected graphs on the same set of (say N) vertices, the problem asks if there is a permutation of the vertices that maps the edges of one graph to exactly the edges of the second. The best known classical algorithm for GI is superpolynomial in N, yet for compelling complexity-theoretic reasons, it is believed not to be NP-complete.

GI reduces to the Hidden Subgroup Problem (HSP), which generalizes integer factorization, discrete logs, and a number of other group-theoretic problems. Efficient quantum algorithms for HSP are known when the base group is abelian. The non-abelian case remains largely unsolved, and is the subject of active research in quantum computation, largely because it includes Graph Isomorphism as a particular instance.

The most fruitful approach to solving HSP has been via creating "coset states", which are uniform superpositions over cosets of the "hidden subgroup". In the case of GI, the hidden subgroup is an order two subgroup of S_N wreath product S_2, if there is an isomorphism between the two graphs, and and is trivial otherwise. A general result due to Ettinger et al. shows that $O(NlogN)$ coset states suffice to distinguish between the two cases, and in fact to find a generating set for the subgroup. However, it takes exponential time to process the coset states. Work over the past year reveals an inherent weakness of this approach [Moore et al. 05, Hallgren et al. 05]: not only are $Omega(NlogN)$ coset states necessary, but any measurement of the states that extracts useful information is necessarily entangled across all the states. In other words, independent measurements of the states (as was possible in the abelian case) are futile.

The meeting set out to alternatives to the standard coset state approach for efficiently solving Graph Isomorphism. An approach completely orthogonal to this, but possibly less daunting, would be to place Graph Non-Isomorphism in the complexity class QMA (the quantum analogue of NP).

Lattice problems

A lattice is an additive subgroup of R^N, generated by all integer linear combinations of a basis of linearly independent vectors. Many seemingly unrelated problems in mathematics and computation can be recast in terms of lattices, e.g., integer programming, polynomial factorization, and integer factorization. Computational problems related to lattices have the feature that random instances of the problem (according to specific distributions) are at least as hard as the worst case instance. This makes them ideal candidates for building cryptographic schemes (and a number of such schemes have been proposed [Ajtai and Dwork'03, Regev'04]).

The Shortest Vector problem (SVP) asks for the shortest non-zero vector in the lattice generated by a basis for R^N (in the Euclidean norm). The Closest Vector Problem (CVP) asks for the lattice point that is closest (in Euclidean norm) to a given vector in R^N. Both these problems have been studied extensively, and are known to be NP-hard. The best classical algorithm for SVP runs in time $2^O(N)$, and the best polynomial time algorithms for either problem only approximate the solution to within exponential factors. Approximation to within polynomial factors are possible via constant round interactive proofs, even co-NP proofs, and such approximation is believed not to be NP-hard.

A number of connections have been established between quantum computation and lattice problems. The first reduces approximation of SVP to an instance of the Hidden Subgroup Problem (mentioned above) over the dihedral group. The second uses quantum algorithms to reduce worst case instances of SVP to average case instances of "learning with error". Perhaps the most interesting connections are the upper bounds on the quantum and classical complexity of approximating SVP and CVP [Regev and Aharonov'03-04]. While we do not yet have efficient quantum algorithms for approximating these problem to within polynomial factors, it has was shown that approximation to within $\sqrt{N}$ is in QMA (the quantum analogue of the complexity class NP). Moreover, using the insights gained from this work, the result was improved to show that such approximation is also possible with *classical* proofs, i.e., in NP.

We would like to further investigate connections of lattices with quantum computation. In particular, we would like to study the extent to which SVP and CVP can be approximated in quantum polynomial time, or with quantum proofs. We suspect that there are further lessons for classical algorithms to be learnt from investigating these problems with techniques from quantum computation.

Other group theoretic problems

To date, the hidden subgroup problem has dominated research efforts devoted to quantum algorithms for computational problems in groups. While (as described above) there are good reasons for this focus, there are other natural types of problem in groups, for which, for some of the same reasons, it is reasonable to search for an efficient quantum algorithm. The workshop was to be a useful venue for exploring some of these problems and the means with which one might approach them.

Proceedings of the meeting

Schur tranform: Andrew Childs

Andrew Childs presented work on using the Schur transform to distinguish coset states in the standard approach to the hidden subgroup problem (joint work with Aram Harrow and Pawel Wocjan, quant-ph/0609110, to appear in STACS 2007). This transform exploits the permutation symmetry of many copies of the coset state, decomposing the global state into subspaces labeled by partitions. Andrew explained why simply measuring the partition (”weak Schur sampling”) provides very little information about the hidden subgroup. In fact, even a combination of weak Fourier sampling and weak Schur sampling fails to identify the hidden subgroup. He also explained how the problem is connected to a quantum version of the collision problem, and used this connection to prove tight bounds on how many coset states are required to solve the hidden subgroup problem by weak Schur sampling.

During the workshop, Andrew made some progress on extending these tight bounds to cover the case of weak Fourier-Schur sampling.

Hidden quadratic structures: Leonard Schulman

Leonard Schulman described a class of algebraic problems which might be amenable to solution by quantum computers in spite of being hard for classical computers. In this class of problems the input is a mixture of superpositions, each of which is uniform over the roots of a particular multivariate finite-field quadratic equation. Two instantiations Leonard discussed are: (a) The quadratic is a sphere of unknown radius, and the mixture is over all translates of this sphere; the problem is to determine the radius. (b) The quadratic is an axis-parallel ellipsoid of unknown eccentricity, and the mixture is over all scalings of the ellipsoid; the problem is to determine the eccentricity.

In working sessions during the workshop, several participants suggested promising variants of the problem, and there was much brainstorming about various attacks (e.g., quantum random walks). Also during the workshop, Andrew Childs was able to obtain numerical evidence in support of the conjecture that the sphere problem is (at least information-theoretically) amenable to solution by quantum computers.

After the workshop, Umesh Vazirani, Childs and Schulman continued work on this class of problems and were able to prove that quantum computers can solve various cases of it. Childs will be giving a presentation on this work at QIP (Quantum Information Processing) 2007, and a paper is in the writing stages.

Open problem session: John Watrous

John Watrous discussed two open problems in quantum computation. The problems were not new – the intention was to take a fresh look at interesting problems in the light of more recently developed techniques.

The first problem asks whether or not it is possible to compute the greatest common divisor of two positive integers using a logarithmic or poly-logarithmic depth quantum circuit. The analogous question for classical circuits is a long-standing open question in theoretical computer science. The fact that the quantum Fourier transform can be performed using logarithmic depth quantum circuits (Cleve and Watrous, FOCS 2000) might be a useful tool for addressing this problem.

The second problem concerned the quantum complexity of the Group Order problem. In this problem, one is given generators of a finite group along with a means to compute products and inverses. The black-box group model reflects this situation. The goal of the problem is to compute the order (or size) of the generated group. This is a candidate for a difficult computational problem even for quantum computers – although the

Abelian case is easily handled using Shor's algorithm, thus far quantum algorithms are of little help in the non-Abelian case. The question is this: is the problem in QMA? In other words, is it possible to prepare a quantum state that, although not necessarily efficiently preparable, could efficiently convince someone with a quantum computer that the order of a given group was some given value N?

Quantum one-way functions: Umesh Vazirani

Umesh Vazirani discussed recent work done by himself along with Cris Moore and Alex Russell on quantum one-way functions. The goal is to find a function that can be efficiently computed in the forward direction by a classical computer, but which is difficult to invert even using a quantum computer. Such functions have potential to be very important in cryptographic applications when security is required against adversaries having quantum computers. The specific class of functions that was proposed is related to the notorious hidden subgroup problem that has frustrated quantum algorithm designers for several years. In this case, however, the difficulty of the Hidden Subgroup problem is beneficial, because the difficulty of inverting a given function is closely related to solving instances of the problem.

A paper describing these results has recently appeared on the quant-ph preprint archive.

Two quantum algorithms: Ashwin Nayak

Ashwin Nayak presented two works, one on the use of quantum walks in search algorithms (by Frederic Magniez, A.N., Jeremie Roland, and Miklos Santha), and the other on the learnability of quantum states (by Scott Aaronson).

There are several ways of devising quantum processes analogous to random walks, and these have been used fairly successfully in designing search type algorithms. The algorithms extend the search technique due to Grover to "structured databases" and provide polynomial speed-up over the best classical algorithms. Ashwin described a quantum walk based algorithm that may be defined for an arbitrary ergodic Markov Chain. It combines the benefits of two previous approaches (due to Ambainis and Szegedy, 2004) while guaranteeing the better form of run time. Ashwin also pointed out an open question regarding the speed-up in hitting time achievable by considering the quantum analogue of a non-reversible Markov chain. The question asks if a certain "discriminant matrix" has singular value gap that is significantly larger than the eigenvalue gap of the Markov chain.

The work due to Aaronson may be seen as the poor man's version of quantum state tomography. State tomography involves estimating the exponentially many variables that determine the state of a quantum system. Often we are more interested in the outcome of making a certain measurement on a quantum state, rather than in explicitly knowing the entire state. A typical example is in the setting of a two-party communication protocol, where the parties wish to compute a bivariate function f(x,y) by sending only one message. Here, a quantum state encoding one input is sent as the message. In a classical simulation of the protocol, we may avoid sending the exponential size description of the message, since the other party need only recover the result of the measurement made on input y. Aaronson showed how a polynomial number of observations may be used to form a reasonable hypothesis on the state. The hypothesis would allow outcomes of typical measurements to be predicted correctly with high probability.

Closest vector problem: Niel de Beaudrap

Niel de Beaudrap discussed the possible use of Gaussian states over lattices in algorithms for the closest vector problem (CVP). The problem asks for the lattice point that is closest (in Euclidean norm) to a given vector in R^N. Gaussian states have occurred in most recent results on quantum and classical algorithms (or interactive proofs) for lattice problems and in cryptosystems based on lattices. Niel described a quantum algorithm that is in a limited sense, a worst case to average case reduction. This was used by Aharonov and Regev in a polynomial-time classical algorithm for the $\sqrt{N}$-gap CVP with pre-processing. During the meeting, Niel and Ashwin Nayak identified the key difficulties in converting the reduction to a full-fledged polynomial-time algorithm for $\sqrt{N}$-Gap-CVP. Work on the problem is still under way.

List of Participants

Childs, Andrew (California Institute of Technology)
de Beaudrap, Niel (University of Waterloo)
Hallgren, Sean (NEC Laboratories America, Inc.)
Nayak, Ashwin (University of Waterloo and Perimeter Institute)
Schulman, Leonard (California Institute of Technology (Caltech))
Vazirani, Umesh (University of California, Berkeley)
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Research in Teams Reports

Chapter 32

Saari's Conjecture (06rit303)

February 11–February 25, 2006

Organizer(s): Florin Diacu (University of Victoria, Canada), Toshiaki Fujiwara (Kitasato University, Japan), Ernesto Pérez-Chavela (UAM–Iztapalapa, Mexico), Manuele Santoprtete (University of California at Irvine, USA)

Overview of the Field

Saari's conjecture is notoriously difficult. Donald Saari proposed it in 1970 in the following form [13]: *In the Newtonian n-body problem, if the moment of inertia, $I = \sum_{k=1}^{n} m_k |q_k|^2$, is constant, where $q_1, q_2, \ldots, q_n$ represent the position vectors of the bodies of masses $m_1, \ldots, m_n$, then the corresponding solution is a relative equilibrium*. In other words: Newtonian particle systems of constant moment of inertia rotate like a rigid body.

There have been many attempts to solve this problem. Some of them even led to the publication of incorrect proofs, such as those in [10, 11]. More recently, the interest in this conjecture has grown considerably due to the discovery of the "figure eight" solution (see [2]), which—as numerical arguments show—has an approximately constant moment of inertia but is not a relative equilibrium.

Still, there have been a few successes in the struggle to understand Saari's conjecture. McCord proved that the conjecture is true in the case of three bodies with equal masses [9]. Llibre and Piña provided an alternative proof of this case, but they never published it [6]. Moeckel obtained a computer-assisted proof for the Newtonian three-body problem for any values of the masses [7, 8]. Diacu, Pérez-Chavela, and Santoprete showed that the conjecture is true for all n in the collinear case for any potential that depends only on the mutual distances between point masses [3]. There have also been results, such as [1, 4, 5, 12, 14, 15], that consider the conjecture in contexts different from the Newtonian one.

Even more difficult than Saari's conjecture is its following extension. In the Newtonian n-body case, define the configurational measure of the particle system to be the function $UI^{1/2}$, where U is the Newtonian potential. The extended Saari's conjecture can then be stated as follows: *Every solution of constant configurational measure is homographic*. In particular, if the moment of inertia is constant, then the potential is constant, therefore every homographic solution is a relative equilibrium, so Saari's conjecture follows from its extension.

The extended Saari's conjecture covers new territory. While in the original Saari's conjecture collisions are excluded (because they lead to an unbounded potential, which contradicts the constant moment of inertia assumption) and the motion remains bounded (because the moment of inertia is constant), both collision and unbounded orbits may occur in the extended version of the conjecture.

Scientific Progress Made

During the two weeks spent at BIRS, we succeeded to solve the extended Saari's conjecture in some important cases, especially in the case of three bodies. Our results can be outlines in eight theorems, as follows.

Theorem 1 shows that, for homogeneous potentials of order $a < 2$, the extended Saari's conjecture is true for any total-collision solution of the planar or spatial n-body problem. Theorem 2 validates the extended Saari's conjecture for $0 < a < 2$ for any type of collision in the n-body case. Theorem 3 proves the extended conjecture correct in the rectilinear case for $0 < a < 2$. Theorem 4 shows the extended conjecture to be always valid in the collinear case. Theorem 5 proves that, for $0 < a < 2$, the extended conjecture is true in the three-body problem if the solutions stay away from the paths that make them scatter asymptotically towards rectilinear central configurations. Theorem 6 proves the extended conjecture correct in the Newtonian three-body problem with equal masses and non-negative energy. Theorem 7 shows that for any given initial configuration of three bodies, the extended conjecture is valid if the chosen angular momentum is large enough. Finally, Theorem 8 shows that if the angular momentum is chosen first, then the extended conjecture is true if the initial positions are taken close enough to an equilateral triangle of a certain size.

The key tool for obtaining these results is provided by what we call Fujiwara coordinates, which have been earlier developed by one member of this team. These coordinates allowed us to regard the conjecture from a new point of view and thus obtain the results outlined above.

Outcome of the Collaboration

The outcome of this "research in teams" collaboration has been very positive. Without the opportunity to meet for two weeks together and work on this problem in the excellent conditions offered by BIRS, we might have not obtained these results. Moreover, this is the beginning of what we hope will be a long and fruitful collaboration.

List of Participants

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Chapter 33

Partial Unconditionality in Banach Spaces (06rit099)

Mar 04 - Mar 18, 2006

Organizer(s): Thomas Schlumprecht (Texas A&M University)

Research Accomplished

The problem of partial unconditionality, namely does there exist a K so that given any weakly null normalized sequence (x_n) in a Banach space and a $d > 0$, some subsequence is Elton unconditional for d with constant K, touches many areas of mathematics as evidenced in our original proposal. One such area is that of discrete approximation: given a normalized sequence, when can one be assured that every element of the space can be well approximated by a finite linear combination of the elements with coefficients chosen from a discrete alphabet? It is easy to see that the unit vector basis of c_0 has this property. In fact this property played a key role in W.T.Gowers' proof that c_0 satisfies "the ultimate Ramsey property". However other bases also possess this property as witnessed by use of the sigma-delta algorithm in signal processing applied to the summing basis. We were able to verify however that an unconditional basis with the discrete approximation property which is Elton unconditional must be the c_0 basis. This then led us to a study of such bases. Here are the main results we achieved.

Throughout, X will denote a separable infinite-dimensional Banach space and (e_i) will denote a seminormalized basis of X. Our definitions and results can also be formulated for more general dictionaries, but for the sake of simplicity we will only consider bases.

Definition 1 A seminormalized sequence (e_i) has the (ε, δ) -*Coefficient Quantization Property* (abbr. (ε, δ)-CQP) if for every $x = \sum_{i\in E} a_i e_i \in X$ (where E is a finite subset of $\mathbb{N}$) there exist $n_i \in \mathbb{Z}$ ($i \in E$) such that

$$\|x - \sum_{i\in E} n_i \delta e_i\| \leq \varepsilon. \tag{33.1}$$

(e_i) has the CQP if (e_i) has the (ε, δ)-CQP for some $\varepsilon > 0$ and $\delta > 0$.
(b) A dictionary (e_i) has the (ε, δ)-*Net Quantization Property* (abbr. (ε, δ)-NQP) if for every $x \in X$ there exist a finite subset $E \subset \mathbb{N}$ and $n_i \in \mathbb{Z}$ ($i \in E$) such that

$$\|x - \sum_{i\in E} n_i \delta e_i\| \leq \varepsilon. \tag{33.2}$$

(e_i) has the NQP if (e_i) has the (ε, δ)-NQP for some $\varepsilon > 0$ and $\delta > 0$.

Note that in the definition of the NQP we do not insist that vectors are approximated by vectors with the same support.

Theorem 2 *Suppose that $c_0 \hookrightarrow X$ and that X has a basis. Then X has a normalized, bounded basis which has the $(\varepsilon, c\varepsilon)$-CQP for all $\varepsilon > 0$, where c is an absolute constant (independent of X and ε).*

Theorem 3 *Let (e_i) be a semi-normalized basic sequence with the CQP. Then (e_i) has a subsequence that is equivalent to the unit vector basis of c_0 or to the summing basis of c_0.*

Theorem 4 *Let (e_i) be a normalized monotone basis for a Banach space E. Given $\eta > 0$ there exists a Banach space U with a normalized monotone basis (u_i) with the following properties:*

(a) (u_i) *has the* $(\varepsilon, \varepsilon/3)$*-NQP;*

(b) there exists a subsequence (u_{n_i}) *of* (u_i) *that is* $(1+\eta)$*-equivalent to* (e_i).

Theorem 5 *Let (e_i) be a seminormalized basis with the NQP. Then every subsequence of (e_i^*) has a further subsequence equivalent to the unit vector basis of ℓ_1.*

The aforementioned results have been written up in [1].

In addition we began to study the same problem for frames. Indeed a frame workshop at BIRS during our FRG provided us with an opportunity to learn about frames and deduce certain things.

Since vectors in Banach space do not have unique representation with respect to frames it turned to be much harder to formulate results similar to Theorems 2, 3, 4 and 5 as the following example shows.

Example 6 There is a tight frame (x_i) of normalized vectors (actually the union of two orthonormal bases) in a separable Hilbert space H so that for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $x \in H$ there is a sequence $(n_i) \in \mathbb{Z}$, having finite support (i.e. the set $\{i \in N : n_i \neq 0\}$ is finite)

$$\Big\|x - \sum_{i=1}^{\infty} n_i \delta x_i\Big\| < \varepsilon.$$

This example shows that one needs to state the quantization problem for frames differently.

Question 7 *Assume H is a separable Hilbert space. Is there a normalized frame (x_i) of H with the following property: There is a constant C and for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $x \in H$ there is a family $(n_i) \subset \mathbb{Z}$, having finite support, so that*

a) $\max_{i \in \mathbb{N}} |n_i| \leq C\|x\|$

b) $\big\|x - \sum_{i=1}^{\infty} n_i \delta x_i\big\| < \varepsilon$.

We were able to prove in several special cases, that such a frame does not exist, but the general problem is still open.

List of Participants

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Chapter 34

Multi-Parameter Nehari Theorems (06rit302)

March 20 – 24, 2006

Organizer(s): Michael T. Lacey (Georgia Institute of Technology), Jill C. Pipher (Brown University), Stefanie Petermichl (University of Texas at Austin), Brett D. Wick (Vanderbilt University)

Nehari Theorems

The Nehari Theorem characterizes bounded Hankel operators. Let us state the result on $L^2(\mathbf{R})$. Let M_b be the operator of pointwise multiplication by b, $M_b\varphi = b \cdot \varphi$. Consider the standard decomposition $L^2(\mathbf{R}) = H^2(\mathbf{R}) \oplus H^2_-(\mathbf{R})$ of $L^2(\mathbf{R})$ into the Hardy spaces of analytic and anti analytic functions. Let $P_\pm$ be the orthogonal projections of L^2 onto H^2 and H^2_- respectively. A *Hankel operator with symbol* b maps H^2 into itself and is given by $H_b := P_+ M_b \overline{\varphi}$. This definition depends only on the analytic part of b. The Nehari Theorem [6] asserts that the bounded Hankel operators are exactly those which admit a bounded symbol.

Theorem 1 *H_b is bounded if and only if there is a bounded function β with $P_+\beta = P_+ b$, and*

$$\|H_b\| = \inf\{\|\beta\|_\infty \mid P_+\beta = P_+ b\}. \tag{34.1}$$

This Theorem is one of the foundations of modern operator theory. Coifman, Rochberg and Weiss [3] characterized real valued H^1 in several variables by way of a variant of Nehari's Theorem, this time stated in the language of commutators and the dual to H^1, BMO.

Theorem 2 *[Coifman, Rochberg and Weiss] Fix a dimension $d > 1$. Let R_j, $1 \le j \le d$, be the Riesz transforms on $\mathbf{R}^d$. We have the equivalence*

$$\sup_j \|[M_b, R_j]\|_{2\to 2} \simeq \|b\|_{\mathrm{BMO}}. \tag{34.2}$$

The latter space is real one parameter $\mathrm{BMO}(\mathbf{R}^d)$.

Multi-parameter Setting

We are interested in multi-parameter extensions of the results described above. Some results in this setting are already known, and for the purposes of this note, we restrict ourselves to the two parameter setting of Ferguson and Lacey [4], and stress that the higher parameter setting (which requires new ideas) is discussed in Lacey and Terwilleger [5].

The function theoretic setting for this Theorem is the Hardy space $H^2(\mathbf{R}\otimes\mathbf{R})$, consisting of functions f of two complex variables, analytic in each variable separately, with values on the boundary of $\mathbf{C}_+\otimes\mathbf{C}_+$ that are square integrable. This is a closed subspace of $L^2(\mathbf{R}\otimes\mathbf{R})$, and we let $P_{+,+}$ be the orthogonal projection of L^2 onto this Hardy space. It is worth emphasizing that the complex domain is the product of two disks which is *not* pseudoconvex. It has boundary given by the product of two flat domains $\mathbf{R}\otimes\mathbf{R}$, hence the relevance of two parameter Harmonic Analysis.

The Hankel operators we consider are $H_b\varphi := P_{+,+}M_b\overline{\varphi}$, considered as an operator from H^2 to itself. This definition only depends upon the jointly analytic part of b, namely $P_{+,+}b$. These are the so called 'little Hankel operators' as the projection $P_{+,+}$ is the 'smallest' reasonable projection to use.

Theorem 3 *[Ferguson and Lacey [4]] A Hankel operator H_b is bounded if and only if it admits a bounded symbol. Namely, there is a bounded function β with $P_{+,+}\beta = P_{+,+}b$, and*

$$\|H_b\| = \inf_{\beta}\{\|\beta\|_\infty \mid P_{+,+}\beta = P_{+,+}b\}\,. \tag{34.3}$$

It is to be stressed that the relevant Hardy spaces here are on *product domains*, which do not fall in the scope of the elaborate theory built up around the classical Hardy spaces. In particular, the dual to $H^1(\mathbf{R}\otimes\mathbf{R})$ is a BMO$(\mathbf{R}\otimes\mathbf{R})$ space identified by S.-Y. Chang and R. Fefferman in a famous series of papers [1, 2].

Scientific Progress Made

Our focus has been to obtain a multi-parameter extension of the Coifman, Rochberg and Weiss result, and the Lacey Terwilleger result. This result, once established, would yield Nehari Theorems for certain Bergman spaces, and novel Div-Curl Lemmas. Namely, the principal result of our meeting is this Theorem.

We are concerned with product spaces $\mathbf{R}^{\vec{d}} = \mathbf{R}^{d_1}\otimes\cdots\otimes\mathbf{R}^{d_t}$ for vectors $\vec{d} = (d_1,\ldots,d_t)\in\mathbf{N}^t$. For Schwartz functions b, f on $\mathbf{R}^{\vec{d}}$, and for a vector $\vec{j} = (j_1,\ldots,j_t)$ with $1\le j_s\le d_s$ for $s = 1,\ldots,t$ we consider the family of commutators

$$C_{\vec{j}}(b,f)(x) =:= [\cdots[[M_b, R_{1,j_1}], R_{2,j_2}],\cdots](f)(x) \tag{34.4}$$

where $R_{s,j}$ denotes the jth Riesz transform acting on $\mathbf{R}^{d_s}$.

Theorem 4 *We have the estimates below, valid for* $1<p<\infty$.

$$\sup_{\vec{j}}\|C_{\vec{j}}(b,\varphi)\|_p \simeq \|b\|_{\mathrm{BMO}}. \tag{34.5}$$

By BMO*, we mean Chang–Fefferman* BMO*.*

Many of the techniques of proof used by Coifman Rochberg and Weiss are simply not available in the higher parameter setting. Many of the techniques of the Lacey Terwilleger approach apply, but they are not enough to conclude the proof of the Theorem. The argument of Lacey and Terwilleger relies at several points on the fact that the Hilbert transform is a difference of Fourier projections. And so several new methods must be brought to bear on the problem.

List of Participants

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Pipher, Jill (Brown University)
Wick, Brett (Vanderbilt University)

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Chapter 35

Homology stability of moduli of vector bundles over a curve (06rit100)

April 8 – April 16, 2006

Organizer(s): Donu Arapura (Purdue University), Ajneet Dhillon (University of Western Ontario)

Overview

Arapura and Dhillon spent an intensive and productive week at BIRS, under the auspices of the *Research in Teams* programme. During this period, they were able to produce an outline of a research project, described below, on the moduli of bundles over a curve. Although some more work will be needed to flesh out the details, a finished paper is expected to result from this research in the near future.

Mathematical Details

Let C be a smooth projective curve of genus $g \geq 2$ over field of complex numbers, and let $G = G_n$ be a classical group (i.e. one of GL_n, SL_n, SO_n, Sp_n). The research project involves the study of the moduli stack $Bun_G(C)$ (respectively moduli space $M_G(C)$) of (stable) principal G-bundles over C. When $G = GL_n$, these objects can be identified with the moduli stack or space of vector bundles. The basic goal is understand the Hodge structure, and the underlying motive, on the cohomology of $Bun_G(C)$ and $M_G(C)$ as C varies. This can be reduced to a series of subproblems:

1. Construct a relative theory of motives in the spirit of André [1].

2. Use an Atiyah-Bott type isomorphism [3, 6] to “compute” the motive of $Bun_G(C)$ in terms of the motive of C. Apply this to the universal curve.

3. Find good estimates to relate the cohomology and motive of $Bun_G(C)$ to that of $M_G(C)$. For vector bundles, suitable estimates have been found in [2, 5]. In general, some of the basic tools are contained in [4].

Since the estimates in 3 should grow with n, this can be used to show that $H^*(M_{G_n}(C))$ stabilizes, as expected.

List of Participants

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Chapter 36

Curvature and instability of flows of ideal incompressible fluid (06rit311)

May 20 – May 27, 2006

Organizer(s): Alexander Shnirelman (Concordia University)

Overview of the Field

Serious interaction between the dynamics of an ideal incompressible fluid and the (infinite-dimensional) differential geometry began since the now classical works of V.Arnold who first regarded the flows of an ideal incompressible fluid in a bounded domain M as geodesics on the group $SDiff(M)$ of volume preserving diffeomorphisms of M. This group is an infinite-dimensional manifold with a right-invariant Riemannian metric defined by the kinetic energy. V.Arnold [1] computed the curvature of $SDiff(M)$ in case M is the 2-dimansional torus, and found that it can assume either sign, depending on direction. In this connection he asked whether the negative sign of curvature is connected with instability of steady flows. Further, D.Ebin in his important work with J. Marsden [3] has systematically built the differential geometry of $SDiff(M)$ as a part of global analysis. In further works of V.Yudovich, B.Khesin and other people [2] the curvature of $SDiff(M)$ was found via external geometry of M, which was a great simplification. G.Misiolek [6] investigated the role of positive curvature and established the existence of conjugate points on some geodesics in $SDiff(M)$. Further, D.Ebin, G.Misiolek and S.Preston [4] proved that in 2-dimensional case the exponential map on $SDiff(M)$ is Fredholm, while in 3-d case this is not true (which is again connected with positive curvature). This work included the deep analysis of functional-analytic properties of the curvature operator. On another hand, A.Shnirelman [7] considered the global geometric properties of the group $SDiff(M)$. He proved that it has finite diameter in the dimension of M is greater than 3 (while in 2-dimensional case the diameter is infinite, which was proved by Y. Eliashberg and T.Ratiu [5]). Using his results, A.Shnirelman shows that not all pairs of fluid configurations can be connected by the shortest geodesic. For the exponential map on $SDiff(M)$ in the 2-d case he proved that this map has a rigid geometric structure: it is a Fredholm Quasiruled map [8].

This domain has a good potential for growth which is ensured by the existence of open problems. Some of them are described below.

Recent Developments and Open Problems

The stability theory of flows of ideal incompressible fluid is a testbed of all new ideas and methods of mathematical fluid dynamics. One of this approaches which has lead already to deep results is the study of fluid flows from the viewpoint of infinite-dimensional differential geometry. Ideal incompressible fluid inside a bounded domain M is an example of an infinite-dimensional Lagrangian system. Without external forces it moves along geodesics on the group $SDiff(M)$ of volume-preserving diffeomorphisms of M, which is an infinite-dimensional Riemannian manifold. Hence, the stability of the flows should be connected with the Riemannian curvature of $SDiff(M)$, as it was pointed out by V.Arnold. However, the question is far from certain. For example, it was found by V.Yudovich that for any parallel flow in a channel the curvature in any 2-d direction containing velocity itself is negative (this result was considerably extended by A.M.Lukatsky, G. Misiolek and S. Preston). However, stability properties of such flows depend on the velocity profile and are quite different for different profiles.

The differential geometry of $SDiff(M)$ is complicated enough. The curvature can assume either signs and be zero, depending on the direction. The space $SDiff(M)$ is homogeneous, but it is not symmetric. To the contrary, it is extremely far from symmetry. Therefore our ”symmetric” intuition may be misleading, and in some situations negative curvature can stabilize the flow, if it varies in certain way. Likewise, positive curvature can be destabilizing. Directions of zero curvature and asymptotically flat geodesic subspaces should play important role in the stability and in the long time behavior of the flows.

The question of interconnection between the stability of fluid flows and the curvature and, possibly, other differential-geometric properties of the space SDiff(M) is far from clear. The curvature, being typically negative in all directions containing the velocity field, is not bounded off from zero. The solution of the Jacoby equations (the linearized Euler-Lagrange equations) can tend to the asymptotically zero-curvature direction, thus neutralizing the effect of negative curvature on the perturbation growth. This phenomenon lies beyond the sheer negativity of curvature, it is possibly associated with the fact that $SDiff(M)$ is not a symmetric space. However, little progress is done in this direction.

Still another problem is the study of the structure of the Jacoby equations themselves. The typical solution is growing (or decreasing) algebraically, rather than exponentially (if there is no unstable eigenvalues). Such behavior may be associated with the presence of Jordan cells in the linearized operator (it is ”made of” a continuum of low-dimensional Jordan cells). This idea is far from realization.

Another problem is the structure of singularities of the exponential map on the group SDiff(M). We expect that the Fredholm structure of this map enables to use topological methods (like the degree theory) in the study of the Euler equations. The interesting question is, whether the variant of condition C of Palais-Smale holds for the action functional on $SDiff$ in the 2D case. If true, this could lead to a Morse theory of geodesics in this context.

Presentation Highlights

Our group worked in the permanent exchange of ideas, and there were no formal presentations. I'd highlight the impressive demonstration of the use of Maple for bulky asymptotic computations by Steve Preston.

Scientific Progress Made

During the meeting D. Ebin and S. Preston worked on an interesting idea of a finite-codimensional approximation to the Euler equations. This work is in its beginning and may bring some interesting results.

Outcome of the Meeting

The main outcome of the meeting is the intensive exchange of ideas between the participants. It helped to find and root out some misconceptions, and to formulate some questions. One of the results is the beginning of systematic common work of S. Preston and A. Shnirelman on the problems of the motion of an ideal inextensible thread, which is peculiarly close to the fluid, while appear simpler. Here the combination of differential geometry, functional analysis, dynamics, and even computer simulations works even better than for the fluid (however some features of this system are almost opposite to the fluid).

List of Participants

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Chapter 37

Inflation from string theory (06rit102)

Jul 08 – Jul 22, 2006

Organizer(s): James M. Cline (McGill University)

Overview of the Field

String cosmology is a rapidly changing field, which is still in its early days. One of the most promising applications is to inflation, the theory of superluminal expansion of the early universe, which is receiving spectacular confirmation from experiments, like WMAP, which measure the cosmic microwave background (CMB) fluctuations. Many string theorists hope that this wealth of new data can help to give some empirical tests of string theory, which are so far lacking in particle physics experiments.

Although it may be unlikely that the CMB by itself can decide whether inflation is due to string theory or conventional field theory, string theory is also tightly constrained by its own rich mathematical structure, which has made it challenging to build successful models of inflation within realistic and fully consistent vacuum states of string theory. Through a combination of the experimental constraints and those imposed from the need for internal mathematical consistency, it may be possible to eventually pinpoint the most likely models of stringy inflation, and to better focus the strategies of experimental verification.

Recent Developments and Open Problems

One of the most distinctive proposals for inflation from string theory is that inflation is due to the relative motion of D-branes, higher dimensional objects intrinsic to string theory, within the compact internal dimensions which are equally essential. The main realizations involve D3- and anti-D3 branes (where 3 denotes the spatial dimension) and D3 and D7 branes. The major challenge is to find ways of making the potential between the branes sufficiently flat to support a long period of inflation, during which the size of the universe grew by a factor of at least e^{60}. This question cannot be fully addressed until stability of the size of the extra dimensions has been assured by some mechanism, since the dynamics of this compact space has a strong effect on the force between the branes. Warped compactifications in type IIB string theory have provided a mathematically rigorous framework for stabilizing the extra dimensions, and have thus made the problem of constructing a flat potential mathematically well-posed.

To this point, flat inflaton potentials do not seem to arise in generic compactifications. Instead, it appeared necessary to tune parameters of the compactification (like fluxes of gauge fields, or rank of the gauge groups) in a very special way to achieve flatness. However, even these attempts have suffered from a lack of mathematical rigor. It was known that there exist corrections to the superpotential describing the force on a D3 brane in a warped throat which could potentially be tuned to give a flat potential [1], but these corrections had never been explicitly calculated, only parametrized. Thus the details of how flatness could be achieved in a specific string background remain to be elucidated.

Scientific Progress Made

We were fortunate in that an important breakthrough was made by authors at Princeton University in computing the aforementioned superpotential corrections [2], just when our meeting at BIRS began. We quickly absorbed the content of this work so that we could begin to apply it to the D3-$\overline{\text{D3}}$ and D3-D7 systems. The goal is to see whether it is indeed possible to tune these corrections in the way that was originally envisioned, to flatten the inflaton potential to an acceptable level.

In the process of this investigation, we made two serendipitous discoveries which enhance the mathematical consistency of the basic framework for the two systems which we are studying. These discoveries pertain to the description of forces acting on the D3 branes within a supergravity framework. In the D3-D7 system we were able to compute in the language of supergravity what form the potential for the D3 brane takes, as a result of forces associated with stabilization of the extra dimensions, and from supersymmetry-breaking fluxes on the D7 brane.

In the the context of the D3-$\overline{\text{D3}}$ system, we discovered a new way of *uplifting* the brane potential from negative to positive values, as needed to obtain inflation rather than anti-de Sitter space solutions. This uplifting was achieved in the seminal reference [3] using the supersymmetry-breaking effect of the $\overline{\text{D3}}$-brane, but we have discovered, as a result of the newly computed superpotential corrections, that uplifting can be achieved even without the $\overline{\text{D3}}$-brane, without the explicit breaking of supersymmetry. This is an improvement over the original construction of [3] in terms of keeping approximations under control.

Outcome of the Meeting

Based on the above progress, we have prepared drafts of two papers which are nearing completion [4], and which set the stage for further, more detailed study, of the actual inflationary process in these systems. Our meeting at BIRS thus laid the groundwork for continued collaboration on inflation within string theory, sharpening our understanding of the framework within which to construct specific realizations. The meeting was an ideal venue for defining the problem; that step required intensive interaction. We are now able continue working together from a distance and we hope to make further significant progress during the coming year.

List of Participants

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Chapter 38

Threshold Dynamics, with applications to image processing and computer vision (06rit314)

Jul 29 – Aug 05, 2006

Organizer(s): Steve Ruuth (Simon Fraser University)

Overview

Many important models of image processing and computer vision involve curvature dependent functionals. In segmentation, the Kass, Witkin, Terzopoulos method [8] of snakes originally calls for minimizing an active contour energy that involves integrating curvature squared along the curve. In segmentation with depth, the 2.1D Sketch model of Nitzberg, Mumford, and Shiota [10] involves the integral of a function of the curvature along the free discontinuity set. In the image inpainting application of Bertalmio, Sapiro, Caselles, and Ballester [2], Chan, Shen, and Kang [4] proposed generating the missing image information in the inpainting domain by minimizing Eulers elastica energy along image isophotes. Finally, in computer graphics, curvature dependent functionals have been proposed for surface denoising and smoothing.

One of the most successful techniques for minimizing variational models in image processing that involve unknown contours has been the level set method of Osher and Sethian [11]. Once the energies in question are written in terms of a level set representation for the unknown curves, optimality conditions (Euler-Lagrange equations) can be obtained in terms of the level set function, and a gradient descent procedure can be carried out. This involves solving non-linear, degenerate, fourth order parabolic PDE and can be computationally very expensive.

The idea behind threshold dynamics is to alternate the solution of a linear parabolic PDE, such as the heat equation, and thresholding to generate geometric motion of interfaces. The original idea is due to Merriman, Bence, and Osher [9], who proposed a technique for approximating the motion by mean curvature of an interface by alternating the solution of the heat equation (i.e. convolution by the Gaussian kernel) and thresholding. Convergence was proved by Evans [6], and by Barles and Georgelin [1]. There have been various generalizations of their method to other curvature dependent velocities, and a highly accurate version was developed by Ruuth in [12]; these offer an alternative to level set based techniques that require the solution of nonlinear second order equations.

On the novel side, this workshop proposes further investigation of non-standard thresholding that involves the geometry of the solution instead of only the pointwise values of the solution. To our knowledge, there has not been any existing working algorithms or analysis if the thresholding procedure is replaced, for example, by redistancing (reshaping the solution into the distance function of the shape it represents). This may be an important strategy to improve upon the existing algorithms and to solve more general problems.

Finally, there is a close link between variational approaches for image processing and more general inverse problems. The threshold dynamics approach described above can potentially be a useful tool for building up efficient solvers for inverse problems that require certain types geometric regularizations. Furthermore, in problems involving optimal shapes, there are on-going discussions on suitable representations and the efficiency of their respective numerical algorithms. Typically these criticisms consist of algorithms being too costly or unable to obtain globally optimal shapes. Threshold dynamics offers a different way of building and analyzing the problems that may potentially lead to robust and efficient inverse problem solvers.

Motivated by the Merriman, Bence, Osher scheme, recently Esedoglu and Tsai proposed a technique for minimizing the piecewise constant versions of the Mumford-Shah segmentation functional that were introduced by Chan and Vese [3]. This new algorithm involves alternating the solution of a linear parabolic PDE and simple thresholding. It leads to a very efficient minimization of Chan and Veses Mumford-Shah energies. Based on this successful application of threshold dynamics ideas to an important image processing problem, we believe it is prudent to ask what other image processing and computer vision problems might benefit from this approach.

An important class of models in image processing and computer vision involve curvature dependent functionals. The minimization of these functionals involve the solution of fourth order geometric pdes. The numerical solution of such pdes with standard level set methods can be very costly.

Overview of the Research in Teams Event

Recently, Grzibovskis and Heintz [7] have found a threshold dynamics that approximates gradient flow for an important curvature dependent functional known as the Willmore energy. This energy consists of the integral of the square of a surface's mean curvature over that surface. Furthermore, it constitutes an essential part certain variational image models for segmentation with depth, disocclusion, and image inpainting. As a first step in bringing threshold dynamics to bear upon higher order models of image processing and computer vision, we recently generalized the work of Grzibovskis and Heintz to the reconstruction of an occluded binary image [5].

In this program we continued our earlier work by considering replacements to the characteristic function representations used in traditional threshold dynamics. We anticipate that by introducing new techniques to threshold dynamics we can broaden the appeal of the methods while simultaneously achieving improved accuracy to high order flows.

Developments and Progress

The first issue addressed by the BIRS event was how to evolve a curve with a normal velocity equal to the local curvature of the curve, according to a combination of convolution and redistancing[1]. "Motion by curvature" is precisely the motion that arises from a surface tension driven interface motion, and is central to many models in image processing as a method to regularize or smooth reconstructed images. For the prototype case of a circle, a consistent combination of convolution and redistancing was obtained. This result lead to a number of questions:

- Should the convolution and redistancing steps be combined into one? Will this give faster iterations and smoother solutions, thereby giving better results?
- Can we carry out extrapolation in the timestep size to obtain higher order methods, thereby obtaining more accurate approximations of mean curvature motion?
- Can combinations of different convolutions be considered which, taken together, give higher order convergence?
- Can combinations of different convolutions achieve more general high order motions, specifically the Willmore flow?

[1] Redistancing is the process of determining a signed distance function to a curve from an earlier approximation.

The answers we obtained for these questions follow.

Combining Convolution and Reinitialization Steps

A central question is whether to carry out convolution and reinitialization together as separate terms in the same PDE, or to carry out each separately in a traditional threshold dynamics fashion. It was found that by combining both we were able to obtain small errors for basic curvature flow, but that the combined approach was inferior when higher order accurate methods are sought. This further suggests that the combined approach would not be well-suited for obtaining motions for high order PDEs. A focus was therefore made on separate reinitialization and convolution steps.

Richardson Extrapolation and Methods for Canceling Error Terms

While consistent results were obtained with the basic approach, the results were low order accurate, specifically first order in the timestep size. Richardson extrapolation gave much better accuracy, as did combining different convolutions to cancel the dominant error terms. The former approach was extremely simple, however, the latter had the advantage of allowing clear extensions to the evolution by Willmore flow and flow by surface diffusion.

Willmore and Surface Diffusion Flows

Carrying out convolutions with multiple kernels of different widths, allows the construction of combinations which give Willmore flow and surface diffusion flow. Methods for both of these motion laws were obtained and found to be consistent for the prototype case of a circle.

Conclusions and Outstanding Issues

We have developed new algorithms based on combinations of convolution and redistancing. While both of these components are standard numerical techniques, the combinations considered are new. We obtained a variety of interesting motion laws including curvature flow (accurate to second order), Willmore flow and surface diffusion flow. The algorithms have the advantages that

- They use combinations of standard algorithms to obtain complicated flows.
- Stability of the methods is very good; indeed often better than existing algorithms. Thus, the approach holds strong promise for computing steady state flows in applications such as image processing.

Outstanding issues are primarily related to the speed of the algorithms. Traditional redistancing is slow to obtain accurate solutions, and the number of reinitialization steps was comparable to the number of grid points in a coordinate direction. We are interested in improving on this component in future work.

List of Participants

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Chapter 39

The path partition conjecture for oriented graphs (06rit126)

August 5 – August 12, 2006

Organizer(s): Jean Dunbar (Converse College), Marietjie Frick (University of South Africa), Ortrud R. Oellermann (University of Winnipeg), Susan van Aardt (University of South Africa)

Overview of the Field

The vertex set and arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively, and the number of vertices in a digraph D is denoted by $n(D)$. A *directed* cycle (path, walk) in a digraph will simply be called a cycle (path, walk). A graph or digraph is called *hamiltonian* if it contains a cycle that visits every vertex, *traceable* if it contains a path that visits every vertex, and *walkable* if it contains a walk that visits every vertex.

A digraph D is called *strong* (or *strongly connected*) if every vertex of D is reachable from every other vertex. Thus a digraph D of order bigger than 1 is strong if and only if it contains a closed walk that visits every vertex. A maximal strong subdigraph of a digraph D is called a *strong component* of D and a maximal walkable subdigraph of D is called a *walkable component* of D.

A longest path in a digraph D is called a *detour* of D. The order of a detour of D is called the *detour order* of D and is denoted by $\lambda(D)$.

If (a, b) is a pair of positive integers, a partition (A, B) of the vertex set of a digraph D is called an (a, b)-*partition* if $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.

If a digraph D has an (a, b)-partition for every pair of positive integers (a, b) such that $a + b = \lambda(D)$, then D is called λ-*partitionable*. The Directed Path Partition Conjecture (DPPC) states:

DPPC: Every digraph is λ-partitionable.

A directed version of the PPC that is perhaps stronger than the DPPC was stated by Bondy [2]. (His conjecture requires $\lambda(D\langle A\rangle) = a$ and $\lambda(D\langle B\rangle) = b$ while the DPPC only requires $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.)

Results supporting the DPPC appear in [1], [8] and [11]. The analogous conjecture for undirected graphs, known as the Path Partition Conjecture (PPC), was first formulated in 1981 and is still an open problem. (Cf [3]-[9], [12], [13] for results supporting the PPC.) A digraph D is called *symmetric* if for every $xy \in E(D)$ the arc yx is also in $E(D)$. The PPC is, obviously, equivalent to the conjecture that every symmetric digraph is λ-partitionable.

An *oriented* graph is a digraph with no cycle of length 2. We can therefore obtain an *oriented* graph D from a graph G by assigning a direction to each edge of G. We call such a digraph D an *orientation* of G. The DPPC restricted to oriented graphs states:

OPPC: Every oriented graph is λ-partitionable.

The DPPC implies both the PPC and the OPPC. We do not know the relationship between the OPPC and the PPC.

Recent Developments and Open Problems

The *detour deficiency* of a digraph D is defined as $p(D) = n(D) - \lambda(D)$. A digraph is p-*deficient* if its detour deficiency is p. These concepts have analogous definitions for graphs. The PPC has been proved for all graphs with detour deficiency $p \leq 3$, and for $p \geq 4$ it has been proved for all p-deficient graphs of order at least $10p^2 - 3p$ (see [3], [9]). Moreover, it is shown in [10] that if a graph G is 1-deficient or 2-deficient, then even the weaker condition $a + b = \lambda(G) - 1$ guarantees that G is λ-partitionable.

For oriented graphs the situation is very different. The OPPC has not even been settled for 1-deficient oriented graphs. Moreover, it is shown in [15] that, for every $p \geq 0$ and every pair $a, b \geq 5$, there exists a strong, p-deficient oriented graph D such that $a + b = \lambda(D) - 1$ and D has no (a, b)-partition. Thus, if the OPPC is true, it will be *best possible* in a very strong sense. These observations underline the importance of settling the OPPC for 1-deficient oriented graphs. We call this special case of the conjecture the OPPC(1).

M. Nielsen suggested a new approach for solving the OPPC(1). He defined an oriented graph D to be k-*traceable* for some k, $1 \leq k \leq n$ if every induced subdigraph of D of order k is traceable. A similar concept can be defined for graphs. It is readily seen that if a graph G of order n is k-traceable for some $k \in \{2, 3, \ldots, \lceil \frac{n}{2} \rceil\}$, then G is hamiltonian. This is not the case for oriented graphs. In fact, for every $n \geq 6$ we can construct a nonhamiltonian oriented graph of order n that is k-traceable for every $k \in \{5, 6, \ldots, n\}$. We nevertheless suspect that the following conjecture is true.

Traceability Conjecture (TC): Let D be an oriented graph of order n which is k-traceable for some $k \in \{2, 3, \ldots, \lceil \frac{n}{2} \rceil\}$. Then D is traceable.

The OPPC(1) can now be formulated as follows.

OPPC(1): If D is a 1-deficient oriented graph and $a + b = \lambda(D)$, then D is not $(a + 1)$-traceable or D is not $(b + 1)$-traceable.

It is clear from the above formulation that the truth of the TC would imply the truth of the OPPC(1).

Outcome of the Research in Teams Workshop

The focus of the workshop was to approach the OPPC(1) by proving the TC for certain classes of oriented graphs. We showed that if D is a nontraceable oriented graph that is traceable for some $k \in \left\{2, \ldots, \left\lceil \frac{n(D)}{2} \right\rceil \right\}$, then D is walkable and D is a spanning subdigraph of an MNT oriented graph D^* with the same number of strong components as D. We investigated the structure of walkable MNT oriented graphs, and this enabled us to show that the TC need only be considered for oriented MNT graphs having a very special structure. In fact, we showed that proving the TC reduces to showing that if D is an MNT oriented graph with at most three strong components, of which one is non-hamiltonian and the others are tournaments, then for each $k \in \left\{2, \ldots, \left\lceil \frac{n(D)}{2} \right\rceil \right\}$, D is not k-traceable. Using this approach, we proved that the TC, and hence the OPPC(1), holds for oriented graphs with sufficiently small or sufficiently large minimum degree, as well as for oriented graphs whose nontrivial strong components are all hamiltonian.

A paper containing the results of our workshop is in progress. Moreover, the ground work has been laid for future research. Investigating the structure of MNT oriented graphs is in itself an interesting problem, and the Traceability Conjecture for Oriented graphs is an intriguing new conjecture.

List of Participants

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Chapter 40

The topology of hyperkahler quotients (06rit317)

Aug 19 – Aug 26, 2006

Organizer(s): Megumi Harada (McMaster University), Greg Landweber (University of Oregon), Graeme Wilkin (Brown University)

Overview of the Field

The purpose of this workshop was to address a question concerning the computation of topological invariants of hyperkähler quotients. Our approach is based on the successful and well-developed similar theory for the case of *symplectic* quotients, so we begin with a brief account of that theory.

Symplectic geometry is the mathematical framework of classical mechanics. A symplectic manifold is a manifold equipped with a *symplectic form*, i.e. a non-degenerate closed differential 2-form, which is the geometric data needed to translate a Hamiltonian function on the system to the dynamics of the system. Examples of symplectic manifolds are any 2-dimensional surface equipped with its area form, cotangent bundles T^*M, toric varieties, and flag manifolds. A symplectic manifold is Kähler if there is also a complex structure compatible with the symplectic form; when there are *three* Kähler structures on M, with associated compatible complex structures interacting like the quaternions, then M is *hyperkähler*. Many hyperkähler manifolds appear naturally in physics and representation theory. Examples from physics are $T^*\mathbb{P}^1$ with the Eguchi-Hansen metric, K3 surfaces, and moduli spaces of Higgs bundles over a Riemann surface [7]; examples arising in representation theory are quiver varieties, as studied by Nakajima [14].

In the theory of hyperkähler or symplectic quotients, we are primarily concerned with a situation in which there is a symmetry of the system, as encoded by the action of a compact Lie group G. Symplectic manifolds with an action of a Lie group G and a corresponding moment map, which is a suitably compatible collection of Hamiltonian functions, are called Hamiltonian G-spaces. For a hyperkähler manifold M, we require that there be a moment map $\mu_i : M \to \mathfrak{g}^*, i = 1, 2, 3,$ for *each* of the three Kähler structures. Given a symplectic Hamiltonian G-space, the symplectic quotient is defined as $M//G := \mu^{-1}(0)/G$. The reduced space inherits a symplectic structure from M. In the hyperkähler case, we take the hyperkähler quotient $M////G$ to be the quotient by G of the intersection of the zero-level sets of all three moment maps; this is again hyperkähler.

Hyperkähler quotients, and hyperkähler geometry in general, has recently attracted much attention due to its relationship between many other fields of mathematics. The topological invariants of hyperkähler manifolds, such as rational cohomology or integral K-theory, are often quite interesting. For example, the K-theory of quiver varieties give geometric realizations of representations of certain algebras associated to quivers. There are also close connections between the cohomology of hyperkähler analogues of toric varieties and the combinatorial theory of hyperplane arrangements.

We now give an overview of our approach towards the computation of the topology (more specifically, the rational cohomology ring) of hyperkähler quotients. There is a "meta-principle" for computing such invariants of Hamiltonian quotients of various types, which we call here the *Kirwan method*. Let M be a Hamiltonian G-space of some type (symplectic or hyperkähler, and M_G the appropriate Hamiltonian quotient of M by G. Then the Kirwan method consists of the following three steps:

"The Kirwan method":

1. **"Meta-Theorem" (Kirwan surjectivity):** For M and M_G as above, there is a natural ring homomorphism
$$\kappa : H_G^*(M;\mathbb{Q}) \to H^*(M_G;\mathbb{Q})$$
which is *surjective*. In particular, in order to compute $H^*(M_G;\mathbb{Q})$, it suffices to compute $H_G^*(M;\mathbb{Q})$ and $\ker(\kappa)$.

2. Compute $H_G^*(M;\mathbb{Q})$.

3. Compute $\ker(\kappa)$.

The point of this method is that one can often compute the last two objects, $H_G^*(M;\mathbb{Q})$ and $\ker(\kappa)$, using *equivariant* techniques which are unavailable on the quotient. In the symplectic case, this "Kirwan method" has been well-developed; in particular, Step (1) in this case was proven by Kirwan [9], and various explicit solutions of Steps (2) and (3) can be found e.g. in [8, 17, 3, 4]. Thus, our research program is to develop the Kirwan method for hyperkähler quotients. The focus of our BIRS workshop was in the proof of Step (1) for this hyperkähler case.

Recent Developments and Open Problems

Kirwan's proof of Step (1) in the case of symplectic quotients involves showing that the norm-square $\|\mu\|^2$ of the symplectic moment map $\mu : M \to \mathfrak{g}^*$ gives rise to an equivariantly perfect Morse-type stratification of M, which gives surjectivity since the 0-level set of μ is the absolute minimum of the norm-square.[1] We propose to prove an analogue of Kirwan surjectivity in the setting of finite-dimensional hyperkähler quotients using Morse-type methods similar to Kirwan's proof. There are already specific known examples where such a hyperkähler analogue of Kirwan surjectivity result does hold [10, 11]. Moreover, in the specific infinite-dimensional case of the moduli space of Higgs bundles over a Riemann surface, Daskalopoulos, Weitsman, and Wilkin have developed several new Morse-theoretic techniques using the norm-square of the moment map to obtain new Kirwan surjectivity results in rational Borel-equivariant cohomology [2, 18].

In the case of a hyperkähler manifold with the action of a group G which is Hamiltonian with respect to each of the three Kähler structures (a *hyperhamiltonian* group action), there are three moment maps $\mu_i : M \to \mathfrak{g}, i = 1, 2, 3$ (one for each of the Kähler structures).

In an unpublished draft manuscript, Kirwan suggested that one could first use the Morse theory of $\|\mu_2\|^2 + \|\mu_3\|^2 = \|\mu_{\mathbb{C}}\|^2$ (where $\mu_{\mathbb{C}} = \mu_2 + i\mu_3 : M \to \mathfrak{g} \otimes \mathbb{C}$) to construct a map $H_G^*(M) \to H_G^*(\mu_{\mathbb{C}}^{-1}(0))$, and then use the Morse theory of the function $\|\mu_1\|^2$ on the space $\mu_1^{-1}(0)$ to construct a map $H_G^*(\mu_{\mathbb{C}}^{-1}(0)) \to H_G^*(\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))$. It would then remain to show that both of these maps are surjective. There are two main technical difficulties in carrying out the second step, firstly that the gradient flow of $\|\mu_1\|^2$ on $\mu_{\mathbb{C}}^{-1}(0)$ might not converge (we need convergence to construct a Morse theory on this space), and secondly that the space $\mu_{\mathbb{C}}^{-1}(0)$ is singular so we would need to provide some extra analysis for the Morse theory to work. Both of these difficulties are new to the hyperkähler situation; the first does not arise in the presence of only *one* moment map (assuming we take the preimage of a regular value), and the second does not arise since in the symplectic or Kähler case one usually assumes that the moment map is proper, so the level set is compact.

Nevertheless, despite these difficulties, Wilkin has made Kirwan's second approach work in the infinite-dimensional case of the moduli spaces of Higgs bundles. In particular, Wilkin (in collaboration with his Ph.D. supervisor Georgios Daskalopoulos and Jonathan Weitsman)

[1] There are technical difficulties arising from the fact that $\|\mu\|^2$ is not, in fact, Morse; this is the technical and important contribution of Kirwan's proof, which has had wide applications.

1. proved the gradient flow converges, despite the non-properness of the moment maps [18],
2. showed there exists a Morse-type theory for the norm-square $\|\mu_{\mathbb{R}}\|^2$ on $\mu_{\mathbb{C}}^{-1}(0)$ [2], and
3. showed how to use the singularities in the preimage $\mu_{\mathbb{C}}^{-1}(0)$ to obtain the correct formula for the Poincaré polynomial of the moduli space by developing a theory which can be described as "Morse theory in a stratified sense" [2].

We intend to follow the approach of Daskaopoulous, Weitsman, and Wilkin, and in particular to prove finite-dimensional analogues of their theorems to obtain a general surjectivity result in the hyperkähler case.

Progress made at the BIRS workshop and outcomes of the meeting

The Morse theory of the norm-square of the moment maps

- As in the outline of Wilkin's work in Section 40 above, we first need to prove that the gradient flow of $\|\mu_1\|^2$ converges to a critical point. A result of Lojasiewicz in [12] shows that this problem reduces to proving that the gradient flow remains in a compact set. As a first test case, we proved explicitly that for the case of S^1 acting on $T^*\mathbb{C}^n$, the flow indeed stays in a compact set. We also discussed how Hitchin proved the gradient flow convergence in [7] for the case of Higgs bundles, where he uses the fact that the finite-time gradient flow lies on a $G^{\mathbb{C}}$ orbit to compute estimates along the flow. We can describe the case of quiver varieties via a setup similar to Hitchin's. During the BIRS workshop we computed simple examples of quivers and came up with specfic conjectures of analytic estimates on the gradient flow which would suffice to prove its convergence in the case of quiver varieties.

- We also made progress at the BIRS meeting in understanding the singularites arising in the Morse theory of $\|\mu_1\|^2$. Previous to the meeting, we proved the following theorem about the Morse index of $\|\mu_1\|^2$.

  Theorem. *Let M be a finite-dimensional hyperkähler manifold, and let $f(x) = \|\mu_1(x)\|^2$ on M. At a critical point $x \in \mu_{\mathbb{C}}^{-1}(0) \subset M$ let $N(x) \subset T_xM$ denote the negative eigenspace of f, and let $L(x)$ denote the linearisation of the complex moment map $\mu_{\mathbb{C}}$. Then $N(x) \subseteq L(x)$.*

  This is the first step in relating the Morse index calculations of $\|\mu_1\|^2$ on the smooth manifold M (where Kirwan's results show that the index is well-defined) to the Morse index calculations on the singular space $\mu_{\mathbb{C}}^{-1}(0)$. To carry out the approach of [2] in our case, we need to show that the negative directions at a critical point are *contained within* the space $\mu_{\mathbb{C}}^{-1}(0)$, not just the linearisation of this space. During the BIRS workshop we computed several concrete examples and showed that the negative directions are indeed contained within $\mu_{\mathbb{C}}^{-1}(0)$ in each case. Using these examples, we formulated explicit strategies to prove the more general cases.

- We were also able to prove the following theorem regarding the critical sets of the functional $\|\mu_1\|^2$ on the space $\mu_{\mathbb{C}}^{-1}(0)$ for quiver varieties.

  Theorem. *At a critical point of $\|\mu_1\|^2$ the quiver splits into sub-quivers. In particular, each connected component of the set of non-minimal critical points can be expressed as the product of quiver varieties of simpler quivers.*

  Hence we can inductively build up the critical sets by studying quivers with simpler structures. This is analogous to the well-known setting for the Yang-Mills functional, where a holomorphic bundle splits into sub-bundles at a critical point (see for example [1]). This fact (for Higgs bundles) is used heavily in [2], which leads us to believe that in the case of quiver varieties many of the methods of [2] for Higgs bundles will hold.

Alternative Approaches

During the BIRS workshop, we also discussed possible alternative approaches to our problem. In particular, we discussed the possibility of first taking the Kähler quotient $N := T^*\mathbb{C}^n//_\alpha G$ with respect to the real moment map $\mu_{\mathbb{R}}$, and then further restricting to $\mu_{\mathbb{C}}^{-1}(0)$, as $T^*\mathbb{C}^n////_{(\alpha,0)}G = N \cap \mu_{\mathbb{C}}^{-1}(0)$. With this method, the Kähler quotient N should be a smooth manifold, and we expect that its relation to the hyperkähler quotient $M////G$ can be obtained using Morse theory for the norm square of the complex moment map $\|\mu_{\mathbb{C}}\|^2$.

In this case, we hope to use the S^1-action rotating the fibers of $T^*\mathbb{C}^n$ and its corresponding moment maps on the Kähler and hyperkähler quotients in order prove that the cohomology of the Kähler quotient surjects onto the cohomology of the hyperkähler quotient. Using Morse theory to build both of the quotients simultaneously, we note that the minimal level sets of the S^1-moment maps are the same in both cases. As we pass each higher critical level, we hope to prove that surjectivity still holds, and in order to show this we have formulated the following conjecture:

Conjecture. *For each connected component C of the S^1-fixed set of $T^*\mathbb{C}^n//G$, the restriction to $\mu_{\mathbb{C}}^{-1}(0)$ induces a surjection in cohomology, $H^*(C) \twoheadrightarrow H^*(C')$, and the two Morse indices agree: $\lambda_C = \lambda_{C'}$.*

We verified that this argument works for hyperpolygon spaces by performing explicit computations based on [11]. In this case, the S^1-fixed sets in both the Kähler and hyperkähler quotients are compact projective spaces, and our above conjecture holds. If this argument works in general, then we can establish the surjectivity from the Kähler quotient to the hyperkähler quotient $H^*\big(T^*\mathbb{C}^n//_{\alpha}G\big) \twoheadrightarrow H^*\big(T^*\mathbb{C}^n////_{(\alpha,0)}G\big)$. To establish the hyperkähler analogue of Kirwan surjectivity, we must further establish Kirwan surjectivity for the Kähler quotient: $H_G^*(T^*\mathbb{C}^n) \twoheadrightarrow H^*\big(T^*\mathbb{C}^n//_{\alpha}G\big)$. Since the spaces involved are non-compact, we must prove that the gradient flow converges in order to apply Kirwan's surjectivity arguments.

A further approach which we discussed at BIRS is to do Morse theory using linear combinations of the S^1-moment map and $\|\mu_{\mathbb{C}}\|^2$. Although we need to restrict to the minimum of $\|\mu_{\mathbb{C}}\|^2$, the S^1-moment map much better behaved. We explored the possibility of starting with one such moment map and perturbing it by adding a multiple of the other, hoping to obtain the same quotient without the Morse-Kirwan difficulties.

Abelianization

Another related topic which we discussed at BIRS is the "abelianization" of hyperkähler quotients. When working with symplectic quotients, one can use the techniques of Tolman-Weitsman [17] and Goldin [3] to compute the cohomology of $M//T$ where T is abelian. For quotients of the form $M//G$ where G is not abelian, we must first abelianize, by restricting from G to a maximal torus T. Working in the hyperkähler case, we studied the following abelianization conjecture of Tamás Hasusel:

Conjecture *(Hausel) Let G be a compact, connected Lie group and T a maximal torus in G. If both of the hyperkähler quotients $T^*\mathbb{C}^n////G$ and $T^*\mathbb{C}^n////T$ are hypercompact, then*

$$H^*(T^*\mathbb{C}^n////G) \cong \frac{H^*(T^*\mathbb{C}^n////T)^W}{\mathrm{Ann}(e_T(\mathfrak{g}/\mathfrak{t})^2)},$$

where $e_T(\mathfrak{g}/\mathfrak{t}) \in H_T^*(\mathrm{pt})$ is the equivariant Euler class of the representation $\mathfrak{g}/\mathfrak{t}$ of T.

In [6], Hausel and Proudfoot prove an S^1-equivariant version of this abelianization theorem based on Martin's proof [13] of the analogous result for symplectic quotients. They use the S^1-equivariance in order to establish integral formulae, which allow them to reproduce Martin's Poincaré duality arguments in the non-compact setting. However, our alternative proof [5] of Martin's theorem does not use Poincaré duality, and we believe that this will allow us to generalize our techniques to the non-compact hyperkähler setting without S^1-equivariance. A close analysis of this conjecture leads us to believe that it is best approached using the algebro-geometric language of holomorphic symplectic quotients, in lieu of hyperkähler quotients.

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Chapter 41

Exact Primal-Dual Regularization of Linear Programs (06rit322)

August 19 – August 28, 2006

Organizer(s): Michael Friedlander (UBC), Dominique Orban (Ecole Polytechnique de Montréal)

In the framework of linear programming, we propose a theoretical justification for regularizing the linear systems used to compute search directions when the latter are (nearly) rank deficient. Our research program is based on the analysis of a primal-dual infeasible algorithm for linear programs (LPs) with explicit primal and dual regularization. The goal is to establish a rigorous connection between proximal-point, regularization, trust-region, and augmented Lagrangian methods. The regularization is termed *exact* to emphasize that, although the LP is perturbed, we are still able to recover a solution of the original LP, independently of the values of the regularization parameters.

General Overview

Consider the primal-dual pair of linear programs (LPs)

$$\begin{array}{llccl} \text{(P)} & \text{minimize}_x & c^T x & \text{subject to} & Ax = b, \quad x \geq 0, \\ \text{(D)} & \text{minimize}_{y,z} & -b^T y & \text{subject to} & A^T y + z = c, \quad z \geq 0, \end{array}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$. In primal-dual interior-point methods for linear programming, the computational kernel lies in the solution of an indefinite system of linear equations to determine search directions. At each iteration, a Karush-Kuhn-Tucker (KKT) system of the form

$$\begin{bmatrix} -D & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \tag{41.1}$$

must be solved; the diagonal matrix D and right-hand side (f_x, f_y) change at each iteration. These systems may be solved "as is" (using direct methods or iterative methods suitable for symmetric indefinite systems), or d_x and d_y may be solved for separately by using the normal equations

$$AD^{-1}A^T d_y = AD^{-1} f_x + f_y \quad \text{and} \quad D d_x = f_x - A^T f_y. \tag{41.2}$$

In either case, near rank deficiency of A, or near singularity of D (diagonal elements that are equal or very close to zero), can give rise to inefficient or unstable solutions of these linear systems. Our research program consists of exploring the theoretical and numerical properties of a primal-dual regularization that perturbs (P) and (D) in ways that favourably affect the numerical properties of the linear algebra subproblems that arise in interior-point methods for LPs. Such regularizations have been used in practical implementations for solving large-scale problems (see, e.g., [AG99,ST96]). Our goal is to provide a strong theoretical justification for their use in practical implementations and also to understand the implications of regularization on the convergence properties of primal-dual interior-point methods.

Our analysis begins with LPs because of the important properties that result from their strong structure. We note, however, that our work can be extended to more general convex quadratic programs and even to nonlinear convex programs under a regularity condition. This important research topic will be the subject of our followup project.

Progress

During our week at BIRS, we examined several aspects of the primal-dual regularization approach. Our first task was to understand the connection of regularization and its algorithmic implications to existing methods.

Connection to augmented Lagrangians. Dual regularization via the ℓ_2 norm turns out to be equivalent to an augmented Lagrangian approach for solving (P), which is based on *approximately* solving the sequence of subproblems

$$\begin{array}{ll} \text{minimize}_x & c^T x + \tfrac{1}{2}\sigma\|r\|_2^2 + y_k^T r \\ \text{subject to} & Ax + r = b, \quad x \geq 0. \end{array} \tag{41.3}$$

An appropriate barrier-formulation of (41.3) leads to a linear system

$$\begin{bmatrix} -D & A^T \\ A & \sigma I \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} \bar{f}_x \\ \bar{f}_y \end{bmatrix}$$

which is a perturbation of (41.1) and has well-defined solutions even if A is rank deficient.

A second crucial implication of the augmented-Lagrangian connection hinges on this fact: the augmented-Lagrangian algorithm, applied to LPs, is finitely convergent. We can therefore deduce the convergence of a primal-regularized or dual-regularized version of the interior-point algorithm that we propose. In particular, this connection allows us to lay the groundwork for proving fast asymptotic convergence of our algorithm. This is a refinement of the analysis carried out by Setiono who, for a similar algorithm, establishes a linear convergence rate only.

Generic regularizations. Before arriving at BIRS, we were only considering regularizations based on the ℓ_2 norm. But the connections to the augmented Lagrangian method that unfolded during our time at BIRS suggested that we should also consider much more general regularization functions. In essence, we could leverage much of the powerful analysis of augmented Lagrangian methods applied to general convex optimization problems.

We therefore began to consider the broad class of LP regularizations

$$\text{minimize}_x \quad c^T x + \phi(x) \quad \text{subject to} \quad Ax = b,\ x \geq 0, \tag{41.4}$$

where ϕ was any function that satisfied

$$\phi(x) - \phi(y) = \frac{\gamma}{2}\|x-y\|^2 + O(\|x-y\|^\alpha) \quad \text{for any} \quad \alpha > 1, \tag{41.5}$$

for some positive constant γ. This requirement of ϕ is reminiscent of the quadratic growth condition that has recently received much attention in the context of constraint qualifications for general nonlinear optimization. We anticipate that our theoretical results will carry over to such functions. In particular, the notion of a Bregman distance—well established in convex programming—seems to possess the desired properties.

Numerical experiments. We implemented a preliminary version of our regularized primal-dual algorithm within the established open-source solver GLPK, which implements both the simplex method and Mehrotra's predictor-corrector method, the latter being a popular variant of the primal-dual interior-point method. We intend to contribute our modification to GLPK back to the optimization community.

The current experiments are based on using a primal-dual ℓ_2 regularization of the classical long-step interior-point method. We have already obtained promising numerical results confirming our theoretical results: that in our regularization scheme, the precise value of the regularization parameter is inconsequential.

For testing purposes, we have selected a test set of degenerate linear programs from the Netlib collection. Those problems have been the center of attention of a large number of algorithms for linear programming and their properties are reasonably well understood by the community. We therefore believe that they form a meaningful test set for our purposes.

Outcome of the Meeting

The convergence theory for primal regularization on either the primal or the dual problem is complete and a realistic large-scale implementation is nearly finalized. Our work at BIRS allowed us to shed light on a few shortcomings of an extension of this theory to the case of simultaneous primal-dual regularizations. In particular, we now believe that while global convergence holds in this case, fast local convergence does not.

We intend to explore the application of our approach, with the necessary adjustments, to more general problem classes, such as convex programs, second-order cone programs and semi-definite programs. The properties that such classes share with linear programs gives good reasons to believe that most results will still hold.

List of Participants

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Chapter 42

Generalized Harish-Chandra Modules of $gl(\infty)$ (06rit072)

Aug 26 – Sep 04, 2006

Organizer(s): Ivan Dimitrov (Queen's University), Ivan Penkov (International University Bremen), Gregg Zuckerman (Yale University)

Due to the absence of Professor Gregg Zuckerman and his replacement by Elizabeth Dan-Cohen, we modified slightly the main topic of our workshop. More precisely, we decided to concentrate on the following two subtopics of the general topic "Generalized Harish-Chandra modules of $gl(\infty)$":

(i) highest weight $gl(\infty)$-modules, existence of maximal submodules in Verma modules induced from general locally finite subalgebras of $gl(\infty)$, and the problem of describing the conjugacy classes of maximal locally finite subalgebras of $gl(\infty)$.

(ii) a geometric realization of highest weight modules of diagonal direct limit Lie algebras as higher cohomology groups of line bundles on flag ind-spaces.

As a work-mode we chose "working sessions" (we did not hold formal talks due to the small size of our group). Topic (i) was a joint topic for the three of us, while topic (ii) was a working topic for Ivan Dimitrov and Ivan Penkov only, as it was part of a joint paper in progress.

Review of our work on topic (i). One of the purposes of the workshop was to introduce our junior participant, Elizabeth Dan-Cohen to some problems which have arisen in the joint work of the two senior participants. The first and most important topic was the existence of unique maximal submodule of Verma module induced from a general locally solvable subalgebra of $gl(\infty)$. The three of us shared all the information which is available on the subject. Roughly speaking, the status quo is that Dimitrov and Penkov have a sufficient condition for the existence of such a submodule, based on results of [DP1], and [BB]; however it is not known whether such a unique submodule exists if the condition is not satisfied. Our general feeling is that there should exist a counterexample to the statement for sufficiently general maximal locally solvable subalgebras, and this was suggested to Dan-Cohen as an excellent thesis problem.

We also discussed in detail the conjugacy problem for maximal locally solvable subalgebras of $gl(\infty)$. Here Dan-Cohen was in leading position because of her experience in the solution of the analogous problem for Cartan subalgebras, [DPS]. We were able to obtain a first partial result: in the special case of a maximal closed generalized flag with exactly one predecessor-successor pair of codimension 1 which is not strongly closed, [DP2], we were able to show that the stabilizers of all such generalized flags are conjugate. This is an encouraging first result in a difficult problem.

Finally, another problem we discussed was the existence of opposite maximal locally solvable subalgebras. Opposite maximal locally solvable subalgebras would be useful in studying Verma modules, as in the case of finite dimensional Lie algebras. Based on our discussion, we modified our conjectured definition for one maximal locally solvable subalgebra to be opposite to another. Oddly enough, the definition is not symmetric, as it uses the notion of a spanning toral subalgebra inside one of the maximal locally solvable subalgebras. While there are many nonobvious examples of maximal locally solvable subalgebras, the existence of an opposite maximal locally solvable subalgebra is not known in general.

Review of our work on topic (ii). Dimitrov and Penkov had been working for several years on the Bott-Borel-Weil theorem for diagonal ind-groups. The main challenge has been to prove that any higher cohomology group $H^i(G/B, \mathcal{L})$ is isomorphic to the algebraic dual of a B-highest weight module. Here $G = \varinjlim G_n$ is a locally simple diagonal ind-group, $B = \varinjlim B_n$ is the direct limit of Borel subgroups with $B_n \cap G_{n-1} = B_{n-1}$, and $\mathcal{L}$ is a G-equivariant line bundle on G/B. In 2005 there had been a significant advance in the problem: through a geometric construction the problem was reduced to a combinatorial problem which was expected to be solvable by the method of [DPW]. Earlier in 2006, after the attempt to make a first draft of the paper, it was discovered that the desired reduction to [DPW] can be carried out only in a particular case, and that the general combinatorial question is therefore open. During this meeting, we were able to achieve a decisive breakthrough in this problem. Our continued work (after the workshop) has showed that indeed the problem is solved and that we have a Bott-Borel-Weil theorem for locally simple diagonal ind-groups. The joint paper of Dimitrov and Penkov is currently in its final stages of preparation.

In summary, we have had a much needed workshop which has stimulated the development of the subject in several different directions. We would like to thank the entire staff at BIRS for their outstanding support.

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Chapter 43

Second duals of measure algebras (06rit316)

Sep 09 – Sep 16, 2006

Organizer(s): Anthony To-Ming Lau (University of Alberta)

Overview of the Field

Let A be a Banach algebra. Then there are two natural products on the second dual A'' of A arising from left and right translations by elements of A; they are called the *Arens products*; we denote these products by $\Box$ and $\Diamond$, respectively. For definitions and discussions of these products, see [2], [4], and [5], for example. We briefly recall the definitions. For $a \in A$, $\lambda \in A'$, and $\Phi \in A''$, define $\lambda \cdot a$ and $a \cdot \lambda$ in A' by

$$\langle b, \lambda \cdot a\rangle = \langle ab, \lambda\rangle , \quad \langle b, a \cdot \lambda\rangle = \langle ba, \lambda\rangle \quad (b \in A) ,$$

and then define $\lambda \cdot \Phi \in A$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a, \lambda \cdot \Phi\rangle = \langle \Phi, a \cdot \lambda\rangle , \quad \langle a, \Phi \cdot \lambda\rangle = \langle \Phi, \lambda \cdot a\rangle \quad (a \in A) .$$

Finally, for $\Phi, \Psi \in A''$, define $\langle \Phi \Box \Psi, \lambda\rangle = \langle \Phi, \Psi \cdot \lambda\rangle \ (\lambda \in A')$, and similarly for $\Diamond$. The *left topological centre* of A'' is defined by

$$\mathfrak{Z}^{(\ell)}(A'') = \{\Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in A'')\} ,$$

and similarly for the *right topological centre* $\mathfrak{Z}^{(r)}(A'')$. The algebra A is *Arens regular* if $\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A''$ and *strongly Arens irregular* if $\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A$. For example, every C^*-algebra is Arens regular [2].

There has been a great deal of study of these two algebras, especially in the case where A is the group algebra $L^1(G)$ for a locally compact group G. Results on the second dual algebras of $L^1(G)$ are given in [1], [7], [16], [17], [18], and [19], for example; a full proof that $L^1(G)$ is always strongly Arens irregular was first given in the case where G is compact in [16], and then in the general locally compact case in [18].

More recently, the three participants [5] have studied the second dual of a semigroup algebra; here S is a semigroup, and our Banach algebra is $A = (\ell^1(S), \star)$. We see that the second dual A'' can be identified with the space $M(\beta S)$ of complex-valued, regular Borel measures on βS, the Stone–Cech compactification of S. It can be shown that $(\beta S, \Box)$ is itself a subsemigroup of $(M(\beta S), \Box)$; properties of the latter algebra are intimately related to those of the semigroup $(\beta S, \Box)$, a subtle and much-studied mathematical object, even in the case where S is the obvious semigroup $(\mathbb{N}, +)$ [14].

Let G be a locally compact group. The measure algebra $M(G)$ of G has also been much studied (see [13], [23], [2], for example). This algebra is the multiplier algebra of the group algebra $L^1(G)$. Even in the case where G is the circle group $\mathbb{T}$, the Banach algebra $M(G)$ is very complicated; its character space is 'much larger' than the dual group $\mathbb{Z}$ of $\mathbb{T}$ [12].

Recent Developments and Open Problems

Let A be a Banach algebra which is strongly Arens irregular, and let V be a subset of A''. Then V is *determining for the topological centre* if $\Phi \in A$ for each $\Phi \in A''$ such that $\Phi \,\Box\, \Psi = \Phi \,\Diamond\, \Psi \ \ (\Psi \in V)$. Recently it has become clear that various 'small' subsets of A'' are determining for the topological centre in the case of some of the above algebras.

For example, it is shown in [5, Chapter 12] that, in the case where S is an infinite, weakly cancellative and nearly right cancellative semigroup (which includes the case where S is a group), there is a subset V of A'' of cardinality 2 that is determining for the topological centre of $\ell^{\,1}(S)''$. Independently, a similar results has recently been proved by Filali and Salmi [8]. An extension of these results to weighted convolution algebras is contained in [4]; for example, it is proved that, if ω is a weight on a countable, infinite group G such that ω is diagonally bounded by c on an infinite subset of G (in the sense of [4], etc.), and if $n \in \mathbb{N}$ with $n > c$, then there is a subset of $\ell^{\,1}(G)''$ of cardinality n that is determining for the topological centre.

We now consider which subsets of A'' are determining for the topological centre in the case where $A = L^1(G)$ for a locally compact (non-discrete) group G. The spectrum Φ of $L^\infty(G)$ is naturally a subset of the space A''. A theorem contained within [16] shows (in our terminology) that Φ is determining for the topological centre of A'' whenever G is a compact group. A different approach to this topological centre problem has been recently given by Neufang in [21]. Here it is shown that a certain family of Hahn–Banach extensions of the elements of G, regarded as characters on $LUC(G)$, are determining for the topological centre of A''.

The question whether or not the Banach algebra $M(G)$ is strongly Arens irregular for each locally compact group was raised in [9]. Some related results are given in [10], where it is shown that, in the case where G is comapct, $M(G)''$ uniquely determines G.. The main question was resolved positively for non-compact groups G satisfying certain cardinality conditions by Neufang in [22]. Our proposal stated that we planned to study the Banach algebra $M(G)$, and in particular seek to show that $M(G)$ is strongly Arens irregular for each compact group G.

Presentation Highlights

Since this was a workshop for three people assembled for 'Research in teams', there were no formal presentations.

Scientific Progress Made

We made progress in two related areas.

First, let $A = L^1(G)$ for a locally compact group G, and let Φ be the spectrum of L^∞. We now know that Φ is determining for the topological centre of A'' for each locally compact group G. This appears to give a shorter proof of the fact that A is always strongly Arens irregular than was known before. Indeed, we have proved that various subsets of Φ are determining for the topological centre, but we cannot yet say exactly which subsets of Φ have this property.

Second, consider the measure algebra $M(G)$. We see that the second dual of $M(G)$ is naturally presented as a space $M(\widetilde{G})$ of measures on a certain hyperstonean space $\widetilde{G}$, and we have characterized $M(G)$ as the space of normal measures in $M(\widetilde{G})$, essentially as in [6]. Our approach to these matters seems to be somewhat different from and more direct than that of earlier work, and allows us to identify easily important subsets of $\widetilde{G}$. We are studying which subsets of $\widetilde{G}$ are semigroups with respect to the map $\Box$. We have various partial results; in particular, we have identified various subsets which are semigroups, and, by using the *spine* of a group algebra (see [23] and [15]), we have proved that, for many compact groups G, $\widetilde{G}$ is not a semigroup; this was previously known [20] for all non-compact groups G.

Outcome of the Meeting

The three participants have written a draft paper on the matters described above; we expect to submit it for publication when more complete results are achieved.

Lau will visit the other two authors in England (supported by a grant from the London Mathematical Society) in November 2007; we hope to make further progress on this paper during the visit.

Dales will be a *PIMS Distinguished Visiting Professor* in Edmonton in November/December 2007; we hope to continue our joint work at that time.

List of Participants

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Chapter 44

Classification of smooth 4-manifolds (06rit320)

Oct 07 – Oct 21, 2006

Organizer(s): Ronald Fintushel (Michigan State University), Ronald Stern (University of California, Irvine)

Overview of the Field

Despite spectacular advances in defining invariants for simply-connected smooth and symplectic 4-dimensional manifolds and the discovery of important qualitative features about these manifolds, we seem to be retreating from any hope to classify simply-connected smooth or symplectic 4-dimensional manifolds. The subject is rich in examples that demonstrate a wide variety of disparate phenomena. Yet it is precisely this richness which, at the time of our work at BIRS, gives us little hope to even conjecture a classification scheme.

Recent Developments and Open Problems

Below is a list of operations that are effective constructing and altering the smooth structure on a given 4-dimensional smooth manifold. The open problem is to determine if this is the complete list.

Surgery on a torus.; This operation is the 4-dimensional analogue of Dehn surgery. Assume that X contains a homologically essential torus T with self-intersection zero. Let N_T denote a tubular neighborhood of T. Deleting the interior of N_T and regluing $T^2\times D^2$ via a diffeomorphism $\varphi : \partial(T^2\times D^2)\to\partial(X-\mathrm{int}\, N_T)=\partial N_T$ we obtain a new manifold X_φ, the result of surgery on X along T. The manifold X_φ is determined by the homology class $\varphi_*[\partial D^2]\in H_1(\partial(X\smallsetminus N_T);\mathbf{Z})$. Fix a basis $\{\alpha,\beta,[\partial D^2]\}$ for $H_1(\partial(X\smallsetminus N_T);\mathbf{Z})$, then there are integers p, q, r, such that $\varphi_*[\partial D^2]=p\alpha+q\beta+r[\partial D^2]$. We sometimes write $X_\varphi=X_T(p,q,r)$. It is often the case that $X_\varphi=X_T(p,q,r)$ only depends upon r, e.g. T is contained in a cusp neighborhood, i.e. α and β can be chosen so that they bound vanishing cycles in $(X-\mathrm{int}\, N_T)$. We will sometimes refer to this process as a *generalized logarithmic transform* or an *r-surgery* along T.

If the complement of T is simply connected and $t(X)=1$, then $X_\varphi=X_T(p,q,r)$ is homeomorphic to X. If the complement of T is simply connected and $t(X)=0$, then X_φ is homeomorphic to X if r is odd, otherwise X_φ has the same c and χ_h but with $t(X_\varphi)=1$.

Knot surgery. This operation is the 4-dimensional analogue of sewing in a knot complement along a circle in a 3-manifold. Let X be a 4-manifold which contains a homologically essential torus T of self-intersection 0, and let K be a knot in S^3. Let $N(K)$ be a tubular neighborhood of K in S^3, and let $T\times D^2$ be a tubular neighborhood of T in X. Then the knot surgery manifold X_K is defined by

$$X_K = (X \smallsetminus (T \times D^2)) \cup (S^1 \times (S^3 \smallsetminus N(K))$$

The two pieces are glued together in such a way that the homology class $[\mathrm{pt} \times \partial D^2]$ is identified with $[\mathrm{pt} \times \lambda]$ where λ is the class of a longitude of K. If the complement of T in X is simply connected, then X_K is homeomorphic to X.

Fiber sum. This operation is a 4-dimensional analogue of sewing together knot complements in dimension 3, where a knot in dimension 4 is viewed as an embedded surface. Assume that two 4-manifolds X_1 and X_2 each contain an embedded genus g surface $F_j \subset X_j$ with self-intersection 0. Identify tubular neighborhoods N_{F_j} of F_j with $F_j \times D^2$ and fix a diffeomorphism $f : F_1 \to F_2$. Then the fiber sum $X = X_1 \#_f X_2$ of (X_1, F_1) and (X_2, F_2) is defined as $X_1 \smallsetminus N_{F_1} \cup_\varphi X_2 \smallsetminus N_{F_2}$, where φ is $f \times$ (complex conjugation) on the boundary $\partial(X_j \smallsetminus N_{F_j}) = F_j \times S^1$. We have

$$(c, \chi_h)(X_1 \#_f X_2) = (c, \chi_h)(X_1) + (c, \chi_h)(X_2) + (8g - 8, g - 1)$$

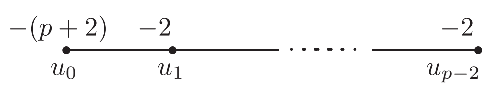
Also $t(X_1 \#_f X_2) = 1$ unless F_j is characteristic in X_j, $j = 0, 1$.

Branched covers. A smooth proper map $f : X \to Y$ is a *d-fold branched covering* if away from the critical set $B \subset Y$ the restriction $f|X \smallsetminus f^{-1}(B) : X \smallsetminus f^{-1}(B) \to Y \smallsetminus B$ is a covering map of degree d, and for $p \in f^{-1}(B)$ there is a positive integer m so that the map f is $(z, x) \to (z^m, x)$ in appropriate coordinate charts around p and $f(p)$. The set B is called the *branch locus* of the branched cover $f : X \to Y$. In the case of *cyclic* branched covers, i.e. when the index-d subgroup $\pi_1(X \smallsetminus f^{-1}(B)) \subset \pi_1(Y \smallsetminus B)$ is determined by a surjection $\pi_1(Y \smallsetminus B) \to \mathbf{Z}_d$, and B is a smooth curve in Y, then $e(X) = d\, e(Y) - (d-1)\, e(B)$ and $\sigma(X) = d\, \sigma(Y) - \frac{d^2-1}{3d} B^2$, and it follows that

$$(c, \chi_h)(X) = d(c, \chi_h)(Y) - (d-1)e(B)(2, \frac{1}{4}) - \frac{(d^2-1)}{3d} B^2(3, \frac{1}{4})$$

Blowup. This operation is borrowed from complex geometry. Form $X \# \overline{\mathbf{CP}}^2$, where $\overline{\mathbf{CP}}^2$ is the complex projective plane $\mathbf{CP}^2$ with the opposite orientation.

Rational blowdown. Let C_p be the smooth 4-manifold obtained by plumbing $p-1$ disk bundles over the 2-sphere according to the diagram



Then the classes of the 0-sections have self-intersections $u_0^2 = -(p+2)$ and $u_i^2 = -2$, $i = 1, \ldots, p-2$. The boundary of C_p is the lens space $L(p^2, 1-p)$ which bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X then the rational blowdown manifold $X_{(p)}$ of [FS1] is obtained by replacing C_p with B_p, i.e., $X_{(p)} = (X \smallsetminus C_p) \cup B_p$.

Connected sum. Another operation is the *connected sum* $X_1 \# X_2$ of two 4-manifolds X_1 and X_2. We call a 4-manifold *irreducible* if it cannot be represented as the connected sum of two manifolds except if one factor is a homotopy 4-sphere. Keep in mind that we do not know if there exist smooth homotopy 4-spheres not diffeomorphic to the standard 4-sphere S^4 and that we have very little understanding of the uniqueness of connect sum decompositions of a reducible 4-manifold.

Prior to this meeting we understood that knot surgery on a given smooth 4-manifold X was obtained via a sequence of logarithmic transformations on null-homologous tori in X. A problem considered during this meeting, but not resolved, was to determine if two homeomorphic simply-connected smooth 4-manifolds are related via a sequence of log transformations on null-homologous tori.

Scientific Progress Made during the Research in Teams meeting

As a focal point for the start of our our meeting we concentrated on smooth 4-manifolds with small Euler characteristic. In the past few years there has been significant progress on the problem of finding exotic smooth structures on the manifolds $P_m = \mathbf{CP}^2 \# m\,\overline{\mathbf{CP}}^2$. The initial step was taken by Jongil Park, [P], who found the first exotic smooth structure on P_7, and whose ideas renewed the interest in this subject. Peter Ozsvath and Zoltan Szabo proved that Park's manifold is minimal [OS], and Andras Stipsicz and Szabo used a technique similar to Park's to construct an exotic structure on P_6 [SS]. Shortly thereafter, the organizers of this meeting produced a new method for finding infinite families of smooth structures on P_m, $6 \le m \le 8$ [FS3], and Park, Stipsicz, and Szabo showed that our techniques can be applied to the case $m = 5$ [PSS].

One goal of this meeting was to better understand the underlying mechanism which produces infinitely many distinct smooth structures on P_m, $5 \le m \le 8$. All these constructions start with the elliptic surface $E(1) = P_9$, perform a knot surgery using a family of twist knots indexed by an integer n [FS2], then blow the result up several times in order to find a suitable configuration of spheres that can be rationally blown down [FS1] to obtain a smooth structure on P_m that is distinguished by the integer n. During this meeting we explained how this can be accomplished by surgery on nullhomologous tori in a manifold R_m homeomorphic to P_m, $5 \le m \le 8$. In other words, we found a nullhomologous torus Λ_m in R_m so that $1/n$-surgery on Λ_m preserves the homeomorphism type of R_m, but changes the smooth structure of R_m in a way that depends on n. Presumably, R_m is diffeomorphic to P_m, but we have not yet been able to show this in general. Our hope is that by better understanding Λ_m and its properties, one will be able to find similar nullhomologous tori in P_m, for $m < 5$.

In addition we developed a technique to construct interesting 4-manifolds called *reverse engineering*. The idea here is to start with a smooth 4-manifold X with non-trivial Seiberg-Witten invariants and with non-trivial first betti number. In the case of complex surfaces, such manifolds are called *irregular* surfaces. The goal would then be to find tori with trivial normal bundle with the property that the inclusion induced homomorphism on H_1 has kernel at most **Z**. In this case there is a log transform on this torus that results in a manifold X' that has betti number one less than that of X. We showed that X' has infinitely many distinct smooth structures. As a test, we applied this construction to the product of a genus two surface with itself and the symmetric product of a genus three surface. In the first case there results infinitely many distinct smooth manifolds with the same integral homology as $S^2 \times S^2$ and in the second case infinitely many distinct smooth manifolds with the same integral homology as P_3.

Concerning the problem to determine if two homeomorphic simply-connected smooth 4-manifolds are related via a sequence of log transformations on null-homologous tori we made further progress and developed a new surgical technique to alter smooth structures that will be developed in further work of the organizers.

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Summer School Reports

Chapter 45

PIMS/UNAM Algebra Summer School (06ss100)

Jul 01 – Jul 06, 2006

Organizer(s): Alejandro Adem (PIMS), James Carrell (University of British Columbia), Jose Antonio de la Pena (Universidad Nacional Autonoma de Mexico)

The objective of the summer school was to bring together researchers working in algebrac topics from the PIMS universities and UNAM in order to help establish collaborations. A number of expository lectures were given, on a wide range of topics covering representation theory, algebraic geometry, number theory and combinatorics. These talks helped establish a common language and a convergence of themes during the workshop, to the benefit of the students and postdocs who attended. The research talks were lively and of very high quality, leading to plenty of discussions.

A number of very useful interactions arose from the meeting, and there are tentative plans for organizing a sequel in Mexico. The following specific research topics were covered at the workshop:

- **Non-commutative blowing up and representations of** $SL(2, C)$
- **Schur algebras for classical groups**
- **Moduli spaces as a classification tool in the representation theory of finite dimensional algebras**
- **Preprojective algebras and cluster algebras**
- **Decomposition of modules over rings with several objects**
- **Quantized coordinate rings**
- **Quantum projective 3-spaces**
- **Donaldson-Thomas type invariants via microlocal geometry**
- **The total coordinate ring of a normal projective variety**
- **Constructing examples of automorphic representations**
- **Relative Gorenstein dimensions and stable equivalence**
- **Derived endo-discrete artin algebras**
- **Monomial Rees algebras and the Mengerian property**
- **Primitivity in twisted homogeneous coordinate rings**

- **Arithmetical invariants of algebraic cycles**
- **Deciding the existence of rational points on curves**
- **Artin-Schelter regular algebras and categories**

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