Banff International Research Station Proceedings 2004

PIMS

MSRI
Mathematical Sciences Research Institute

MITACS

Contents

Five-day Workshop Reports

Chapter 1

Interactions Between Model Theory and Geometry (04w5534)

March 14–18, 2004

Organizer(s): Jan Denef (Katholieke Universität Leuven), Deirdre Haskell (McMaster University), Ehud Hrushovski (Hebrew University), Angus Macintyre (University of Edinburgh), Anand Pillay (University of Illinois, Urbana-Champaign), Patrick Speissegger (McMaster University)

Model theory is in a period of rich activity. Advances in pure model theory are finding immediate applications, in particular to the model theory of fields, while the applications are themselves motivating the abstract developments. Applied model theory is using ideas and methods from other parts of mathematics, ranging from homology theory to complex analytic geometry. These two strands of research were exhibited at the BIRS workshop. The workshop was used as an opportunity to exhibit and elucidate two large pieces of technical work which have been in the process of development for several years. The first of these, "Homology in o-minimal theories" is a prime example of the way in which a mathematical tool can be developed to apply in a wider model-theoretic context. The second tutorial, "Imaginaries in valued fields" illustrates the way in which an applied context (in this case, an algebraically closed valued field) has motivated a theoretical development (the notion of stable domination). The three afternoon sessions presented recent research developments in three different active areas of research in applied model theory.

A. Berarducci: Tutorial on o-minimality

The principal aim of the tutorial was to describe the "transfer approach" in the study of definable groups and definable manifolds in an o-minimal structure, where by "transfer approach" I essentially mean a reduction to the classical (i.e. locally compact) case. The 5 talks of the tutorial focused in particular on work in collaboration with M. Otero in this direction, as well as on related results and conjectures by other authors, which I describe in the sequel.

The definable sets in any o-minimal structure M admit a cell decomposition analogous to the cylindric cell decomposition for semialgebraic subsets of $\mathbb{R}^n$ with the difference that cells are bounded by graphs of definable continuous functions rather than semialgebraic functions. The analogy with semialgebraic geometry may however be misleading, since in general an o-minimal structures may not admit an easy description of its definable continuous functions (hence of its definable sets). The problem is the lack of quantifier elimination theorems or model completeness theorems.

We are interested in definable groups in an o-minimal structure M, namely groups whose underlying set is definable and with a definable group operation. (We will assume in the sequel that M expands a field, although for many results this assumption is not necessary.) Such groups have been studied by many authors and the results so far obtained suggest a close analogy between definable groups and classical Lie groups.

In particular it follows from the results of A. Pillay [26] that if the underlying set of the o-minimal structure is the real line, then every definable group is indeed a Lie group, while Y. Peterzil, A. Pillay and S. Starchenko [24] obtained matrix representation theorems for definable groups which confirm the analogy with Lie groups. Despite these results, many fundamental questions on definable groups remain open. The difficulty is that many tools available in the study of classical Lie groups are not available in the o-minimal context, mainly due to the fact that definable groups and definable manifolds may fail to be locally compact (if the structure M is not locally compact).

This motivates the search for "transfer theorems" which allow, for various problems, a reduction to the case when the underlying set of M is the real line.

A considerable amount of work in this direction was motivated by the following problem. It is known that every compact Lie group has torsion elements (namely elements of finite order) and Peterzil and Steinhorn [25] asked whether the same holds for every "definably compact" group (such groups need not be compact in the classical sense). It was known (Strzebonski [31]) that if a prime p divides the "o-minimal Euler characteristic" $E(G)$ of a definable group G, then G has an element of order p. (The o-minimal Euler characteristic of a set is defined as the number of its even dimensional cells minus the number of its odd dimensional cells, with respect to any given cell decomposition.) This and other results by the same author establish a beautiful analogy between finite groups and definable groups, with the o-minimal Euler characteristic playing the role of the cardinality. It follows in particular from the above mentioned result that if $E(G) = 0$ then G has elements of any prime order. On the other hand the analogy with Lie groups suggests the conjecture (now verified) that if G is definably compact then $E(G) = 0$.

After the development of o-minimal homology, initiated in the Ph.D. thesis of A. Woerheide [32], a new notion of Euler characteristic became available in the o-minimal category, defined as the alternating sum of the ranks of the o-minimal homology groups. The two notions of Euler characteristic are not equal in general, but they do coincide for closed and bounded sets (any definably compact group admits an embedding as a closed and bounded subset of M^n for some n). This opened the way for the homology approach in the study of definable groups.

In a paper with M. Otero [1] we obtain some "transfer results" establishing natural isomorphisms which connect the o-minimal versions of the homology groups and the fundamental group with the corresponding classical notions, and we apply the result to show that every definable manifold of dimension not equal to 4 corresponds in a natural way to a classical topological manifold. (We could not eliminate the dimension assumption, but it turned out to be harmless in the intended applications to groups.) Using these results we then obtain [2] a proof that for every definable compact group G we have $E(G) = 0$ (hence G has torsion). A different solution to the same problem had been obtained by M. Edmundo (our work is independent). Other results by Edmundo allow to count the number of torsion elements of a given order and give information on the o-minimal co-homology rings.

Recently Anand Pillay ([27] stated an insightful conjecture which, if solved positively, would greatly clarify the relationship between Lie groups and definable groups. The conjecture states that one can associate in a canonical way to every definable group G a classical Lie group G/G^{00}. Here G^{00} is the smallest "type definable subgroup of bounded index" (whose existence is part of the conjecture) and G/G^{00} has a suitable "logic topology" studied by Lascar and Pillay. The conjecture states in addition that if G is definably compact then G/G^{00} resembles G very closely. In particular the o-minimal dimension of G coincides with the dimension of G/G^{00} as a Lie group and these two groups have the same torsion. The results in[27] establish the conjecture for every definable groups of dimension one and for every definably simple group. Other partial results were obtained in our recent preprint with M. Otero [4]. Thanks to these results and the exchange of ideas and collaborations made possible during the Banff meeting, we now know that the type definable subgroups of bounded index satisfy the DCC (in analogy with the closed subgroups of a Lie group) and therefore G^{00} exists and G/G^{00} is a Lie groups. What remains open is the computation of the dimension of G/G^{00}.

The analogy between definable groups and Lie groups suggests the possibility of defining an analogue of the Haar measure in any definably compact group. With this aim in mind in the preprint [3] we define a finitely additive real valued measure in every o-minimal expansion M of a field and we show that the definable subsets of M^n contained in the finite part of M^n are measurable (if the underlying set of M is the real line one obtains the Jordan measure). This entails that we can indeed define the analogue of the Haar measure for certain definably compact groups, but the general problem remains open.

D. Macpherson: Tutorial on imaginaries in valued fields

A tutorial was presented on recent work on elimination of imaginaries in algebraically closed valued fields, together with extensions and applications. This consisted of 5 lectures, two by Dugald Macpherson, two by Deirdre Haskell, and one by David Lippel. Material from [15], [16] and the survey [17] was presented, together with work from a manuscript in preparation of E. Hrushovski and B. Martin.

A complete first order theory T has *elimination of imaginaries* (e.i.) if, for every $M \models T$, every $n \geq 1$, and every $\emptyset$-definable equivalence relation E on M^n, there are $m \geq 1$ and an $\emptyset$-definable function $f = f_E : M^n \to M^m$ such that, for $x, y \in M^n$, xEy holds if and only if $f(x) = f(y)$. The E-classes are called *imaginaries*. Close model-theoretic study of a structure usually requires one to understand the imaginaries, and e.i. ensures that each imaginary is coded by a tuple in the structure. Elimination of imaginaries can always be guaranteed by adjoining a new sort M^n/E for each n, E as above, to obtain M^{eq}; in the process, however, one can lose sight of definability. An intermediate step is to adjoin to M certain specific, well-understood, sorts from M^{eq}, and prove elimination of imaginaries with these sorts.

Suppose that K is an algebraically closed field equipped with a non-trivial valuation $v : K \to \Gamma$, where Γ is an ordered group. Let $R := \{x \in K : v(x) \geq 0\}$ be the valuation ring of M, $\mathcal{M}$ its maximal ideal, and $k := R/\mathcal{M}$ be the residue field. The model theory of (K, v, Γ) was investigated by Abraham Robinson in [28]. From his work a quantifier elimination can also be obtained, and it can be shown that Γ is o-minimal, k is strongly minimal, and both are stably embedded. However, imaginaries are rather complicated. The main theorem of [15] is that one obtains e.i. after adding, for each $n \geq 1$, two sorts S_n and T_n. Here, S_n is the set of all R-lattices in K^n, that is, free rank n R-submodules of K^n. If $A \in S_n$ then $\mathcal{M}A$ is an R-submodule of A, and T_n is the union, over all $A \in S_n$, of the k-vector spaces $\mathrm{red}(A) := A/\mathcal{M}A$; in particular, there is a map $\pi_n : T_n \to S_n$ whose fibres are n-dimensional k-vector spaces. It is also possible to identify S_n with the coset space $\mathrm{GL}_n(K)/\mathrm{GL}_n(R)$, and to treat T_n similarly. It is known that if all but finitely many of the T_n are omitted, then e.i. no longer holds.

The paper [16] consists of a close model-theoretic study of ACVF, the theory of algebraically closed valued fields with the sorts S_n and T_n (in addition to the 'home sort' K). This theory is certainly not stable (it interprets Γ). However, many ideas from stability theory are applicable. If C is a parameter set, there is a many-sorted structure $\mathrm{Int}_{k,C}$ with a sort for $\mathrm{red}(A)$ for each C-definable lattice A, with all the induced C-definable structure. This structure is ω-stable with elimination of imaginaries, and the elements of any C-definable stable and stably embedded set are coded by tuples from $\mathrm{Int}_{k,C}$.

Certain types over C are 'stable dominated', meaning that their independent extensions are somehow determined by their 'trace' in $\mathrm{Int}_{k,C}$. Some model theory of stable domination is developed in [16] in complete generality, i.e. not specifically for ACVF. In ACVF, it is shown that a type is stably dominated precisely if it is 'orthogonal' to Γ. Given sufficient orthogonality to Γ, all reasonable notions of independence (of which there are several) coincide.

In a recent preprint, Hrushovski and Martin have shown that the p-adic field also has elimination of imaginaries, again with the sorts S_n (but the T_n are not required). The proof uses both the elimination of imaginaries for ACVF, and ideas from its proof, but the result is formulated in greater generality, with many potential applications to further structures. As a consequence, they show that certain power series associated with finitely generated nilpotent groups are rational. If G is such a group, the n-dimensional complex characters χ_1, χ_2 of G are said to be *twist-equivalent* if there is a 1-dimensional complex character ϕ of G such that $\chi_2 = \chi_1 \circ \phi$. By work of Lubotzky and Magid it is known that the number $a_n(G)$ of twist-equivalence classes of n-dimensional irreducible complex characters of G is finite. Hrushovski and Martin show that if p is a prime, the power series $\Sigma_{n \geq 0} a_{p^n} T^n$ is rational. The proof uses some of the methods developed in [14] and extended by du Sautoy, but appears to require an understanding of imaginaries in $\mathbf{Q}_p$.

Many issues arise from the above work: elimination of imaginaries in other valued fields, and their expansions by subanalytic functions or by generic automorphisms or derivations; the structure of definable/interpretable groups and fields; uniformity in p for the p-adic e.i., and uniformity in p for group-theoretic rationality results; further development of the model theory of stable domination. There has been progress on some of these, for example in work by Hrushovski on groups with a stably dominated type, and work by Mellor on imaginaries in real closed valued fields.

In the tutorial, an overview of ACVF was presented. Then aspects of the proof of e.i. were described, with a slightly different treatment due to Lippel. The main ideas of stable domination were sketched, along with some of the independence theory. Finally, it was shown how p-adic e.i. can yield rationality results for finitely generated nilpotent groups.

L. Lipshitz: Session on Rigid Analytic Geometry and p–adic and Motivic Integration

1. Rigid subanalytic sets and rigid analytic quantifier elimination — Leonard Lipshitz

This talk surveyed the current state of knowledge on the subjects in the title. Let K be a complete, algebraically closed valued field. $K^0 = \{x \in K\colon |x| \leq 1\}$. Let $\mathcal{C}$ be a class of analytic function $(K^0)^n \to K,\ n \in \mathbb{N}$. $D\colon K^2 \to K$ is restricted division. $\mathcal{L}_{an}(\mathcal{C})$ is the language of K enriched with symbols for the functions in $\mathcal{C}$ and $\mathcal{L}_{an}^D(\mathcal{C})$ is $\mathcal{L}_{an}(\mathcal{C})$ with D adjoined.

The corresponding (global) semi–analytic (resp. D–semianalytic) subsets of $(K^\circ)^m$ are those defined by quantifier free $\mathcal{L}_{an}(\mathcal{C})$-(resp. $\mathcal{L}_{an}^D(\mathcal{C})$-) formulae. The corresponding subanalytic subsets of $(K^\circ)^m$ are the projections of semi–analytic subsets of $(K^\circ)^{m+n}$.

If K has quantifier elimination in $\mathcal{L}_{an}^D(\mathcal{C})$ (or quantifier simplification (model completeness) in $\mathcal{L}_{an}(\mathcal{C})$) then one obtains a natural theory of subanalytic sets in close analogy to the real and p–adic cases.

Natural classes to consider for $\mathcal{C}$ are

1. $\mathcal{J}$ the strictly convergent power series
2. $\mathcal{O}$ the overconvergent elements of $\mathcal{J}$.
3. $\mathcal{S}$ the separated power series.

• K has QE in $\mathcal{L}_{an}^D(\mathcal{S})$, giving a natural theory of rigid subanalytic sets, including dimension theory and Lojasiewicz inequalities. [18]

• K has quantifier simplification in $\mathcal{L}_{an}(\mathcal{J})$ giving the theorem on the complement for afinoid (i.e. based on $\mathcal{J}$) subanalytic sets. [19]

• K has quantifier elimination in $\mathcal{L}_{an}^D(\mathcal{O})$ giving a natural theory of “strongly” subanalytic sets, which are the images of analytic sets under affinoid proper maps. [29]

• In [12], [13] and [30] it is claimed that K has QE in $\mathcal{L}\mathcal{J}_{an}^D(\mathcal{J})$. A key step (the global flattening theorem) in the proof is incorrect. A counter example to global flattening is given in [21].

2. Henselian Fields with Analytic Structure, Denef-Pas Cell Decomposition, and its Extension to the Analytic Category— Zachary Robinson– (joint work with Leonard Lipshitz and Raf Cluckers) Algebraic cell decomposition for Henselian valued fields has been used by Denef [6] in developing p-adic integration techniques, and more recently by Denef and Loeser [8], and others, in developing the theory of motivic integration. This talk has two parts. The first is an exposition of the algebraic cell decomposition techniques of P. J. Cohen [5], J. Denef [7] and J. Pas [23]. The second part treats the work in progress to extend these methods to handle analytic functions. Here, one must first define carefully the notion of analytic structure on a valued field that may not be complete in a rank one valuation, e.g., a non-standard model of a field that is complete (cf. [9] and [22].) In this setting, Weierstrass Preparation techniques and the completeness of the coefficient ring compensate for the lack of completeness of the domain to yield a suitably rich and general analytic function theory (cf. [20].) One then uses this theory to obtain analogues of classical results of Mittag-Leffler on functions analytic in an annulus over the complex numbers (cf. [10].) This permits a reduction of the problem of analytic cell decomposition to the algebraic case.

3 and 4. A New Framework for Motivic Integration — Raf Cluckers and Francois Loeser In two talks (by R. Cluckers and F. Loeser) a new framework for motivic integration is presented. This is joint work by Cluckers and Loeser. In this new framework, several new concepts, new arguments, new results, and generalised results are introduced. A class of motivic constructible functions is defined, as well as the notion of positive motivic constructible functions, by analogy to a class of p-adic functions one would like to interpolate motivically. For these classes of functions a motivic integral is defined. Essential is that now the integrals may depend on parameters, in other words, given a constructible function, one can integrate some variables out, and end up with a constructible function in the other variables. In the definition of the integrals, no completion process is needed, i.e., no approximation process is used to define the integrals. Instead, a parametrisation process is used where the parameters run over the residue field and the value group (the integers) of the valued field of equicharacteristic 0. Hence, a notion of measure is needed not only on the valued field, but also on the value group (the counting measure), and on the residue field. On the residue field, one takes as measure just formally the class of the set under isomorphisms. When infinite sums occur, we show that these can always be written as geometric power series and hence, their sum exists in certain localisations without needing any completion. The notion of positivity is based on the observation that one can work with semigroups instead of groups, and every element of a semigroup is understood to be positive. We let the motivic measure and the positive constructible functions take values in the semi-Grothendieck group G of isomorphism classes of a certain kind of subsets of vector spaces over the residue field. After inverting additively any element in these semigroups, one can go to Kontsevich's notion of motivic integrals by applying a "forgetful"-morphism from G to (a certain localisation of) the Grothendieck ring of varieties and by completing this. One can also specialise to Denef-Loeser's notion of arithmetic motivic integral by applying the Denef-Loeser map from G to (a certain localisation of) the Grothendieck ring of Chow motives (tensored with the rational numbers) and by completing this. Finally, one can interpolate p-adic integrals for p big enough, i.e., our notion of motivic integrals gives a geometric understanding of p-adic integrals for p big enough. In this framework, a completely general change of variables is obtained. In this generality, a direct image formalism is developed. In the first talk by Cluckers, a general introduction of the new concepts and new framework is given. The notion of positivity, as well as the measures on the valued field, residue field and value group are explained. The proof of change of variables is explained. In Loeser's talk, more exact definitions are given than in the general introduction by Cluckers, the proof of Fubini's theorem is given, and the direct image formalism is explained. The work presented here is available in resume form at the arxiv, and a paper containing all the proofs will be available soon.

T. Scanlon: Session on jets

The portion of the program on jets was organized around the jet space construction and its applications to problems in the model theory of difference and differential fields. Talks in this session were presented by Alexandru Buium, Zoé Chatzidakis, Rahim Moosa, Anand Pillay and Thomas Scanlon.

Alexandru Buium spoke about p-jet spaces of modular curves. A general construction attaches to any smooth scheme X over a p-adic ring a tower of formal schemes called the p-jet spaces of X. When this construction is applied to modular curves a theory emerges that generalizes the theory of p-adic modular forms. Structure theorems can be obtained for the resulting rings of "p-differential" modular forms.

Zoé Chatzidakis spoke about two results related to the jet space methods. First, she reported on work of Bustamante showing that the methods and results of Pillay-Ziegler extend to finite dimensional sets defined in difference-differential fields. Secondly, the jet space methods show that (in appropriate theories) if $\mathrm{SU}(a/c) < \omega$ and $c = \mathrm{Cb}(a/c)$, then $\mathrm{tp}(c/a)$ is internal to the non-locally modular minimal sets. Possible extensions of the result to general supersimple theories were discussed.

Rahim Moosa discussed jet spaces from the point of view developed by Grothendieck in the 1950s. He then discussed how the Campana-Fujiki theorems on complex analytic spaces, which served as precursors to the Pillay-Ziegler theorem, could be understood in this geometric language.

Anand Pillay discussed the category of algebraic D-varieties and algebraic D-groups. Jet space results were used (in joint work with Kowalski) to show that the category of algebraic D-groups has quantifier-elimination. On the other hand, for trivial reasons the category of algebraic D-varieties does not have quantifier elimination. He discussed the issue of finding new "complete" objects among algebraic D-varieties. Some positive answers were given within the context of groups.

Thomas Scanlon spoke about joint work with Moosa and Pillay in which arc spaces are developed for possibly infinite dimensional partial differential varieties to prove a dichotomy theorem for regular types in partial differential fields. Namely, every regular type in a partial differential field is nonorthogonal to some regular generic type of a definable additive group.

C. Steinhorn: Session on o-minimality

O-minimality has been one of the central areas of research in model theory for about twenty years. The wealth of mathematically important examples of o-minimal structures now known combined with the powerful model-theoretic tools that have been developed have led to applications in fields as diverse as representation theory and statistics. The talks in this session were selected to represent a range of topics in and around o-minimality. Starchenko's contribution continues to advance the theme that abelian groups definable in o-minimal structures resemble real Lie groups; Aschenbrenner's and Miller's talks concern new model-theoretic contexts beyond o-minimality in which the definable sets of an ordered structure are still what might be called "tame." The abstracts follow.

Matthias Aschenbrenner, University of Illinois at Chicago. Title: Gaps in H-fields

Abstract: The class of H-fields is a common algebraic abstraction of Hardy fields and of (certain) fields of transseries. In a joint project, Lou van den Dries, Joris van der Hoeven and myself are trying to obtain a model-theoretic understanding of this class. In my talk I will focus on a particularly troublesome phenomenon (gaps) connected with the presence of transexponential elements in Hardy fields.

Chris Miller, Ohio State University Title. Expansions of o-minimal structures by trajectories of definable planar vector fields.

Abstract. An expansion of the real field is said to be o-minimal if every definable set has finitely many connected components. Such structures are a natural setting for studying "tame" objects of real-analytic geometry such as non-oscillatory trajectories of real-analytic planar vector fields. It turns out that even some infinitely spiralling trajectories of such vector fields have a reasonably well-behaved model theory; this motivates the notion of d-minimality, a generalization of o-minimality that allows for some definable sets to have infinitely many connected components. The following trichotomy illustrates why we are interested in this notion. Let $U \subseteq \mathbb{R}^2$ be open and $F\colon U \to \mathbb{R}^2$ be real analytic such that the origin 0 is an elementary singularity of F (i.e., $F^{-1}(0) = \{0\}$ and the Jacobian of F at 0 has a nonzero eigenvalue). Let $g\colon (0,b) \to \mathbb{R}^2$ be a solution to $y' = F(y)$ such that $g(t) \to 0$ as $t \to 0^+$. Then, after possibly shrinking b, the expansion of the real field by the curve $g((0,b))$ either is o-minimal, is d-minimal and not o-minimal, or defines $\mathbb{Z}$.

Sergei Starchenko, University of Notre Dame Title: On torsion free groups definable in o-minimal structures.

Abstract: (Joint work with Y. Peterzil) We consider groups definable in the structure $\mathbb{R}_{an}$ and certain o-minimal expansions of it. We prove: If $\mathbb{G} = \langle G, * \rangle$ is a definable abelian torsion-free group then $\mathbb{G}$ is definably isomorphic to a direct sum of $\langle \mathbb{R}, + \rangle^k$ and $\langle \mathbb{R}^{>0}, \cdot \rangle^m$, for some $k, m \geq 0$. Furthermore, this isomorphism is definable in the structure $\langle \mathbb{R}, +, \cdot, \mathbb{G} \rangle$. In particular, if such $\mathbb{G}$ is semialgebraic, then the isomorphism is semialgebraic. We show how to use the above result to give an "o-minimal proof" to the classical Chevalley theorem for abelian algebraic groups over algebraically closed fields of characteristic zero.

List of Participants

Aschenbrenner, Matthias (University of Illinois, Chicago)
Belair, Luc (Université du Québec à Montréal)
Ben Yaacov, Itay (Massachusetts Institute of Technology)
Berarducci, Alessandro (Università di Pisa)
Bouscaren, Elisabeth (Universite Denis-Diderot Paris 7)
Buium, Alexandru (University of New Mexico)
Chatzidakis, Zoé (CNRS - Université Paris 7)
Cherlin, Gregory (Rutgers, The State University of New Jersey)
Cluckers, Raf (École Normale Supérieure)
Dolich, Alf (University of Wisconsin)
Ealy, Clifton (University of California, Berkeley)
Edmundo, Mario (CMAF-University of Lisbon)
Hart, Bradd (McMaster University)
Haskell, Deirdre (McMaster University)

Hrushovski, Ehud (Hebrew University)
Kowalski, Piotr (University of Illinois, Urbana-Champaign)
Lippel, David (McMaster University)
Lipshitz, Leonard (Purdue University)
Loeser, François (École Normale Supérieure)
Macintyre, Angus (University of Edinburgh)
Macpherson, Dugald (University of Leeds)
Miller, Chris (Ohio State University)
Moosa, Rahim (Massachusetts Institute of Technology)
Mustafin, Yerulan (Universite du Quebec a Montreal)
Peterzil, Yaacov (University of Haifa)
Pillay, Anand (University of Illinois, Urbana-Champaign)
Robinson, Zachary (East Carolina University)
Rolin, Jean-Phillipe (Université de Bourgogne, France)
Scanlon, Thomas (University of California, Berkeley)
Scowcroft, Philip (Wesleyan University)
Singer, Michael (North Carolina State University)
Speissegger, Patrick (University of Wisconsin, Madison, & McMaster University)
Starchenko, Sergei (University of Notre Dame)
Steinhorn, Charles (Vassar College)
Tent, Katrin (Mathematisches Institut der Universität Würzburg)
Tomasic, Ivan (Universite Lyon I)
Wagner, Frank (Universite Claude Bernard Lyon 1)
Wood, Carol (Wesleyan University)
Yaffe, Yoav (McMaster University)

Bibliography

[1] A. Berarducci and M. Otero, O-minimal fundamental group, homology and manifolds, *J. London Math. Soc.* (2) **65** (2002), 1–14.

[2] A. Berarducci and M. Otero, Transfer Methods for O-minimal Topology, *Journal of Symbolic Logic*, (3) **68** (2003), 785–794

[3] A. Berarducci and M. Otero, An additive measure in o-minimal expansions of fields, RAAG Preprint n.66 (11p.; 2003, October 31). To appear in: *Q*uarterly Journal of Mathematics.

[4] A. Berarducci and M. Otero, On a conjecture of Pillay connecting o-minimal groups and Lie groups, RAAG Preprint n.73 (14p.; 2003, December 15).

[5] P. J. Cohen, Decision procedures for real and p-adic fields, *Comm. Pure Appl. Math.*, **22** (1969), 131–151.

[6] J. Denef, The rationality of the Poincaré series associated to the p-adic points on a variety, *Invent. Math.* **77** (1984), no. 1, 1–23.

[7] J. Denef, p-adic semi-algebraic sets and cell decomposition, *J. Reine Angew. Math. 369* (1986), 154–166.

[8] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, *Invent. Math.*, **135** (1999), no. 1, 201–232.

[9] L. van den Dries, Analytic Ax-Kochen-Ershov theorems, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), 379–398, *Contemp. Math.* **131**, Part 3, Amer. Math. Soc., Providence, RI, 1992.

[10] L. van den Dries, D. Haskell and D. Macpherson, One-dimensional p-adic subanalytic sets, *J. London Math. Soc.* (2) **59** (1999), no. 1, 1–20.

[11] M. Edmundo, o-minimal cohomology and definably compact definable groups, Preprint, Dec. 16, 2001, revised Aug. 11, 2003.

[12] T. Gardener, Local flattening in rigid analytic geometry, *Proc. London Math. Soc.* (3) **80** (2000), no. 1, 174–197.

[13] T. Gardener and H. Schoutens, Flattening and subanalytic sets in rigid analytic geometry, *Proc. London math. Soc.* (3) **83** (2001), no. 3, 681–707.

[14] F.J. Grunewald, D. Segal, G.C. Smith, Subgroups of finite index in nilpotent groups, *Invent. Math.* **93** (1988), 185–223.

[15] D. Haskell, E. Hrushovski, H.D. Macpherson, Definable sets in algebraically closed valued fields. Part I: elimination of imaginaries', preprint.
http://www.math.mcmaster.ca/ haskell/index.html.

[16] D. Haskell, E. Hrushovski, H.D. Macpherson, Definable sets in algebraically closed valued fields. Part II: stable domination and independence, preprint.
http://www.math.mcmaster.ca/ haskell/index.html.

[17] D. Haskell, H.D. Macpherson, Definable sets in valued fields, http://www.math.mcmaster.ca/ haskell/index.html.

[18] L. Lipshitz, Rigid subanalytic sets, *Amer. J. Math.* **115** (1993), no. 1, 77–108.

[19] L. Lipshitz and Z. Robinson, Model Completeness and subanalytic sets, *Astérisque* textbf264 (2000), 109–126.

[20] L. Lipshitz and Z. Robinson, Rings of separated power series and quasi-affinoid geometry, *Asterisque* **264** (2000).

[21] L. Lipshitz and Z. Robinson, Flattening and analytic continuation of affinoid morphisms: remarks on a paper of Gardener and Schoutens, URL http://www.math.purdue.edu/∼lipshitz/

[22] L. Lipshitz and Z. Robinson, Uniform properties of rigid subanalytic sets, URL http://www.math.purdue.edu/∼lipshitz/.

[23] J. Pas, Uniform p-adic cell decomposition and local zeta functions, *J. Reine Angew. Math.* **399** (1989), 137–172.

[24] Y. Peterzil, A. Pillay, S. Starchenko, Definably simple groups in o-minimal structures, *Transactions AMS* **352** (2000), 4397–4419

[25] Y. Peterzil and C. Steinhorn, Definable compactness and definable subgroups of o-minimal groups, *J. of London Math. Soc.(2)* **59** (1999) 769–786

[26] A. Pillay, On groups and rings definable in o–minimal structures, *J. Pure and Applied Algebra* **53** (1988), 239-255

[27] A. Pillay, Type–definability, compact Lie groups, and o-minimality, Preprint, September 2003

[28] A. Robinson, *Complete theories*, North-Holland, Amsterdam, 1996.

[29] H. Schoutens, Rigid subanalytic sets, *Compositio Math.* **94** (1994), no. 3, 269–295.

[30] H. Schoutens, Rigid analytic flatificators, *Quant. J.Math. Oxford Ser. (2)* **50** (1995), 321–353.

[31] A. Strzebonski, Euler characteristic in semialgebraic and other o-minimal structures, *J. Pure Appl. Algebra 96* (1994) 173–201.

[32] A. Woerheide, O-minimal homology, PhD Thesis, University of Illinois at Urbana-Champaign, 1996

Chapter 2

Topology of Manifolds and Homotopy Theory (04w5533)

March 20–25, 2004

Organizer(s): Gunnar Carlsson (Stanford University), Ian Hambleton (McMaster University), Erik K. Pedersen (SUNY Binghamton)

The announced purpose of this meeting was to bring together researchers in a wide variety of areas in algebraic and geometric topology, to investigate problems of current interest, and to make new connections. In particular, we hoped for a productive exchange of ideas and viewpoints among mathematicians in active areas of homotopy theory and the topology of manifolds, in order to enrich the future development of both subjects.

There were 37 participants, mostly from universities in the US and Canada, and including 20 young mathematicians (age < 35). Their research interests covered many active areas of our chosen subject, and since the time of the meeting we have received many positive comments from the participants about the breadth and interaction of the program.

The first talk was somewhat philosophical: Matthias Kreck advocated the study of 'real' manifolds occuring 'naturally', such as those related to Lie groups or algebraic varieties. He proposed that simplicity and concreteness should be the characteristics of good problems. This point of view provoked extensive comment and debate throughout the meeting. The closing talk presented the opposite view. Frank Quinn vigorously defended abstraction and the search for unity in complexity. In between of course we had theorems and conjectures of all kinds, as well as a very entertaining problem session. The organizers felt that the most striking feature of our meeting was the productive interaction between the mathematical generations. Two-thirds of the talks were given by the younger researchers.

Here are some of the main topics discussed at the meeting:

1. "Structured homotopy theory", meaning homotopy theory on the category of modules over a ring spectrum (Dundas, Klein, Kitchloo, Mandell, Roendigs)

2. Controlled and equivariant topology, surgery (Crowley, Davis, Friedman, Hughes, Ranicki, Reich, Quinn, Williams)

3. Conjectures of Baum-Connes and Novikov on assembly maps in K-theory and L-theory (Kreck, Lück, Reich, Quinn, Connolly, Goldfarb, Rosenthal)

4. New product structures on the homology of the free loop space of a manifold, moduli spaces (Cohen, Wahl, Ganter)

5. Smoothing of homotopy actions of finite groups on spheres, and free simplicial actions of finite groups on products of spheres (Adem, Davis, Grodal)

6. p-compact groups and the smoothing of finite H-spaces (Bauer, Grodal, Pedersen)

List of Participants

Adem, Alejandro (University of Wisconsini, Madison)
Anderson, Laura (Binghamton University)
Banagl, Markus (University of Cincinnati)
Bauer, Tilman (Westfälische Wilhelms-Universität Münster)
Carlsson, Gunnar (Stanford University)
Cohen, Ralph (Stanford University)
Connolly, Francis X. (University of Notre Dame)
Crowley, Dairmuid (Penn State University)
Davis, James F. (Indiana University)
Dundas, Bjoern (Norwegian University of Science and Technology)
Friedman, Greg (Yale University)
Ganter, Nora (Massachusetts Institute of Technology)
Godin, Veronique (Stanford University)
Goldfarb, Boris (University of Albany)
Grinberg, Anna (University of California, San Diego)
Grodal, Jesper (University of Chicago)
Hambleton, Ian (McMaster University)
Hughes, Bruce (Vanderbilt University)
Kitchloo, Nitu (Johns Hopkins University)
Klein, John (Wayne State University)
Korzeniewski, Andrew (University of Edinburgh)
Kreck, Matthias (University of Heidelberg)
Lawson, Tyler (Stanford University)
Lueck, Wolfgang (Universitat Muenster)
Mandell, Michael A. (University of Chicago)
Nicas, Andrew (McMaster University)
Noohi, Behrang (University of Western Ontario)
Paulo Santos, Joao (Massachusetts Institute of Technology)
Pedersen, Erik (SUNY Binghamton)
Quinn, Frank (Virginia Polytechnic Institute and State University)
Ranicki, Andrew (University of Edinburgh)
Reich, Holger ((Universitat Muenster)
Roendigs, Oliver (University of Western Ontario)
Rosenthal, David (McMaster University)
Scull, Laura (University of British Columbia)
Wahl, Nathalie (Aarhus Universitet)
Williams, Bruce (University of Notre Dame)

Chapter 3

Orthogonal Polynomials; Interdisciplinary Aspects (04w5530)

March 27–April 1, 2004

Organizer(s): Percy Deift (Courant Institute of Mathematical Sciences), Lance Littlejohn (Utah State University), David Sattinger (Yale University), Jacek Szmigielski (University of Saskatchewan)

Introduction

Since 1984, the community of researchers in orthogonal polynomials and special functions have held regular international symposia in Europe. Indeed, these meetings, their venues, and the year of the symposium:

1. First International Symposium on Orthogonal Polynomials, Special Functions and Applications, Bar-Le-Duc, France; 1984.

2. Second International Symposium on Orthogonal Polynomials, Special Functions and Applications, Segovia, Spain; 1986.

3. Third International Symposium on Orthogonal Polynomials, Special Functions and Applications, Evian, France; 1992.

4. Fourth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Delft, The Netherlands, 1994.

5. Fifth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Patras, Greece, 1999.

6. Sixth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Rome, Italy, 2001.

7. Seventh International Symposium on Orthogonal Polynomials, Special Functions and Applications, Copenhagen, Denmark, 2003.

The Eighth International Symposium on Orthogonal Polynomials, Special Functions and Applications is currently in the planning stage and will be held in Munich, Germany in July, 2005. In addition, there have been several large Spanish national conferences that have attracted a large international audience, including Laredo (1987, 1992), Gijon (1989), Vigo (1990), Granada (1991), and Sevilla (1997).

In North America, there has been one large NATO-sponsored meeting on orthogonal polynomials (Columbus, Ohio) and several smaller special sessions of national and regional AMS meetings dedicated to orthogonal polynomials and related topics. The organizers of the BIRS Workshop on Orthogonal Polynomials; Interdisciplinary Aspects felt it was necessary to organize a mid-size conference in this area with a North American venue to further invigorate North American interest as well as to bring the core of researchers in integrable systems closer together to orthogonal polynomials and special functions.

Areas of Participation

The following areas of mathematics were represented at the BIRS Workshop on Orthogonal Polynomials; Interdisciplinary Aspects together with the names of the people who gave seminars in these areas; there were 26 talks (each 50 minutes duration) given during the five days of this workshop by participants from 17 different countries.

1. Lie algebras and Darboux transform theory (Luc Vinet, Francisco Marcellan)
2. Integrable Systems (John Harnad, Hans Lundmark, Mark Adler, Marco Bertola)
3. Moment theory (Christian Berg, Andreas Ruffing)
4. Classical theory of OPS's and Special Functions (Richard Askey, Christian Berg, Lance Littlejohn, Mourad Ismail)
5. Asymptotic theory of orthogonal polynomials (Alexei Borodin, Ken McLaughlin)
6. Spectral theory of Differential Operators (Lance Littlejohn)
7. The Bochner-Krall Classification Problem (Dong Won Lee)
8. Riemann-Hilbert Problems (Percy Deift, Arno Kuijlaars, Marteen Vanlessen, Ken McLaughlin, Jeff Geronimo, Peter Miller)
9. Random Matrices (Percy Deift, Alexei Borodin, Arno Kuijlaars, Marco Bertola)
10. Multiple Orthogonality (Walter van Assche, Jorge Arvesu)
11. q-orthogonal polynomials (Andreas Ruffing, Sergei Suslov, Jorge Arvesu, Natig Atakishiyev)
12. Matrix orthogonal polynomials (Alberto Grunbaum)
13. Multivariate Discrete Orthogonal Polynomials (Yuan Xu)
14. Orthogonal Polynomials and Special Functions in Mathematical Physics (Mourad Ismail, Anatol Odzijewicz, Mark Adler)

In addition to the participants listed above, Professor Madan Mehta (Service de Physique Theorique) presented some interesting open problems in the area and there were three graduate students in attendance at the workshop: Keivan Mohajer (University of Saskatchewan), Davut Tuncer (Utah State University), and Tomohiro Takata (Kyoto University).

Orthogonal Polynomials: General Overview

The last thirty years has seen a remarkable rebirth of research in the theory of special functions and orthogonal polynomials. Essentially invigorated by Richard Askey's series of NSF lectures [1] at Virginia Tech in 1974, research in various areas of orthogonal polynomials and their applications has kept a steady pace ever since. As a result, key contributions in these subjects during this time period have come from researchers in more than thirty countries.

In the classical theory of orthogonal polynomials in the real variable x, a sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials, where the degree of each p_n is exactly n, is said to be an orthogonal polynomial sequence with respect to the Borel measure μ (possibly signed) if

$$\int_{-\infty}^{\infty} p_n p_m d\mu = k_n \delta_{n,m} \quad (k_n \neq 0; n, m \in \mathbb{N}_0).$$

If the measure μ is positive, this is the positive-definite case, then each $k_n > 0$; it is particularly this case that several classical textbooks have been written, most notably by Szegö [20] and Chihara [6]. Standard topics in these books include a detailed and fundamental general theory of orthogonal polynomials with several detailed examples ranging from discrete measures (e.g. Charlier, Krawtchuck), continuous measures (e.g. the classical orthogonal polynomials of Jacobi, Laguerre, Hermite), and signed measures (Bessel polynomials). Furthermore, topics in these books include an in depth discussion of the three-term recurrence relation for orthogonal polynomials, connections to the theory of continued fractions and the theory of moments, asymptotic properties of the classical orthogonal polynomials and their zeros, inequalities associated with the classical orthogonal polynomials, various characterizations of the classical orthogonal polynomials, approximation and expansion properties, and various classical applications of orthogonal polynomials to numerical analysis and mechanical quadrature methods as well as to the theory of electostatics. Moreover, complex orthogonal polynomials are discussed in [19].

The past thirty years may, arguably, be called the golden years of orthogonal polynomials. Indeed, the topics listed above have been greatly enriched and generalized and, furthermore, the subject has witnessed the openings of several new avenues of research. Several key results and open problems have fallen during this time period. Indeed, among many results, we mention that a key result leading to the solution of the Bieberbach conjecture involved a new inequality involving certain Jacobi polynomials. Major developments in asymptotic theory have been made during this period, including the solution of the Freud conjecture. The long standing Bessel moment problem was also elegantly solved during this period. The electrostatic interpretation of the roots of a large class of orthogonal polynomials has also been achieved in recent years.

This time period also saw the important development of the theory of q-orthogonal polynomials and basis hypergeometric series leading to the development of the Askey tableau in orthogonal polynomials and a new meaning for 'classical' orthogonal polynomials. These q-orthogonal polynomials and new special functions have proved useful in both theory and applications to mathematical physics and statistical mechanics.

Interpreting certain problems in terms of Riemann-Hilbert problems has led to some impressive new results, as well as some powerful new tools, in the asymptotic theory of orthogonal polynomials; further remarks on this and connections to the theory of random matrices are described below in the next section.

Sobolev orthogonality, that is orthogonality with respect to bilinear forms of the type

$$(p, q)_M := \sum_{k=0}^{M} \int_{\mathbb{R}} p^{(k)} \overline{q}^{(k)} d\mu_k,$$

made significant strides during this period as has the theory of matrix orthogonal polynomials. Key advances have also been made in the past fifteen years to the characterization of orthogonal polynomial sequences to differential equations, in particular to the so-called $BKS(N, M)$ classification problems, and to the spectral theory of the associated operators. A sequence of polynomials $\{p_n\}$ belongs to the $BKS(N, M)$ class if $\{p_n\}$ is orthogonal with respect to a bilinear form of the type $(\cdot, \cdot)_M$ above and each p_n is an eigenfunction of a fixed differential equation of order N. Left-definite spectral analysis, whose roots can be traced to fundamental work of Hermann Weyl, has proved promising in establishing new orthogonality results for polynomials in the $BKS(N, M)$ sets as well as helped to obtain new characterization results for this class of polynomials. Development of the theory of Darboux transforms has also shed considerable light on the $BKS(N, M)$ classification problem and given us a powerful tool for constructing new orthogonal polynomials in these classes.

During the past few years, significant advances in the theory of multiple orthogonal polynomials have been made with some very interesting, and important applications including a new proof of the irrationality of $\zeta(3)$; there is much hope that further classical results and new applications to number theory will be found using the tools that have recently been developed in this area. The theory of multivariable orthogonal polynomials or orthogonal polynomials in several variables, with respect to discrete and continuous measures, is a subject of much recent interest and significant developments.

The last few years has seen a new group of researchers, namely people in integrable systems, enter the orthogonal polynomial scene. As further described below, several integrable systems can be solved using orthogonal polynomials and moment theory. One of the central reasons for organizing this BIRS meeting was to bring this group together with a core of researchers in orthogonal polynomials for an exchange of ideas and for further collaboration. To this end, this meeting was hugely successful!

Riemann Hilbert Problems and Orthogonal Polynomials

The introduction of random matrices to theoretical physics dates back to Wigner in the 1950's, and was motivated by the attempt to explain resonances in the scattering of slow neutrons off heavy nuclei. Physical observations made it clear that a statistical theory was needed to explain the events. A particularly salient feature of the data was the "repulsion of energies" – on average, the resonance energies stayed "far" apart. In earlier work with von Neumann in 1927, Wigner had shown that the degeneracy of the eigenvalues of a Hermitian matrix was a "rare" event (of co-dimension 2), and this led Wigner to suggest the eigenvalues of matrices distributed according to some probabilistic law as an appropriate statistical model for the resonance energies. It was believed that the phenomenon of resonance scattering was universal, subject only to some general symmetry restrictions, and so the statistics should be largely independent of the details of the theory. Early work in the theoretical physics community on the theory of random matrices was due to Wigner, Dyson, as well as Mehta and Gaudin.

The theory of random matrices is concerned with the distribution of the eigenvalues of ensembles of matrices distributed according to some probability measure. The Unitary Ensembles (UE's) consist of $N \times N$ Hermitian matrices with the measure

$$\frac{1}{Z_N} e^{-N \mathrm{Tr}\, V(M)} dM, \tag{3.1}$$

where M is an $N \times N$ Hermitian matrix; dM denotes the Lebesgue product measure on the (algebraically independent) entries of M, and V is a real-valued function growing sufficiently rapidly at infinity. The term "Unitary Ensembles" refers to the fact that the measure in (3.1) is invariant under unitary conjugation: $M \to UMU^*$. In the particular case $V(x) = x^2$ the ensemble is called the Gaussian Unitary Ensemble (GUE). The function Z_N is the partition function for the ensemble:

$$Z_N = \int e^{-N \mathrm{Tr} V(M)} dM.$$

The principal goal of random matrix theory is to calculate the basic statistical quantities for the eigenvalues of matrices distributed according to a given probability distribution and to evaluate these quantities in the limit as $N \to \infty$. A statement of the universality of the theory for large N is as follows: Note first that for any $a < b$,

$$\lim_{N\to\infty} \frac{1}{N} \text{Exp} \sharp \{ \text{ eigenvalues in } (a,b)\}$$

exists and equals $\int_a^b \rho(x)dx$ for some $\rho(x) \geq 0$, the so-called density of states. We consider a point $x = x_0$ of positive density, $\rho(x_0) > 0$, and then rescale the matrices so that the expected number of eigenvalues per unit length at x_0 is equal to 1. Then for $\theta > 0$ and for "reasonable" V's (polynomial V's are certainly included!),

$$\lim_{N\to\infty} \Pr \left\{ \text{no eigenvalues in } \left(x_0 - \frac{\theta}{N\rho(x_0)}, x_0 + \frac{\theta}{N\rho(x_0)} \right) \right\} = \det(I - S_\theta), \tag{3.2}$$

where S_θ is the Fredholm integral operator on $L^2(-\theta, \theta)$ with kernel

$$S_\theta(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)},$$

and $\det(I - S_\theta)$ denotes the Fredholm determinant. In other words, under appropriate scaling, the probability that there are no eigenvalues in an appropriately scaled interval is universal, and in particular is independent of the choice of V.

Orthogonal polynomials play a significant role in the theory of Unitary Ensembles. By classical theory, there is a sequence of polynomials $p_{j,N} = \gamma_{jN} x^j + \dots$, $\gamma_{jN} > 0$, $j = 0, 1, 2, \dots$, which are orthonormal with respect to the measure $e^{-NV(x)}dx$, that is,

$$\int_{-\infty}^{\infty} p_{j,N} p_{k,N} e^{-NV(x)} dx = \delta_{jk}, \quad j, k \geq 0,$$

and it turns out that in the basic contributions [17], [16], Gaudin and Mehta computed the probability distribution of the eigenvalues of matrices in UE's directly in terms of these orthogonal polynomials. Set

$$\phi_{j,N}(x) = e^{-\frac{N}{2}V(x)} p_{j,N}$$

and define

$$K_N(x, y) = \sum_{j=0}^{N} \phi_{j,N}(x) \phi_{j,N}(y).$$

By the Christoffel-Darboux identity,

$$K_N(x, y) = e^{-\frac{N}{2}(V(x)+V(y))} \frac{\gamma_{N-1,N}}{\gamma_{N,N}} \frac{p_{N,N}(x) p_{N-1,N}(y) - p_{N,N}(y) p_{N-1,N}(x)}{x - y}, \tag{3.3}$$

where $\gamma_{j,N}$ is the leading coefficient of $p_{j,N}$ as above. The basic statistical quantities for the eigenvalues in the Unitary Ensemble can be computed in terms of $K_N(x, y)$. For example, the m-point correlation function $R_{m,N}$ is given by

$$R_{m,N}(x_1, \dots x_m) = \det ||K_N(x_j, x_k)||_{1 \leq j,k \leq m}; \tag{3.4}$$

and the gap probability for any $a < b$ by,

$$\Pr\{\text{no eigenvalues in } (a, b)\} = \sum_{m=0}^{N} \frac{(-1)^m}{m!} \int_a^b \dots \int_a^b R_{m,N}(x_1, \dots, x_m) dx_1 \cdots dx_m. \tag{3.5}$$

Also, for the density of states we have

$$\rho(x) = \lim_{N \to \infty} \frac{1}{N} K_N(x, x).$$

Universality for the (scaled) kernel K_N takes the form (for $\rho(x_0) > 0$),

$$\lim_{N \to \infty} \frac{1}{N\rho(x_0)} K_N \left(x_0 + \frac{u}{N\rho(x_0)}, x_0 + \frac{v}{N\rho(x_0)} \right) = \frac{\sin \pi(u - v)}{\pi(u - v)},$$

which in turn leads to the V-independent limit (3.2) above, via (3.5).

From (3.3), (3.4), and (3.5) it is clear that the computation of the limiting eigenvalue statistics as $N \to \infty$ requires precise knowledge of the asymptotic behaviour of the associated orthogonal polynomials $p_{j,N}$ as $N \to \infty$. Universality for random matrix theory is thus a direct consequence of the asymptotic properties of the polynomials.

The discovery by Fokas, Its, and Kitaev [14] that the orthogonal polynomials can be obtained as the solution of a Riemann-Hilbert problem, was a significant step toward this goal. The authors in [9], [12] have shown the steepest descent method for Riemann-Hilbert problems to be a powerful tool in obtaining the asymptotic behaviour of orthogonal polynomials, and hence in proving universality (see also [5]). The steepest descent method for Riemann Hilbert problems was introduced by Deift and Zhou in 1993 [11], and further developed together with Venakides [13] to include fully nonlinear oscillations. Inverse scattering problems in the theory of completely integrable systems are often formulated as Riemann-Hilbert problems, and the asymptotic behaviour of solutions to integrable equations has been obtained by the application of the steepest descent method to the associated Riemann-Hilbert problems. (cf. the bibliography in [10].)

The situation for unitary ensembles is now in a fairly satisfactory state, but the proof of universality remains open for many other types of ensembles.

Orthogonal Polynomials and Integrable Systems

A number of discrete integrable systems, such as the Toda flows, [18], [3], multipeakon flows [2], relativistic Toda lattice, [7], [8], can be solved in terms of orthogonal polynomials or orthogonal rational functions. See the bibliography in [4].

We illustrate this approach with a recent example from integrable systems. The Camassa-Holm equation,

$$u_t - \frac{1}{4}u_{xxt} + \frac{3}{2}(u^2)_x - \frac{1}{8}(u_x^2)_x - \frac{1}{4}(uu_{xx})_x = 0,$$

is formally an isospectral deformation of the differential operator $D^2 - zm - 1$, where $2m = (4 - D^2)u$. By a Liouville transformation the eigenvalue problem for this differential equation on the real line is carried into the acoustic spectral problem [2]

$$D^2\varphi = zg(y)\varphi(y,z), \quad -1 \le y \le 1, \quad \varphi(\pm 1, z) = 0. \tag{3.6}$$

The special case of multi-peakons (non-smooth solitons) is obtained when g is a sum of Dirac delta functions:

$$g(y) = \sum_{j=1}^{n} g_j \delta(y - y_j), \qquad -1 < y_1 < \cdots < y_n < 1.$$

The discrete spectral problem associated with (3.6) was studied by Krein [15]. The inverse problem recovers the masses and spacings between the mass points from the spectral data, and is realized succinctly as a direct application of Stieltjes' solution of the classical moment problem [2]. Stieltjes original solution applies to positive masses, but the solution extends immediately to both positive and negative masses.

The solution of the inverse problem is directly related to the construction of a system of orthogonal polynomials relative to a positive measure: Let $\varphi(y,z)$ be a solution of (3.6). The Weyl function is defined by

$$W(z) = \frac{\varphi_y(1,z)}{\varphi(1,z)}.$$

The Weyl function can be constructed knowing the spectral data, that is the eigenvalues and coupling coefficients of the left and right wave functions of (3.6). These give respectively the location of the poles and the associated residues of the Weyl function. The Weyl function can thus be represented as the Stieltjes transform

$$\frac{W(z)}{z} = \int_{-\infty}^{\infty} \frac{1}{z - \lambda} d\mu(\lambda),$$

where $d\mu$ is a positive discrete measure. This positive measure has a sequence of orthogonal polynomials

$$\int_{-\infty}^{\infty} P_j(\lambda,t)P_k(\lambda,t)e^{-2t/\lambda} d\mu(\lambda) = \delta_{jk}.$$

For fixed t the $P_j(\lambda,t)$ satisfy a second order recursion relation

$$\lambda P_j(\lambda,t) = b_j P_{j+1}(\lambda,t) + d_j P_j(\lambda,t) + b_{j-1}P_{j-1}(\lambda,t), \quad 1 \le j \le n-1,$$

where the coefficients b_j are given. The multipeakon solutions may be expressed in terms of the orthogonal polynomials $P_j(0,t)$ as follows:

$$g_j = -\frac{1}{b_{n-j}P_{n-j+1}(0)\, P_{n-j}(0)}$$

$$y_j = 1 + b_{n-j}\Big(P'_{n-j}(0)P_{n-j+1}(0) - P_{n-j}(0)P'_{n-j+1}(0)\Big),$$

where the primes denote differentiation with respect to λ. The impossibility of triple collisions follows directly from the classical Christoffel-Darboux formula in orthogonal polynomials [2].

A bijective map from a discrete string problem to Jacobi matrices [3] gives a unified picture of the Toda, Jacobi, and multipeakon flows, and leads to explicit solutions of the Jacobi flows in terms of orthogonal polynomials.

Exit Comments of Participants

M. Adler (Brandeis University): The work I discussed involved understanding the Dyson process, Airy and Sine processes and deriving p.d.e.'e for correlation probabilities. Implicitly it involved deriving equations for the coupled 2-matrix modelling, which tied in with work of Harnad and Bertola, discussed at the conference. The Dyson process is an Ornstein–Uhlenbeck process on Hermitian matrices arranged so as to be stationary, by taking the equilibrium measure at infinity for initial data, and correlating probabilities at two distinct times, which precisely corresponds to computing a coupled two-matrix integral and then deriving PDE's in the coupling constant and the endpoints of the 2 spectral intervals in question. Scaling according to the edge or the bulk in the classical Hermite case then leads to PDE's for the Airy and Sine processes. Now the coupled two-matrix integral is the focal point for Harnad and Bertola's work, where now they integrate over spectral sets which may even be curves in the complex plane, which of course leaves the realm of Hermitian matrices. They however are interested in isomonodromy properties of the two-matrix integral, so it is entirely permissible to make such deformations and study their properties. It turns out that all of the isomonodromic data can be computed via residue calculations made on various differential forms on an associated spectral curve, which is basically very good news, as that means all of the data can be effectively computed in any example.

C. Berg (Copenhagen): My talk on 'Orthogonal Polynomials associated to positive definite matrices' focused on several new characterizations of indeterminate moment problems in terms of Hankel matrices. This raised the question of how this was related to Riemann-Hilbert problems and brought me in contact with Ken McLaughlin. He had treated asymptotic questions for weights of the form $w(x) = exp(-|x|^a)$, which for $0 < a < 1$ corresponds to indeterminate moment problems. He did not know about the problems of determining the order of the entire functions in the Nevanlinna matrices for these problems, and one can hope that this can be done by Riemann-Hilbert methods.

In the talk by Natig Atakishiev was presented some duality results for certain systems of q-orthogonal polynomials. To several people in the audience it was felt that one needed a precise notion of duality in this area, and after several hours of discussions and thoughts I think that he and I have reached such a notion and this could lead to a joint paper.

Jeff Geronimo, whom I did not meet since the Columbus meeting 1989, is now interested in orthogonal polynomials in two variables, a subject I had touched upon years back, so we gave fruitful exchange of information.

I had opportunity to discuss my ongoing project with Mourad Ismail on Kibble-Slepian kind of formulas - we came further ahead.

M. Ismail (University of Central Florida):

1. The presence of a mix of mathematicians from integrable systems and orthogonal polynomials was a great idea. I hope in future people like Charles Dunkl, Dennis Stanton, Erik Koolink, Tom Koornwinder, H. Rosengren, who do special functions and orthogonal, will be invited.

2. I did get mathematical ideas from the meeting which will be part of the paper I am writing on the tau function. It was nice to listen and talk to others.

3. The BIRS environment is ideal for these meetings and workshops and the limitation on the size makes it easy to interact with others.

4. I was impressed that two of the organizers did not give talks to provide space for others. This is a nice touch.

J. Harnad (Concordia University):

1. Summary of my presentation: Biorthogonal Polynomials and 2-Matrix Models

   A survey was given of results relating the spectral theory of coupled pairs of random matrices and the theory of biorthogonal polynomials. This was largely based on joint work with Marco Bertola and

Bertrand Eynard, and included the following topics: 1) In the case of measures involving exponentials of polynomial potentials, the associated Christroffel Darboux kernels, which are projectors onto spaces spanned by the first N biorthogonal polynomials, determine the spectral properties of coupled unitarily dialgonalizable random matrices, with spectra supported on curves in the complex plain. All correlation functions are given by determinantal formulae involving these kernels, and the gap probabilities are given by Fredholm determinants of the associated integral operator, supported on the complement of the region considered. 2) The associated systems of differential-deformation-difference equations (obtained by "folding" the first order differential deformation equations satisfied by the biorthogonal polynomial sequences through use of the recursion relations) satisfied on dual "windows" of consecutive biorthogonal polynomials of sizes equal to the degrees of the potentials are compatible as overdetermined systems. These therefore admit joint fundamental systems, integral representations of which were given, as well as their large x and y asymptotics. 3) By virtue of the compatibility of these sequences of differential-deformation equations, the generalized monodromies of each of the dual sequences of differential equations with respect to the dual variables are invariant under deformations in the polynomial potentials determining the measures, and independent of the integer determining the positions of the "window" . 4) A theorem of "spectral duality" was given, stating that the spectral curves defined by the characteristic polynomials of the corresponding Lax matrices for the dual systems, as well as those satisfied by the sequences of Fourier Laplace transforms of the biorthogonal polynomials, were all identical. Due to lack of time, a new, simplified proof and generlization of a theorem of Soshnikov concerning Janossy densities and their relation to gap probabilities in multi-matrix models was not presented in the seminar, but was discussed in private sessions with other participants.

2. Comments on presentations by other participants.

   The quality and level of interest of presentations by other participants was very high. The background and specialized interests of the various participants included several different orientations. There were those whose primary interest was in the classical theory of orthogonal polynomials, with measures that are both continuous and discrete, and their placement within the general framework of special functions - particularly, of hypergeometric and q-hypergeometric type, as well the classical moment problem. There was also considerable interest in generalizations, such as biorthogonal systems or multiple orthogonal polynomials, as well as the applications of orthogonal polynomials, e.g. to random matrices and other probabilistic and combinatorial systems, Thiere were those particularly interested in the study of large N asymptotics, either in random matrices, or in the representation theory of Lie groups, in relation to asymptotics of orthogonal polynomials. There were also many interested in applications to integrable systems, or the use of methods developed in that area, such as the matrix Riemann-Hilbert (inverse spectral) and dbar methods. The interactions between these various perspectives was particularly useful and stimulating. The talks having the greatest immediate interest in relation to my own work were those on multiple orthogonal polynomials and their applications (by Van Assche and Kuijlaars), those on large N asymptotics and universality (by Deift, McLaughlin and Vanlessen) and on the dbar steepest descent method for Riemann-Hilbert problems (Miller). Of course, the talk by Marco Bertola, which was based on joint work that we had done on the isomonodromic deformation problems arising in the study of semi-classical orthogonal polynomials, and the relation of the isomonodromic tau function, the spectral curve and the partition function for the corresponding matrix models was of immediate interest. Other talks of considerable interest, where I felt that I learnt something new, included those on the relation between the random matrix recursion relations and various Markov processes (Askey and Grunbaum), the analysis of the indeterminancy problem in reconstruction of measures from moments (Berg), the problem of Darboux and Christoffel transformations in terms of LU factorization of the recursion matrices, (Marcellan), the relation between matrix model processes and Brownian motion (Adler)and the large N asymptotics for representations of the unitary group (Borodin).

3. Interaction with other participants.

   Perhaps the most important aspect of this meeting was the opportunity it provided for interacting with others in the area - some of whom I had known before, but would not have otherwise had an opportunity to interact with or discuss recent developments with at this time, and others, whom I had only known through their published works, or indirectly, e.g. from lectures given by others. The most fruitful interactions were with Arne Kuijlaars, whose work on applications of multiple orthogonal polynomials, both to random matrices with external coupling (jointly with Pavel Bleher) and to 2-matrix models and the Riemann-Hilbert problem for biorthogonal polynomials (joint work with Ken McLaughlin) relates closely to my own work. Our group (Bertola, Eynard, Harnad) had previously developed a version of the Riemann-Hilbart problem for biorthogonal polynomials based on the notion of duality described above, but recently Kuijlaars and McLaughlin found another one, based on the relation to multi-orthogonal polynomials. Through discussions initiated at this meeting, we seem to have clarified the relation between these two approaches. This will very likely lead to new work developing this link. There was also another possible joint project initiated, based on using the interpretation of KP tau function as a determinant over an infinite Grassmannian, to obtain new tau functions from the deformation theory of multiple orthogonal polynomials. The ongoing open question of large N asymptotics for biorthogonal polynomials remains the most important objective, and this meeting gave a chance for the two groups most actively involved in resolving this problem to interact, and develop new ideas for its eventual solution. This included very helpful discussions with Ken McLaughlin, who is perhaps the most experienced in application of Riemann-Hilbert methods to the asymptotic analysis of orthogonal polynomials arising in matrix models, through his joint work with Deift, Krichebaur, Venakides and Zhou, one of the main pioneering works in this direction. There were also useful discussions with other workshop participants - in particular, with Alexei Borodin, who has also done some work on multi-level determinantal ensembles (of which coupled chains of random matrices are an example) and the computation of Janossy densities and gap distributions through the use of multi-orthogonal functions. Useful references and information on other developments relating more broadly to the theory of isomonodromic deformations, which we are also working on, followed from these discussions. There were also useful discussions with Mourad Ismail, on the relation of our work on recursion relations, isomonodromic deformations, and tau functions to the general framework of orthogonal polynomials, as well as many smaller discussions with other workshop participants, in which useful information was exchanged.

A. Kuijlaars (Leuven): It was an excellent workshop and I thank the organizers for bringing together a wonderful group of people, coming from a variety of different areas. What brought us together was our interest in orthogonal polynomials and special functions, which in one way or another played a crucial role in our research.

The interaction between people with different backgrounds was therefore an important goal of the workshop and it was achieved wonderfully well.

Having a background in approximation theory, I had inspiring discussions with people like Mark Adler, Marco Bertola and John Harnad, who work in integrable systems. With Marco and John I talked at length about a problem from random matrix theory that we are all interested in, but that we approach from different backgrounds. The interaction with them was very fruitful.

There were excellent talks at the workshop. Personally I enjoyed the talks by Percy Deift and Alexei Borodin very much, since they showed how orthogonal polynomials can play a role in seemingly unrelated areas. There were many other great talks as well.

It was a good idea of the organizers to invite a number of graduate students and recent Ph.D.'s as well. I am sure the workshop has been a great experience for them and will stimulate them in their future research into orthogonal polynomials and their applications.

M. Vanlessen, (Leuven): At the workshop I presented a talk about universality questions for eigenvalue correlations at the origin of the spectrum, which was joint work with Arno Kuijlaars. In this talk I focused on unitary ensembles with a singularity at the origin of the spectrum. This singularity appears through a factor $|detM|^{2\alpha}$ in the ensemble. The aim of the talk was to show how the Riemann-Hilbert approach could be used to get a universality class (in terms of Bessel functions) at the origin of the spectrum for this ensemble.

The Riemann-Hilbert approach was first applied by Deift, Kriecherbauer, McLaughlin, Venakides and Zhou. Therefore, for my point of interest, it was useful that Percy Deift and Ken McLaughlin were present at the workshop. Especially the talk of Percy was very interesting since he talked about universality questions for orthogonal and symplectic ensembles, which was one of my proposed projects for my postdoc application.

The presence of Alexei Borodin was also useful because of his knowledge about discrete orthogonal polynomial ensembles. This is because Ken McLaughlin and I are currently very interested in the Gamma kernel (of Alexei) and we had some useful conversations with Alexei.

K. McLaughlin (University of North Carolina): A number of presentations pertained to the asymptotic analysis of Riemann-Hilbert problems, and applications to a variety of different areas of mathematics.

The asymptotic analysis of orthogonal polynomials is directly connected to the statistical behaviour of eigenvalues of random Hermitian matrices. In order to describe the limiting statistical behaviour of the eigenvalues when the size of the matrices grow to infinity, it is required to understand the so-called "semi-classical" asymptotic properties of orthogonal polynomials. In the late 90s, the problem of determining such global asymptotic behaviour of orthogonal polynomials was re-cast in terms of Riemann-Hilbert problems [12], [9], and several of the presentations concerned new results for the asymptotic analysis of Riemann-Hilbert problems. These results, in turn, yield new results for random matrices.

Here are three examples: (1) Maarten Vanlessen described a new universality class for the local statistical behaviour of eigenvalues of random Hermitian matrices (work with A. Kuijlaars). (2) Peter Miller described an extension of the above results to the asymptotic analysis of Riemann-Hilbert ($\bar{\partial}$) problems, which establish asymptotic properties of orthogonal polynomials under much weaker assumptions on the orthogonality weights (work with K. McLaughlin). (3) Arno Kuijlaars and Walter Van Assche gave presentations describing the asymptotic analysis of some 3×3 Riemann-Hilbert problems, multiple orthogonal polynomials, and applications to random matrices with external sources.

In other directions, new universality results were stated (for the first time) concerning the limiting statistical behaviour of random symmetric matrices (rather than Hermitian) by P. Deift (with D. Gioev). Ken McLaughlin presented joint work with V. Pierce and N. Ercolani in which continuum limits of the Toda lattice are used to obtain explicit formulae for coefficients in a recently established expansion for the partition function of random matrices. Alexei Borodin spoke about the connection between discrete orthogonal polynomials and asymptotic representation theory.

What links this subset of presentations is the interplay between the asymptotic behaviour of orthogonal polynomials, and applications. Here are a few examples of the cross-fertilization that occurred during this meeting:

1. Researchers investigating the asymptotic behaviour of Riemann-Hilbert problems associated to "multiple orthogonal polynomials" got together with researchers investigating biorthogonal polynomial asymptotics, and discovered new connections between seemingly disparate Riemann-Hilbert problems appearing in their respective work.

2. A new Gamma kernel, identified in asymptotic representation theory, is being investigated from the point of view of universal behaviour of discrete orthogonal polynomial ensembles, using recent results from Riemann-Hilbert analysis and asymptotic representation theory.

List of Participants

Adler, Mark (Brandeis University)
Arvesu, Jorge (Universidad Carlos III de Madrid)
Askey, Richard (University of Wisconsin, Madison)
Atakishiyev, Natig (Universidad Nacional Autonoma de Mexico)
Berg, Christian (University of Copenhagen)
Bertola, Marco (Concordia University)
Borodin, Alexei (California Institute of Technology)
Deift, Percy (Courant Institute of Mathematical Sciences)
Geronimo, Jeff (Georgia Institute of Technology)
Grunbaum, Alberto (University of California, Berkeley)
Harnad, John (Concordia University)
Ismail, Mourad (University of Central Florida)
Kuijlaars, Arno (Katholiek Universiteit Leuven)

Lee, Dong Won (Kyungpook National University)
Littlejohn, Lance (Utah State University)
Lundmark, Hans (Linköping University)
Marcellan, Francisco (Universidad carlos III de Madrid)
McLaughlin, Ken (University of North Carolina)
Mehta, Madan (Service de Physique Théorique, Saclay)
Miller, Peter (The University of Michigan)
Mohajer, Keivan (University of Saskatchewan)
Odzijewicz, Anatol (University of Bialystok)
Ruffing, Andreas (Munich University of Technology)
Sattinger, David (Yale University)
Suslov, Sergei (Arizona State University)
Szmigielski, Jacek (University of Saskatchewan)
Takata, Tomohiro (Kyoto University)
Tuncer, Davut (Utah State University)
Van Assche, Walter (Katholieke Universiteit Leuven)
Vanlessen, Maarten (Katholieke Universiteit Leuven)
Vinet, Luc (McGill University)
Xu, Yuan (University of Oregon)

Bibliography

[1] ASKEY, R. *Orthogonal polynomials and special functions*. SIAM Publications, Philadelphia, PA, 1975.

[2] BEALS, R., D. H. SATTINGER, & J. SZMIGIELSKI. Multipeakons and the classical moment problem. *Adv. in Math.*, **154**:229–257, 2000.

[3] BEALS, R., D. H. SATTINGER, & J. SZMIGIELSKI. Peakons, Strings, and the finite Toda lattice. *Comm. Pure & Applied Math.*, **LIII**:1–16, 2000.

[4] BEALS, R., D.H. SATTINGER, & J. SZMIGIELSKI. Continued fractions and integrable systems. *Journal of Applied and Computational Mathematics*, **153**:47–60, 2002. Proceedings of the Sixth Symposium on Orthogonal Polynomials, Special Functions, and Integrable Systems, Rome, June, 2001.

[5] BLEHER, P. & ITS, A.R. Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. *Ann. of Math. (2)*, 150(1):185–266, 1999.

[6] CHIHARA. T.S. *An introduction to orthogonal polynomials*. Gordon and Breach, New York, 1978.

[7] COUSSEMENT, J., KUIJLAARS, A. B. J. & VAN ASSCHE, W. Direct and inverse spectral transform for the relativistic Toda lattice and the connection with Laurent orthogonal polynomials. *Inverse Problems*, 18(3):923–942, 2002.

[8] COUSSEMENT, J. & VAN ASSCHE, W. An extension of the Toda lattice: a direct and inverse spectral transform connected with orthogonal rational functions. *Inverse Problems*, 20(1):297–318, 2004.

[9] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.*, 52(11):1335–1425, 1999.

[10] DEIFT, P. *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*. American Mathematical Society, Courant Institute of Mathematics, Providence, R.I., New York, 2000.

[11] DEIFT, P. AND ZHOU, X. A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation. *Ann. of Math. (2)*, 137(2):295–368, 1993.

[12] DEIFT, P., KRIECHERBAUER, T., MCLAUGHLIN, K. T-R, VENAKIDES, S. & ZHOU, X. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.*, 52(12):1491–1552, 1999.

[13] DEIFT, P., VENAKIDES, S. & X. ZHOU. New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems. *Internat. Math. Res. Notices*, (6):286–299, 1997.

[14] FOKAS, A.S., A.R. ITS, AND A.V.KITAEV. The isomondromy approach to matrix models in 2d quantum gravity. *Comm. Math. Physics*, 147:395–430, 1992.

[15] KREIN, M. G. On inverse problems for an inhomogeneous string. *Dokl. Akad. Nauk. SSSR*, **82**:669–672, 1952.

[16] MEHTA, MADAN LAL. *Random matrices*. Academic Press Inc., Boston, MA, second edition, 1991.

[17] MEHTA, M. L. & GAUDIN, M. On the density of eigenvalues of a random matrix. *Nuclear Phys.*, 18:420–427, 1960.

[18] MOSER, J. Finitely many mass points on the line under the influence of an exponential potential–an integrable system. In *Dynamical systems, theory and applications*, volume **38** of *Lecture Notes in Phys.*, pages 467–497, Berlin, 1975. Springer.

[19] STAHL,H. & V. TOTIK. *General Orthogonal Polynomials*. Cambridge University Press, London, 1992.

[20] SZEGÖ, G. *Orthogonal polynomials*, volume 23 of *Colloquim Publications*. American Mathematical Society, Providence, R.I., 3 edition, 1967.

Chapter 4

Model Reduction Problems and Matrix Methods (04w5513)

April 3–8, 2004

Organizer(s): Anne Greenbaum (University of Washington), Gene Golub (Stanford University), Jim Varah (University of British Columbia)

This workshop will focus on techniques from numerical linear algebra and ordinary differential equations for model reduction problems in dynamical systems and control. A goal is to bring together people from industry and other areas of academia working on problems involving model reduction and numerical analysts studying applicable solution techniques. Of particular interest are very large nonsymmetric systems of linear equations and eigenvalue problems. Preconditioners play an important part in such algorithms, and here questions of inner/outer iterations must be addressed. Effects of finite precision arithmetic are similar to those of inexact solution of a preconditioning matrix, and recent work in this area also will be included.

List of Participants

Antoulas, Thanos (Rice University)
Ascher, Uri (University of British Columbia)
Bai, Zhaojun (University of California, Davis)
Benner, Peter (Technische Universität Chemnitz)
Boley, Daniel (University of Minnesota)
Chan, Raymond (Chinese University of Hong Kong)
Cuyt, Annie (University of Antwerp)
Daniel, Luca (Massachusetts Institute of Technology)
Elfadel, Abe (IBM T.J. Watson Research Center)
Freund, Roland (Bell Laboratories)
Gallivan, Kyle (Florida State University)
Gemignani, Luca (University of Pisa)
Golub, Gene (Stanford University)
Greenbaum, Anne (University of Washington)
Greif, Chen (University of British Columbia)
Gu, Ming (University of California, Berkeley)
Gugercin, Serkan (Virginia Polytechnic Institute)
Hui, Huang (University of British Columbia)
Kwok, Felix (Stanford University)
MacKinnon-Cormier, Sarah (University of British Columbia)
Machorro, Eric (University of Washington)

Meerbergen, Karl (Free Field Technologies)
Megretski, Alexandre (Massachusetts Institute of Technology)
Olsson, Henrik (Royal Institute of Technology)
Petzold, Linda (University of California, Santa Barbara)
Reichel, Lothar (Kent State University)
Roychowdhury, Jaijeet (University of Minnesota)
Sameh, Ahmed (Purdue University)
Sorensen, Dan (Rice University)
Stykel, Tatjana (Technische Universität Berlin)
Van Dooren, Paul (Universite Catholique de Louvain)
Varah, Jim (University of British Columbia)
Verdonk, Brigitte (University of Antwerp)
White, Jacob (Massachusetts Institute of Technology)
Willcox, Karen (Massachusetts Institute of Technology)
Ye, Qiang (University of Kentucky)

Chapter 5

Analytic and Geometric Aspects of Stochastic Processes (04w5023)

April 10–15, 2004

Organizer(s): Martin Barlow (University of British Columbia), Alexander Grigoryan (Imperial College, London), Elton Hsu (Northwestern University)

The conference was attended by about 35 participants, including Ph.D. students, postdoctoral fellows, young researchers and international leaders in stochastic analysis and related fields. What follows is an attempt to focus on some of the key topics discussed at the meeting both in the lectures and in the informal meeting rooms.

Brownian Motion and Classical Analysis

The original development of this area arises from the link between Brownian motion, the heat equation, and classical potential theory. One should see this link as acting in both directions – that is, both sets of objects are of mathematical interest, and one can exploit these connections to (e.g.) use Brownian motion to study harmonic functions on domains in $\mathbb{P}^d$, or properties of the heat equation to refine estimates on Brownian motion. With the late P.A. Meyer, we rather regret the view (common in the USA at points in the last century) that regards work on stochastic processes as only interesting if it leads to some result in analysis.

Chris Burdzy spoke on several problems inspired or related at the technical level to the "hot spots" conjecture of J. Rauch, which was made in 1974. This was that the second Neumann eigenfunction attains its maximum at the boundary for every Euclidean domain. Progress on the conjecture was slow, but it is now known (see [2], [1]) that while it is false in general, it is true for some classes of domains. To study this question by probabilistic methods, one needs to construct reflecting Brownian motion in the domain D. There are various possible constructions; Burdzy discussed a 'strong' construction via the Skorohod equation, and stated a theorem which shows that strong existence and uniqueness hold if the domain is Lipschitz with the Lipschitz constant less than 1.

He then discussed the question of when the second Neumann eigenvalue is simple: this holds in long and thin domains, in domains with bottlenecks, and in lip domains. Next, he considered the location of the nodal line (zero line) of the second Neumann eigenfunction. Coupling methods give some information about the location of this line: for example, in obtuse triangles. Finally he presented an explicit formula for the Lyapunov exponent for the flow of reflected Brownian motions in a smooth domain.

M. van den Berg spoke on properties of the Wiener sausage $W(t)$ in $\mathbb{R}^3$. This is the random set $B(s)+k$,where $B(.)$ is a Brownian motion , $s \in [0,t]$,and where k runs over a fixed compact set K. This volume of this set is of fundamental importance in physics e.g. in the modelling of polymers, diffusion of matter. In probability theory it provides a simple, non trivial example of a non Markovian process. Moreover its expectation shows up in the calculation of the amount of heat which has emanated from the compact K (kept at temperature 1) into its complement (initial temperature 0). F. Spitzer [11], J.F.Le Gall [10], S. Port and others have obtained the first few terms in the asymptotic series for this expectation as $t \to \infty$. Donsker and Varadhan [7] studied the large deviation properties of the volume and van den Berg , Bolthausen and den Hollander [3] studied the moderate deviations of the volume . The results in [3] were then used to obtain the moderate deviations for the intersection volume of two and three independent Wiener sausages.

In his talk he summarised some properties for the expected volume of n Wiener sausages in Euclidean space $\mathbb{R}^m$. The case where $n = 2$ can be reduced to the case of a single sausage. The planar case for any n has been studied extensively by J. F. Le Gall in his St. Flour lecture notes [10]. The cases where $n > 2$ and $m > 3$ gives finite expectation as $t \to \infty$. His main result was for the case $n = m = 3$: the first three terms in the asymptotic series, of order $\log t$, order 1, and order $(\log t)/t^{1/2}$ were obtained together with a remainder $O(t^{-1/2})$ (which is sharp for the ball). The proof relies on repeated use of the strong Markov property together with a last exit time decomposition .

Diffusions, Stochastic Differential Equations and Calculus on Manifolds

The connection between Markov processes, potential theory and differential equations is much broader than just that between Brownian motion and classical analysis. In this section we describe connections between diffusions and various geometric objects.

Thierry Coulhon talked on "Riesz transform on manifolds, and heat kernel regularity": this is joint work with Pascal Auscher, Xuan Thinh Duong, and Steve Hofmann. The aim was to give a necessary and sufficient condition for the two natural definitions of homogeneous first order L^p Sobolev spaces to coincide on a large class of Riemannian manifolds, for p in an interval (q_0, p_0), where $2 < p_0 \leq \infty$ and q_0 is the conjugate exponent to p_0. On closed manifolds, these definitions are well-known to coincide for all $1 < p < \infty$. For non-compact manifolds, and again for $p_0 = \infty$, a sufficient condition was asked for by Robert Strichartz in 1983 and many partial answers have been given since. The condition proposed was in terms of regularity of the heat kernel: more precisely in terms of integral estimates of its gradient. This allowed him to treat manifolds which satisfy the doubling property and natural heat kernel bounds, as well as one with locally bounded geometry where the bottom of the spectrum of the Laplacian is positive.

Bruce K. Driver talked on joint work with his student Tai Melcher (who also attended the meeting), on "Hypoelliptic heat kernel inequalities". In the last twenty years or more, a fairly complete and very beautiful theory has been developed applying to elliptic operators on Riemannian manifolds. This theory relates properties of the solutions of elliptic and parabolic equations to properties of the Riemannian geometry. These geometric properties are determined by the principal symbol of the underlying elliptic operator. The following is a typical example of this type of result:

Theorem 1 (Bakry, Ledoux, Emery,...) *Suppose (M, g) is a complete Riemannian manifold, and ∇ and Δ are the gradient and Laplace-Beltrami operators acting on $C^\infty(M)$. Let $|v| := \sqrt{g(v, v)}$ for all $v \in TM$, Ric denote the Ricci curvature tensor, and k denote a constant. Then the following are equivalent:*

1. $\mathrm{Ric}(\nabla f, \nabla f) \geq -2k|\nabla f|^2$ *(or equivalently $\Gamma_2(f, f) \geq -2k\Gamma_1(f, f)$ for all $f \in C_c^\infty(M)$),*
2. $\left|\nabla e^{t\Delta/2} f\right| \leq e^{kt} e^{t\Delta/2} |\nabla f|$, *for all $f \in C_c^\infty(M)$ and $t > 0$,*
3. $\left|\nabla e^{t\Delta/2} f\right|^2 \leq e^{2kt} e^{t\Delta/2} |\nabla f|^2$, *for all $f \in C_c^\infty(M)$ and $t > 0$, and*
4. *there is a function $K(t) > 0$ such that $K(0) = 1$, $\dot{K}(0) =: 2k$ exists, and*
$$|\nabla e^{t\Delta/2} f|^2 \leq K(t) e^{t\Delta/2} |\nabla f|^2, \tag{5.1}$$
for all $f \in C_c^\infty(M)$ and $t > 0$.

In his talk, we explored the possible of extension of Theorem 1 to hypoelliptic operators of the form

$$L = \sum_{i=1}^n X_i^2, \tag{5.2}$$

where $\{X_i\}_{i=1}^n$ is a collection of smooth vector fields on M satisfying the Hörmander bracket condition. When L is not elliptic, Theorem 1 can no longer hold because, roughly speaking, the "Ricci curvature" is no longer bounded from below. Nevertheless it is reasonable to ask if inequalities of the form (5.1) might still hold. To be more precise, let $\nabla := (X_1, \ldots, X_n)$, $p \in [1, \infty)$ and $t > 0$ and let $K_p(t)$ be the best constant such that

$$|\nabla e^{tL/2} f|^p \leq K_p(t) e^{tL/2} |\nabla f|^p \text{ for all } f \in C_c^\infty(M). \tag{5.3}$$

The question then becomes; when is $K_p(t) < \infty$? In this regard the following theorem was demonstrated in the talk.

Theorem 2 (T. Melcher and B. Driver) *Let G be $\mathbb{R}^3$ (equipped with the Heisenberg group multiplication),*

$$X := \partial_x - \frac{1}{2} y \partial_z,\ Y := \partial_y + \frac{1}{2} x \partial_z \text{ and } L := X^2 + Y^2.$$

Then for all $p \in (1, \infty)$,

1. *$K_p(t)$ is independent of t,*
2. *$K_p(t) = K_p < \infty$,*
3. *$K_p > 1$ and in particular, $K_2 \geq 2$.*

Results analogous to Theorem 2 will (in Tai Melcher's thesis) be generalized to any nilpotent Lie group with a collection of left invariant vector fields satisfying Hörmander's condition. The case $p = 1$ is still open.

Yves Le-jan talked on research motivated by physics. There have been very few studies of stochastic processes done in a relativistic framework up to now, at least by mathematicians. The idea was to show that some techniques used to define and study stochastic processes on Riemannian manifolds can be transferred to the framework of general relativity.

He recalled the definition, due to Dudley, of a relativistic diffusion. He then formulated an SDE representation of the solution using the trivial frame bundle on the Minkowski space. This SDE can then be extended to the general relativistic setting in a canonical way. The example of the Swatzschild space was studied in more detail, using several barrier functions to show the transience of the process.

Takashi Kumagai talked on "Characterization of sub-Gaussian heat kernel estimates on graphs and measure metric spaces", based on joint work with M.T. Barlow, R.F. Bass, and T. Coulhon. The motivation of the study of sub-Gaussian heat kernel estimates is from analysis on fractals. It is known that the heat kernels for Brownian motions on various "regular" fractals (such as the Sierpinski gasket) enjoy sub-Gaussian estimates. From the estimates, many properties of the processes can be deduced; for instance, law of iterated logarithms, Green kernel estimates etc. So, it is natural and important to ask whether such estimates are stable under perturbations.

He discussed various conditions which are necessary and sufficient for sub-Gaussian heat kernel estimates to hold:

1. A generalized parabolic Harnack inequality,
2. volume doubling + an elliptic Harnack inequality + some hitting time estimate (or some resistance estimate) (due to Grigor'yan-Telcs, [9])
3. volume doubling + a Poincaré inequality + cut-off Sobolev inequality.

It can be proved that (3) is stable under a bounded perturbation of the operator, and under a rough isometry. Under a stronger volume growth condition, a simpler equivalent condition can be given in terms of electrical resistance. As an application, he described quenched heat kernel estimates for simple random walks on the incipient infinite clusters on Galton-Watson branching processes.

Jump Processes

From an analytic viewpoint, non-local operators arise when one looks at $-(-\Delta)^{\alpha}$ for $\alpha \in (0,1)$. These correspond to jump processes, the most familiar being the class of stable processes.

Zhen-Qing Chen talked on "SDEs Driven by Stable Processes"; joint work with R. Bass and K. Burdzy. Stochastic differential equation (SDE) driven by Brownian motion plays a central role in the theory of modern probability and its applications. In the last few years there has been intensive interest in the study of processes with jumps. Much of the motivation has come from mathematical physics and from financial mathematics: in many applications jump processes (such as stable processes) provide more realistic models than continuous processes do. So it is quite natural to study SDE system driven by stable processes.

Given this, it is somewhat surprising that SDE systems with continuous coefficients driven by stable processes have not previously been studied in a systematic fashion. In the first part of his talk, Chen reported on recent progress on the existence of a strong solution, and pathwise uniqueness for 1-dimensional SDEs driven by a symmetric stable processes. In the second part of the talk, he discussed the existence of weak solution and weak uniqueness for systems of SDEs driven by either by n-independent copies of a 1-dimensional symmetric stable processes, or by a symmetric stable processes in R^n. The approach uses the martingale problem method, and requires estimates for pseudodifferential operators with singular state-dependent symbols.

Renming Song discussed "Potential Theory of Special Subordinators and Subordinate Killed Brownian motions". The technique of 'subordination' was introduced by Bochner [4], and allows the construction of stable processes from a Brownian motion and an independent increasing Lévy process (called a 'subordinator'). However, subordination of processes in a domain D have only been studied fairly recently.

Let D be a bounded open set in $\mathbb{R}^d$, $d \geq 3$, and let $\Delta|_D$ be the Dirichlet Laplacian in D. This operator is the infinitesimal generator of the semigroup $(P_t^D : t \geq 0)$ corresponding to the process $X^D = (X_t^D : t \geq 0)$, Brownian motion killed upon exiting D. Let $S = (S_t : t \geq 0)$ be an $\alpha/2$-stable subordinator independent of X^D, where $0 < \alpha < 2$, and let $Z_\alpha^D = (Z_\alpha^D(t) : t \geq 0)$ be the process X^D subordinated by S: so that $Z_\alpha^D(t) := X^D(S_t)$. The infinitesimal generator of the semigroup of Z_α^D is the fractional power $-(-\Delta|_D)^{\alpha/2}$ of the negative Dirichlet Laplacian.

The study of the process Z_α^D was initiated in [17]. In [18] the domain of the Dirichlet form of Z_α^D was identified when D is a bounded smooth domain and $\alpha \neq 1$. In [20] and [19], the process Z_α^D was studied in detail and sharp upper and lower bounds on the jumping function and the Green function of Z_α^D were established when D is a bounded $C^{1,1}$ domain. One of the most intriguing aspects of the potential theory of Z_α^D was discovered in [17], and completely described in [16]. Let us introduce another subordinate process, $Z_{2-\alpha}^D$, obtained by subordinating killed Brownian motion X^D by an independent $(1-\alpha/2)$-stable subordinator. Let G^D, G_α^D and $G_{2-\alpha}^D$ denote the potential operators of X^D, Z_α^D and $Z_{2-\alpha}^D$, respectively. Then the following factorization identity holds true:

$$G^D = G_\alpha^D G_{2-\alpha}^D = G_{2-\alpha}^D G_\alpha^D \,. \tag{5.4}$$

Song discussed applications of this identity to a Harnack inequality, and to the identification of the Martin boundary for Z_α^D.

The Laplace exponent of the $\alpha/2$-stable subordinator is $\phi(\lambda) = \lambda^{\alpha/2}$, $\lambda > 0$. Clearly, $\lambda/\lambda^{\alpha/2} = \lambda^{1-\alpha/2}$ is the Laplace exponent of the $(1-\alpha/2)$-stable subordinator. The existence of a "dual" subordinator of this type is the key for the factorization (5.4). Song then described a more general family of 'special subordinators' for which this kind of duality holds, and showed that the main results of [17] and [16] remain valid when S is only assumed to be a special subordinator.

Masayoshi Takeda talked on "Gaugeability for Symmetric α-Stable processes and and it's Applications". Let $\mathbb{M}^\alpha = (\mathbb{P}_x, X_t)$ be a symmetric α-stable process on $\mathbb{R}^d$. Assume that $\mathbb{M}^\alpha$ is transient, and denote by $G(x,y)$ the Green function of $\mathbb{M}^\alpha$. Let μ be a smooth measure and A_t^μ the continuous additive functional corresponding to μ. (If μ has a density f then $A_t^\mu =\in_0^t f(X_s)ds$.) The measure μ is said to be *gaugeable* if

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\exp(A^\mu(\infty))\right] < \infty. \tag{5.5}$$

For Brownian motion on $\mathbb{R}^d$, Zhao [15] introduced the class of Green-tight measures and Chen [5] generalized this to jump processes. Let $K_{d,\alpha}^\infty$ denote this class for the stable process $\mathbb{M}^\alpha$. In [14] and [5] an analytic condition for a measure $\mu \in K^{\infty}_{d,\alpha}$ to be gaugeable was obtained. Let

$$\lambda(\mu) = \inf \left\{ \mathcal{E}^{(\alpha)}(u,u) : \ u \in \mathcal{F}^{(\alpha)}, \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1 \right\}.$$

Then the gaugeability of μ is equivalent to $\lambda(\mu) > 1$.

Takeda gave three applications of this fact: to the differentiability of spectral functions, the ultracontractivity of Schrödinger semigroups, and the behaviour of branching symmetric α-stable processes.

Infinite Dimensional Analysis

Maria Gordina talk used stochastic differential equations (SDEs) in infinite-dimensional spaces to construct and study heat kernel measures (a noncommutative analogue of Gaussian or Wiener measure) on the infinite-dimensional manifolds. In general these infinite-dimensional groups are not locally compact, and therefore do not have an analogue of the Lebesgue (volume) measure. The motivation comes from several fields. Infinite-dimensional spaces such as loop groups and path spaces appear in physics, for example, in quantum field theory and string theory. One goal is to formalize some of the notions used in physics, such as Gaussian measures on certain infinite-dimensional spaces.

Her talk described the construction of the Gaussian measures on certain groups of infinite matrices, and gave some analytical properties of these measures. In addition, she presented new results on Riemannian geometry of these groups, which show that these groups are drastically different from their finite-dimensional analogs.

Shigeki Aida's interest is in analysis in infinite dimensional spaces and the interplay between the analysis and the geometry on such infinite dimensional spaces. In particular, the analysis on loop spaces is natural object but the basic property of the differential operator on the spaces are not well understood. For example, it is not clear when the Dirichlet forms on loop spaces satisfies a Poincaré's inequality, or when the Dirichlet forms satisfy log-Sobolev inequalities. As the terminology suggests, 'weak Poincaré inequalities' (WPIs) are weaker than Poincaré's inequalities, but nevertheless this property is stronger than irreducibility. WPIs hold on the loop spaces over simply connected compact manifolds in general. These inequalities contain a function which describes the degree of the ergodicity of the diffusion semi-group. However, explicit estimates on this function are not known in general.

In his talk, he proved WPIs on domains in Wiener spaces which are inverse images of open sets in $\mathbb{R}$ by continuous functions of Brownian rough paths. First, WPIs are established for ball like sets in the sense of rough paths and next the results are extended to "connected domains". This result is applicable to Dirichlet forms on loop spaces and connected open subsets of path spaces over compact Riemannian manifolds by using Lyons' continuity theorem of the solution of SDE. We still need more to obtain the estimate on the function in WPI in the case of loop spaces.

Xuemei Li talked on 'Asymptotics of Exponential Barycentres of mass transported by a random flow on Cartan Hadamard manifold.' This is joint work with M. Arnaudon. They considered the motion of a mass moving according to the law of a random flow. This can be used to model the motion of passive tracers in a fluid, e.g. the spread of oil spilt in an ocean. Such motion can be assumed to obey a stochastic flow where particles at nearby points are correlated. The evolution of pollution clouds in the atmosphere or a gas of independent particles, on the other hand, can be described as blocks of masses moving according to the laws of independent stochastic flows. We study the dynamics of masses transported by stochastic flows by investigating the motion of its centre of mass. As the media in which the liquid travels is not necessarily homogeneous or flat it makes sense to work on a non linear space, e.g. on a manifold diffeomorphic to the flat space but with different geometric structure.

The talk considered the mass pushed forward by a random flow in the sense of Kunita. The state spaces under consideration are Cartan Hadamard manifolds. Under suitable conditions on the flow and on the initial measure, the Barycentre can be shown to be a semi-martingale and is described by a stochastic differential equation. They showed that under suitable conditions an unstable flow satisfying the law of large numbers pushes the exponential barycentre of a discrete mass to the Busemann Barycentre of the limiting measure on the visibility boundary.

List of Participants

Aida, Shigeki (Osaka University)
Banuelos, Rodrigo (Purdue University)
Barlow, Martin (University of British Columbia)
Bauer, Robert (University of Illinois, Urbana-Champaign)
Burdzy, Chris (University of Washington)
Chen, Zhen-Qing (University of Washington)
Coulhon, Thierry (Universite de Cergy-Pontoise)
Deuschel, J-D. (Technische Universitat Berlin)
Driver, Bruce (University of California, San Diego)
Elworthy, Kenneth D. (University of Warwick)
Gordina, Maria (University of Connecticut)
Hambly, Ben (University of Oxford)
Hsu, Elton (Northwestern University)
Kassmann, Rolf Moritz (University of Connecticut)
Kumagai, Takashi (Cornell University)
LeJan, Yves (Universite Paris Sud)
Levin, David (University of Utah)
Li, Xue-Mei (The Nottingham Trent University)
Limic, Vlada (University of British Columbia)
Lyons, Terry (University of Oxford)
McDonald, Patrick (New College of Florida)
Melcher, Tai (University of California, San Diego)
Mendez, Pedro J. (University of Utah)
Metz, Volker (Universitat Bielefeld)
Neel, Robert (Harvard University)
Popescu, Ionel (Massachusetts Institute of Technology)
Rivasplata, Omar (University of Alberta)
Saloff-Coste, Laurent (Cornell University)
Song, Renming (University of Illinois, Urbana-Champaign)
Sturm, Anja (University of British Columbia)
Sturm, Karl-T. (Universitat Bonn)
Takeda, Masayoshi (Tohoku University)
Teufl, Elmar (Graz University of Technology)
Virag, Balint (University of Toronto)
van den Berg, Michiel (University of Bristol)

Bibliography

[1] R. Banuelos, K. Burdzy. On the ”hot spots” conjecture of J. Rauch. *J. Funct. Anal.* **164** (1999), no. 1, 1–33.

[2] K. Burdzy,W. Werner. A counterexample to the ”hot spots” conjecture. *Ann. of Math. (2)* **149** (1999), no. 1, 309–317.

[3] M. van den Berg, E. Bolthausen, F. den Hollander. On the volume of the intersection of two Wiener sausages. *Ann. of Math. (2)* **159** (2004), no. 2, 741–782.

[4] S. Bochner. Subordination of non-Gaussian stochastic processes. *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962) 19–22.

[5] Z.Q. Chen, *Gaugeability and Conditional Gaugeability*, Trans.Amer.Math.Soc., **354** (2002), 4639-4679.

[6] Z.Q. Chen and S.T. Zhang, *Girsanov and Feynman-Kac type transformations for symmetric Markov processes*, Ann. Inst. Henri Poincare, **38** (2002), 475-505.

[7] M.D. Donsker, S.R.S. Varadhan. Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28** (1975), no. 4, 525–565.

[8] A. Grigor'yan and M. Kelbert, *Recurrence and transience of branching diffusion processes on Riemannian manifolds*, Ann. Probab. **31** (2003), 244-284.

[9] A. Grigor'yan, A. Telcs. Harnack inequalities and sub-Gaussian estimates for random walks. *Math. Annalen* **324** (2002), 521–556.

[10] J.-F. Le Gall. Some properties of planar Brownian motion. École d'Été de Probabilités de Saint-Flour XX – 1990, 111–235, Lecture Notes in Math., **1527**, Springer, Berlin, 1992.

[11] F. Spitzer. Electrostatic capacity, heat flow, and Brownian motion. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3**, (1964) 110–121.

[12] M. Takeda, *Subcriticality and conditional gaugeability of generalized Schrödinger operators*, J. Funct. Anal., **191**(2002), 343-376.

[13] M. Takeda and K. Tsuchida, preprint, 2004

[14] M. Takeda and T. Uemura, *Subcriticality and Gaugeability for Symmetric α-Stable Processes*, to appear in Forum Math.

[15] Z. Zhao, *Subcriticality and gaugeability of the Schrödinger operator*, Trans. Amer. Math. Soc., **334** (1992), 75-96.

[16] J. Glover, Z. Pop-Stojanovic, M. Rao, H. Šikić, R. Song and Z. Vondraček, Harmonic functions of killed Brownian motions, *J. Funct. Anal.*, to appear, 2004.

[17] J. Glover, M. Rao, H. Šikić and R. Song, Γ-potentials, in *Classical and modern potential theory and applications* (Chateau de Bonas, 1993), 217–232, Kluwer Acad. Publ., Dordrecht, 1994.

[18] N. Jacob and R. Schilling, Some Dirichlet spaces obtained by subordinate reflected diffusions, *Rev. Mat. Iberoamericana*, **15**(1999), 59–91.

[19] R. Song, Sharp bounds on the density, Green function and jumping function of subordinate killed BM, *Probab. Th. Rel. Fields* (2004), **128** (2004), 606-628.

[20] R. Song and Z. Vondraček, Potential Theory of Subordinate Killed Brownian Motion in a Domain, *Probab. Th. Rel. Fields*, **125**(2003), 578–592.

Chapter 6

Celestial Mechanics (04w5012)

April 17–22, 2004

Organizer(s): Florin Diacu (University of Victoria), Donald Saari (University of California, Irvine)

There were 22 participants to the First BIRS Celestial Mechanics Workshop, organized by Florin Diacu (Canada) and Donald Saari (USA).

The programme of the workshop consisted of 45-minute talks followed by 15 minutes of discussions. Often those discussions were continued in the evening in smaller groups.

The workshop was opened on Sunday by Alain Chenciner of Paris. During the past few years, Alain has focused on studying the existence of choreographic solutions using variational methods. His proof of the existence of the Figure Eight solution together with Richard Montgomery [1] has revolutionized the field. Many researchers have adopted their methodology, seeking new periodic solutions in the n-body problem and for related systems of differential equations.

In his talk, Alain presented some recent work written with Jacques Féjoz and Richard Montgomery. He showed the existence of three families of relative periodic solutions which bifurcate out of the Figure Eight solution of the equal-mass three-body problem : the planar Hénon family, the spatial Marchal P_{12} family and a new spatial family. Each family corresponds to a different breaking of the $D_6 \times Z_2$ symmetry of the Eight solution in 3-space. Alain described this result as well as some of its developments.

The end-of-the-talk discussions revolved around questions related to the nature of the symmetries. For example, Jeff Xia pointed out that he had already obtained some more general results relative to one type of symmetry, but admitted that they did not contain the other rotation types. Alain's talk was very well received and imposed a very high standard for the entire workshop. A preliminary version of the paper, [2], is available at:

http://www.imcce.fr/Equipes/ASD/person/chenciner/chen_preprint.html

The second talk of the morning was given by Ernesto Lacomba. He talked about symbolic dynamics in the rectilinear restricted 3-body problem. This joint work with Sam Kaplan extended some ideas Sam had developed in his doctoral thesis in connection with a 2-body problem with a bumper.

They generalized these results to a symmetric rectilinear restricted 3-body problem for which the equal mass primaries perform elliptic collisions, while the infinitesimal body moves in the line between the primaries. Symbolic dynamics could be applied to mark the time between two consecutive elliptic collisions. Since any nonhomothetic solution performs binary collisions, the basic behaviour of the solution can be studied through its successive intersections with 2 two-dimensional strips, corresponding to regularized binary collisions. Thus Ernesto and Sam obtained a singular global Poincaré section. In this way, they were able to describe all possible itineraries an orbit may have.

The last talk of the morning was that of Montserrat (Montse) Corbera of the Vic University near Barcelona. She presented several new results on the global existence of subharmonic orbits in the Sitnikov Problem. This consists of the motion of 3 bodies, two of them of equal mass, moving in a plane on circular or elliptic orbits of eccentricity e, and the third of infinitesimal mass, moving on the z axis perpendicular to the plane of the other two that passes through their centre of mass. A solution is said to be an (m, n)-orbit if it is $2m\pi$-periodic and there are exactly $2n$ zeroes of z in the time interval $[0, 2m\pi)$.

Montse wrote this paper in collaboration with Jaume Llibre (present at the meeting) and Pedro Torres. The main result of the talk was that for all m natural numbers, there exist at least two $(m, 1)$-orbits for any value of the eccentricity $0 < e < 1$. Moreover, for all m, n natural numbers, there is a positive number $e_{m,n}$ such that the problem has at least an (m, n)-orbit for any value of the eccentricity $e < e_{m,n}$. The proof used a version of the Poincaré-Birkhoff theorem proved in [3]. The discussions focused on the central theorem and on possible generalizations.

The first talk of the afternoon session was that of Richard Montgomery of the University of California in Santa Cruz. The title of his talk was "Fitting hyperbolic pants to a three-body problem," and his results were inspired by the the Figure Eight solution he and Chenciner had discovered a few years earlier [1]. He considered bounded zero-angular momentum solutions to the $1/r^2$ potential (not Newton's $1/r$) three-body problem. He showed that upon modding out by the symmetries of scaling, translation and rotation, this problem is equivalent to the geodesic flow for a certain metric on the pair of pants, namely the thrice-punctured two-sphere. The sphere is the shape sphere. The punctures are collisions. The metric is the Jacobi-Maupertuis metric at zero energy. It is complete and noncompact. His main result was that if all masses are equal, then the Gaussian curvature of the metric is everywhere negative, except at two points, the Lagrange points. A number of dynamical consequences directly follow, such as the uniqueness of the $1/r^2$ figure-eight solution, and the existence of a complete symbolic dynamics description (symbols are syzygies [4]) for the non-collision bounded solutions. Other papers relevant to his talk are [5], [6], [7].

It is interesting to note that the excellent internet connection in Room 159 at BIRS was of great help during Richard's talk. He pointed at his website (which anyone with a laptop could access) and at several papers, including the preprint of the present talk. These were great additions to the talk and helped deepening the understanding of his results. The discussions that followed showed the clear necessity of an ad-hoc session on the Figure Eight solution. This took place with 6 participants on Tuesday night, after dinner.

The last talk of the day was that of Dan Offin of Queens University. He talked about the variational-stability method for some N-body problems. He showed showed how the variational method can be extended to the variational-stability method for existence and stability type of periodic solutions in certain subsystems of the N-body problem. These include, the isosceles 3-body problem, and the equal mass symmetric 4-body problem. Then he showed that the instability of absolutely minimizing periodic orbits in these systems has implications for the existence of mountain pass critical orbit sand orbits homoclinic to minimizing-type orbits.

The variational method has long traditions in celestial mechanics since Poincaré introduced it in 1896 to obtain periodic orbits in what we call today Manev-type potentials ($1/r + 1/r^2$). The proof of the existence of the Figure Eight solution, of a few more choreographic solutions as well as the numerical discovery of hundreds of periodic orbits in the last few years, have led to some intense research, and several of the researchers present at this meeting work in this direction. Therefore the discussions that followed after Dan's talk focused on technical aspects related to the variational method.

It was a fortunate decision to have Dan talk on Sunday since the same night his wife gave birth to healthy son in Kingston, Ontario, and Dan had to leave on the first flight he could book in Calgary. We missed him, but such circumstances need no further comment.

Monday, the second day of the meeting, had several talks on central configurations. The subject is extremely important in celestial mechanics, it showed up in almost every talk, and two more presentations in different days were dedicated to it.

The first talk of the day was that of Ernesto Pérez-Chavela of Departamento de Matemáticas UAM-Iztapalapa, Mexico City, now on sabbatical leave at the University of Victoria. He talked about symmetrical central configurations in 4-body problems. More precisely, he studied planar central configurations with an axis of symmetry containing two of the particles. The central configurations can be concave or convex, depending if one mass is in the interior of the convex hull of the other three or not. If three of the masses are equal (of unit mass), the axis of symmetry contains the mass m, and we find the total number of central configurations. If two of the masses are equal, and not taking into account the permutations between the equal masses, then there is exactly one convex central configuration. Ernesto also proved the existence of several concave central configurations. The references relevant to his talk are [8], [9] and [10].

The discussions focused on the question of existence of infinitely many central configurations for N given masses. This is an open problem left about 60 years ago. Ernesto as well as Gareth Roberts have shown that for certain types of potentials and/or for negative masses, there exists a continuous set of solutions, but in the Newtonian case the problem is open for more than 4 masses. However, recently important progress was done in this direction, as will become clear from other talks.

The second talk of the day was that of Ken Meyer of the University of Cincinnati. He talked about elliptic central configuration solutions of the N-body problem. This was a paper written in collaboration with Dieter Schmidt and Klaatu. At the beginning of the talk, Ken presented a few scenes from a science-fiction movie: “The Day the Sun Stood Still,” released in 1951. The movie is about an alien who comes to the Earth and tries to save the earthlings from a collision with a comet. One of the scenes shows the alien (as a middle-aged man) with a boy, knocking at the door of a famous professor. On the blackboard in the professor’s study are written the equations of motion of the 3-body problem. It was amusing to listen to the dialogue and see that it was not totally nonsensical relative to mathematics, as it usually is in such movies.Then Ken got into the real mathematics and showed how a planar central configuration of the N-body problem gives rise to a solution where each particle moves on a specific Keplarian orbit while the totality of the particles move on a homothety motion. The totality of such solutions forms a 4-dimensional symplectic subspace. He gave a symplectic coordinate system which is adapted to this subspace and its symplectic complement. If the Keplerian orbit is elliptic, the solution of the N-body problem is called an elliptic central configuration solution. In his coordinate system, the linear variational equations of such a solution decouple into three subsystems. One subsystem simply gives the motion of the centre of mass, another is Kepler’s problem and the third determines the non-trivial characteristic multipliers. Using these coordinates, Ken studied the linear stability of for several cases when $N = 3, 4, 5$. The discussions focused on the stability question, which is fundamental in celestial mechanics. More about this when discussing Gareth Robert’s talk. The last talk of the morning session was given by Patricia Yanguas of Pamplona, Spain. She had obtained the results she presented together with her husband, Jésus Palacián (also present at the meeting) as well as with the colleagues M. Iñarrea, V. Lanchares, A. I. Pascual and J. P. Salas. The title of the talk was “Dynamics of Charged Particles in Planetary Magnetospheres: Periodic Orbits, Two-Dimensional Tori and Bifurcations,” and it presented a study of the dynamics of a charged particle orbiting around a rotating magnetic planet. The system is modelled by the Hamiltonian of the two-body problem perturbed by an axially-symmetric function which goes to infinity as soon as the particle approaches the planet. The perturbation consists in a magnetic dipole field and a corotational electric field. When the perturbation is weak compared to the Keplerian part of the Hamiltonian, the authors averaged the system with respect to the mean anomaly up to first order in terms of a small parameter defined by the ratio between the magnetic and the Keplerian interactions. After truncating higher-order terms, they used invariant theory to reduce the averaged system by virtue of its continuous and discrete symmetries, determining also the successive reduced phase spaces. Once the original system is reduced, they studied the flow of the resulting system in the most reduced phase space describing all equilibria and their stability, as well as the different classes of bifurcations. Finally, they connected the analysis of the flow on these reduced phase spaces with the one corresponding to the original system. More details about this work can be found in [11]. Other relevant references are: [12], [13] and [14]. Since Richard Cushman had done some work in this direction, an interesting discussion about the main results took place at the end of the talk.

The first talk of the afternoon session was that of Alain Albouy, who presented his results about Alain Albouy some (possibly) new “hidden symmetries” in the Kepler problem and Lambert’s theorem. The so-called SO(4) symmetry of the Kepler problem is usually associated to the Gyorgyi-Moser correspondence of this problem with the geodesic flow on the sphere. This correspondence is not time-preserving.Alain showed that there is another symmetry which does not change the time, and discussed the relation with the classical Lambert theorem. Alain pointed at the study [15], which inspired his research. Several questions occurred during the talk and the end-of-the-talk discussions tried to find answers to some of those questions.

The last talk of the day was that of Gareth Roberts of the College of the Holy Cross, near Boston. He talked about some work in progress about the linear stability of the Figure-Eight orbit [1]. This is an interesting topic, which has preoccupied him since Carles Simó came up with numerical evidence that the Figure Eight solution has a very small zone of stability [17]. Gareth had done previous work on the linear stability of the elliptic Lagrangean triangle solutions of the 3-body problem, so he wanted to use this experience in this new case [16]. At the time of his presentation he had no definite results, but he was able to point out the directions and the plan of his research as well as directions he had tried and which seemed to lead nowhere. The discussions focused on the evaluation of the chances of the possible directions of attack.

The first talk of Tuesday morning was that of Jaume Llibre of Barcelona, Spain. He presented some new results he had obtained on certain families of periodic orbits of the Sitnikov problem. This was a continuation of previous work he had done with Montse Corbera and P.J. Torres, [19], [18]. The main goal of this talk was to present a study of the families of symmetric periodic orbits of the elliptic Sitnikov problem for all values of the eccentricity in the interval [0,1). The basic tool for proving our results was the global continuation method of the zeros of a function depending on one-parameter provided by Leray and Schauder and based in the Brouwer degree.

Since the 1960s, when Sitnikov came up with his problem in order to prove the existence of oscillatory solutions in the 3-body problem, the equations of motions have been studied intensively. Jaume's work presented the latest in this direction. The end-of-the-talk discussions were related to technical details in proving the main result.

The second talk of the day was that of Chris McCord of Cincinnati, who presented his latest results on collinear blow-ups and the integral manifolds of the spatial n-body problem. For the past decade, Chris and Ken Meyer had been exploring the dependence of the topology of the integral manifolds (as measured by their homology groups) on the energy and angular momentum. They had analyzed various special cases: spatial 3-body for all energies; planar n-body for all energies; spatial n-body for positive energy. In addition to whatever intrinsic interest these studies may have had, they had also served to isolate the obstacles to solving the general problem: the spatial n-body problem for all energies. It has emerged that all of the obstacles centre around the collinear configurations. By introducing a blow-up of the configuration space at the collinear configurations, Chris was able to understand how the discontinuities at the collinear configurations change the behaviour of the integral manifolds. This in turn allowed him to develop Morse-theoretic formulae for the homology of the spatial integral manifolds. The discussions focused on the perspectives these results open to the understanding of the global dynamics of the n-body problem.

The last talk of the morning was that of Joe Gerver from Rutgers University. He talked about infinite spin and noncollision singularities. It is well known that as n bodies approach a collision in a system with Newtonian potential, they must approach the central configuration manifold. It is not known whether they must approach a single point on this manifold. In particular, a central configuration remains central if it is rotated, and it is an open question whether a set of bodies can revolve an infinite number of times as it approaches a collision in a Newtonian system. (This can of course occur with an inverse squared potential.) Joe presented a possible model for a Newtonian infinite spin collision in the case when the collision is not isolated; instead other bodies, which are involved in a noncollision singularity which occurs simultaneously with the collision, approach the colliding bodies arbitrarily closely, but keeping moving away again. Joe's results in this direction follow his previous work on noncollision singularities [20] and [21]. The discussions focused on other possibilities of achieving such a scenario.

The first talk of the afternoon was that of Manuele Santoprete of the University of California at Irvine, who presented his results as well as some he obtained with Florin Diacu and Ernesto Pérez-Chavela on the qualitative properties of the anisotropic Manev problem. Manuele had just received his Ph.D. degree at the University of Victoria under the supervision of Florin Diacu and the day before the meeting started he learned that he had been awarded the Governor General's Gold Medal at the University of Victoria, for the best dissertation presented in 2003. Anisotropic problems, describing the interaction of two bodies, started to arouse a good deal of interest in the 1970s, when Martin Gutzwiller proposed the Anisotropic Kepler Problem to study connections between classical and quantum mechanics. In recent years other anisotropic potentials have been introduced, as for example the anisotropic Manev problem and the Kepler problem with anisotropic perturbations. In this talk Manuele described some qualitative properties of the anisotropic Manev problem and of the Kepler problem with anisotropic perturbations.In particular he studied collisions, near collision solutions, and the mechanisms responsible for the appearance of chaos. His techniques are a nice combination of dynamical and variational techniques. Papers relevant to his talk are [22] and [23]. The discussions focused on the differences and similarities of the anisotropic and nonisotropic cases as well as on the unusual case of a disconnected infinity manifold

The last talk of the day was that of Ilias Kotsireas of the Wilfried Laurier University, who presented his results about symmetries of polynomial and differential equations. Many systems of polynomial and differential equations arising in celestial mechanics, exhibit various kinds of symmetries that can best be described group theoretically by finite and Lie group actions. Computational approaches to solving systems with symmetries are available for both the polynomial and the differential case. The eigenvalue method for solving polynomial systems is an ideal paradigm to study the effect of the symmetries on the complexity of the method. The eigenvalue method is simplified considerably in the presence of symmetries, in the sense that the sizes of the matrices involved are diminished considerably. Certain aspects of the interplay between methods for solving polynomial and differential systems can be exploited effectively via a well-known polynomial/differential morphism. Ilias's talk was a tour-de-force on how very complicated computations can be performed in celestial mechanics using a computer. The essential references to his talk are [24], [25] and [26].

On Wednesday the first talk was that of Jeff Xia of Northwestern University. He presented his latest results on action-minimizing periodic and quasi-periodic solutions of the n-body problem. These solutions extend the classic Euler and Moulton relative equilibria. This interesting new development can be found in detail in [27] and [28]. The discussions focused on the perspectives this research opens for further investigations.The second talk of the day was that of Richard Cushman of Utrecht, Holland and University of Calgary. He presented his latest results on monodromy in the swing spring. Richard discussed the three degree of freedom classical mechanical system of an elastic pendulum which is tuned to be in 1:1:2 resonance. This explains the following motion: start the pendulum springing in the vertical direction. After a while it begins to swing in a plane and then returns to the springing motion. During successive cycles the swing planes make the same angle with the vertical direction. Richard's explanation used of the concept of monodromy for a Liouville integrable system. The references relevant to his talk are: [29], [30], [31], [32] and [32].

The last talk of the morning and of the day (since the afternoon was dedicated to trips and relaxation) was that of Marshall Hampton of the University of Minnesota at Minneapolis. He presented his latest results on new central configurations in the 5-body problem. Marshall showed the existence of a class of planar 5-body central configuration which contradict an assertion of W. L. Williams in his 1938 paper, "Permanent configurations in the problem of five bodies", in which he claims that there are no central configurations with positive masses and which have 2 masses in the interior of a triangle. His methodology combined ingenious geometrical, algebraic and analytical techniques, see [33], citeSma and [35]. The discussions focused on these results as well as on the recent proof of Rick Moeckel of Minneapolis on the finiteness of the central configurations in the 4-body problem with positive masses.

Thursday, the last day of the meeting, started with Don Saari's talk on analysing central configurations. Don, who is now at the University of California at Irvine, has written a few decades ago a famous paper on the role and properties of central configurations, and in his talk he used a geometric approach to see how central configurations are described. Many traditional results followed from his approach, and it appeared that several new results are forthcoming. The discussions focused on the potential of this new approach.

The meeting was closed by Ed Belbruno of Princeton, who showed how the theoretical results most of the members of this group have obtained can be used in space science. Ed's talk was about the existence of chaos associated with weak ballistic capture and about low energy lunar transfer. A theory to achieve low energy transfers using ballistic capture (with no fuel required), called weak stability boundary theory, was successfully used in 1991 to resurrect a Japanese lunar mission and successfully bring the spacecraft, Hiten, to the Moon using a new type of lunar transfer. This was one of the more spectacular applications of celestial mechanics, and although well known in the aerospace community, was not as known in the dynamical systems/celestial mechanics community, until much more recently.This is because the underlying mathematics of the dynamics of the capture and the transfer itself were not really understood and were understood more from a numerical point of view. A proof has recently been obtained which explains, in part, the capture process. This is accomplished by two ingredients: one is to estimate a special region near the secondary mass point (Moon) in the restricted three-body problem where "weak capture" occurs, and the other is to prove that there exists a hyperbolic invariant set within this region, instrumental in the capture process. This result solves a problem investigated by Alekseev in 1981 in his last published paper.Ed also mentioned a number of applications in astrodynamics and dynamical astronomy. The relevant references to his talk are: [36], [37] and [38].

Overall this was a highly stimulating meeting of which all participants benefited greatly. Everybody has been impressed with the facilities at BIRS, the efficiency of the staff and with the way the meeting was run.

List of Participants

Albouy, Alain (Institut de mécanique céleste et de calcul des éphémérides, Paris)
Belbruno, Ed (Princeton University)
Buck, Gregory (Saint Anselm College)
Chenciner, Alain (Institut de mécanique céleste et de calcul des éphémérides, Paris)
Corbera, Montserrat (Universitat de Vic, Spain)
Cushman, Richard (Utrecht University & University of Calgary)
Diacu, Florin (University of Victoria)
Gerver, Joseph (Rutgers University)
Hampton, Marshall (University of Minnesota)
Kotsireas, Ilias (Wilfred Laurier University)
Lacomba, Ernesto (University Autonoma Metropolitana, Mexico)
Llibre, Jaume (Universitat Autonoma de Barcelona)
McCord, Chris (University of Cincinnati)
Meyer, Kenneth (University of Cincinnati)
Montgomery, Richard (University of California, Santa Cruz)
Offin, Dan (Queens University)
Palacian, Jesus (Universidad Publica de Navarra, Pamplona, Spain)
Perez, Ernesto (Universidad Autnoma Metropolitana-Iztapalapa, Mexico)
Roberts, Gareth (College of the Holy Cross, USA)
Saari, Donald (University of California, Irvine)
Santoprete, Manuele (University of California, Irvine)
Xia, Zhihong (Jeff) (Northwestern University)
Yanguas, Patricia (Universidad Publica de Navarra, Pamplona, Spain)

Bibliography

[1] A. Chenciner and R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Annals of Math.*, **152**, (2000), 881-901.

[2] A. Chenciner, J. Féjoz and R. Montgomery, Rotating Eights I : the three Γ_i families, preprint 2003.

[3] W. Ding, A generalization of the Poincaré–Birkhoff theorem,*Proc. Amer. Math. Soc.* **88** (1983), 341–346.

[4] Richard Montgomery, Infinitely Many Syzygies, *Archives for Rational Mechanics and Analysis*, **164** (2002), 311–340.

[5] H. M. Morse, A one-to-one representation of geodesics on a surface of negative curvature, *Am. J. Math*, **43**, (1921) , 33-51, 1921.

[6] D. Mumford, C. Series, and D. Wright,*Indra's pearls. The vision of Felix Klein.* Cambridge University Press, New York, 2002.

[7] T. Fujiwara, H. Fukuda, A. Kameyama, H. Ozaki, M.Yamada, Synchronised Similar Triangles for Three-Body Orbit with Zero Angular Momentum, arXiv:math-ph/0404056.

[8] A. Albouy and A. Chenciner, The n-body problem and mutual distances,*Inventiones Mathematicae* **131** (1998), 151–184.

[9] W.D. MacMillan and W. Bartky, Permanent configurations in the problem of four bodies,*Trans. Amer. Math. Soc.* **34** (1932), 838–875.

[10] C. Simó, relative equilibrium solutions in the four body problem, *Celestial Mechanics* **18** (1978), 165–184.

[11] M. Iñarrea, V. Lanchares, J. F. Palacián, A. I. Pascual, J. P. Salas and P. Yanguas,The Keplerian regime of charged particles in planetary magnetospheres (2003), preprint.

[12] M. Braun, Particle motions in a magnetic field, *J. Differential Equations* **8** (1970), 294–332.

[13] R. H. Cushman and D. A. Sadovskií, Monodromy in the hydrogen atom in crossed fields, *Phys. D* **142** (2000), 166–196.

[14] H. R. Dullin, M. Horányi and J. E. Howard, Generalizations of the Størmer problem for dust grain orbits, *Phys. D* **171** (2002), 178–195.

[15] G. Györgyi, Kepler's Equation, Fock Variables, Bacry's Generatorsand Dirac Brackets, *Il nuovo cimento* **53A** (1968), 717–736.

[16] G. Roberts, Linear stability of the elliptic Lagrangean triangle solutions in the three-body problem,*Journal of Differential Equations* **182** (2002), 191–218.

[17] C. Simo, New families of solutions in N-body problems.In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*,Progr. Math., **201**, 101–115, Birkhuser, Basel, 2001.

[18] M.Corbera, J. Llibre and P.J. Torres, Global existence of subharmonic orbits in the Sitnikov problem, preprint 2004.

[19] M.Corbera and J. Llibre, Periodic orbits of the Sitnikov problem via a Poincaré map,*Celestial Mechanics and Dynamical Astronomy* **77** (2000), 273–303.

[20] J. L. Gerver, The existence of pseudo collisions in the plane, *Journal of Differential Equations* **89** (1991), 1–68.

[21] J. L. Gerver, Noncollision singularities: do four bodies suffice?, *Experimental Mathematics* **12** (2003), 187–198.

[22] F. Diacu and M. Santoprete, On the global flow of the anisotropic Manev problem, *Physica D* (to appear).

[23] F. Diacu and M. Santoprete, Nonintegrability and chaos in the anisotropic Manev problem, *Physica D* **156** (2001), 39–52.

[24] I. S. Kotsireas, Central configurations in the Newtonian N-body problem of celestial mechanics. In *Symbolic computation: solving equations in algebra, geometry, and engineering(South Hadley, MA, 2000), Cont. Math.* **286**, 71–98, American Mathematical Society, 2001.

[25] R. Corless, K. Gatermann and I. S. Kotsireas, Using symmetries in the eigenvalue method for polynomial systems, in preparation.

[26] H. J. Stetter, *Numerical Polynomial Algebra*,SIAM, 2004.

[27] K. Chen, T. Ouyang and Z. Xia, Action-minimizing solutions in the n-body problem, preprint, 2003.

[28] M. Salomone and Z. Xia, non-planar minimizers and rotational symmetry in the n-body problem, preprint, 2004.

[29] H. Dullin, A. Giacobbe, and R. Cushman, Monodromy in the resonant swing spring, *Physica D* **190** (2004), 15–37.

[30] R.H. Cushman and L. Bates, *Global aspects of classical integrable systems*, Birkhäuser, Basel, 1997.

[31] , R. Cushman, A. Gicobbe, D.A. Sadovskii, and B.I. Zhilinskii, *Monodromy of the quantum 1:1:2 resonant swing-spring*, preprint Université du Littoral April 2004.

[32] R.H. Cushman, H. Dullin, D.D. Holm, A. Giacobbe, M. Joyeux, D.A. Sadovskii, and B.I. ZHilinskii, *The* CO_2 *molecule as quantum realization of the* $1:1:2$ *resonant swing-spring system with monodromy*, (submitted to Physics Review Letters).

[33] D. N. Bernstein, The number of roots of a system of equations, *Fun. Anal. Appl.* **9** (1975), 183-185.

[34] S. Smale, Mathematical problems for the next century,*Mathematical Intelligencer* **20** (1998), 7-15.

[35] G. Zeigler, *Lectures on Polytopes*,Springer Graduate Texts in Mathematics 152, Springer Verlag, New York,1995.

[36] E. Belbruno, *Capture Dynamics, and Chaotic Motions in Celestial Mechanics*, Princeton University Press, Princeton, 2004

[37] E. Belbruno, Existence of Chaos Associated with Weak Capture, and Applications to Low Energy Transfers. In *Astrodynamics, Space Missions, and Chaos,(E. Belbruno, et al,, ed.)*, Annals, **1017**, 1–10,New York Academy of Sciences, 2004.

[38] M. Adler, To the Planets on a Shoestring,*Nature* **408** (2000), 510–512.

Chapter 7

BIRS Workshop on Mathematics and Creative Writing (04w5555)

April 17–22, 2004

Organizer(s): Marjorie Senechal (Smith College), Chandler Davis (University of Toronto)

In the spring of 2004, tucked in between workshops on mathematical logic and foundations, manifolds and cell complexes, Fourier analysis, numerical analysis, probability theory and stochastic processes, mechanics of particles and systems, game theory, economics, social and behavioural sciences, dynamical systems and ergodic theory, and quantum theory, BIRS held its second five day experimental workshop on creative scientific writing. The first took place in September, 2003. Curious colleagues have asked us, and continue to ask, what these workshop were like and what they accomplished. In the self-interview that follows, we address these and other questions.

An Interview with and by the Organizers: Marjorie Senechal (Smith College) and Chandler Davis (University of Toronto)

Why hold workshops on creative scientific writing at BIRS or anywhere else? Mathematics is an art form, so isn't mathematical writing creative? Alas, the population that recognizes the creativity in a mathematical or scientific paper is smaller by many orders of magnitude than the number who remember their Latin. By *creative scientific writing* we mean something else: mathematical and scientific ideas as subjects for poetry, drama, short stories, novels, nonfiction, comic books, essays, and film.

Why would anyone write about science and mathematics in this way? Does anyone do it? Mathematics is part of world culture, part of the human spirit. It's as fit a subject for art, music, and literature as any other. As for who, some mathematicians write poetry, fiction, nonfiction, or drama. And there are non-mathematician poets, fiction writers, nonfiction writers and dramatists whose work engages mathematicians and mathematical ideas. Our first workshop had fifteen participants, all highly accomplished, and the second had twenty, ditto.

But aren't you mixing apples and eggs? Talk about mixing! But, with garlic and salt, apples and eggs make an excellent omelet. We assumed from the start—and now we firmly believe—that non-mathematicians who write creatively about mathematics and mathematicians, and mathematicians engaged in creative writing, have a lot to teach and learn from one another.

Okay, but isn't it confusing to mix all those literary genres? On the contrary! Creative writing is often sparked by cross-genre insights. For example, in our workshops a poet helped a fiction writer find a better way to tell the end of his story. A mathematician nonfiction writer helped a dramatist extend the ideas of her play, ideas a filmmaker sitting in on their discussions recast in doggerel form. A novelist had insightful comments on poetry. Of course, it helped a lot that we pressed everyone to circulate his or her work in advance. By the time we arrived in Banff, we'd read it all, thought about it, and were eager to comment.

Why should BIRS take the lead in encouraging this? Call it "outreach" if you like, part of the larger effort of mathematicians everywhere in these days of dwindling funds to explain who we are and what we do — and why it matters. Or, if you prefer, an effort to engage scientists and mathematicians in a wider world of discourse. The need to create a body of literature around mathematics and science is widely acknowledged by mathematicians and non-mathematicians alike.

But is there an audience for creative scientific writing, as you describe it? The popularity of plays like *Proof* and biographies like *A Beautiful Mind* and *The Man Who Loved Only Numbers* show that there's a large and growing public eager to share in the great ideas of mathematics and science. The creative writer's job is not to coerce them to eating these things like medicine hidden in jam, but to convey these ideas through literature instead of formalism.

Yet except for obvious examples, like those you cited, creative writing about the content of mathematics is extremely rare and creative writing about the activity of mathematical creation is even rarer. That's why we organised the workshops: to encourage practitioners who engage this content in their work. To give them opportunities to discuss important issues, to learn what others are doing, to encourage each other, to critique current work, to welcome young writers into the field, to spark collaborations, to forge networks and build community.

Then the creative writing workshops' goals are the same as any other BIRS workshop! Yes, but as we noted in our report to BIRS after the first workshop, our program is, of necessity, highly experimental. In the first workshop we followed the standard practice of assigning each participant an hour lecture slot. But that didn't always give people the detailed, line-by-line, feedback some hoped for. And a few people read work they'd already published, so feedback was moot. We found we needed to set aside time for other things too. So, for the second workshop we modified the format in various ways.

How did you organise the time? Well, a typical day went like this:
8–9: Breakfast in Corbett Hall, BIRS's headquarters
9–10: Reflections: the full group meets to discuss, orally or in writing, issues raised or works presented the previous day; further ideas and inspirations.
10–11 and 11–12: Two presentations of works-in-progress to the full group, followed by discussion.
12–1:30: Lunch
1:30–4:30: Time free for writing
4:30–6: Parallel sessions — as many as anyone wished — on works-in-progress in small groups, two to ten, for line-by-line comments and editing
6–7:30: Dinner in the Banff Centre dining hall
7:30–9: Discussions of general issues, or public readings with participants in the Banff Centre's workshop Writing With Style.

Tell us about the evening discussions. The first was called, "What, Why, and For Whom?" It covered a lot of ground, from lamenting math phobia and emphasizing the need for better science and mathematics education, to considering the many forms that outreach can take. And we lamented the worm in the apple: Proof, A Beautiful Mind and other popular works wouldn't have been so successful had the mathematician character been sane.

Audiences always prefer demented geniuses, or flawed ones. Not only scientists and mathematicians. Think of Amadeus, about Mozart, and all those films about van Gogh. It's true, it's very difficult to portray intellectual creativity of any kind. But the scientific/mathematical nut is tougher to crack — sorry, wrong metaphor. I mean, the mad composer or painter or writer can be shown composing or painting or writing madly, furiously, but in the end he or she produces something the public can hear or see or read. While a mathematician, mad or sane, produces a mystifying theorem. But on the other hand, the play Copenhagen was a great success and the novel Einstein's Dreams conveys the scientific creative process in a beautiful way. And Arcadia, a funny and chaotic play whose leitmotif is chaos theory, is a modern classic. The mathematical formalism is symbolized in its structure.

Using a mathematical structure to talk about math — that reminds me of a sonnet by Edna St. Vincent Millay, "Euclid alone has looked on Beauty bare." The poem's strict form mirrors deductive geometry's austere beauty. "Fortunate they Who, though once only and then but far away, Have heard her massive sandal set on stone." Would you say the sonnet form has mathematical affinities? One of us would, the other wouldn't. But that's a discussion topic for a future workshop. Back to your earlier question: our discussion the last night was, "Where Do We Go From Here?"

The last night? Then tell us first about the public readings. Well, as you know, BIRS is located in the world-renowned Banff Centre. With studios nestled in the woods, outstanding mentors, excellent performance spaces and a fine library, the Banff Centre nurtures aspiring, mid-career, and established musicians, painters, photographers, writers, and actors. Artists love Banff. And Banff loves the artists: the centre's world-class exhibitions, public readings, and performances enhance Banff's appeal to tourists year-round. The BIRS leadership hopes BIRS will interact with the Centre. So in organising our workshops, we worked closely with Carol Holmes and Edna Alford of the Banff Centre's Writing and Publishing Program. Their "Writing in Style" workshop and our second workshop took place the same week. On two evenings, we merged the two groups for informal public readings. A few participants in other BIRS and Banff Centre programmes attended too. We hosted an evening of poetry, with eleven readers from both groups. They hosted a prose reading evening, with fewer readers of course, but again from both groups.

And were these readings successful? Very. Participants in the two groups met one another and some of their conversations continued at meals the next day. Another important benefit was the opportunity for writers, in both groups, to read their work to and get responses from audiences outside their usual orbits.

So where do we go from here? In many directions! Workshop participants plan to stay in touch, and to keep each other informed of the progress of their work. We will share information about publishers and agents. Someone suggested we ask BIRS to link our publications to its website. The Mathematical Intelligencer already encourages creative writing in mathematics, but we want it to do even more. We hope to hold another workshop at BIRS in the future, in close association with Banff Centre writing programs, and to publish an anthology under their auspices.

I can see it now: a hefty tome, the year's Best Creative Writing in Mathematics. Yes, the hottest item in the bookstore, its sales topping the year's best short stories, best essays, best mystery stories, best political fiction, best non-required reading, best recipes, best science and nature writing, best spiritual writing, best sports writing, best travel writing, and best erotica.

Hors de doute. Merci.

List of Participants

Abate, Marco (Universita di Pisa)
Brandts, Wendy (Independent Physicist and Writer)
Davis, Chandler (University of Toronto)

Diacu, Florin (University of Victoria)
Granville, Andrew (Universite de Montreal)
Granville, Jennifer (Ohio University)
Grosholz, Emily (The Pennsylvania State University)
Hansen, Vagn L. (Technical University of Denmark)
Holmes, Philip (Princeton University)
Maddow, Ellen (Playwright and Composer)
Moody, Robert (University of Alberta)
Priestley, William McGowen (Mac) (University of the South)
Senechal, Marjorie (Smith College)
Suri, Manil (University of Maryland Baltimore County)
Tasic, Vladimir (University of New Brunswick)
Thomas, Robert (University of Manitoba)
Tuffley, Chris (University of California, Davis)
Wali, Kameshwar (Syracuse University)
Zwicky, Jan (University of Victoria)

Chapter 8

Microeconometrics of Spatial and Grouped Data (04w5036)

April 24–29, 2004

Organizer(s): Thomas Lemieux (University of British Columbia), David Card (University of California, Berkeley)

The objective of the workshop is to bring together a group of applied economists and econometricians who share a common interest for economic problems in which the group or spatial dimension play an important role. Each of the two groups will bring a unique expertise on these issues. The contribution of the applied researchers will be to present the economic models and the complex data sets they need to use to estimate these models. The econometricians will present innovative research showing how standard estimation and inference procedures can be adapted to settings where group or spatial effects are important. The presentations will be concentrated in the morning to leave enough time for intensive but free exchanges in the afternoon. The hope is that applied researchers will come out of the workshop with a much better sense of which econometric tools can be used in their research given the complex nature of the underlying data. By contrast, econometricians will have a unique opportunity to see the types of models and data being used in cutting-edge empirical research. This should suggest interesting avenues for their future research. One final objective of the workshop is to encourage new joint research projects between participants, and in particular between applied researchers and econometricians.

The model for this particular workshop is a Summer Symposium that David Card and Daniel McFadden organized on quasi-experimental methods at Berkeley in August 1999. This particular symposium brought together a group of applied economists and econometricians interested in this particular topic. The Banff International Research Station provides a unique opportunity to apply this successful model to new econometric and applied issues that have emerged since 1999.

List of Participants

Anderson, Siwan (University of British Columbia)
Battistin, Erich (Institute for Fiscal Studies)
Boozer, Michael (Yale University)
Bugamelli, Matteo (Bank of Italy)
Card, David (University of California, Berkeley)
Cipollone, Piero (Bank of Italy)
Clark, Damon (University of California, Berkeley)
Coelli, Michael (University of British Columbia)
Duflo, Esther (Massachusetts Institute of Technology)

Firpo, Sergio (University of British Columbia)
Friesen, Jane (Simon Fraser University)
Graham, Bryan (Harvard University)
Ichino, Andrea (European University Institute)
Kling, Jeffrey (Princeton University)
Krauth, Brian (Simon Fraser University)
Lee, David (University of California, Berkeley)
Lemieux, Thomas (University of British Columbia)
Martorell, Paco (University of California, Berkeley)
McEwan, Patrick (Wellesley College)
Miguel, Edward (University of California, Berkeley)
Milligan, Kevin (University of British Columbia)
Moretti, Enrico (University of California, LA)
Oreopoulos, Philip (University of Toronto)
Riddell, Craig (University of British Columbia)
Rothstein, Jesse (Princeton University)
Sakata, Shinichi (University of British Columbia)
Verhoogen, Eric (University of California, Berkeley)

Chapter 9

Singular Cardinal Combinatorics (04w5523)

May 1–6, 2004

Organizer(s): Matt Foreman (University of California, Irvine), Claude Laflamme (University of Calgary), Stevo Todorcevic (University of Toronto and CNRS Paris)

From May 1 to May 6, 2004 24 set theorists met at the Banff International Research Station to discuss Singular Cardinal Combinatorics. Descriptions of the contents of their talks will be published in a Proceedings that will appear in the Notre Dame Journal of Symbolic Logic.

During the workshop, several important new results were announced and explained, and there were problem sessions held (some with significant amounts of prize money attached to particular problems, see the last section for details). To summarize the direction of the conference we will present here an annotated collection of representative problems with some references. Where the problems were novel, attribution is attempted and it is noted where there is money attached to particular problems.

Three closely related themes dominated the discussion: stationary sets and stationary set reflection, variations of square and approachability and the singular cardinals hypothesis. Underlying most of the discussion were ideas from Shelah's PCF theory. Important subthemes were mutual stationarity, Aronszajn trees and superatomic Boolean Algebras.

The Singular Cardinals Hypothesis and Hilbert's First Problem

In 1871, Cantor showed that for every cardinal κ the cardinality of the collection of subsets of κ (which we call 2^κ) is at least the cardinal successor of κ (which we call κ^+). For infinite cardinals, it is independent of the usual assumptions of mathematics (the axioms "ZFC") whether $2^\kappa = \kappa^+$. Indeed the question of whether cardinality of all subsets of the natural numbers is equal to the first uncountable cardinal was the first problem on the famous list of problems presented by Hilbert at the 1900 International Congress of Mathematics. Partial information on this question is given by *Konig's Theorem* which says that the cofinality of 2^κ is at least κ^+.

Godel showed that in the Constructible Universe L, the *Generalized Continuum Hypothesis* holds; namely for all infinite cardinals $\kappa, 2^\kappa = \kappa^+$. For regular cardinals Konig's theorem is all one can say: it is a theorem of Easton that if $V \models GCH$ then for all monotone functions $f : OR \to OR$ such that $f(\alpha) \geq \alpha$ and $cf(\aleph_{f(\alpha)}) > \aleph_\alpha$ there is a generic extension of V where $2^{\aleph_\alpha} = \aleph_{f(\alpha)}$ for all α where $\aleph_\alpha$ is regular.

At singular cardinals the situation turns out to be quite different. Silver proved that if λ is a singular cardinal of uncountable cofinality and for a stationary collection of $\kappa < \lambda, 2^\kappa = \kappa^+$ then $2^\lambda = \lambda^+$. ([13]) This was improved by Galvin and Hajnal to get general bounds on the power of a singular cardinal of uncountable cofinality in terms of the behaviour of the power of smaller singular cardinals ([7]). At the conference, Gitik announced recent results along this line, that are summarized in his paper for the proceedings.

This left the problem of cardinals with countable cofinality quite open. Magidor ([9]) showed that Silver's theorem is false for cardinals of countable cofinality: assuming large cardinals it is consistent for $2^{\aleph_\omega} > \aleph_{\omega+1}$ with the GCH holding below $\aleph_\omega$. After this result it was generally thought that the behaviour of the power of singular cardinals of cofinality ω was as arbitrary as that of regular cardinals.

However in the late 1980's S. Shelah proved a series of results getting cardinal bounds on the behaviour of the power function at singular cardinals by studying reduced products of cardinals below the singular cardinal. This ultimately led to a powerful general tool, known as PCF theory ([12]). This theory has had many applications outside the study of cardinal arithmetic, constructing examples of Jonnson algebras on successor of singular cardinals, and providing interesting examples in set theoretic topology and algebra.

PCF Theory Problems

We will say that a set A is an *interval of regular cardinals* if it is the intersection of an interval of cardinals with the regular cardinals. A will be called *progressive* iff $|A| < \min(A)$. If A is a set of regular cardinals then $PCF(A)$ is defined to be:

$$\{cof(\prod A/D) : D \text{ is an ultrafilter on A}\}.$$

Shelah showed that if A is a progressive interval of regular cardinals with supremum λ then

$$cf(\langle[\lambda]^{|A|^+}, \subset\rangle) = \max PCF(A).$$

In particular $\max PCF(A)$ always exists. As an immediate corollary one sees that if $|A| < \kappa < \lambda$ and κ is regular then

$$[\lambda]^\kappa = 2^\kappa \times \max PCF(A).$$

In particular, if λ is a singular strong limit cardinal of cofinality κ that is not a cardinal fixed point then $2^\lambda = 2^\kappa \times \max PCF(A)$.

It remains to bound the cardinality of $PCF(A)$. Shelah did this by proving the remarkable theorem that if A is a progressive interval of cardinals then

$$(\dagger) \qquad |PCF(A) \leq |A|^{+3}.$$

Putting these results together we get the following corollary:

Theorem(Shelah) Suppose that λ is a singular cardinal of cofinality κ and is not a cardinal fixed point. Then

$$2^\lambda < \max((2^\kappa)^+, \aleph_{\kappa^{+4}}(\lambda)).$$

In particular if $\aleph_\omega$ is a strong limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Despite significant progress by Gitik, Shelah, Woodin and others, it is not known if these bounds are optimal. Our first questions relate to this:

Question 1 Is it consistent to have a progressive set A such that $|PCF(A)| > |A|$?

Question 2 Is it consistent that

$$\max PCF\{\aleph_n : 1 \leq n < \omega\} > \aleph_{\omega_1}?$$

Question 3 Is it possible that

$$\{\kappa < \lambda : \max PCF(\kappa) \geq \lambda\}$$

be uncountable?

Question 4 Is it possible that

$$\{\kappa : cf(\kappa) > \omega \text{ and } \max PCF(\kappa) \geq \lambda\}$$

be infinite?

The assumption that the answers to questions 3 and 4 are "no" is known as the Shelah *weak hypothesis*. (These questions are well known, but relayed to the author by M. Gitik.)

PCF Structures

There are several collections of axioms that have been proposed to capture the essence of PCF theory. Indeed Shelah's original bound $(\dagger)$ was proved by summarizing results about the behaviour of real PCF structures and showing that any structure satisfying his summary had to have small cardinality.

Jech ([8]) found a very weak collection of axioms that suffice to prove Shelah's bound. Here our intention is different. We want to find as strong a collection of axioms as possible and see if they can prove a better bound.

This project then has two directions: the first is to establish whether a better bound on the size of PCF structures can be proved. The second is to find a "complete" axiomatization of PCF structures. We will use here an axiomatization due to Magidor (with aid from Foreman). It appeared in print in the Ph.D. thesis of John Ruyle (1998).

The PCF Topology

Inherent in the axiomatization is the PCF topology. The operation $A \mapsto PCF(A)$ is a closure operator and hence there is a natural topology associated with the PCF operation. For simplicity we will restrict ourselves to progressive sets A of regular cardinals that have no limit points that are cardinal fixed points.

Explicity: $A \subset PCF(A)$ and for all $B, C \subset PCF(A)$,

1. If $B \subset C$ then $PCF(B) \subset PCF(C)$
2. $PCF(B \cup C) = PCF(B) \cup PCF(C)$.
3. $PCF(PCF(B)) = PCF(B)$.

The PCF topology is compact Hausdorff, 0-dimensional and scattered. Via Stone duality there is a direct connection between locally compact Hausdorff, 0-dimensional, scattered spaces and superatomic Boolean Algebras. Namely given such a space X, the regular open sets form a superatomic Boolean algebra whose Stone space is the original space X.

To review:

Let B be a Boolean Algebra. Define a transfinite sequence of ideals in B by setting:

- J_0 to be the ideal generated by the atoms of B
- $J_{\alpha+1}$ the ideal generated by the atoms of B/J_α and J_α
- for limit α, $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$.

B is *superatomic* iff whenever J_α is a proper ideal, B/J_α is atomic. (We will use the jargon "SBA" for superatomic Boolean algebra.)

If one traces through the proof of Stone duality, it is immediate that the atoms of B/J_α correspond canonically with the isolated points in the α^{th} Cantor-Bendixson derivative of the Stone space of B.

We now give some more definitions necessary to formulate the PCF axioms:

1. The *height* of B is the least $\alpha, J_\alpha = B$.
2. The *rank* of $b \in B$ is the least $\alpha, b \in J_\alpha$.
3. c_α is defined to be the cardinality of $\{b \in B : \text{rank of } b = \alpha\}$.
4. The *cardinal sequence* of B is $\langle c_\alpha : \alpha < \text{height of } B\rangle$.

There is a standard mechanism for building SBA's involving well-founded partial orderings. Let $<^*$ be a well-founded partial ordering on a set T. For $t \in T$, let $b_t = \{s : s <^* t\}$.

An *SBA ordering* will be a pair $(<^*, i)$ such that $<^*$ is a well-founded ordering on a set T and

$$i : [\theta]^2 \to [\theta]^{<\omega}$$

is such that

1. for all $s, t, i(s,t)$ is a minimal set such that;

$$b_s \cap b_t = \bigcup_{u \in i(s,t)} b_u$$

(so if $i(s,t) = \{u_0, \ldots u_n\}$ then

$$b_s \cap b_t = b_{u_0} \cup \cdots \cup b_{u_n}.)$$

2. For all $t \in T$, α less than the $<^*$-rank of t,

$$b_t \cap \{s : \text{rank}(s) = \alpha\}$$

is infinite.

Other authors call SBA orderings *"selectors"* or *"admissible partial orderings"*. Given an SBA ordering on a set T we can topologize T by taking basic open sets to be of the form:

$$b_t \backslash (b_{u_0} \cup b_{u_1} \cup \ldots b_{u_n}).$$

The following proposition is standard:

Proposition Let $(<^*, i)$ be an SBA ordering on a set T and endow T with the topology above. Then:

1. T is locally compact, Hausdorf, 0-dimensional and scattered.
2. $T \subset b_{u_0} \cup b_{u_1} \cdots \cup b_{u_n}$, for some u_i's then T is compact.
3. The α^{th} Cantor-Bendixson derivative of T is $\{t$: the $<^*$-rank of t is at least $\alpha\}$.
4. The algebra of clopen subsets of T is an SBA with cardinal sequence

$$c_\alpha = |\{t : \text{ the rank of } t = \alpha\}|.$$

We are now in a position to give the PCF axioms:

Definition An δ-*PCF structure* is an SBA partial ordering $<^*$ on a successor ordinal θ satisfying:

PCF1 $\nu <^* \mu$ implies $\nu \in \mu$.

PCF2 $\overline{\delta} = \theta$.

PCF3 If $I \subset \theta$ is an interval, the $\overline{I}$ is also an interval.

PCF4 For each $\nu < \theta$ of uncountable cofinality, there is a closed unbounded $C_\nu \subset \nu$ such that $\overline{C_\nu} \subset \nu + 1$.

PCF5 θ is compact with the $<^*$ topology.

The main point of the axioms is that the work of Shelah shows that the PCF axioms are true:

Theorem (Shelah, [12])Let A be a progressive interval of regular cardinals of order type δ. Then there is an ordering $<^*$ on $PCF(A)$ which makes $PCF(A)$ into a PCF structure.

(Hint: To define $<^*$, find a "transitive" collection of generators $\langle b_\alpha : \alpha < \max PCF(A) \rangle$ for the PCF ideals on $PCF(A)$ and define $\beta <^* \alpha$ iff $\beta \in b_\alpha$.)

We now are in a position to state the main open questions involving PCF structures.

Question 5 Do the PCF axioms capture ALL of PCF theory? (PCF completeness)

Question 6 What PCF structures consistently exist?

We need some more background to make these questions explicit:
Let $(\theta, <^*)$ be a δ-PCF structure. Let $\langle c_\alpha : \alpha < ht(<^*)\rangle$ be the cardinal sequence of $(\theta, <^*)$. Then:

1. ($|\delta|$-tightness/localization) If $A \subset \theta$ and $\alpha \in \overline{A}$ then there is a $B \in [A]^{|\delta|}$ such that $\alpha \in \overline{B}$. (In fact, using results of Todorcevic, if $\delta = \omega$ the topology is "sequential".)

2. If X is closed then $\sup X \in X$.

3. For $\xi < ht(<^*), c_\xi \leq |\xi|$.

4. If $\theta = \kappa + 1$, then there is a closed unbounded set of $\xi < \kappa$ such that $c_\xi \leq |\delta|$.

These facts show a close connection between PCF structures and the literature about cardinal sequences for SBA's, especially those that have each $c_\alpha = \omega$. Using the work of Baumgartner and Shelah ([1]) and extending work of Velickovic, Ruyle proved that if $\langle c_\alpha : \alpha < \omega_2\rangle$ is a cardinal sequence with $c_\alpha = \omega$ on a closed unbounded set, then there is a cardinal preserving forcing for adding an SBA on $\omega_2 + 1$ with this cardinals sequence (and a little further). Moreover, if $\langle c_\alpha : \alpha < \gamma < \omega_2\rangle$ is a cardinal sequence where $c_\alpha = \omega$ for $\alpha < \omega_1$ and $|c_\alpha| \leq \omega_1$, then there is a PCF algebra of height $\gamma + 1$ with this cardinal sequence.

Question 7 Is it consistent that there is an ω-PCF algebra of size ω_3? (If not, there is a better bound on $2^{\aleph_\omega}$.)

This requires some new SBA techniques as there are no known examples of SBA's of height ω_3+1 which have each countable level countable, and in which there are a closed unbounded collection of levels of cardinality ω_2 that are countable.

Question 8 Is it consistent that there are ω-PCF algebras of height δ for all $\delta < \omega_3$? What about $\delta = \iota + 1$ where ι is the first indecomposible ordinal above ω_2?

Question 8 may not require new SBA techniques, as Martinez, in work exposited at the workshop, has showed it consistent that there are thin SBA algebras of all heights less than ω_3.

The question of "PCF completeness" is a little vaguer, and may involve all of the difficulties of the SCH itself. However here is a concrete version of the question that may be somewhat easier:

Question 9 Assuming large cardinals, is it true that if $\mathfrak{A}$ is a PCF structure then there is a forcing extension which produces a κ such that $\mathfrak{A}$ is isomorphic to a closed subset of $PCF(\kappa) \cap \{\text{regular cardinals}\}$?

This subset should be of the form $PCF(A)$ where A is a progressive subset of the regular cardinals of κ.

We conclude with a problem of Todorcevic about PCF structures. Topological results of Todorcevic can be used to show that PCF structures are *sequential*. This leads to the question:

Question 10 What is the *sequential rank* of $PCF(\{\aleph_n : n > 1\})$?

In his talk, Martinez gave a collection of problems about the structure of SBA's that are not necessarily PCF algebras. These problems will appear in the proceedings of the conference.

Stationary Set Reflection, Variations of Square, Scales and Aronszajn Trees

In 1989 Woodin and others asked whether the failure of the Singular Cardinals hypothesis at a cardinal κ of cofinality ω implied the existence of an Aronszajn tree on κ^+. The existence of special Aronszajn trees was proved by Jensen in the 1970's to be equivalent to the existence of a weak square sequence, so Woodin's question seems closely related to questions about square sequences of various types. Investigations of square properties in inner models for large cardinals led to the isolation of certain square properties weaker than conventional square. ([11]) These turned out to have direct relations to previously known combinatorial properties such as weak square and very weak square ([5].) In this section we present some background and state some problems that remain open.

We begin first by motivating Woodin's question: As noted in the previous paragraph, Jensen showed that there is a special Aronszajn tree on κ^+ iff $\square^*_\kappa$ holds. Shelah showed that there are no Aronszajn trees on κ^+ if κ is a limit of countably many strongly compact cardinals. Using this work, Magidor and Shelah ([10]) showed that if it is consistent that there is a 2-huge cardinal then it is consistent that there is no Aronszahn tree on $\aleph_{\omega+1}$.

Lacking any evidence to the contrary these results suggest that the failure of existence of Aronszahn trees on successors of cofinality ω cardinals is tied to being a limit of strongly compact cardinals. Since results of Solovay ([14]) show that the SCH holds above a strongly compact cardinal Woodin's question seems quite natural. We list it in the following form:

Question 11 If there are no Aronszajn trees on $\aleph_{\omega+1}$ and $\aleph_\omega$ is a strong limit, is it true that $2^{\aleph_\omega} = \aleph_{\omega+1}$?

Cummings, Foreman and Magidor initiated a program of giving an affirmative answer to Woodin's question. The philosophy was to try to use PCF theory to construct Aronszajn trees. It has the following components:

1. Isolate PCF properties that are consequences of square.
2. Show that they imply the existence of A-trees
3. Show that they follow from the failure of SCH

Figure 9.1 is a summary of the results of this program. This diagram includes results from ([5],[4],[2],[3]). Some of the arrows and non-arrows in the diagram were the main contents of the series of talks given by Cummings and Magidor at the workshop.

Recent results of Gitik and Sharon deal a major blow to this program when they showed:

Theorem (Gitik, Sharon) From appropriate large cardinals follows the relative consistency of:

1. λ is singular strong limit of cofinality $\omega, 2^\lambda > \lambda^+$ and the approachability property fails.
2. There is a singular strong limit cardinal λ, and $\langle \lambda_i : i \in \omega \rangle$ cofinal in λ with $PCF(\lambda_i : i \in \omega) = \{\lambda_i : i \in \omega\} \cup \{\lambda^+\}$ but no very good scale on $\langle \lambda_i \rangle$ of length λ^+.
3. λ is a singular strong limit cardinal, $2^\lambda > \lambda^+$ and every stationary subset of λ^+ reflects.

In particular these results show that one cannot hope to prove (for example) that the failure of the SCH implies the approachability property or that there is a very good scale. Both of these latter propositions were viewed as candidates for a property intermediate between the failure of the SCH and the existence of Aronszajn trees.

There are some potential loopholes in the Gitik/Sharon results though. Their arguments can be improved to make λ into $\aleph_{\omega^2}$, but are not yet known to apply to $\aleph_\omega$. Thus, they may not be directly relevant to Question 11. There are examples of properties (such as the equivalence between the approachability property and Very Weak Square) that hold at $\aleph_\omega$, but not at $\aleph_{\omega^2}$. A very strong conjecture might be that the following question has an affirmative answer:

Question 12 If $2^{\aleph_\omega} > \aleph_{\omega+1}$, then $\square^*_{\aleph_\omega}$ holds.

Moreover, in the second result, the sequence $\langle \lambda_i : i \in \omega \rangle$ is not the generator b_{λ^+}. In particular, the following remains open:

Question 13 If λ has cofinality ω, is it true that there is some sequence $\langle \lambda_i : i \in \omega \rangle$ cofinal in λ which has a very good scale of length λ^+.

The problem of the relation between scale properties and Aronszajn trees seems interesting on its own merits. A typical question here might be:

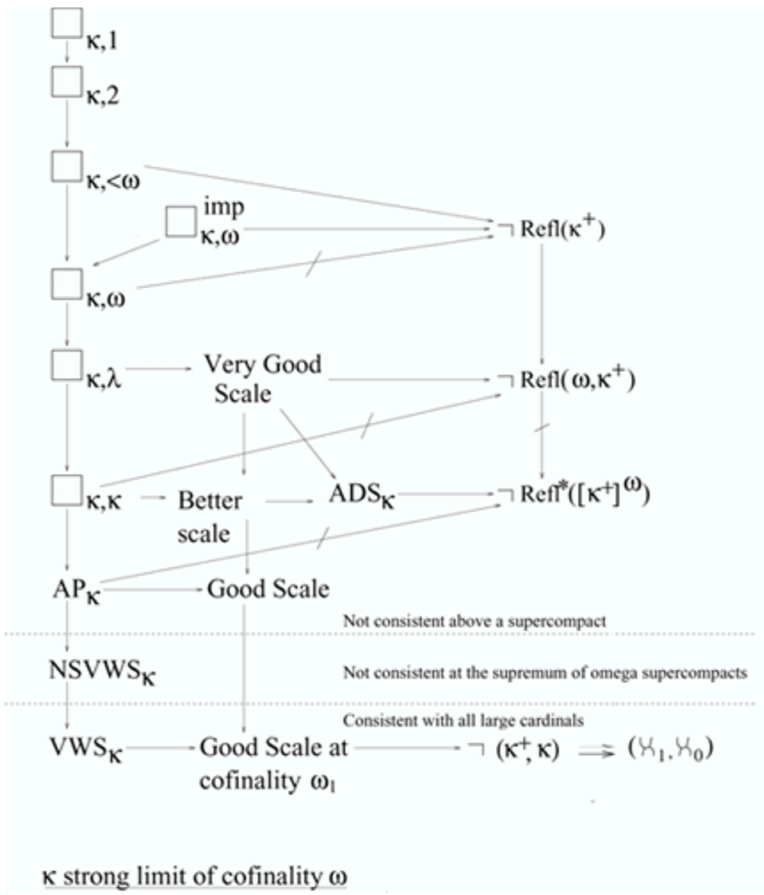


Figure 9.1: Squarelike consequences of PFA

Question 14 If λ has cofinality ω and there is some sequence $\langle \lambda_i : i \in \omega \rangle$ cofinal in λ which has a very good scale of length λ^+ is it necessarily true that there is an Aronszajn tree on λ^+?

Affirmative answers to both questions 13 and 14 yield a solution to Woodin's question.
A variation of questions 13 and 14 is:

Question 15 If λ^+ has cofinality ω and the approachability property holds at λ^+, is it necessarily true that there is an Aronszajn tree on λ^+? If the SCH fails at λ does the approachability property hold?

We note that the diagram leaves many problems open (and there are "obvious" arrows that we have not included in the diagram).

$I[\lambda]$ and Partial Squares

Shelah's ideal $I[\lambda]$ was an important topic in the workshop. This ideal can be defined as follows:

Definition Let λ be a regular cardinal. Let $\vec{X} = \langle a_\alpha : \alpha < \lambda \rangle$ be a sequence of bounded subsets of λ. Define $A(\vec{X})$ (the ordinals *approachable with respect to X*) as the collection of all $\beta < \lambda$ such that there is a set $C \subset \beta$ such that:

1. C is unbounded in β and the order type of C is the cofinality of β.

2. For all $\gamma < \beta$ there is an $\alpha < \beta$ such that $C \cap \gamma = a_\alpha$.

This ideal is normal and λ-complete and turns out to have close connections to forcing, especially for arguments that show (λ, ∞)-distributivity.

If $\lambda = \kappa^+$ and $[\kappa^+]^{<\kappa^+}$ has cardinality κ^+, then $I[\kappa^+]$ contains a stationary set S such that $I[\kappa^+]$ is generated by the non-stationary ideal restricted to $\kappa \backslash S$. Without the cardinal arithmetic assumption, it was a longstanding open problem whether $I[\kappa^+]$ contained a stationary subset of $\kappa^+ \cap cof(\kappa)$. This was recently settled by Mitchell who showed that at ω_2 this need not be the case. His techniques also show that it is consistent that $I[\omega_2]$ is not generated by a single set over the non-stationary ideal. Mitchell's results will appear in the proceedings of this conference. While it appears promising it is not completely clear that Mitchell's techniques generalize to ω_3. Thus we ask the following question which might not remain open for long:

Question 16 For regular $\kappa \geq \omega_2$ must $I[\kappa^+]$ contain a stationary subset of $\kappa^+ \cap cof(\kappa)$?

Because of its close connection to forcing it would be very useful to know the answers to the following questions:

Question 17 Can $I[\omega_2]$ be ω_3-saturated? Can $I[\omega_2] \subset J$ for some ω_3-saturated ideal J on ω_2?

The *approachability property* mentioned above is the statement that $I[\lambda]$ is not a proper ideal. If square holds, then the square sequence itself is a witness to $\lambda \in I[\lambda]$. In general, $I[\lambda]$ can be viewed as those sets on which there is a defective square sequence, with its timing out of order.

We now define a closely related notion. If $S \subset \lambda$ then a *partial square sequence* on S is a sequence of sets $\langle C_\alpha : \alpha \in S \rangle$ such that

1. C_α is an unbounded subset of α of order type the cofinality of α.
2. If β is a limit point of both C_α and C_γ ($\alpha, \gamma \in S$) then $C_\alpha \cap \beta = C_\gamma \cap \beta$.

Shelah showed that if $\mu < \kappa$ are regular then $\kappa^+ \cap cof(\mu) = \bigcup_{\delta \in \kappa} S_\delta$ where each S_δ carries a partial square sequence. In particular, $\kappa^+ \cap cof(\mu) \in I[\kappa^+]$.

At successors of singular cardinals, this type of question appears quite open. In particular we would like to know the following:

Question 18 Is it provable in ZFC that there is a partial square sequence on a stationary subset of $\aleph_{\omega+1} \cap cof(\omega_1)$? On other cofinalities?

In contrast to the successors of regular cardinals, it is always the case that $I[\kappa^+]$ contains a stationary set: if κ is singular and $\mu < \kappa$ is regular, then $I[\kappa^+]$ contains a stationary subset of $cof(\mu)$. Indeed in most cofinalities it not known if $I[\kappa^+]$ can be a proper ideal. At $\aleph_{\omega+1}$ it is consistent that there is a stationary subset of $\aleph_{\omega+1} \cap cof(\omega_1)$ that does not belong to $I[\aleph_{\omega+1}]$, but this is not known at other cofinalities. This is our next question:

Question 19 Does $I[\aleph_{\omega+1}]$ contain a closed unbounded set relative to cofinality ω_2?

A related question is:

Question 20 At successors of singular cardinals, is $I[\lambda]$ generated by a single set over the non-stationary ideal?

In the same vein, it would be interesting to understand the relationship between the collection of approachable points in successors of singular cardinals and other natural stationary sets. A typical question here might be described as follows. If $b_{\aleph_{\omega+1}}$ is the generator for $PCF(\{\aleph_n : n \in \omega\})$ at $\aleph_{\omega+1}$, then relative to a closed unbounded set any two continuous scales agree on the collection of good points. Hence the collection of "good points" form a well-defined stationary set (modulo the closed unbounded filter). An extreme form of a question relating canonical structure would be:

Question 21 Is $I[\aleph_{\omega+1}] = NS \restriction \{\text{Good Points}\}$?

We note that it is known that $I[\aleph_{\omega+1}]$ ([12], [2], [3]) includes $NS \restriction \{\text{Good Points}\}$ and that if square holds below $\aleph_\omega$, then the two ideals coincide.

At the workshop Eisworth gave a collection of problems involving a "recipe" for generating ideals from square like principles and his contribution to the proceedings will list these questions.

Stationary Sets

In [6] Foreman and Magidor began to develop a theory of stationary sets for singular cardinals of countable cofinality. We work on the $\aleph_n$'s for simplicity. Since a subset $A \subset \aleph_\omega$ naturally gives rise to a sequence of subsets $S_n = A \cap \omega_n$ we deal with sequences of subsets of the ω_n's directly.

Let θ be a large regular cardinal and $S \subset PP(\theta)$. Let $\langle S_n : m \leq n \in \omega \rangle$ be a sequence of sets with $S_n \subset \omega_n$. Then the sequence S_n is S-*stationary* iff

$$\{N : \text{for all } n \geq m, \sup N \cap \omega_n \in S_n\} \in S$$

Define $\chi_N(n) = \sup N \cap \omega_n$. Then we can restate this as saying that $\chi_N \in \prod_{m \leq n} S_n$. To illustrate the definition we give two important examples:

Example 1 $S = \{A \subset \theta : A \text{ is stationary}\}$. For this example we call the sequence *mutually stationary*.

Example 2 $S = \{A \subset \theta : A \text{ is stationary and consists of tight structures}\}$, where N is *tight* iff $N \cap \prod \omega_n$ is cofinal in $\prod(N \cap \omega_n)$ (i.e. $N \cap \prod \omega_n$ is cofinal below χ_N.) This is called *tight stationarity*.

We note that there are many other interesting examples taken by varying S. One is obtained by taking S to be the internally approachable structures.

The theory of mutual stationarity and its variants is still in its infancy despite some success. In particular there are a large number of embarrassing problems still completely open. (Welch, in his proceedings article, gives another collection.)

Question 22 Is there a ZFC example of a sequence of stationary sets $\langle S_n \subset \omega_n : n \in \omega \rangle$ such that $\langle S_n \rangle$ is not mutually stationary? For concreteness we may demand that $S_n \subset cof(\omega_1)$. Find a *combinatorial* property that implies the existence of such a set.

Foreman and Magidor showed that such a sequence exists in L and Welch, Schindler and others have extended their results to certain inner models for large cardinals. The question of the existence of such sequences is open even in many well-studied inner models.

Solovay showed that every stationary subset of a regular cardinal κ can be slit into κ many disjoint stationary subsets. Foreman and Magidor showed that a tightly stationary sequence of sets consisting of ordinals of a fixed cofinality μ can be split into μ many disjoint tightly stationary sequences. For mutual stationarity we do not know if we can split a sequence into even two disjoint mutually stationary sequences:

Question 23 Suppose that $\langle S_n : n \in \omega \rangle$ is mutually stationary. Are there $\langle S_n^0, S_n^1 : n \in \omega \rangle$ such that

- S_n is the disjoint union of S_n^0, S_n^1
- $\langle S_n^i \rangle$ is mutually stationary for $i = 0, 1$.

A subproblem for Question 23 would be to isolate the appropriate Fodor's Theorem. We note that the natural conjecture would be that if $\langle S_n : m \leq n \in \omega\rangle$ is mutually stationary, then each S_n can be partitioned into ω_n disjoint subsets $\langle S_n^\alpha : \alpha < \omega_n\rangle$ such that for every function $f \in \prod_{m\leq n\in\omega}\omega_n$ the sequence $\langle S_n^{f(n)} : m \leq n\rangle$ is mutually stationary.

There are a whole host of related problems. We note the following definitions, which we give for sets of cardinality ω_1, again for concreteness. Let $N \prec H(\lambda)$ have cardinality ω_1. Then N is:

1. N is *internally unbounded* iff $N \cap [N]^{\aleph_0}$ is unbounded in $[N]^{\aleph_0}$.
2. N is *internally stationary* iff $N \cap [N]^{\aleph_0}$ is stationary in $[N]^{\aleph_0}$.
3. N is *internally club* iff $N \cap [N]^{\aleph_0}$ contains a closed unbounded set in $[N]^{\aleph_0}$.
4. N is *internally approachable* iff $N = \bigcup_{\alpha<\omega_1} N_\alpha$ where each N_α is countable and for $\beta \in \omega_1, \langle N_\alpha : \alpha < \beta\rangle \in N$.

Under certain circumstances, such as the CH, these properties are all equivalent. It is not clear in general what the relation is.

Question 24 Give examples separating the properties 1)-4).

Many properties in set theory propagate through successor cardinals, but require special hypothesis to pass through limit cardinals. (This is one of the main reasons for the workshop.) There are however some properties where the propogation is not clear. We give one example that would seem to require useful new ideas:

Question 25 Suppose that κ is regular, $N \prec H(\theta)$ and $N \cap [N \cap \kappa]^{\aleph_0}$ is stationary. Is $N \cap [N \cap \kappa^+]^{\aleph_0}$ stationary?

General Combinatorial Problems

We list here several problems that were asked at the conference. The first is due to Hajnal who announced a \$250 (US) prize for *any significant progress* on the problem.

Question 26 Does $\omega_2 \to (\alpha)^2_\omega$ for $\omega_1 + 1 < \alpha < \omega_2$?

We note that it is also and interesting problem to determine what happens at successors of singular cardinals.

Cummings reminded the audience of the following 2 closely related questions:

Question 27 Is it consistent that there is a forcing that makes $\aleph_{\omega+1}$ into ω_2?

Question 28 Is it consistent that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_2, \omega_1)$?

In the presence of Woodin cardinals a positive answer to question 28 yields a positive answer to question 27.

Schimmerling (as explicated in his contribution to the Proceedings) noted the following question:

Question 29 Is it consistent to have the GCH, weak square and no Suslin trees on $\aleph_{\omega+1}$? What about $\square^\omega_{\aleph_\omega}$?

Question 30 (Steel) Let M be the canonical minimal iterable extender model with a Woodin limit of Woodin cardinals λ. Let N be a derived determinacy model obtained by forcing over M with the Levy collapse making $\lambda = \omega_1^N$. (Thus N satisfies AD_R.) Prove or refute: Θ is regular in N.

Reward: \$200

The next two questions were asked with significant cash prizes:

Question 31 (Steel) Prove or refute (in Peano Arithmetic): if ZFC + "there is a singular strong limit cardinal κ such that $\square_\kappa$ fails" is consistent, then ZFC + "there is a superstrong cardinal" is consistent.

Reward: $300 for a refutation. For a proof, $4000 - $500x, where x is the time in years from May 1, 2004 to the date of submission of a correct, complete manuscript. UC Berkeley faculty are not eligible for the reward.

Question 32 (Woodin) Suppose that there is an extendible cardinal. Must HOD compute the successor correctly for some (uncountable) cardinal?

Prize:

$$\$1000[max(min(n, 10 - n), 1)]$$

where

$$n = (\text{calender year of submission}) - 2004.$$

Terms: Collect if a correct proof is given for either "yes", or if a correct proof is given that the failure implies the consistency with ZFC of the large cardinal I0 of Kanamori's book. (Details: Clay rules)

List of Participants

Abraham, Uri (Ben Gurion University of the Negev, Israel)
Asperó, David (University of Bristol)
Bagaria, Joan (University of Barcelona)
Cummings, James (Carnegie-Mellon University)
Dzamonja, Mirna (University of East Anglia)
Eisworth, Todd (University of Northern Iowa, Cedar Falls)
Foreman, Matthew (University of California, Irvine)
Gitik, Moti (Tel-Aviv University)
Hajnal, Andras (Rutgers, The State University of New Jersey)
Ishiu, Tetsuya (University of Kansas)
Kojman, Menachem (Ben Gurion University of the Negev, Israel)
Larson, Jean (University of Florida)
Liu, Andreas (University of California, Berkeley)
Magidor, Menachem (Einstein Institute of Mathematics)
Martinez, Juan Carlos (University of Barcelona)
Mitchell, William (University of Florida)
Schimmerling, Ernest (Carnegie Mellon University)
Shioya, Masahiro (University of Tsukuba)
Steel, John (University of California, Berkeley)
Thompson, Katherine (Carnegie Mellon University)
Todorcevic, Stevo (University of Toronto and CNRS Paris)
Welch, Philip (University of Bristol)
Woodin, W. Hugh (University of California, Berkeley)

Bibliography

[1] J. E. Baumgartner and S. Shelah. Remarks on superatomic Boolean algebras. *Ann. Pure Appl. Logic* **33(2)** (1987), 109–129.

[2] J. Cummings, M. Foreman, and M. Magidor. Canonical structure in the universe of set theory: Part one. *Ann. Pure and Appl. Logic*, To Appear.

[3] J. Cummings, M. Foreman, and M. Magidor. Canonical structure in the universe of set theory: Part two. *Ann. Pure and Appl. Logic*, To Appear.

[4] J. Cummings, M. Foreman, and M. Magidor. Squares, scales and stationary reflection. *J. Math. Log.* **1(1)** (2001), 35–98.

[5] M. Foreman and M. Magidor. A very weak square principle. *J. Symbolic Logic* **62(1)** (1997), 175–196.

[6] M. Foreman and M. Magidor. Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\varkappa(\lambda)$. *Acta Math.* **186(2)** (2001), 271–300.

[7] F. Galvin and A. Hajnal. Inequalities for cardinal powers. *Ann. of Math. (2)* **101** (1975), 491–498.

[8] T. Jech. Singular cardinals and the pcf theory. *Bull. Symbolic Logic* **1(4)** (1995), 408–424.

[9] M. Magidor. On the singular cardinals problem. II. *Ann. Math. (2)* **106(3)** (1977), 517–547.

[10] M. Magidor and S. Shelah. The tree property at successors of singular cardinals. *Arch. Math. Logic* **35(5-6)** (1996), 385–404.

[11] E. Schimmerling. A finite family weak square principle. *J. Symbolic Logic* **64(3)** (1999), 1087–1110.

[12] S. Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[13] J. Silver. On the singular cardinals problem. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 265–268. Canad. Math. Congress, Montreal, Que., 1975.

[14] R. M. Solovay. Strongly compact cardinals and the GCH. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 365–372, Providence, R.I., 1974. Amer. Math. Soc.

Chapter 10

Mathematical Structures in Economic Theory and Econometrics (04w5536)

April 1–6, 2004

Organizer(s): Ivar Ekeland (University of British Columbia), Pierre-Andre Chiappori (University of Chicago)

In the past years, progress in economic theory and econometrics has relied on increasingly sophisticated mathematical tools. Some current problems we aim to discuss in this workshop are:

Principal-agent problems

Such problems are typical of optimization under asymmetric information. The principal submits a contract, which the agent may accept or refuse. There are several types of agents, and agents of different types have different tastes. The principal does not know the type of the agents (it knows only the distribution of types), and must therefore deal with the fact that agents may lie to her about their type. In other words, contracts must be drawn in such a way that no agent has an incentive to lie about his type.

Such incentive-compatible contracts are well understood in the case where the type is one-dimensional. When there are several parameters describing the type, the mathematical situation is much more complicated. In fact, one runs into problems in the calculus of variations with global convexity constraints: instead of minimizing an integral criterion over all functions satisfying some boundary condition, one minimizes over convex functions only.

After a seminal paper by Rochet and Chone, such problems have been investigated by others, and striking results have been obtained by Carlier and Lachand-Robert. It is now clear that there is an intimate relation between this problem and others, of a more mathematical type, namely the optimal transportation problem and the problem of approximating numerically convex functions.

Economic geography

There has been much literature on the economics of transportation, but very little where the location of production and distribution centres appeared endogeneously, as a result of the model. In a recent paper, Lucas and Rossi-Hansberg made a breakthrough. They constructed a mathematical model for the structure of cities where the distribution of business districts and residential areas where explained by purely economic arguments, either as an equilibrium problem, or of a planning problem. To do this, they modelled transportation costs as “iceberg” costs, in the time-honoured fashion of Samuelson. Carlier and Ekeland changed the model, expressing transportation costs as monetary costs, which brought the whole problem into the general mathematical framework of optimal transportation.

There is clearly a great potential for economic modelling: not only the structure of cities, but also international trade, are domains of potential applications.

Hedonic models in econometrics

Hedonic models were first started by Rosen, but their impact was limited for a long time by they were seen as fundamentally indeterminate: the underlying parameters were thought to be difficult, or impossible, to extract from the data. Recent work by Heckman and Nesheim has shown that this indeterminacy, far from being a general property of hedonic models, is an artefact of linearity. In other words, hedonic models will be identifiable provided they incorporate some nonlinearity.

This opens several mathematical doors. One the one hand, transversality theory: the concept of genericity is new to econometrics, and a systematic exploration of generic properties of various models seems quite promising. On the other, optimal transportation (again): as a central feature of hedonic models, there is a continuum of products, and for each quality the market must clear. Mathematically, this means that the distribution of buyers and the distribution of sellers have the same image in the space of product qualities. In the case where the matching is one-to-one, we recover the classical mass transportation framework: the distribution of sellers then is the image of the distribution of buyers. In most economics situations, however, the matching is not one-to-one (bunching), and interesting mathematical problems appear.

Analysis of demand functions

It is an old question whether economic theory is testable, and if so, whether one can recover the individual utilities from the data. The question of course branches into several subquestions, depending on what kind of data is used, individual or collective. It is now apparent that the answer to such questions goes through the exterior differential calculus developed by Elie Cartan at the beginning of the 19th century. Indeed, they translate into systems of nonlinear PDEs which are quite different from the ones which appear in physics. One of the main (and very difficult) questions which appear in this connection is whether these systems, which can be shown to have solutions in the analytic framework, using the Cartan-Kahler theorem, also have solution in the indefinitely differentiable framework.

Collective behaviour in economics

The formation of prices in financial markets is a typical example of collective behaviour. The precise understanding of such behaviour is still far away, but there are interesting clues. First, the theory of rational anticipations, where the criterion for anticipations to be held collectively is that they should be self-fulfilling. Second, neighbourhood models, coming from statistical physics (spin glasse) also have been used to explain collective behaviour of investors. Finally, recent work by Schuman shows how speculative bubbles can evolve from the fact that investors pay attention to different sources of information and speculate on the possibility of reselling their assets to other with different (and, from their point of view, information). These models involve a variety of techniques, the most basic one being stochastic optimal control.

List of Participants

Ahmad, Najma (University of Toronto)
Chiappori, Pierre-Andre (University of Chicago)
Dana, Rose-Anne (University of Paris-Dauphine)
Ekeland, Ivar (University of British Columbia)
Guesnerie, Roger (College de France (Paris))
Jara, Pedro (DELTA-EHESS, DIM-U. de Chile)
Lazrak, Ali (University of British Columbia)
Matzkin, Rosa (Northwestern University)
McCann, Robert (University of Toronto)
Mossay, Pascal (Universidad de Alicante)

Nesheim, Lars (University College London)
Oberman, Adam (Unviersity of Texas)
Pakzad, Reza (PIMS, University of British Columbia)
Peters, Mike (University of British Columbia)
Scott, L. Ridgway (University of Chicago)
Shneyerov, Art (University of British Columbia)
Trokhimtchouk, Maxim (University of Toronto)
Wooders, Myrna (University of Warwick)

Chapter 11

Knots and Their Manifold Stories (04w5037)

May 8–13, 2004

Organizer(s): Kent Orr (Indiana University), Cochran Tim (Rice University), Rolfsen Dale (University of British Columbia)

A confluence of several strong currents in mathematics has invigorated knot theory and ancillary areas of 3 and 4-dimensional manifolds.

This workshop will bring together a multidisciplinary community to investigate the connections between what at one time seemed disparate areas of mathematical research. The underlying root and impetus has been knot theory. Numerous questions, both classical and modern have arisen. Among these are the four-dimensional topological surgery conjecture, which has been long connected to classical knot theory, the knot concordance problem, the classification of classical knot groups, classification of four dimensional homology cobordism, computation of localization and completion in homotopy theory, the L-theory of localized rings, the classification of knots and three manifolds via finite type and quantum invariants, classical interpretations of these modern invariants, and numerous other classical problems in knots and manifold theory, high and low dimensional.

The first (and oldest) such current is that of knots and Higher-dimensional manifolds, including surgery and homotopy theory and especially localization and completion of groups, rings and modules. The second current is that of topological 4-manifolds, specifically including the topological techniques of M. Freedman and A. Casson that distinguish this field from higher-dimensional manifolds. The third current is that of combinatorial (quantum) knot theory which began with work of V. Jones, E. Witten, M. Konsevitch and V. Vassiliev. The fourth current is that of von Neumann algebras and L^2 homology. Some of the earliest and most exciting applications of surgery theory were classification results for higher-dimensional knots and links in the work of A. Haefliger, J. Levine and M. Kervaire. S. Cappell and J. Shaneson developed surgery with coefficients to approach codimension-two placement, tangential structures with coefficients were classified in part by Taylor and Williams, and localization was introduced by Vogel and Le Dimet. Their work on a theoretical classification of concordance classes of links underscored the necessity of considering noncommutative localization. But to a great extent the difficulty of the algebra of noncommutative rings and localizations of modules has, until recently, obstructed the transfer of these algebraic techniques to low-dimensional situations.

When applied to 4-manifolds and classical knot concordance this new toolbox was found to be inadequate. The inability to obtain the embedded 2-disks whose existence were predicted by homotopy theory presented seemingly insurmountable barriers. Moreover, in low-dimensions the fundamental groups of the relevant spaces were usually large and nonabelian and demanded closer attention. A. Casson and C. Gordon first proved the inadequacy of the higher-dimensional tools in the context of classical knot concordance. They introduced new tools, including the Atiyah-Singer G-signature theorem, and showed that it was necessary and profitable to consider nonabelian covering spaces. The appearance of the G-signature theorem hinted at possible further use of analytic invariants. This was underscored in more recent work of M. Farber and J. Levine on homology cobordism of manifolds using eta invariants associated to finite-dimensional unitary representations of the fundamental group.

In topological 4-manifolds, the higher-dimensional techniques were augmented by remarkable techniques of M. Freedman, A. Casson and F. Quinn, involving infinite (but convergent) topological constructions called Casson towers. They also approximated embedded 2-disks with towers of iterated embedded surfaces called gropes. These topological constructions have, until recently, remained largely outside the scope of algebraic or analytical understanding, (although gropes were seen from the outset to be a topological reflection of the algebra of commutator series of the fundamental group). Although both the Casson-Gordon invariants and the Casson-Freedman towers were intrinsic to topological four-manifolds the connections between these two phenomena remained obscure. On the other hand, the field of combinatorial/quantum knot theory began with work of V. Jones on von Neumann algebras. Moreover noncommutative algebra underlay the work of Drinfeld that in turn suggested the primary invariant of quantum knot theory, the Kontsevitch integral. Together with ideas of E. Witten from physics and analysis, these suggested the primary invariants of quantum 3-manifolds, those of N. Reshitikin and V. Turaev. The field seemed to turn increasingly combinatorial and 2-dimensional, with major advances by many researchers in understanding the algebra of knots and 3-manifolds via planar projections of links, and through the algebra of trivalent (Feynman) diagrams. Connections with 4-manifolds and with higher-dimensional techniques seemed particularly elusive. Moreover the field struggled to find topological interpretations of these powerful new invariants. It remains open whether or not these represent complete invariants for knots.

Recently the merging of these currents accelerated in exciting ways.

In the recent work of T. Cochran, K. Orr and P. Teichner on classical knot concordance, the high-dimensional, 4-dimensional and von Neumann currents come together. They defined new invariants using noncommutative localization of modules and L^2 techniques - specifically the von Neumann rho invariants of Atiyah-Cheeger-Gromov. Moreover the relationship between the topology of gropes and the algebra of the derived series of the fundamental group is underscored, and a connection is established between the algebraic/homotopy-theoretic techniques (including those of Casson-Gordon) and the grope and tower constructions. Recent work of L. Rozansky, S. Garoufalidis and A. Kricker has combined quantum knot theory with several of these other currents. They have shown that the Konsevitch integral for knots and for boundary links (which is defined over the rationals) satisfies a certain hidden "integrality" property, only described through the use of the language of surgery and of noncommutative localization of group rings. This has enabled them to ?lift? the Kontsevitch integral to a potentially more powerful invariant. This invariant has already been shown by Rozansky, Garoufalidis and Teichner to give strong new results about Alexander polynomial one knots (for example) and to suggest an organization of the Kontsevitch invariant (and hence of all Vassiliev finite type invariants) that better reflects the topology of knots. Finally, using ideas of K. Habiro, J. Conant and P. Teichner show that filtering knots by *the size of the gropes they bound in 3-space* provides a possible connection between the 3- and 4-dimensional worlds. Specifically they show one version of this filtration is respected by the Kontsevich integral and captures much of its topological content; whereas another version yields a filtration consistent with recent tower filtrations of the classical knot concordance group. Several other provocative connections remain unexplored. First, many of the above constructions can be indexed by families of trivalent (Feynman) diagrams. Work of R. Schneiderman shows that these diagrams also parametrize certain higher-order intersection data among 2-spheres in 4-manifolds, refining the classical intersection theory of higher-dimensional manifolds to a more subtle theory, more effective in dimension four. Second, earlier researchers such as Culler-Shalen, Casson, Levine and Farber were able to address non-commutativity through studying finite dimensional complex representations of the fundamental group. Some techniques of algebraic geometry were then used profitably. Is there a possible interaction between these techniques and the regular representations into unitary operators on (infinite-dimensional) Hilbert space used to obstruct slicing knots? Lastly, the families of higher-order Alexander polynomials for 3-manifolds M, defined by S. Harvey using noncommutative algebra, surprisingly obstruct the existence of symplectic structures on MxS^1 even when Seiberg-Witten invariants fail. This hints at a hierarchy of higher-order Seiberg-Witten invariants. Can the methods of Osvath and Zabo be refined (perhaps by looking at equivariant intersection theory) to a nonabelian world that will reflect the above algebraic invariants?

The common source is knot theory. In its broadest context, this workshop seeks insights across fields.

Homotopy theory and localization, surgery theory, functional analysis, topological four manifolds, classical knot theory in high and low dimensions, combinatorial and geometric three manifold theory, representation theory, physics and quantum topology are all finding a common area for shared discourse rooted in knot theory and illustrated in the discussion above. Such connections demand deeper exploration only available through the collaborative enterprise and through the dissolution of the artificial barriers between fields. The workshop will reinforce this interdisciplinary emphasis and will encourage the interaction of researchers across these fields. We anticipate only five talks a day, with the first day of the workshop consisting of expository talks on each of the four topics mentioned above - higher dimensional manifolds and knots, topological four manifolds and tower constructions, quantum invariants of knots and three manifolds, and analytic invariants.

This workshop proposal is timed to benefit and complement the PIMS thematic program in knot theory and 3-manifolds at UBC in the summer of 2004, sponsored by PIMS and organized by Dale Rolfsen, who is a coorganizer of this proposed workshop. This benefit may prove especially effective if the workshop occurs either the week before or after the Vancouver program.

List of Participants

Cha, Jae Choon (Information and Communications University)
Chang, Stanley (Wellesley College)
Clay, Adam (University of British Columbia)
Cochran, Tim (Rice University)
Crowley, Diarmuid (Penn State University)
Denecke, Klaus (Potsdam University)
Fenn, Roger (University of Sussex)
Friedl, Stefan (University of Munich)
Gee, Carol (Rice University)
Gilmer , Patrick (Louisiana State University)
Gordon, Cameron (University of Texas)
Harvey, Shelly (Massachusetts Institute of Technology)
Heap, Aaron (Rice University)
Hempel, John (Rice University)
Kania-Bartoszynska, Joanna (Boise State University and National Science Foundation)
Kim, Taehee (Rice University)
Klein, John (Wayne State University)
Krebes, David (University of Calgary)
Krushkal, Slava (University of Virginia and Microsoft Research)
Leidy, Constance (Rice University)
Levine, Jerome (Brandeis University)
Linnell, Peter (Virginia Tech)
Melvin, Paul (Bryn Mawr College)
Orr, Kent (Indiana University)
Ranicki, Andrew (The University of Edinburgh)
Rhemtulla, Akbar (University of Alberta)
Rolfsen, Dale (University of British Columbia)
Schneiderman, Rob (Courant Institute)
Sheiham, Des (International University Bremen)
Stanford, Ted (New mexico State University)
Watson, Liam (University of British Columbia)
Yurasovskaya, Ekaterina (University of British Columbia)
Zvengrowski, Peter (University of Calgary)

Chapter 12

New Developments on Variational Methods and Their Applications (04w5033)

May 15–20, 2004

Organizer(s): Kung-Ching Chang (Peking University), Jingyi Chen (University of British Columbia), Changfeng Gui (University of Connecticut), Paul Rabinowtiz (University of Wisconsin at Madison)

Introduction

The calculus of variations has repeatedly proved itself to be a powerful and far-reaching tool for advancing our understanding of mathematics and its applications. This is no doubt due to the fact that variational methods are not merely techniques for solving individual, albeit very important, problems but are often "variational principles", i.e. they are manifestations of very general laws of nature which are valid in diverse branches of science and engineering.

Modern variational approaches to non-linear problems were initiated by mathematicians like H. Poincare, G. Birkhoff and his student M. Morse. E.g. Morse revealed the deep relationship between the number and types of critical points of functions and the topology of their level sets. Around the same time, variational approaches were also being developed and used by Ljusternik and Schnirelmann to establish the existence of 3 distinct closed geodesics on any compact surface of genus zero. These methods and results –which also mark the beginning of global analysis– were finite dimensional in nature. The development of tools to deal with infinite dimensional problems of nonlinear partial differential equations and geometry accelerated in the 1960's. The type of compactness required, often embodied in the Palais-Smale condition, was much studied and considerable progress was made. Subsequently in the mid 1980's and afterwards an understanding of how the the Palais-Smale condition can breakdown in less compact situations (like unbounded domains, limit exponent problems, etc.) emerged and now one can use this understanding to get existence results in such settings. Novel minimization arguments have also been developed and there is considerable current activity in refining and extending them so as to overcome the limitations to their applicability to present-day variational problems, ranging from geometry to pattern recognition and from superconductivity to phase transitions, etc. The subject has come of age in the last forty years, and a number of surveys and monographs have described much of the progress.

The goal of our workshop was to discuss some of the recent developments while emphasizing new applications to nonlinear problems. More often than not, progress is driven by specific applications. Novel variational techniques developed by groups or individuals concerned with these applications often do not make their way to others who may be using similar variational methods but on different types of problems. This workshop brought together senior experts such as Ivar Ekeland, Maria Esteban, Louise Nirenberg, Paul Rabinowitz, etc. and leading young researchers working in different areas of variational methods such as Yanyan Li, Yiming Long, Eric Séré, Peter Sternberg, etc. These areas include abstract variational methods such as Novikov Morse theory and nonsmooth critical point theory, geometric PDEs, and nonlinear problems from applied fields such as superconductivity and phase transitions. Participants with different areas and background had an opportunity to exchange ideas on topics ranging from abstract theories to applications so that novel variational methods can find more applications and new theories can be developed. Below are the main themes of the workshop:

Phase Transitions and Superconductivity

There are many problems in these applied areas that can be studied by variational methods. Indeed recently several new ideas such as the renormalized energy method, new reduction methods, new perturbation methods have been developed through the study of individual problems in these areas. Variational methods give a very good understanding of physical phenomena such as concentration, vortex formation and their dynamics. In this workshop, many participants spoke about their new results and ideas in this area.

One such topic is gamma-convergence in Ginzburg-Landau models of superconductivity and Allen-Cahn models of phase transition. Sylvia Serfaty presented a method to prove convergence of gradient-flows of families of energies which gamma-converge to a limiting energy. It provides lower bound criteria to obtain the convergence, which correspond to a sort of C^1-order gamma-convergence of functionals. They then apply this method to establish the limiting dynamical law of a finite number of vortices for the heat-flow of the Ginzburg-Landau energy in dimension 2. In this case, the limiting objects whose dynamics they study are the limiting vortices of the maps u_ϵ, and the limiting energy is a “renormalized energy”, defined on the finite-dimensional space of possible vortex-locations. They prove that the conditions above are satisfied and thus re-obtain with a different method the result of Lin and Jerrard-Soner, that the limiting vortices follow the gradient-flow of the renormalized energy. They also obtain the analogous new result for the full Ginzburg-Landau model with magnetic effects. One extension of this method is to push it to “second order” to compare the C^2 structures of the energy-landscapes and F near critical points. This gives necessary conditions for stable/unstable critical points of the Ginzburg-Landau energy functional to converge to stable/unstable critical points of F. This is again applied in the case of Ginzburg-Landau to obtain stability results on the limiting vortex-configurations, and a nonexistence result of nontrivial stable

critical points when there are Neumann boundary condition and no magnetic field . Another extension is to apply it to Ginzburg-Landau vortex-dynamics with suitable space-time rescalings, which allow one to continue studying dynamics at times of collisions of vortices. This, coupled with a new estimate (in the case of Ginzburg-Landau) to the vortex-distances, allows one to give energy-dissipation rates at collision time and optimal estimates on those collision-times, and under certain assumptions, to extend the limiting dynamics after collision.

Montero discussed the weak Jacobians of Jerrard and Soner and showed they can be viewed as linear functionals that act on Hölder continuous, compactly supported vector fields in Ω. He and his collaborators use the limiting behaviour of E_ε to identify a geometric condition on Ω that guarantees the existence of local minimizers of E_ε. This condition essentially amounts to the existence of a line segment in Ω, with endpoints in $\partial\Omega$, that locally minimizes length. He also showed an existence result for G_ε in 3-d simply connected domains when the applied field h_{ap} is not too big. In particular, for the case $h_{ap} = 0$, this provides what is perhaps the first existence result via Ginzburg-Landau theory of permanent currents in the presence of vortices.

X. Ren studied gamma convergence in a dipolymer model. A molecule in a diblock copolymer is a linear sub-chain of A monomers grafted covalently to another sub-chain of B monomers. The different type sub-chains tend to segregate locally, resulting in micro-domains rich in A and B monomers. These micro-domains form morphology patterns/phases in a larger scale. The Ohta-Kawasaki free energy of a diblock copolymer melt is a functional of the A monomer density field $u(x)$. When there is high A monomer concentration at x, $u(x)$ is close to 1; when there is high concentration of B monomers at x, $u(x)$ is close to 0. A value of $u(x)$ between 0 and 1 means that a mixture of A and B monomers occupies x. The re-scaled, dimensionless free energy of the system is

$$I(u) = \int_D \{\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{\epsilon\gamma}{2}|(-\Delta)^{-1/2}(u-a)|^2 + W(u)\}\,dx,$$

which is defined in the admissible set

$$X_a = \{u \in W^{1,2}(D) : \overline{u} = a\}$$

where $\overline{u} = \frac{1}{|D|}\int_D u\,dx$ is the average of u in D. a is a fixed constant in $(0,1)$. It is the ratio of the number of the A monomers to the number of all the monomers in a chain molecule. One can take

$$W(u) = \frac{1}{4}(u^2 - u)^2.$$

The two parameters ϵ and γ characterize the system. Ren considered the parameter range

$$\epsilon \to 0,\ \gamma \sim 1.$$

He studied two solutions: the spot solution and the ring solution of K interfaces, both in a unit disc. The spot solution models a cell in a cylindrical phase of the diblock copolymer and the ring solution models a defective lamellar phase. Using the Γ-convergence theory he showed that the spot solution exists for all $\gamma > 0$ and there exists $\gamma_1 > 0$ such that the ring solution exists for $\gamma > \gamma_1$.

Next he considered the stability of these solutions by analyzing their critical eigenvalues. He showed that there exists $\gamma_0 > 0$ such that the spot solution is stable if $\gamma < \gamma_0$ and unstable if $\gamma > \gamma_0$. For the ring solution, there exists $\gamma_2 > \gamma_1$ such that the ring solution is stable if $\gamma \in (\gamma_1, \gamma_2)$ and unstable if $\gamma > \gamma_2$. Finally he made a comparison between the diblock copolymer problem and the Cahn-Hilliard problem, which is obtained by setting $\gamma = 0$ in the definition of I.

Glotov studied the 'variable thickness' Ginzburg-Landau equations describing type-II superconducting thin films. The convergence of the order parameter was discussed in the literature in a paper by Chapman, Du, and Gunzburger. Glotov and his collaborators focussed their attention to the equation for the magnetic potential and obtained results on convergence of various quantities involved in the latter equation. They also showed that the limiting order parameter is a minimizer of the two-dimensional thin-film energy. The limiting problem, among other properties, has an advantage, from a computational point of view, of being restricted to a bounded domain. The regularity of the solutions to the three-dimensional problem presents another interesting question for us. Using regularity, they obtain uniform convergence of the three-dimensional minimizers. This in turn allows them to conclude, thanks to the description of the vortex structure for minimizers of the two-dimensional thin-film energy available from the work of Ding and Du, that the three-dimensional minimizers exhibit vortices and their degree is preserved as the thickness of the film tends to zero.

Alama and his collaborators consider the following variational problem arising from a two-dimensional model for rotating Bose–Einstein Condensates (BEC.)Let $a = a(r)$ be a real-analytic radially symmetric function in the plane, with the property that

$$\mathcal{A} = \{x \in \mathbf{R}^2 :\ a(|x|) > 0\}$$

is an *annulus*, and such that a vanishes linearly at each edge of the annulus $\mathcal{A}$. Examples include $a(r) = -b_0 + b_1 r^2 - b_2 r^4$ with appropriately chosen coefficients. Let $\Omega \in \mathbf{R}$,$x = (x_1, x_2) \in \mathbf{R}^2$, $x^\perp = (-x_2, x_1)$,and $\epsilon > 0$. They study minimizers $u \in H_0^1(\mathcal{A}; \mathbf{C})$ of the energy functional

$$E_\epsilon(u) = \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega\mathbf{x}^\perp \cdot (iu, \nabla u) + \frac{1}{4\epsilon^2}(|u|^2 - a(x))^2 \right\}\, dx,$$

in the singular limit as $\epsilon \to 0$. In the context of BEC, u is the quantum wave-function, Ω is the angular speed of rotation, and $-a(r)$ gives a potential well imposed to"trap" the condensate (by means of lasers) in a bounded region of space. The choice of an annular trap here is meant to simulate certain current experiments for BEC.

Alama showed how the annular topology of the condensate domain affects the presence and location of vortices as a function of the angular speed Ω. His results concern both fixed rotation Ω (independent of ϵ and rotations which grow with ϵ. When Ω is fixed, it is proved that minimizers converge to anon-zero (radially equivariant) solution away from the hole, while the hole itself plays the role of a "Giant Vortex" with degree increasing with Ω. Alama also considered angular velocities of the form

$$\Omega = \omega_0 |\ln \epsilon| + \omega_1 \ln |\ln \epsilon|,$$

with ω_0, ω_1 constant. He showed that there is a critical value of the coefficient $\omega_0 = \omega_0^*$ such that whenever $\omega_0 < \omega_0^*$, minimizers have no vorticity in the interior of the annulus $\mathcal{A}$, but when $\omega_0 = \omega_0^*$ and ω_1 is large enough, then vortices begin to appear inside $\mathcal{A}$. The location of these vortices is completely determined by the coefficient a: they lie on one or several concentric circles in $\mathcal{A}$ whose radii attain a given minimization problem involving $a(r)$. This method involves deriving sharp upper and lower bounds on the energy of minimizers via a vortex-ball construction as in the work of Sandier–Serfaty. In order to determine the number and location of the vortices for supercritical rotations they must take into account the effect of the Giant Vortex in constructing the upper and lower bounds.

Sternberg discussed various basic questions for the Ginzburg-Landau and Allen-Cahn equations that remain unanswered. One such question is on the monotonicity (i.e. increasing in one direction) of local minimizers for the Allen-Cahn energy in a convex domain. Another such question is the smoothness of the zero level set of solutions to Allen-Cahn energy.

Bates and his collaborators study a global minimizer of the van der Waals' free energy functional with nonlocal interaction. Short-range repulsive and long-range attractive interactions appear in, e.g., the van der Waals' forces (often modeled by the Lennard-Jones potential). For this model since there is no gradient term, the underlying space is not restricted to differentiable functions and critical points are possibly discontinuous functions. Indeed, monotone discontinuous heteroclinic critical points were constructed in previous work by the authors and others. For instance, they discovered families of critical points, discontinuous along arbitrarily prescribed interfaces, which are seemingly stable, since the formal second variation is positive. However, there is no variational sufficiency condition for minimizers, since if the functional is defined on the natural space L^2 or $u_0 + L^2$, it is only $C^{1,1}$ and the discontinuous solutions which usually form nonsmooth continua in L^2 are in general not local minimizers. Bates studied the case when the Fourier transform, $\hat{J} \leq 1$, which assures that the energy functional is bounded below by 0. If $J \geq 0$, a monotone global minimizer can be constructed using monotone rearrangements. However, if J changes sign, monotonicity methods are not applicable, and in general the global minimizer will not be monotone. For this case ideas from concentration and convexification techniques are employed.

Bubbles, Spikes and Concentration

Many variational problems arises from geometry particularly in the study of the Yamabe problem, Kahler-Einstein manifolds, minimal surfaces, scalar curvature, harmonic maps, etc. A typical difficulty is the lack of compactness, i.e., some kind of bubble or singularity appears. This area is very active and has been a major source of new ideas in variational methods.

Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 3$, and let

$$A_g := \frac{1}{n-2}(Ric_g - \frac{R_g}{2(n-1)}g)$$

denote the Schouten tensor of g, where Ric_g and R_g denote respectively the Ricci tensor and the scalar curvature of g. Let $\lambda(A_g) = (\lambda_1(A_g), \cdots, \lambda_n(A_g))$ denote the eigenvalues of A_g with respect to g. Let V be an open convex subset of R^n which is symmetric with respect to the coordinate axes. Assume that $\emptyset \neq \partial V$ is smooth and satisfies

$$\nu(\lambda) \in \{\mu \in R^n | \mu_i > 0, \forall 1 \leq i \leq n\}, \qquad \forall\, \lambda \in \partial V,$$

and

$$\nu(\lambda) \cdot \lambda > 0, \qquad \forall\, \lambda \in \partial V.$$

Let

$$\Gamma(V) := \{s\lambda \mid \lambda \in V,\ 0 < s < \infty\}$$

be the cone with vertex at the origin generated by V.

Conjecture. *Let* (M^n, g), V *and* $\Gamma(V)$ *be as above. Assume that*

$$\lambda(A_g) \in \Gamma(V), \qquad \textit{on } M^n.$$

Then there exists a smooth positive function $u \in C^\infty(M^n)$ *such that the conformal metric* $\hat{g} = u^{\frac{4}{n-2}} g$ *satisfies*

$$\lambda(A_{\hat{g}}) \in \partial V, \qquad \textit{on } M^n.$$

For $V = \{\lambda \in R^n \mid \sum_{i=1}^n \lambda_i > 1\}$, the conjecture is the Yamabe Conjecture in the positive case. Yanyan Li discussed recent joint work on this conjecture including some proofs of results on the existence and compactness of solutions as well as some Liouville type theorems.

Min Ji consider the famous Nirenberg problem: which positive function R can be the scalar curvature of some metric g which is pointwise conformal to g_0? Writing $g = e^u g_0$, the problem is equivalent to the solvability of the following PDE:

$$-\triangle_{g_0} u + 2 - R\, e^u = 0, \qquad \text{on} \quad S^2. \tag{12.1}$$

Several years ago Moser proved the solvability for R an even function and this was followed by much further research. Equation (12.1) can be reduced to a variational problem. The corresponding functional is bounded from below, however it has no minimum if R is not a constant. Most subsequent work involved attempts to look for minimax type of solutions. A.Chang–P.Yang and others made further important progress. Min Ji gave a nice general framework for getting solutions.

Wei and his coauthors consider the following nonlinear elliptic equation

$$\Delta u - \mu u + u^q = 0 \text{ in } \Omega, \;\; u > 0 \;\text{ in } \Omega \;\text{ and }\; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{12.2}$$

where Ω is a bounded and smooth domain in R^N, $\mu > 0$ and $q = \frac{N+2}{N-2}$. Problem (12.2) has been studied by many authors in recent years . Wei mentioned the following results of Gui-Wei: Let $q < \frac{N+2}{N-2}$. Given arbitrary two positive integers K, l, there exists a $\mu_{k,l}$ such that for $\mu > \mu_{k,l}$, there exists a solution to (12.2) with $k-$interior spikes and $l-$boundary spikes. In his talk, Wei showed similar phenomena for the critical exponent case. Wei's first result concerns the case of μ large and $N \geq 7$. (This is joint work with C.-S. Lin.) They showed that at a positive nondegenerate local minimum point Q_0 of the mean curvature, (they may assume that $Q_0 = 0$), for any fixed integer $K \geq 2$, there exists a $\mu_K > 0$ such that for $\mu > \mu_K$, the above problem has a $K - bubble$ solution u_μ concentrating at the same point Q_0. More precisely, they show that u_μ has K local maximum points $Q_1^\mu, ..., Q_K^\mu \in \partial\Omega$ with the property that $u_\mu(Q_j^\mu) \sim \mu^{\frac{2}{N-2}}, Q_j^\mu \to Q_0, j = 1, ..., K,$ and $\mu^{\frac{3-N}{N}}(Q_1^\mu, ..., Q_K^\mu)$ approach an optimal configuration of the following problem

$(*)$ *Find out the optimal configuration that minimizes the following functional:*

$$R[Q_1, ..., Q_K] = c_1 \sum_{i=1}^K \varphi(Q_j) + c_2 \sum_{i \neq j} \frac{1}{|Q_i - Q_j|^{N-2}}.$$

where $c_1, c_2 > 0$ are two generic constants and $\varphi(Q) = Q^T \mathbf{G} Q$ with $\mathbf{G} = (\nabla_{ij} H(Q_0))$. This result shows that the bubbling accumulations phenomenon can occur for $N \geq 7$. (When $N = 3$, it was proved by Y.Y. Li that no bubbling accumulations can occur.)

Wei's second result concerns μ and the lower dimension case $N = 4, 5, 6$. (This is joint work with O. Rey.) They show that for $N = 4, 5, 6$ and any positive integer K such that $K \neq 2$, there exists $\mu_K > 0$ such that for $0 < \mu < \mu_K$, the above problem has a nontrivial solution which blows up at some K interior points in Ω, as $\mu \to 0$. The locations of the blowing up points are related to the domain geometry. No assumption on the symmetry or the geometry or the topology of the domain is needed.

Sharp Inequalities and Symmetry of Extreme Functions

Inequalities are a crucial part of mathematics. Many inequalities have a variational formulation. It is very useful to get sharp inequalities by finding the extreme functions and their properties, which often involve symmetries and underlying invariance of the functionals. In the workshop, many participants discussed the relationship of sharp inequalities and symmetry.

Z.Q. Wang and his coauthors consider a family of weighted Hardy-Sobolev type inequalities due to Caffarelli, Kohn and Nirenberg: There is $S(a,b) > 0$ such that for all $u \in C_0^\infty(R^N)$, the inequality

$$\int_{R^N} |x|^{-2a}|\nabla u|^2\, dx \geq S(a,b)\left(\int_{R^N} |x|^{-bq}|u|^q\, dx\right)^{2/q} \tag{12.3}$$

holds for $N \geq 3$: $-\infty < a < \frac{N-2}{2}, 0 \leq b-a \leq 1$ and $q = \frac{2N}{N-2+2(b-a)}$. These inequalities extend to $D_0^{1,2}(R^N) := \overline{C_0^\infty(R^N)}^{||\cdot||}$ with respect to the norm $||u||_a^2 = \int_{R^N}|x|^{-2a}|\nabla u|^2 dx$, and have the associated Lagrange equation $-div(|x|^{-2a}\nabla u) = |x|^{-bq}u^{q-1}$, which is a prototype of more general anisotropic type nonlinear elliptic PDEs with multiple singularities and degeneracies. Wang then discussed:

• **Symmetry and symmetry breaking of extremal functions.** Here due to the work of Aubin(1976), Talenti(1976), Lieb(1983), and Chou-Chu(1993), for $a \geq 0, a \leq b < a+1$, all extremal functions of the inequalities are radially symmetric. Some recent work have partially clarified the symmetry property of extremal functions for the remaining parameter region. More precisely:

Theorem *(Catrina-Wang, 2001) There is a function $h(a)$ defined for $a \leq 0$, satisfying $h(0) = 0$, $a < h(a) < a+1$ for $a < 0$, and $a+1-h(a) \to 0$ as $-a \to \infty$, such that for (a,b) satisfying $a < 0$ and $a < b < h(a)$, the extremal functions for $S(a,b)$ are non-radial.*

A more precise result was given by Felli-Schneider(2003) who showed
$h(a) = 1 + a - \frac{N}{2}(1 - \frac{N-2-2a}{\sqrt{(N-2-2a)^2+4(N-1)}})$.

As a more recent result we have:

Theorem *(Lin-Wang, 2004) For (a,b) satisfying $a < 0$ and $a < b < h(a)$, any extremal function u to $S(a,b)$ is axially symmetric about a line through the origin. Moreover, up to a rotation, $u(x)$ only depends on the radius r and the angle θ_N between the x_N-axis and $\vec{ox}$, and on each sphere $\{x \in R^N \mid |x| = r\}$, u is strictly decreasing as the angle θ_N increases.*

Next Wang spoke on:

• **Sharp versions of the improved Hardy inequalities.** When restricted to bounded domains, on the right hand side of (12.3) one can add additional terms leading to Hardy-Sobolev inequalities with remainder terms. The following is the improved weighted Hardy inequality which gives the sharp version of the improved Hardy inequality due to Brezis-Vazguez(1997) and Vazguez-Zuazua(2000), as well as generalizes their results to the weighted versions. These inequalities are useful tools for elliptic and parabolic equations having singular potentials.

Theorem *(Wang-Willem, 2003) Let $N \geq 1$, $a < \frac{N-2}{2}$, and $\Omega \subset\subset B_R(0)$ for some $R > 0$. Then there exists $C = C(a,\Omega) > 0$ such that for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx - \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega |x|^{-2(a+1)}u^2 dx \geq C\int_\Omega \left(\ln\frac{R}{|x|}\right)^{-2}|x|^{-2a}|\nabla u|^2 dx.$$

When $0 \in \Omega$, the inequality is sharp in the sense that $\left(\ln\frac{R}{|x|}\right)^{-2}$ can not be replaced by $g(x)\ln\left(\frac{R}{|x|}\right)^{-2}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$.

• **Further questions.** i.) The symmetry of extremal functions for parameters $a \leq 0, h(a) \leq b < a+1$. ii.) Related issues for the L^p versions of the weighted Hardy-Sobolev inequalities.

Congming Li and his coauthors studied the well-known Hardy-Littlewood-Sobolev inequality:

$$\int_{R^n}\int_{R^n} f(x)|x-y|^{\alpha-n}g(y)dxdy$$
$$\leq C(n,s,\alpha)||f||_r||g||_s.$$

Here $f \in L^r(R^n)$, $g \in L^s(R^n)$, $0 < \alpha < n$ and $\frac{1}{r} + \frac{1}{s} = \frac{n+\alpha}{n}$. They were mainly interested in the study of non-negative solutions to the associated Euler-Lagrange equations which can be transformed to the following system of integral equations in R^n:

$$\begin{cases} u(x) = \int_{R^n} |x-y|^{\alpha-n} v(y)^q dy \\ v(x) = \int_{R^n} |x-y|^{\alpha-n} u(y)^p dy \end{cases}$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$. First, under the natural integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, They prove that all the solutions are radially symmetric and monotone decreasing about some point. In the special case $p = q$, they classified all the solutions which solved a open problem posed by E. Lieb.

Congming Li also presented some of his joint work on regularity, radial symmetry, and monotonicity of solutions to this and some related systems which include subcritical cases, super critical cases, and singular solutions in all cases; and obtain qualitative properties for these solutions.

Burchard used symmetrization to gain compactness and study related inequalities. Indeed, lack of compactness is a principal analytical difficulty in the study of functionals on unbounded domains. For symmetric functionals, the existence of minimizers can often be established by first restricting the problem to radially symmetric functions with the help of a rearrangement inequality, and then using the additional compactness properties of symmetric functions, as captured by the Strauss radial lemma [1977], to find a convergent minimizing sequence. This strategy was used in the determination of the sharp Sobolev constants by Talenti [1976], in the analysis of the sharp Hardy-Littlewood-Sobolev inequalities by Lieb [1983], and for the study of ground states for many functionals of Mathematical Physics.

Certain dynamical stability problems can also be reduced to the study of related variational problems. Here, it is the compactness of arbitrary minimizing sequences, not just the existence of minimizers, that plays the key role. In a series of famous papers, Lions [1984] introduced a general abstract *concentration compactness* principle which has lead to many applications. In order to apply this principle to a specific problem, some additional analysis is usually needed. In recent joint work with Y. Guo [2004], Burchard closely examines the role of translations for minimizing sequences of two classes of functionals that appear in many applications of the concentration compactness principle: convolution integrals of the form

$$\mathcal{I}(f) = \int_{R^n} \int_{R^n} f(x) K(|x-y|) f(y)\, dx dy$$

with some strictly decreasing, positive definite kernel K, and gradient integrals of the form

$$\mathcal{J}(g) = \int_{R^n} \Phi(|\nabla g(x)|)\, dx$$

with some strictly convex, increasing integrand Φ. Special cases are the Coulomb kernel in three dimensions, and the p-norm of the gradient. They show that the difference between a minimizing sequence and the corresponding sequence of symmetrized functions is characterized by appropriate translations. Besides the interest of their results in classical analysis, this characterization suggests a practical two-step procedure for establishing compactness on an unbounded domain. *Step 1.* Show convergence of all symmetric minimizing sequences. *Step 2.* Show convergence up to translations for general minimizing sequences, assuming that their symmetrizations converge. The first step implies the existence of minimizers; it is also a necessary ingredient in the proof that these minimizers are dynamically stable under symmetric perturbations. They focus on the second step, which implies dynamical stability under more general perturbations. They discuss applications to symmetric galaxy configurations appearing in recent work of Guo and Rein [1999-2001], and to functionals with additional scaling symmetries.

Technically, their results are inspired by *asymmetry* inequalities, which estimate the difference between a function or a body and a symmetric one by a related geometric quantity. The most powerful result in that direction, due to Hall [1992], states that a body whose surface area is close to the surface area of a ball of the same volume is in fact close (in symmetric difference) to a suitable translate of the ball. They expect that asymmetry inequalities should hold for large classes of symmetric functionals, including the Coulomb electrostatic energy. They hope that their approach can give another perspective on concentration compactness for symmetric functionals.

On another front, McKenna discussed the symmetry of approximate solutions. Over the past quarter century, one field of intense research activity has been the study of what symmetry properties the solution of a nonlinear elliptic boundary value problem can inherit from the domain on which it is being solved. A classic paper is that of Gidas-Ni-Nirenberg, in which a typical result of the type we have in mind is: a positive solution of the boundary value problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{12.4}$$

must be radially symmetric if Ω is a ball. More recently, a related area has been attracting growing attention, namely how does one approximate solutions of this type of nonlinear boundary value problem? Typically, the work in this area relies on a suitable discretization of (12.4), (most commonly by finite-differences), and then uses theoretical ideas from nonlinear analysis such as monotonicity methods, mountain pass algorithms, or linking methods, to develop an approximate or exact solution to the discretized problem.

McKenna addressed the so-far-neglected question: if the partial differential equation (12.4) has inherited certain symmetry properties from the domain, to what extent does the discretized problem also inherit these symmetry properties?

This leads to the study of the most natural discretization of (12.4), namely,

$$\begin{aligned} &u_{i+1} - 2u_i + u_{i-1} = h^2 f(u_i), \quad u_i > 0, \quad i = -(N-1)\dots N-1, \\ &u_{-N} = u_N = 0, \end{aligned} \tag{12.5}$$

where $h = L/N > 0$ is the mesh-size of an equidistant mesh on $[-L, L]$. Suppose that $f : [0,\infty) \to R$ is a given function. A solution of (12.5) is represented as a vector $u = (u_{-N}, \dots, u_N) \in R^{2N+1}$ with $\|u\|_\infty = \max_{i=-N\dots N} |u_i|$. The first natural conjecture would be that the discrete approximate solution u_i would have a maximum at $j = 0$, and be symmetric about 0 in the sense that $u_{-j} = u_j$. This would exactly reflect the symmetry properties of the analogous continuous problem. This conjecture is false. Roughly speaking McKenna's result states *as $h \to 0$, the solution becomes more and more symmetric about the origin and the maximum $\to$ towards the origin.* Thus, the correct result is that for a sufficiently small space step, the solution will be "approximately" symmetric about the origin. They hope to prove an analogous result in the partial differential equation setting.

Hamiltonian Systems and Mathematical Physics

Let $V \in C^2(R^n, R)$ and $h > 0$ such that $\Omega \equiv \{q \in R^n | V(q) < h\}$ is bounded, open and connected. Consider the following given energy problem of the second order Hamiltonian system:

$$\ddot{q}(t) + V'(q(t)) = 0, \quad \text{for } q(t) \in \Omega,) \tag{12.6}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \qquad \forall t \in R,) \tag{12.7}$$

$$\dot{q}(0) = \dot{q}(\frac{\tau}{2}) = 0,) \tag{12.8}$$

$$q(\frac{\tau}{2} + t) = q(\frac{\tau}{2} - t), \qquad q(t + \tau) = q(t), \quad \forall t \in R.) \tag{12.9}$$

A solution (τ, q) of (12.6)-(12.9) is called a *brake orbit* on Ω. Two orbits q and $p : R \to R^m$ are said to be *geometrically distinct*, if $q(R) \neq p(R)$. Denote by $\mathcal{J}(\Omega)$ and $\tilde{\mathcal{J}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively.

In 1948, H. Seifert proved ${}^\#\mathcal{J}(\Omega) \geq 1$ provided V is analytic, Ω is homeomorphic to the unit ball in R^n, and $V'(q) \neq 0$ for $q \in \partial\Omega$. Then he conjectured that ${}^\#\tilde{\mathcal{J}}(\Omega) \geq n$ holds under the same conditions. Since then many studies have been carried out for brake orbits. Specially in 1983-1984, K. Hayashi, H. Gluck-W. Ziller, and V. Benci proved independently that ${}^\#\mathcal{J}(\Omega) \geq 1$, if V is C^1, $\bar{\Omega} = \{V \leq h\}$ is compact, and $V'(q) \neq 0$ for all $q \in \partial\Omega$. In 1987, P. Rabinowitz proved the corresponding result for first order Hamiltonian systems. For multiplicity results concerning Seifert's conjecture, there are only the papers of E. van Groesen in 1985, A. Szulkin in 1989, and A. Ambrosetti-V. Benci-Y. Long in 1993, in which ${}^\#\tilde{\mathcal{J}}(\Omega) \geq n$ was proved under various pinching conditions on the hypersurface $\partial\Omega$.

Yiming Long and his students study the multiplicity of brake orbits without any pinching conditions. Their main result is the following:

Theorem. *For $n \geq 2$ and $V \in C^2(R^n, R)$, suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$, and $V''(q)$ is positive definite for all $q \in R^n \setminus \{0\}$. Then for any given $h > 0$ and $\Omega \equiv \{q \in R^n | V(q) < h\}$, there holds*

$$^{\#}\tilde{\mathcal{J}}(\Omega) \geq 2. \tag{12.10}$$

Bolotin studied another class of problems for (12.6). Consider the 3-body problem in R^2 where we have a sun of mass 1, Jupiter of mass ε and an asteroid of negligible mass. Let $u(t)$ be the elliptic T-periodic orbit of the Jupiter. The motion of the asteroid is described by a Lagrangian system (L_ε) with

$$L_\varepsilon(q, \dot{q}, t) = |\dot{q}|^2/2 + |q + \varepsilon u(t)|^{-1} + \varepsilon |q - u(t)|^{-1}.$$

The system (L_ε) is a singular perturbation of the Kepler problem (L_0).

Fix $m, n \in N$. Let Π be the set of chains $c = (c_i)_{i=1}^n$ of collision curves $c_i : [t_{i-1}, t_i] \to R^2 \setminus \{0\}$ such that $c_i(t_{i-1}) = u(t_{i-1})$, $c_i(t_i) = u(t_i)$. The time moments $t_0 < \cdots < t_{n-1}$ are independent variables and $t_n = t_0 + mT$. Thus Π is an open set in $W_0^{1,2}([0,1], R^{2n}) \times R^n$. Critical points of the action functional

$$I(c) = \sum I(c_i), \qquad I(c_i) = \int L_0(c_i(t), \dot{c}_i(t), t)\, dt$$

are chains of collision orbits of system (L_0) such that the relative Hamiltonian $h = H_0 - \dot{q} \cdot \dot{u}(t)$ does not change at collisions: $h_i^+ = h_i^- = h_i$. We say that $c = (c_i)_{i=1}^n$ is a *nondegenerate collision chain* if it is a nondegenerate critical point of I and at each collision the direction of the relative velocity $v = \dot{q} - \dot{u}(t)$ changes: $v_i^+ \nparallel v_i^-$. Bolotin's main result is:

Theorem 1 *For any nondegenerate periodic collision chain $c = (c_i)_{i=1}^n$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a unique mT-periodic solution of system (L_ε) which is $O(\varepsilon)$-close to $c_i(t)$ for $t_{i-1} \leq t \leq t_i$.*

Such shadowing periodic orbits were called periodic solutions of the second kind by Poincaré. However, Poincaré didn't prove their existence. A similar result holds for infinite collision chains. Take N open bounded sets $U_k \subset R^2$ such that for each $(t_1, t_2) \in U_k$ there exists a collision orbit $c : [t_1, t_2] \to R^2$ of (L_0) with $c(t_1) = u(t_1)$, $c(t_2) = u(t_2)$, smoothly depending on (t_1, t_2). In particular $t_1 < t_2$ are not conjugate along c. Then $I(c) = S_k(t_1, t_2)$ is a smooth function on U_k. Sequences $\varkappa = (k_i)_{i \in Z}$ and $\tau = (t_i)_{i \in Z}$ such that $(t_{i-1} - Tm_i, t_i - Tm_i) \in U_{k_i}$, $m_i \in Z$, define a collision chain $c = (c_i)_{i \in Z}$. Set

$$A_\varkappa(\tau) = \sum I(c_i) = \sum S_{k_i}(t_{i-1}, t_i).$$

The functional is formal but its derivative $A'_\varkappa(\tau) \in l_\infty$ is well defined. A collision chain $c = (c_i)_{i \in Z}$ corresponding to the critical point τ is called nondegenerate if the second derivative $A''_\varkappa(\tau) : l_\infty \to l_\infty$ has a bounded inverse and the changing direction condition is uniform in i. Then for small $\varepsilon \in (0, \varepsilon_0)$ there exists an orbit of (L_ε) shadowing the chain c.

If S_k satisfies the twist condition $D^2_{t_1 t_2} S_k \neq 0$, critical points of $A_\varkappa$ correspond to orbits of compositions $f_{k_n} \circ \cdots \circ f_{k_0}$ of symplectic maps $f_{k_i} : (t_{i-1}, h_{i-1}) \to (t_i, h_i) \in (R/TZ) \times R$ with generating functions S_{k_i}. Such random dynamical systems have rich hyperbolic dynamics even if every map f_k is integrable. This makes it possible to construct many nondegenerate collision chains and hence periodic and chaotic shadowing orbits for system (L_ε).

Turning from classical to quantum mechanics, the Dirac-Fock equations are the Euler-Lagrange equations corresponding to the Dirac-Fock energy functional in a "sphere" of $L^2(R^3, C^4)^N$, N being a positive integer. This model corresponds, in an approximate way, to the search of stationary states for relativistic atoms and molecules. The Dirac operator being unbounded, both from above and from below, the corresponding energy functional is highly indefinite. However, this model "should contain" a notion of ground state if it is to describe a physical situation in which "minimal energy" solutions should exist and correspond to "most probable" configurations for the physical system. Moreover, the nonrelativistic limit of these equations (in the high light speed limit) can be shown to be the Hartree-Fock equations, for which ground state solutions exist under reasonable conditions (the Hartree-Fock energy is bounded from below). Maria Esteban first described the Dirac-Fock equations, and how taking the nonrelativistic limit leads us to the Hartree-Fock equations.

Then, she concentrated in showing how for high light speeds, various different variational problems are equivalent to the one that she uses to show existence of solutions. From this, one can obtain a physically relevant notion of ground state solution for this model. By doing so, what is really shown is that the critical points that are physically relevant all lie in a subset of the "sphere" defined by a nonlinear constraint. What is indirectly shown is that the Dirac-Fock energy is bounded from below in that set while it is not in the whole "sphere", i.e. that its minimum is reached there and that the minimizers are critical points of the energy which correspond to the solutions that previously found by using an unconstrained variational argument.

Eric Séré also presented his work on existence of a stable polarized vacuum in the Bogoliubov-Dirac-Fock approximation. According to Dirac's ideas, the vacuum consists of infinitely many virtual electrons which completely fill up the negative part of the spectrum of the free Dirac operator D^0 (this model is called the "Dirac sea"). In the presence of an external field, these virtual particles react and the vacuum becomes polarized. In this work, Séré and his coauthors consider a nonlinear model of the vacuum derived from QED, called the Bogoliubov-Dirac-Fock model (BDF). In this model, the vacuum is represented by a bounded self-adjoint operator Γ on $L^2(R^3)$. An energy of this vacuum is defined. A stable vacuum is a minimizer of this BDF energy functional, under some convex constraints. Séré showed the existence of a minimizer of the BDF energy in the presence of an external electrostatic field and proved that this minimizer is a projector, which solves a self-consistent equation of Hartree-Fock type. This minimizer is interpreted as the polarized Dirac sea.

Other Aspects and Applications of Variational Methods

In addition to the topics discussed above, some new methods and applications related to variational problems were presented in the workshop.

Nassif Ghoussoub developed a theory of anti-self dual Lagrangians and new variational formulations of boundary value problems and evolution equations. Its antecedents are old work of Brezis and Ekeland. His theory of anti-self dual Lagrangians allows for surprising variational formulations and resolutions for many boundary value and initial value problems which normally cannot be obtained as Euler-Lagrange equations of action functionals. Examples include non-potential operator equations (like nonlinear transport and others involving first order differential operators), as well as certain dissipative evolution equations (like the heat equation, porous media, other gradient flows and the Navier-Stokes equations).

Ivar Ekeland and Louis Nirenberg studied a very interesting variational problem from economics. When computing conditional expectations by Monte-Carlo methods, one tries to minimize the mean variance of the error. Applying Malliavin calculus to the problem, one is led to a novel type of Sobolev space, consisting of all functions on the positive orthant of R^n, such that every derivative not containing terms in $(d^p)/(dx_i)^p$ with $p = 2$ or more is square integrable. The last derivative with this property is $d^n/(dx_1)...(dx_n)$. They show that this is a bona fide Sobolev space, and They consider the problem of minimizing a quadratic form on that space under boundary conditions. they show existence, uniqueness and regularity of the minimizer.

List of Participants

Alama, Stanley (McMaster University)
Bartsch, Thomas (University of Giessen)
Bates, Peter (Michigan State University)
Bisgard, Jim (University of Wisconsin-Madison)
Bolotin, Sergey (University of Wisconsin)
Buffoni, Boris (Federal Swiss Institute of Technology-Lausanne)
Burchard, Almut (University of Virginia)
Ekeland, Ivar (University of British Columbia)
Esteban, Maria (Universite Paris IX-Dauphine)
Farber, Michael (University of Durham)
Ghoussoub, Nassif (University of British Columbia)
Glotov, Dmitry (Purdue University)
Gui, Changfeng (University of Connecticut)

Hajaiej, Hichem (University of Virginia)
Jerrard, Robert (University of Toronto)
Ji, Min (Chinese Academy of Sciences)
Jiang, Meiyue (Peking University)
Kang, Xiaosong (University of Toronto, Fields Institute)
Li, Congming (University of Colorado)
Li, Yanyan (Rutgers University)
Long, Yiming (Nankai University)
McKenna, Joe (University of Connecticut)
Montero, Alberto (McMaster University)
Nirenberg, Louis (Courant Institute)
Rabinowitz, Paul (University of Wisconsin, Madison)
Ren, Xiaofeng (Utah State University)
Sere, Eric (Universite Paris IX-Dauphine)
Serfaty, Sylvia (Courant Institute of Mathematical Sciences)
Speicher, Regina (University of Connecticut)
Sternberg, Peter (Indiana University)
Stredulinsky, Ed (University of Wisconsin-Richland)
Wang, Zhi-Qiang (Utah State University)
Wei, Juncheng (Chinese University of Hong Kong)
Zhang, K. W. (University of Sussex)

Chapter 13

Mathematical Foundations of Scientific Visualization, Computer Graphics, and Massive Data Exploration (04w5043)

May 22–27, 2004

Organizer(s): Torsten Möller (Simon Fraser University), Bernd Hamann (University of California, Davis), Robert Russell (Simon Fraser University)

In the last 15 years the profound impact of scientific computing upon virtually every area of science and engineering has been well established. The increasing complexity of the underlying mathematical models has also highlighted the critical role to be played by Scientific Visualization. It therefore comes as no surprise that Scientific Visualization is one of the most active and exciting areas of Mathematics and Computing Science, and indeed one which is only beginning to mature.

The importance of more rigorous mathematical approaches is becoming self apparent. At the last few Siggraph and Visualization conferences (the main conferences in the fields of graphics and visualization), an increasing number of mathematically oriented tutorials have been offered and received an enthusiastic reception. Examples are tutorials on “Multiresolution Techniques for Surfaces and Volumes” (Visualization 2001), “From Transfer Functions to Level Sets: Advanced Topics in Volume Image Processing” (Visualization 2001), “Level Set and PDE Methods for Computer Graphics” (Siggraph 2002), “A Practical Guide to Global Illumination Using Photon Mapping” (Siggraph 2002), “Mathematical Optimization in Graphics and Vision” (Siggraph 2002), “An Introduction to the Kalman Filter” (Siggraph 2001), “Geometric Signal Processing on Large Polygonal Meshes” (Siggraph 2001), “Using Tensor Diagrams to Represent and Solve Geometric Problems” (Siggraph 2001).

A primarly objective of the workshop is to gather together the main researchers in the mathematical areas relevant to the recent advances in order to discuss the research challenges facing this field in the next several years. The workshop shall cover five main thrusts:

- PDE’s
  - segmentation
  - level set method
  - mathematical modeling
- Signal Processing + Wavelet methods
  - multi-resolution
  - compression

  - filtering
  - stochastic approaches/noise removal

- Data Approximation
  - splines
  - volume modeling
  - scattered data methods
  - intrinsic surface and volume properties
  - parameterization
  - point cloud fitting

- Topology/Discrete methods
  - combinatorial topology
  - computational geometry
  - differential topology
  - feature and geometry extraction

- Massive Data (the only application track)
  - CFD
  - time-varying
  - multi-variate/multi-valued
  - uncertainty, visualization error metric
  - hardware methods (GPU's, out-of-core methods, parallel and distributed algorithms)
  - information visualization

The format of the workshop will include presentations by the participants as well as brainstorming sessions. Some of the questions to be addressed during the brainstorming sessions include:

- What are the scientifically challenging problems to be tackled in your topic area?
- What are the driving applications in this field?
- Which journals and conferences exist today that are appropriate venues for publishing mathematically oriented methods in this field?
- Which good online resources exist today supporting research in this sub-field. (e.g. example data sets, commercial and free software libraries, publication databases, benchmarking sites, etc.)
- Which scientific domains and sub-fields are needed to solve successfully and elegantly the identified problems?

List of Participants

Alexa, Marc (Technische Universität Darmstadt)
Attali, Dominique (Laboratoire LIS, ENSIEG, Domaine Universitaire)
Banks, David C. (Florida State University)
Bremer, Timo (University of California, Davis)
Carr, Hamish (University of British Columbia)
Ebling, Julia (University of Kaiserslautern)
Edelsbrunner, Herbert (Duke University)

Entezari, Reza (Simon Fraser University)
Hamann, Bernd (University of California, Davis)
Hotz, Ingrid (University of California, Davis)
Johnson, Chris (University of Utah)
Joy, Ken (University of California, Davis)
Kirby, Mike (University of Utah)
Klaus, Mueller (State University of New York at Stony Brook)
Livnat, Yarden (University of Utah)
Ma, Kwan-Liu (University of California, Davis)
Macdonald, Colin (Simon Fraser University)
Machiraju, Raghu (Ohio State University)
Mascarenhas, Ajith (University of North Carolina at Chapel Hill)
Munzner, Tamara (University of British Columbia)
Möller, Torsten (Simon Fraser University)
Nielson, Gregory M. (Arizona State University)
North, Stephen (AT&T Labs)
Pascucci, Valerio (Lawrence Livermore National Laboratory)
Polthier, Konrad (Zuse Institute Berlin)
Prohaska, Steffen (Zuse Institute Berlin)
Quak, Ewald (SINTEF ICT)
Russell, Robert (Simon Fraser University)
Shamir, Ariel (The Interdisciplinary Center, Herzliya)
Silva, Claudio T. (University of Utah)
Snoeyink, Jack (University of North Carolina, Chapel Hill)
Taubin, Gabriel (Brown University)
Tricoche, Xavier (University of Utah)
Weiskopf, Daniel (University of Stuttgart)
Whitaker, Ross T. (University of Utah)
Zhang, Richard (Hao) (Simon Fraser University)
Zhukov, Leonid (California Institute of Technology/Yahoo! Research Labs)
van Wijk, Jack (Technische Universiteit Eindhoven)

Chapter 14

Aperiodic Order: Dynamical Systems, Combinatorics, and Operators (04w5001)

May 29–June 3, 2004

Organizer(s): Michael Baake (Bielefeld), David Damanik (Caltech), Ian Putnam (Victoria), Boris Solomyak (Seattle)

The field of Aperiodic Order is concerned with the structure and properties of point sets that display long-range orientational order, and of all structures that can be derived from such point sets. The latter include tilings, discrete structures in general, measures, operators etc. Although there is no standard monograph on the subject yet, several review volumes on key topics are available by now [48, 53, 8, 56]. This indicates the activity of the field, see also [3] for a guide to further literature.

It was the aim of this workshop to bring people from the various mathematical disciplines together and to exchange the state of the art as well as to communicate open problems.

Model Sets and Diffraction Theory

An important class of ordered Delone sets are Meyer sets and, among them, model sets, compare [41] for details. Model sets are also known as cut and project sets, and admit a rather general formulation in the setting of locally compact Abelian groups, compare [47, 63]. Though they made their appearance in the context of algebraic number theory already in the seventies [47], their importance was only recognized after the discovery of quasicrystals whose spatial structure can be described by model sets, see [69] for a recent review from an experimental perspective.

Mathematically, model sets are defined on the basis of a *cut and project scheme*. The latter is a triple $(G, H, \widetilde{L})$ consisting of locally compact Abelian groups G and H, of which G is also σ-compact, and a lattice $\widetilde{L}$ in $G \times H$ (i.e., a co-compact discrete subgroup)

$$\begin{array}{ccccc} G & \stackrel{\pi_1}{\longleftarrow} & G \times H & \stackrel{\pi_2}{\longrightarrow} & H \\ & & \cup & & \\ & & \tilde{L} & & \end{array} \tag{14.1}$$

such that the natural projections $\pi_1 : G \times H \longrightarrow G$, $(t, h) \mapsto t$ and $\pi_2 : G \times H \longrightarrow H$, $(t, h) \mapsto h$ satisfy the following properties:

- The restriction $\pi_1|_{\tilde{L}}$ of π_1 to $\tilde{L}$ is injective.

- The image $\pi_2(\tilde{L})$ is dense in H.

Let $L := \pi_1(\tilde{L})$ and $^\star : L \longrightarrow H$ be the mapping $\pi_2 \circ (\pi_1|_{\tilde{L}})^{-1}$. Note that * is indeed well defined.

Given a cut and project scheme (14.1) and a compact $W \subset H$, we define $\curlywedge(W)$ by

$$\curlywedge(W) := \{x \in L : x^\star \in W\}.$$

A *model set*, associated with the cut and project scheme (14.1), is a non-empty subset $\varLambda$ of G of the form

$$\varLambda = x + \curlywedge(y + W),$$

where $x \in G$, $y \in H$, and $W \subset H$ is compact with $W = \overline{W^\circ}$. A model set $\varLambda = x + \curlywedge(y + W)$ is called *regular* if the Haar measure of the boundary ∂W of W is zero. A regular model set is called *generic* if $\partial W \cap L^\star = \varnothing$. Any model set is a Delone set. Namely, it is uniformly discrete (as W is compact) and relatively dense (as W has nonempty interior). In fact, they are even Meyer sets, because $\varLambda - \varLambda \subset \curlywedge(W - W)$ and $W - W$ is compact, so that $\varLambda - \varLambda \subset \varLambda + F$ with F a finite set. Moreover, a regular model set has uniform patch frequencies (i.e., the associated dynamical system is uniquely ergodic) and a generic model set is repetitive, see [49] for a review of the properties of model sets.

A prominent feature of regular model sets is their pure point diffraction. If we start from the Dirac comb $\omega = \delta_\varLambda = \sum_{x \in \varLambda} \delta_x$, with δ_x the normalized point measure at x, there exists a natural *autocorrelation*

$$\gamma_\omega = \lim_{r \to \infty} \frac{1}{\mathrm{vol}(B_r)} \tilde{\omega} * \omega|_r$$

where B_r is the ball of radius r around 0 for $G = \mathbb{R}^d$, or a suitable generalization of this concept for general G, see [64] for details. Moreover, $\omega|_r$ is the restriction of ω to B_r and $\tilde{\omega} = \delta_{-\varLambda}$ is the origin inverted variant of ω. This autocorrelation is unique (w.r.t. averaging sequences of van Hove type) which reflects the unique ergodicity of the corresponding dynamical system (see below). What is more, it is always a positive definite and translation bounded measure on G. Consequently, it is transformable, and its Fourier transform, $\hat{\gamma}_\omega$, is a positive measure, called the *diffraction measure* of ω. It describes the outcome of standard diffraction experiments, compare [37, 24].

An important result is that the diffraction measure for the Dirac comb of a regular model set is a pure point measure, or, in other words, that regular model sets are pure point diffractive. In this generality, it was proved by M. Schlottmann in [64], though it has several predecessors [37, 68]. These aspects, and in particular their various relations to dynamical systems, were summarized in the opening lecture by Robert Moody, and reappeared in many other talks throughout the meeting.

The cornerstone of most proofs of this result is the connection to pure point spectra of dynamical systems, to which we will come back below. One alternative proof is known [9] that relates pure point diffraction spectra directly to strong almost periodicity of the autocorrelation measure, the latter being a consequence of a Weyl type result on uniform distribution in model sets [63, 50]. This is also related to recent results of J.-B. Gouéré [34]. Using this approach, one can see in general that a complex translation bounded measure ω on G is pure point diffractive (w.r.t. an averaging van Hove sequence $\mathcal{A} = \{a_n \mid n \in \mathbb{N}\}$, compare [64, 9]) if and only if its autocorrelation γ_ω (obtained w.r.t. $\mathcal{A}$) is strongly almost periodic, compare also [31] for background.

Various generalizations of model sets are studied, such as multi-component model sets (e.g., in the talk by J.-Y. Lee) or deformed model sets (in the talk by D. Lenz), where the latter go considerably beyond the setting of Delone sets of finite local complexity. Of considerable interest is the question for the diffractive properties of Meyer sets. Though it is known [47] that they are subsets of model sets, their diffraction is much more involved. In particular, one can have mixed spectrum (i.e., both pure point and continuous components), and Meyer sets can have positive entropy density (e.g., the union of $2\mathbb{Z}$ with an arbitrary subset of $2\mathbb{Z} + 1$ is Meyer). First general results on the systematic study of Meyer set diffraction were presented by N. Strungaru, building on the more detailed theory of almost periodic measures (strong versus weak, see [31] for details).

Still of interest is the systematic investigation of symmetry, and the development of efficient methods to determine the symmetry of a given Delone set, ranging from translation over point to inflation symmetries. Well-known heuristics from the physics literature, see [46] and references given there, are now reformulated in the context of the cohomology of groups, and the survey talk of B. Fisher showed the present state of affairs, compare [28].

Finally, one common theme of many talks on aperiodic order was the apparent similarity of model sets to lattices. Sometimes, it requires a slight reformulation of the classic concepts, but very often a new and simplifying point of view emerges. The talk by U. Grimm might serve as an illustration, where combinatorial problems of crystallography [5] were reformulated and solved in a unified fashion for lattices and model sets, see [4, 6] for more.

Aperiodic Order and Dynamical Systems

The dynamical systems approach to the theory of aperiodic order has gained prominence in recent years. Let us explain how dynamical systems appear in the simple setting of Delone sets in $\mathbb{R}^d$. Given a Delone set Λ, which can be viewed as a model of an atomic configurations, we can consider its *hull* X_Λ. It can be defined as the closure of the set $\{\Lambda - x : x \in \mathbb{R}^d\}$ of all translates of Λ in the natural (local) topology. In this topology, two sets are close if they almost agree on a large ball around the origin. The group $\mathbb{R}^d$ acts continuously on X_Λ by translations, which is our topological dynamical system. Given an invariant probability measure on X_Λ (it always exists and is often unique; then the system is said to be *uniquely ergodic*), we get a measure-preserving system $(X_\Lambda, \mathbb{R}^d, \mu)$, which may be studied using the tools of ergodic theory. In particular, we can consider the *dynamical spectrum*, that is, the projection spectral measure, or a family of scalar spectral measures, associated with the group of unitary operators on $L^2(X_\Lambda, \mu)$ given by $(U_x f)(\xi) = f(\xi - x)$ for $x \in \mathbb{R}^d$. The spectral type of the dynamical system may be pure point (pure discrete), pure absolutely continuous, pure singular continuous, or a mixture. A key observation made by Dworkin [25] in 1993 is that pure point dynamical spectrum implies pure point diffraction spectrum, which has been widely viewed as the key feature of an ordered (crystalline or quasi-crystalline) structure.

One direction of research in the recent years has been to reverse the implication, that is, to deduce pure point *dynamical* spectrum from pure point *diffraction* spectrum. This was done in a restricted setting of Delone sets of finite local complexity in [42], and more recently, in much greater generality, by J.-B. Gouéré [34] and M. Baake & D. Lenz [7]. In fact, in [7], instead of Delone dynamical systems on $\mathbb{R}^n$, dynamical systems on translation bounded measures on rather general locally compact Abelian groups are considered. This approach via measures is both more general and well-suited for applications. The talks by D. Lenz and J.-B. Gouéré described these achievements, among other things.

Another class of dynamical systems related to aperiodic order is that of tiling dynamical systems. They are defined similarly to Delone dynamical systems, starting with a tiling of the Euclidean space, and considering the translation action on the hull. In recent years, topological methods have been increasingly used to study such systems. Some of these developments were described in the talk on "Tilings, tiling spaces, and topology" by L. Sadun. One of the key questions is: what are the possible perturbations of a tiling and what happens to the dynamical systems under these perturbations? It turns out that the tiling spaces may be homeomorphic (e.g., if the two tiling systems have identical combinatorics), but that their dynamical properties can still differ. In the recent work by A. Clark and L. Sadun [22], the Čech cohomology H^1 of the tiling space is used to determine when the perturbation yields a topologically conjugate system, and when it yields a mutually locally derivable system. The latter notion corresponds to the the existence of a "local code"; unlike in symbolic dynamics, for tiling dynamical systems not every conjugacy is given by a local code.

Symbolic substitution systems form a rich and interesting class of examples, studied in dynamical systems and ergodic theory for several decades. More recently, generalizations to higher dimensions, including substitution tiling systems and substitution Delone sets, largely motivated by the theory of aperiodic order, were introduced and investigated. They provide examples with various spectral types: pure discrete, pure singular, and partially absolutely continuous. But even in the classical, one-dimensional symbolic setting, there remain many open questions. We describe one of them in detail.

Let $\mathcal{A} = \{1, \ldots, d\}$ be a finite alphabet, with $d \geq 2$, and denote by $\mathcal{A}^* = \bigcup_{i=0}^\infty \mathcal{A}^i$ the set of finite words. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^*$; it is extended to a map $\mathcal{A}^* \to \mathcal{A}^*$ by concatenation. The $d \times d$ matrix associated with the substitution ζ is defined by $M_\zeta(i,j) = \ell_i(\zeta(j))$, where $\ell_i(w)$ denotes the number of occurrences of the letter i in the word w. The substitution is *primitive* if there exists a k such that all entries of M_ζ^k are strictly positive. A primitive substitution gives rise to a uniquely ergodic dynamical system: the space is the set of all sequences all blocks of which occur in $\zeta^k(i)$ for some k and i, and the dynamics is given by the shift. One of the outstanding problems is to resolve the *Pisot discrete spectrum conjecture* which is that every Pisot type substitution system has pure point spectrum. The substitution is of *Pisot type* if all the eigenvalues of the matrix M_ζ, except the largest eigenvalue, are strictly between 0 and 1 in absolute value. It is still open, although recently the case of two symbols was settled affirmatively, see [11, 38]. The important special case of unimodular Pisot substitutions, where it is also assumed that $\det(M_\zeta) = 1$, is open, too. There is a related *coincidence conjecture* which we do not describe here. At the workshop, we were fortunate to have several groups present from around the world who are working in this field, and there were many lively discussions of various approaches to the problems. Among the participants were Sh. Akiyama [1], M. Baake and B. Sing [10], and V. Sirvent and Y. Wang [66] who made contributions to this area. J. Kwapisz, in his talk entitled "Geometric coincidence conjecture and pure discrete spectrum for unimodular tiling spaces," described his work in progress, jointly with M. Barge and B. Diamond. Their approach uses the space of "strands", which was first introduced in [11], to represent the dynamical system. One of the new results is that, for unimodular Pisot substitutions, pure point spectrum is equivalent to the model set representation. A geometric approach to the Pisot Substitution Conjecture was pioneered by G. Rauzy [59]; it uses what is now known as the "Rauzy tiling" of the substitution. At the workshop, A. Siegel represented this direction; she described some combinatorial conditions for pure discrete spectrum in her talk, based on [19, 65].

It is impossible to discuss all the recent developments related to substitutions here. There is an excellent recent book on this topic [56] with the chapter on spectral theory written by A. Siegel.

Substitution tilings in $\mathbb{R}^d$ represent a far-reaching generalization of substitution sequences. The role of the alphabet is played by a finite set of *prototiles*. The substitution map replaces a tile by a "patch" of tiles in a consistent manner. We do not go into the details of the definition here. A crucial point comes in deciding how the tiles of the tiling are obtained from the prototiles: (a) using translations only, or (b) using arbitrary Euclidean motions. We have a much better understanding of the class (a), in large part due to the commutativity of the translation group. The class (b), however, is gradually being investigated as well. Its best known example is the *pinwheel tiling* of the plane [58]. The dynamical and diffraction spectrum of this tiling are still poorly understood: it is known that it has no non-trivial discrete spectral component, but we do not know whether the spectrum is singular or it has an absolutely continuous component. At the Problem Session, N. Strungaru described his recent result with R. V. Moody and D. Postnikov [51] which says that the diffraction spectrum is rotation-invariant under the action of S^1.

An important feature of many substitution systems is *unique decomposition*. It can be defined by saying that the substitution map defines a homeomorphism of the tiling space. It was proved in [67] in the translationally-finite setting that the unique decomposition property is equivalent to the tiling being non-periodic (i.e., it should have no translation symmetries). C. Holton reported on the recent progress for non-translationally finite tilings. In joint work with C. Radin and L. Sadun (in preparation), the unique decomposition property was verified under some assumptions, the main one being that the set of relative orientations of a tile in the tiling leaves no subspace of $\mathbb{R}^d$ invariant. On the other hand, a counter-example in $\mathbb{R}^3$ was constructed to demonstrate that the latter condition cannot be dropped.

Combinatorics on Words

A popular way to measure complexity is via subword or pattern complexity. Aperiodic order then manifests itself in low complexity with respect to this measure. In one dimension, investigations in this direction date back at least to the 1930's and have evolved into an independent mathematical subdiscipline, often called "Combinatorics on Words." Currently, this field is particularly active in France, and the French school has put the theory on a firm footing and made it more accessible to a broad audience by the publication of a series of textbooks, [43, 44, 45] (see also [56]). We were happy to have Valérie Berthé and Julien Cassaigne participate in the workshop and report on recent progress of combinatorics on words in dimensions greater than one—a subject that is still in its early stages.

Before summarizing the key results and open problems in higher dimensions, let us briefly discuss the one-dimensional case. Here, one studies words over a finite alphabet. That is, if $\mathcal{A}$ is a finite set, one considers the sets $\mathcal{A}^*, \mathcal{A}^{\mathbb{Z}_+}, \mathcal{A}^{\mathbb{Z}}$ of finite, one-sided infinite, and two-sided infinite words over $\mathcal{A}$, respectively. Given a one-sided or two-sided infinite word w, define its complexity function $p_w : \mathbb{Z}_+ \to \mathbb{Z}_+$ by

$$p_w(n) = \#\{\text{subwords of } w \text{ having length } n\}.$$

It is obvious that p_w is a bounded function if w is (eventually) periodic. A surprising, albeit elementary, result of Hedlund and Morse [36] states that the converse is true and, moreover, there is some minimum growth of the complexity function when w is not eventually periodic. For $w \in \mathcal{A}^{\mathbb{Z}_+}$, the following are equivalent,

(i) w is ultimately periodic, that is, there are $n_0, q \in \mathbb{Z}_+$ such that $w_{n+q} = w_n$ for $n \geq n_0$.

(ii) p_w is bounded.

(iii) There exists $n_1 \in \mathbb{Z}_+$ such that $p_w(n_1) \leq n_1$.

That is, words displaying aperiodic order should have a complexity function that is bounded from below by $n + 1$ but does not grow much faster than that. One is then interested in consequences of low complexity. The case of minimal complexity, $p_w(n) = n + 1$, has been completely analyzed; see [15, 36, 44]. Words w with this complexity function are called *Sturmian* and they have a large number of equivalent descriptions. Aside from the combinatorial description in terms of their complexity function above, they can also be characterized geometrically, in terms of certain balance properties, their palindromic subwords; to mention just a few. Other classes of words, for which strong general structure results are known, are those satisfying $p_w(n) = n + k$ for some k and large enough n, the so-called quasi-Sturmian words (see, e.g., [23, 54]), or $p_w(n) = O(n)$, the words having linearly bounded complexity (cf. [27]).

The Hedlund-Morse result for two-sided infinite words looks slightly more elegant in that ultimate periodicity can be replaced by periodicity. For $w \in \mathcal{A}^{\mathbb{Z}}$, the following are equivalent,

(i) w is periodic, that is, there is $q \in \mathbb{Z}_+$ such that $w_{n+q} = w_n$ for every $n \in \mathbb{Z}$.

(ii) p_w is bounded.

(iii) There exists $n_1 \in \mathbb{Z}_+$ such that $p_w(n_1) \leq n_1$.

In dimensions greater than one, the most basic open problems concern suitable analogues of the results described above, that is, a suitable version of the Hedlund-Morse theorem and a characterization of a suitable class of low-complexity objects. Of course, one has to define a notion of complexity first. A natural way to do this is the following. Given $w \in \mathcal{A}^{\mathbb{Z}^d}$ and $n_1, \ldots, n_d \in \mathbb{Z}_+$, one defines

$$p_w(n_1, \ldots, n_d) = \#\{\text{ subwords of } w \text{ having “shape” } n_1 \times \cdots \times n_d\}.$$

The function p_w on box shapes could be called the box complexity function or, in the case $d = 2$, the rectangle complexity function. In search of an analogue of the Hedlund-Morse theorem, a naive guess could be that if there is some shape $(n_1, \ldots, n_d)$ such that $p_w(n_1, \ldots, n_d) \leq n_1 \times \cdots \times n_d$, then w has a periodicity vector. A simple example found by Sander and Tijdeman [62] shows such a statement cannot hold when $d \geq 3$. The question in $d = 2$ is open, but the answer is conjectured to be affirmative.

Nivat's Conjecture [52]. Let $w \in \mathcal{A}^{\mathbb{Z}^2}$. If there exist $n_1, n_2 \in \mathbb{Z}_+$ such that $p_w(n_1, n_2) \leq n_1 n_2$, then w has a periodicity vector.

However, unlike in the one-dimensional case, Nivat's conjecture is not an equivalence. In fact, there exists a word w, possessing a periodicity vector, for which $p_w(n_1, n_2) > n_1 n_2$ for all pairs (n_1, n_2) [17].

There are partial results saying that if $p_w(n_1, n_2) \leq c\, n_1 n_2$ for some $n_1, n_2 \in \mathbb{Z}_+$ and $c = 1/144$ [26] or $c = 1/16$ [57], then w has a periodicity vector. There are surveys of results and questions centred around Nivat's conjecture by Cassaigne [21] and Tijdeman [70]. Cassaigne's talk at our workshop dealt with these and related issues.

Two-dimensional words of low complexity, and in particular analogues of one-dimensional Sturmian words, have been studied in a number of papers (e.g., [16, 17, 18, 20]). Since Sturmian words in one dimension admit various equivalent descriptions, there are multiple ways to approach such a generalization. It turns out that they lead to different classes of two-dimensional words and hence the situation is more complicated than in one dimension. Valérie Berthé presented an overview of these results at the workshop, together with possible directions for future research.

Topological Aspects of Aperiodic Order

Beginning from a tiling or Delone set in $\mathbb{R}^d$, denoted Λ, one may consider the set of all its translates and endow this with a natural metric. Under the hypothesis of finite local complexity, this metric space is pre-compact. That is, its completion, denoted X_Λ, is a compact metric space. Moreover, the translation action of $\mathbb{R}^d$ extends continuously. In the case that the periodic vectors for the tiling form a spanning set for $\mathbb{R}^d$, the space is just a torus of dimension d.

The topology of the space X_Λ has been the focus of much research. It was observed very early that, for aperiodic tilings, the space, locally, is the product of an open ball in $\mathbb{R}^d$ with a totally disconnected space. A global extension of this result was obtained by Sadun and Williams [61] who showed that the space was a fibre bundle over a torus with totally disconnected fibres.

Anderson and Putnam [2] considered the case of substitution tilings and showed that the space could be written as an inverse limit of spaces which are quite tractable. They are branched oriented manifolds and, if the tiles are polygons meeting edge to edge and vertex to vertex, they are also finite cell complexes. In fact, the inverse limit is stationary in the sense that each space is the same and each map is the same. This result (with non-stationary system) was extended to the general situation in two distinct ways; first by Bellissard, Benedetti and Gambaudo [12] and secondly by Gähler [30] and Sadun [60], the latter paper being based on a talk given by Gähler on a previous meeting. Jean-Marc Gambaudo gave a presentation at the workshop on the former and subsequent generalizations (with Benedetti [14]) to other situations, including tilings of non-Euclidean spaces.

A result of Sadun and Williams [61] states that, under the hypothesis of finite local complexity, the space X_Λ is homeomorphic to one obtained by performing a d-fold suspension of a free minimal action of $\mathbb{Z}^d$ on a Cantor set X. This then connects the subject with the very active area of $\mathbb{Z}^d$ dynamical systems. The construction of Bellissard, Benedetti and Gambaudo may be used in this context to give approximating finite subequivalence relations of the orbit relation for such an action.

The presentation by Anderson and Putnam of the space as an inverse limit made it possible to compute its K-theory and Čech cohomology. (The methods in the general situation also work in principle, but for practical computations they seem unwieldy.) The computation of cohomology and K-theory for projection method tilings can be done using methods of Forrest, Hunton and Kellendonk [29]. The method uses some advanced techniques from algebraic topology: finding resolutions of certain modules and spectral sequences. In his lecture at the workshop, John Hunton sketched the basic ideas. He also discussed the Euler characteristic for these spaces. This can be obtained from the cohomology, of course, but its computation is much simpler. Moreover, examples suggest that there are some interesting questions regarding its sign.

Franz Gähler reported on his work in actually carrying out these cohomology computations (with the title "Examples and counter-examples ..."). In particular, he has developed software to implement the Anderson-Putnam method, as well as to do some calculations using his own technique. This resulting evidence was quite interesting. In particular, he found an example where the cohomology has a torsion component. This contradicted results of Forrest, Hunton and Kellendonk. By the end of the meeting, the problem seemed to have been resolved and a correct version of the Forrest-Hunton-Kellendonk result found, compare [30] for more.

Jean Bellissard showed how one may construct a C^*-algebra from an aperiodic tiling [13]. A different version is due to Kellendonk [39]. By present knowledge, this is – up to strong Morita equivalence – the crossed product construction by the action of $\mathbb{R}^d$ on X_Λ, see [40]. Let us denote this C^*-algebra by A_Λ. It is important from a physical viewpoint because Schrödinger operators (in the tight binding approximation) associated with an electron's movement in an aperiodic material are in this C^*-algebra.

A reasonable amount of information is now known about these C^*-algebras. First of all, their K-theory is computable. By a result of Connes, it is isomorphic to the K-theory of the space X_Λ. The internal structure of the algebras has been investigated by Giordano, Herman, Putnam and Skau [35, 32] in dimension one and by N.C. Phillips [55] in higher dimensions, where the analysis becomes much more difficult. The technique is to make use of the finite approximations of the Bellissard-Benedetti-Gambaudo inverse limit to construct approximating subalgebras which are themselves finite dimensional. Many nice properties of finite dimensional C^*-algebras may then be transferred to the larger algebra. Phillips gave a presentation summarizing these methods.

An alternate approach to this problem is in the work of Giordano, Putnam and Skau to classify minimal

Cantor set dynamics up to orbit equivalence. Skau gave a summary of the past work in the program (in dimension one) and its relations with the structure of the C^*-algebras. Giordano gave a presentation of the current work in dimension two [33]. Here, better finite approximations of the orbit space are obtained by using cocycles for the action, the drawback being that little is currently known about the existence of such cocycles. This problem is equivalent to a more complete understanding of the cohomology of the space X_Λ.

List of Participants

Akiyama, Shigeki (Niigata University)
Baake, Michael (University of Bielefeld)
Barge, Marcy (Montana State University)
Berthe, Valerie (LIRMM, Montpellier)
Carlsen, Toke (University of Copenhagen)
Cassaigne, Julien (IML-CNRS, Marseille)
Damanik, David (California Institute of Technology)
Diamond, Beverly (College of Charleston)
Durand, Fabien (Universite de Picardie Jules Verne)
Edson, Marcia (University of North Texas)
Fisher, Benji (Boston College)
Frank, Natalie (Vassar College)
Gambaudo, Jean-Marc (Universite de Bourgogne)
Giordano, Thierry (University of Ottawa)
Goncalves, Daniel (University of Victoria)
Gouere, Jean-Baptiste (Universite Claude Bernard Lyon 1)
Grimm, Uwe (The Open University)
Gähler, Franz (University of Stuttgart)
Holton, Charles (University of Texas at Austin)
Huck, Christian (Universität Bielefeld)
Hunton, John (University of Leicester)
Kwapisz, Jaroslaw (Montana State University)
Lee, Jeong-Yup (University of Alberta)
Lenz, Daniel (Technische Universität Chemnitz)
Martensen, Brian (University of Texas)
Moody, Robert (University of Alberta, BIRS)
Naumis, Gerardo (Instituto de Fosica, UNAM)
Phillips, N. Christopher (University of Oregon)
Pleasants, Peter A. B. (University of Queensland)
Putnam, Ian (University of Victoria)
Radin, Charles (University of Texas, Austin)
Sadun, Lorenzo (University of Texas, Austin)
Siegel, Anne (IRISA)
Sing, Bernd (Universität Bielefeld)
Sirvent, Victor (Universidad Simon Bolivar)
Skau, Christian (Norwegian University of Science and Technology)
Solomyak, Boris (University of Washington)
Starling, Charles (University of Victoria)
Strungaru, Nicolae (University of Alberta)
Wang, Yang (Georgia Institute of Technology)
Yi, Inhyeop (George Washington University)

Bibliography

[1] Sh. Akiyama, On the boundary of self-affine tilings generated by Pisot numbers, *J. Math. Soc. Japan* **54** (2002), 283–308.

[2] J. E. Anderson and I. F. Putnam, Topological invariants for substitution tilings and their associated C^*-algebras, *Ergodic Th. & Dynam. Syst.* **18** (1998), 509–537.

[3] M. Baake and U. Grimm, A guide to quasicrystal literature, in: [8], 371–373.

[4] M. Baake and U. Grimm, A note on shelling, *Discr. Comput. Geom.* **30** (2003), 573–589; `math.MG/0203025`.

[5] M. Baake and U. Grimm, Combinatorial problems of (quasi-)crystallography, in: *Quasicrystals: Structure and Physical Properties*, ed. H.-R. Trebin, Wiley-VCH, Weinheim (2003), 160–171; `math-ph/0212015`.

[6] M. Baake and U. Grimm, Bravais colourings of planar modules with N-fold symmetry, *Z. Kristallographie* **219** (2004), 72–80; `math.CO/0301021`.

[7] M. Baake and D. Lenz, Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra, *Ergodic Th. & Dynam. Syst.*, to appear; `math.DS/0302061`.

[8] M. Baake and R. V. Moody (eds.), *Directions in Mathematical Quasicrystals*, CRM Monograph Series **13**, AMS, Providence, RI (2000).

[9] M. Baake and R. V. Moody, Weighted Dirac combs with pure point diffraction, *J. Reine Angew. Math. (Crelle)*, to appear; `math.MG/0203030`.

[10] M. Baake and B. Sing, Kolakoski-(3,1) is a (deformed) model set, *Canad. Math. Bull.* **47** (2004), 168–190; `math.MG/0206098`.

[11] M. Barge and B. Diamond, Coincidence for substitutions of Pisot type, *Bull. Soc. Math. France* **130** (2002), 619–626.

[12] J. Bellissard, R. Benedetti and J.-M. Gambaudo, Spaces of tilings, finite telescopic approximations and gap-labelling, *Commun. Math. Phys.*, to appear; `math.DS/0109062`.

[13] J. Bellissard, D. J. L. Herrmann and M. Zarrouati, Hulls of aperiodic solids and gap labeling theorems, in: [8], 207–258.

[14] R. Benedetti and J.-M. Gambaudo, On the dynamics of G-solenoids: applications to Delone sets, *Ergodic Th. & Dynam. Syst.* **23** (2003), 673–691; `math.DS/0208243`.

[15] J. Berstel, Recent results in Sturmian words, in: *Developments in Language Theory*, eds. J. Dassow, G. Rozenberg and A. Salomaa, World Scientific, River Edge, NJ (1996), 13–24.

[16] V. Berthé and R. Tijdeman, Balance properties of multi-dimensional words, *Theor. Comput. Sci.* **273** (2002), 197–224.

[17] V. Berthé and L. Vuillon, Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences, *Discrete Math.* **223** (2000), 27–53.

[18] V. Berthé and L. Vuillon, Suites doubles de basse complexité, *J. Théor. Nombres Bordeaux* **12** (2000), 179–208.

[19] V. Canterini and A. Siegel, Geometric representation of primitive substitutions of Pisot type, *Trans. Amer. Math. Soc.* **353** (2001), 5121–5144.

[20] J. Cassaigne, Double sequences with complexity $mn + 1$, *J. Autom. Lang. Comb.* **4** (1999), 153–170.

[21] J. Cassaigne, Subword complexity and periodicity in two or more dimensions, in: *Developments in Language Theory*, eds. G. Rozenberg and W. Thomas, World Scientific, River Edge, NJ (2000), 14–21.

[22] A. Clark and L. Sadun, When shape matters: deformations of tiling spaces, *Ergodic Th. & Dynam. Syst.*, to appear; `math.DS/0306214`.

[23] E. M. Coven, Sequences with minimal block growth, II, *Math. Systems Th.* **8** (1975), 376–382.

[24] J. M. Cowley, *Diffraction Physics*, 3rd ed., North-Holland, Amsterdam (1995).

[25] S. Dworkin, Spectral theory and X-ray diffraction, *J. Math. Phys.* **34** (1993), 2965–2967.

[26] C. Epifanio, M. Koskas and F. Mignosi, On a conjecture on bidimensional words, *Theor. Comput. Sci.* **299** (2003), 123–150.

[27] S. Ferenczi, Rank and symbolic complexity, *Ergodic Th. & Dynam. Syst.* **16** (1996), 663–682.

[28] B. J. Fisher and D. N. Rabson, Applications of group cohomology to the classification of crystals and quasicrystals, *J. Phys. A: Math. Gen.* **36** (2003), 10195–10214; `math-ph/0105010`.

[29] A. Forrest, J. Hunton and J. Kellendonk, *Topological Invariants for Projection Method Patterns*, Memoirs AMS **159** No. 758, AMS, Providence, RI (2002).

[30] F. Gähler and J. Hunton, Surprises in tiling cohomology, in preparation.

[31] J. Gil de Lamadrid and L. N. Argabright, *Almost Periodic Measures*, Memoirs AMS **85** No. 428, AMS, Providence, RI (1990).

[32] T. Giordano, I. F. Putnam and C. F. Skau, Topological orbit equivalence and C^*-crossed products, *J. reine angew. Math. (Crelle)*, **469** (1995), 51–111.

[33] T. Giordano, I. F. Putnam and C. F. Skau, The orbit structure of Cantor minimal $\mathbb{Z}^2$-systems, in preparation.

[34] J.-B. Gouéré, Quasicrystals and almost periodicity, *Commun. Math. Phys.*, to appear; `math-ph/0212012`.

[35] R. H. Herman, I. F. Putnam and C. F. Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics, *Int. J. Math.* **3** (1992), 827–864.

[36] G. A. Hedlund and M. Morse, Symbolic dynamics, *Amer. J. Math.* **60** (1938), 815–866.

[37] A. Hof, Diffraction by aperiodic structures, in: [48], 239–268.

[38] M. Hollander and B. Solomyak, Two-symbol Pisot substitutions have pure discrete spectrum, *Ergodic Th. & Dynam. Syst.* **23** (2003), 533–540.

[39] J. Kellendonk, Noncommutative geometry of tilings and gap labelling, *Rev. Math. Phys.* **7** (1995), 1133–1180.

[40] J. Kellendonk and I. F. Putnam, Tilings, C^*-algebras and K-theory, in: [8], 177–206.

[41] J. C. Lagarias, Geometric models for quasicrystals. I. Delone sets of finite type, *Discrete Comput. Geom.* **21** (1999) 161–191.

[42] J.-Y. Lee, R. V. Moody and B. Solomyak, Pure point dynamical and diffraction spectra, *Annales H. Poincaré* **3** (2002), 1003–1018; `mp_arc/02-39`.

[43] M. Lothaire, *Combinatorics on Words*, Cambridge University Press, Cambridge (1997).

[44] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge (2002).

[45] M. Lothaire, *Applied Combinatorics on Words*, in preparation.

[46] N. D. Mermin, The symmetry of crystals, in: [48], 377–401.

[47] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North-Holland, Amsterdam (1972).

[48] R. V. Moody (ed.), *The Mathematics of Long-Range Aperiodic Order*, NATO ASI Series C 489, Kluwer, Dordrecht (1997).

[49] R. V. Moody, Model sets: A survey, in: *From Quasicrystals to More Complex Systems*, eds. F. Axel, F. Dénoyer and J. P. Gazeau, EDP Sciences, Les Ulis, and Springer, Berlin (2000), 145–166; `math.MG/0002020`.

[50] R. V. Moody, Uniform distribution in model sets, *Can. Math. Bulletin* **45** (2002), 123–130.

[51] R. V. Moody, D. Postnikoff and N. Strungaru, Circular symmetry of pinwheel diffraction, preprint (2004).

[52] M. Nivat, talk at ICALP, Bologna (1997).

[53] J. Patera (ed.), *Quasicrystals and Discrete Geometry*, Fields Institute Monographs **10**, AMS, Providence, RI (1998).

[54] M. E. Paul, Minimal symbolic flows having minimal block growth, *Math. Systems Th.* **8** (1975), 309–315.

[55] N. C. Phillips, Crossed products of the Cantor set by free minimal actions of $\mathbf{Z}^d$, *Commun. Math. Phys.*, to appear; `math.OA/0208085`.

[56] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, eds. V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel, Lecture Notes in Mathematics **1794**, Springer, Berlin (2002).

[57] A. Quas and L. Zamboni, Periodicity and local complexity, *Theor. Comput. Sci.*, to appear.

[58] C. Radin, The pinwheel tiling of the plane, *Annals Math.* **139** (1994), 661–702.

[59] G. Rauzy, Nombres algébriques et substitutions, *Bull. Soc. Math. France* **110** (1982), 147–178.

[60] L. Sadun, Tiling spaces are inverse limits, *J. Math. Phys.* **44** (2003), 5410–5414.

[61] L. Sadun and R. F. Williams, Tiling spaces are Cantor set fibre bundles, *Ergodic Th. & Dynam. Syst.* **23** (2003), 307–316.

[62] J. W. Sander and R. Tijdeman, The complexity of functions on lattices, *Theor. Comput. Sci.* **246** (2000), 195–225.

[63] M. Schlottmann, Cut-and-project sets in locally compact Abelian groups, in: [53], 247–264.

[64] M. Schlottmann, Generalized model sets and dynamical systems, in: [8], 143–159.

[65] A. Siegel, Pure discrete spectrum dynamical system and periodic tiling associated with a substitution, preprint (2004).

[66] V. Sirvent and Y. Wang, Self-affine tiling via substitution dynamical systems and Rauzy fractals, *Pacific J. Math.* **206** (2002), 465–485.

[67] B. Solomyak, Non-periodicity implies unique composition for self-similar translationally-finite tilings, *Discrete Comput. Geom.* **20** (1998), 265–279.

[68] B. Solomyak, Spectrum of dynamical systems arising from Delone sets, in: [53], 265–275.

[69] W. Steurer, Twenty years of structure research on quasicrystals, *Z. Krist.* **219** (2004), 391–446.

[70] R. Tijdeman, Periodicity and almost periodicity, preprint (2002).

Chapter 15

Semimartingale Theory and Practice in Finance (04w5032)

June 5–10, 2004

Organizer(s): Tom Hurd (McMaster University), Thaleia Zariphopoulou (University of Texas, Austin), Philip Protter (Cornell University), Lane Hughston (King's College London)

The theme of this meeting will reflect these new developments in the foundations of mathematical finance, and the following topics are intended to constitute the main focus of the five-day workshop: a. The mathematical methods of general semimartingale modelling for finance in (i) asset pricing and hedging (ii) portfolio optimization and (iii) optimal stopping problems. b. The consistent statistical estimation and calibration of jump diffusions and purely discontinuous processes with respect to econometric data. c. Theory and implementation of Levy-based stochastic volatility models. d. New term structure models for fixed income and equity dynamics.

It is intended that on the order of twenty of the world's leading probabilists and mathematical finance theorists will be brought to the meeting, balanced with a corresponding number of highly qualified international doctoral and postdoctoral researchers. We intend to adopt a selection procedure that will allow invited senior scientists to identify promising young researchers. The meeting will be appropriately paced with focused talks by invited speakers during the mornings and early afternoons. The late afternoons will typically then be less structured to allow smaller groups to break away to focus on specialized areas of current research.

The finance industry has undergone a prolonged period of intensive mathematization, to the extent that it is now perhaps the leading industrial user of mathematics PhDs and MScs. Will this trend continue through to the next decade and beyond? This workshop will enable the world's experts both to exposite and critically examine the best new mathematical methods in finance. It will also provide a timely opportunity for assessing the future applicability of advanced mathematical methods in finance.

List of Participants

Acciaio, Beatrice (University of Perugia)
Albanese, Claudio (Imperial College London)
Almgren, Robert (University of Toronto)
Becherer, Dirk (Imperial College London)
Berndt, Antje (Cornell University)
Biagini, Sara (University of Perugia and Scuola Normale)
Brendle, Simon (Princeton University)
Carmona, Rene (Princeton University)
Cheridito, Patrick (Princeton University)

Cont, Rama (Ecole Polytechnique)
Davison, Matt (University of Western Ontario)
Di Graziano, Guiseppe (Cambridge University)
Eberlein, Ernst (University of Freiburg)
Frittelli, Marco (Università di Firenze)
Grasselli, Matheus (McMaster University)
Guo, Xin (Cornell University)
Heath, David (Carnegie Mellon University)
Henderson, Vicky (Princeton University)
Hobson, David (University of Bath)
Hughston, Lane (King's College London)
Hugonnier, Julien N. (HEC Montreal)
Hurd, Tom (McMaster University)
Karatzas, Ioannis (Columbia University)
Labbe, Chantal (University of Waterloo)
Macrina, Andrea (King's College London)
Melnikov, Alexander (University of Alberta)
Musiela, Marek (BNP Paribas UK)
Pistorius, Martijn R. (King's College London)
Rafailidis, Avraam (King's College London)
Rindisbacher, Marcel (University of Toronto)
Schachermayer, Walter (Vienna University of Technology)
Schoenbucher, Philipp (ETH-Zentrum)
Schweizer, Martin (ETH-Zentrum)
Seco, Luis (University of Toronto)
Shreve, Steven E. (Carnegie Mellon University)
Tompaidis, Stathis (University of Texas at Austin)
Tourin, Agnes (McMaster University)
Ware, Tony (University of Calgary)
Zariphopoulou, Thaleia (University of Texas at Austin)
Zervos, Mihail (King's College London)

Chapter 16

New Horizons in String Cosmology (04w5021)

June 12–17, 2004

Organizer(s): James Cline (McGill University), Robert Brandenberger (Brown University), Steve Giddings (University of California, Santa Barbara), Brian Greene (Columbia University), Robert Myers (Perimeter Institute), Gordon Semenoff (University of British Columbia)

This workshop was co-sponsored by the Cosmology and Gravity Program of the Canadian Institute for Advanced Research (CIAR).

Because of the aforementioned advances, issues in string cosmology have attracted the attention of string theorists who would not otherwise have attempted to address cosmology, as well as field theorists whose expertise is more on the cosmological issues themselves. Although the main focus of the proposed workshop will be on the mathematical and string theoretic aspects of the subject, it is worthwhile to have input from people in the second group, since the ultimate aim is still to make a connection with observable physics. The purpose of the workshop is therefore to provide an opportunity to discuss the current problems and issues in string cosmology, both at the technical level and at a more conceptual level.

To this end, we are proposing a 5-Day Workshop which will begin by providing overviews of the latest progress in areas of string cosmology, followed by forums to discuss its key problems. The topics and outstanding questions which seem most urgent at present are:

- string theory in time-dependent backgrounds
- identification of the appropriate observables for defining the theory
- time-dependent orbifolds
- AdS/CFT constructions of cosmological string backgrounds
- dynamics of tachyon condensation
- consistency of string theory with deSitter space; alternative ways to get an accelerating universe from string theory
- resolution of spacelike (cosmological) singularities by string theory
- the proposed dS/CFT correspondence; does deSitter space have a finite number of degrees of freedom?
- inflation from string moduli or D-brane interactions
- string theoretic effects on inflationary perturbations

- can noncommutative geometry play a role in the early universe?

One of the goals of the proposed workshop is to try to move closer to having a set of tools or a framework within which one might hope to carry out more rigorous calculations. Despite the remarkable surge of interest amongst both string theorists and cosmologists in exploring the potential overlap of these two fields, and the variety of new ideas which have been generated, the field is still in its infancy and it requires refinement and clarification of the methodology.

The above choice of topics reflects the most important string theoretic issues that emerged from our very successful workshop on string cosmology at the Aspen Center for Physics, in August and September of 2002. The Aspen meeting had a larger phenomenological component than the one presently being proposed, so the proposed BIRS workshop will be complementary.

One measure of the importance of this workshop is the quality of people who have agreed in principle to participate, as enumerated below. They include many of the most highly recognized researchers in modern string theory and early universe cosmology. We are confident that the dialog which occurs at this meeting will positively influence the development of this rapidly progressing field of mathematical physics.

Given the rapid pace of evolution in this interdisciplinary field between mathematical physics and cosmology, the time is right to provide a forum to summarize the different paradigms of string cosmology, to clearly formulate the outstanding challenges, and to outline the most promising avenues to address them. The organizers have in mind to produce a document which will summarize the outcome of this discussion. As of now, no such written account exists which coherently defines the field of string cosmology, and the time is right to create such a reference. The experts in the field who attend the workshop will be asked to contribute. Several publishers have already expressed interest in this idea.

List of Participants

Adams, Allan (Stanford University)
Albrecht, Andreas (University of California)
Battefeld, Thorsten (Brown University)
Bond, Richard (CITA and University of Toronto)
Bousso, Raphael (Harvard University)
Brandenberger, Robert (Brown University)
Burgess, Cliff (McGill University)
Cline, James (McGill University)
Copeland, Edmund (University of Sussex)
Fischler, Willy (University of Texas)
Giddings, Steve (University of California, Santa Barbara)
Greene, Brian (Columbia University)
Gregory, Ruth (University of Durham)
Harvey, Jeff (University of Chicago)
Horava, Petr (University of California)
Jackson, Mark (Columbia University)
Kallosh, Renata (Stanford University)
Kaloper, Nemanja (University of California, Davis)
Khoury, Justin (Columbia University)
Kofman, Lev (Canadian Institute for Theoretical Astrophysics)
Larsen, Finn (Unversity of Michigan)
Linde, Andrei (Stanford University)
Liu, Hong (Massachusetts Institute of Technology)
Lowe, David (Brown University)
Marolf, Don (University of California, Santa Barbara)
Myers, Robert (Perimeter Institute)
Ovrut, Burt (University of Pennsylvania)
Peet, Amanda (University of Toronto)
Quevedo, Fernando (Cambridge University)

Randall, Lisa (Harvard University)
Rozali, Moshe (University of British Columbia)
Schalm, Koenraad (Columbia University)
Semenoff, Gordon (University of British Columbia)
Servant, Géraldine (Enrico Fermi Institute)
Shenker, Stephen (Stanford University)
Shiu, Gary (University of Wisconsin)
Silverstein, Eva (Stanford Linear Accelerator Center)
Steinhardt, Paul (Princeton University)
Stoica, Horace (McGill University)
Tye, Henry (Cornell University)
Weltman, Amanda (Columbia University)
van Raamsdonk, Mark (University of British Columbia)

Chapter 17

Advances in Complexity Theory (04w5100)

July 4–8, 2004

Organizer(s): Stephen Cook (University of Toronto), Arvind Gupta (Simon Fraser University), Russell Impagliazzo (University of California, San Diego), Valentine Kabanets (Simon Fraser University), Madhu Sudan (Massachusetts Institute of Technology), Avi Wigderson (Institute for Advanced Study, Princeton)

Computational Complexity Theory is the field that studies the efficiency of computation. Its major goals are to find efficient algorithms for natural problems in natural computational models, or to show that no efficient solutions exist. The famed "P versus NP" problem (one of the seven open problems of the Clay Institute) is the central problem of this field.

In the last two decades, our understanding of efficient computation has improved significantly through a number of concepts, techniques and results, including:

- Discovery of efficient ways of converting computational hardness into computational randomness (hardness-randomness tradeoffs), and other techniques for eliminating or reducing randomness use in probabilistic algorithms.
- Classification of hardness of approximation algorithms for a number of optimization problems, using the concept of Probabilistically Checkable Proofs (PCP).
- Connections of both items above to old and new problems in coding and information theory, which fertilized both fields.
- Investigations of the complexity of proofs, and their connections to limits on circuit lower bounds on the one hand, and to the complexity of search heuristics on the other.
- Use of quantum computation to get efficient algorithms for classically difficult problems (such as factoring), as well as using quantum arguments to obtain complexity results in the classical model of computation.

Many new developments in these areas were presented by the participants of the workshop. These new results will be described in the following sections of this report, grouped by topic. For each topic, we give a brief summary of the presented results, followed by the abstracts of the talks.

Probabilistically Checkable Proofs

The area of Probabilistically Checkable Proofs (PCPs) and Hardness of Approximation continues to be one of the most active research directions in complexity. The talk by Irit Dinur discussed how to make the original algebraic proof of the PCP Theorem [AS98, ALM$^+$98] more combinatorial (and hence, maybe simpler). Eli Ben-Sasson presented a new construction of shorter PCPs. Finally, Guy Kindler showed optimal conditional in-approximability for the problem MAX-CUT.

IRIT DINUR, **Assignment testers: Towards a combinatorial proof of the PCP Theorem** (joint work with Omer Reingold)

In this talk we look back into the proof of the PCP Theorem, with the goal of finding new proofs that are "more combinatorial" and arguably simpler. For that we introduce the notion of an assignment tester, which is a strengthening of the standard PCP verifier, in the following sense. Given a statement and an alleged proof for it, while the PCP verifier checks correctness of the *statement* the assignment-tester checks correctness of the statement *and the proof*. This notion enables simpler composition that is truly modular, i.e., one can compose two assignment-testers without any assumptions on how they are constructed. A related notion was independently introduced in [Ben-Sasson et al., *STOC'04*]. Based on this notion, we present two main results: 1. The first is a new proof of the PCP Theorem. This proof relies on a rather weak PCP given as a "black box". From this, we construct combinatorially the full PCP, relying on composition and a new combinatorial aggregation technique. 2. Our second construction is a "standalone" combinatorial construction showing "NP $\subset$ PCP [polylog 1]". This implies, for example, that approximating max-SAT is quasi-NP-hard.

ELI BEN-SASSON, **Simple PCPs with poly-log rate and query complexity** (joint work with Madhu Sudan)

We give constructions of PCPs of length $n \cdot \mathrm{poly}(\log n)$ (with respect to circuits of size n) that can be verified by making $\mathrm{poly}(\log n)$ queries to bits of the proof. These PCPs are not only shorter than previous ones, but also simpler. Our (only) building blocks are Reed-Solomon codes and the Bivariate Low Degree Test of Polischuk and Spielman. First, we present a novel reduction of SAT to the following problem. Given oracle access to a string of length $n' = n \cdot \mathrm{poly}(\log n)$, verify whether it is close to being an evaluation of a univariate polynomial of degree $n'/10$. While somewhat similar reductions have been extensively used in previous PCP constructions, our new reduction favours over them in its simplicity. Notice the degree of the polynomial is larger than the size of the original SAT problem. Thus, testing low degree of this string seems to cost more queries than required for reading the original satisfying assignment in its entirety! To overcome this, we present a short PCP of Proximity for certain Reed-Solomon codes. For these codes, verifying that a string of length n' is close to an evaluation of a degree $n'/10$ polynomial can be done with $\mathrm{poly}(\log n')$ queries into the string and into an additional proof of length $n' \cdot \mathrm{poly}(\log n')$. Such PCPs of proximity also gives rise to locally testable codes with poly-logarithmic rate and query complexity.

GUY KINDLER, **Conditional optimal in-approximability results for MAX-CUT** (joint work with Subhash Khot, Elchanan Mossel, and Ryan O'Donnell)

In this talk we give evidence that it is hard to approximate the maximal cut in a given graph to within a factor of $\alpha + \epsilon$, for all $\epsilon > 0$. Here $\alpha = .878567..$ denotes the approximation ratio achieved by the Goemans-Williamson algorithm [GW95], which means that we achieve an essentially optimal factor. Our result relies on two conjectures: (1) A widely-believed conjecture we fondly call "Majority is Stablest"; this conjecture leads to a long-code test that queries two bits, and whose soundness/completeness factor is exactly α. (2) The Unique Games conjecture of Khot [Khot02]. Our results suggest (even for non-believers in the above conjectures) that the geometric structure imposed on the MAX-CUT problem by the Goemans-Williamson algorithm may in fact be intrinsic to it. They also raise several interesting questions of both complexity-theoretic and geometric nature.

Pseudorandomness

Pseudorandomness is the area concerned with explicit constructions of various "random-like" combinatorial objects. New constructions of one type of such objects, *randomness extractors*, have been reported by Ronen Shaltiel, Russell Impagliazzo, and Guy Kindler. The work described in the talk by Impagliazzo relied on some tools from Combinatorial Number Theory. A tutorial on Combinatorial Number Theory was given by Avi Wigderson. Finally, Pavel Pudlak described a new explicit construction of Ramsey graphs with better parameters than previously known; interestingly, the (yet unpublished) results on extractors described in the talk by Kindler actually yield the construction of Ramsey graphs with even better parameters.

RONEN SHALTIEL, **Deterministic extractors for bit-fixing sources by obtaining an independent seed** (joint work with Ariel Gabizon and Ran Raz) [GRS04]

An (n, k)-bit-fixing source is a distribution X over n bit strings such that there is a subset of k variables in $X_1, ..., X_n$ which are uniformly distributed and independent of each other, and the remaining $n-k$ indices are fixed. A deterministic bit-fixing source extractor is a function E which given an arbitrary (n, k)-bit-fixing source outputs m bits which are statistically-close to uniform. Recently, Kamp and Zuckerman gave a construction of deterministic bit-fixing source extractor which extracts $\Omega(k^2/n)$ bits, and requires $k > \sqrt{n}$. In this paper we give constructions of deterministic-bit-fixing source extractors that extract $(1 - o(1))k$ bits whenever $k > (\log n)^c$ for some constant $c > 0$. Thus, our constructions extract almost all the randomness from bit-fixing sources and work even when k is small. For $k >> \sqrt{n}$ the extracted bits have statistical distance $2^{-n^{\Omega(1)}}$ from uniform, and for $k < \sqrt{n}$ the extracted bits have statistical distance $k^{-\Omega(1)}$ from uniform. Our technique gives a general method to transform deterministic bit-fixing source extractors that extract few bits into extractors which extract almost all the bits.

AVI WIGDERSON, **Gems of Combinatorial Number Theory**

We describe three theorems from Combinatorial Number Theory, and give their proofs. These theorems are related to the recent extractors obtained by Barak, Impagliazzo and Wigderson [Barak et al., *FOCS'04*](described in another talk of this workshop).

The extensive research area of Combinatorial Number Theory often deals with the structure of sets of (commutative) groups, and its evolution under the group operation. The theorems below are prime examples, not only being basic and powerful, but also due to their ingenious proofs that utilize ideas and tools from seemingly unrelated areas.

Let A, B be subsets of size m in an Abelian group. We use the notation $A + B = \{a + ba \in A, b \in B\}$ (here + is the group operation; later we'll use both addition and multiplication over the Reals). Further if $G = (A, B; E)$ is a bipartite graph on A, B, we let $A +_G B = \{a + ba \in A, b \in B, (a, b) \in E\}$.

The theorems below will hold for all choices of m and sets A, B (and C) of this size.

Theorem [Ruzsa, Plunneke]: For every k, if $|A + B| = km$, then $|A + A| \le k^2 m$.

Theorem [Gowers]: For every k and graph $G = (A, B; E)$ with $|E| \ge m^2/k$, if $|A +_G B| \le km$, then there exist subsets $A' \subseteq A$ and $B' \subseteq B$ such that $|A' + B'| \le k^8 m$

Theorem [Erdos-Szemeredi, Elekes]: Let A, B, C be subsets of size m of the real numbers. Then $|AB + C| \ge m^{3/2}$

RUSSELL IMPAGLIAZZO, **Extracting randomness using few independent sources** (joint work with Boaz Barak and Avi Wigderson) [BIW04]

Randomness is prevalent in computer science, and is widely used in algorithms, distributed computing, and cryptography. Perhaps the main motivation and justification for the use of randomness in computation is that randomness does exist in nature, and thus it is possible to sample natural phenomena (such as radioactive decay) in order to make random choices in computation. However, there is a discrepancy between the type of random input that we expect when designing randomized algorithms and protocols, and the type of random data that can be found in nature. While randomized algorithms and protocols expect a stream of independent uniformly distributed random bits, in many cases, the sampled natural data is not distributed according to the uniform distribution.

We consider the problem of extracting truly random bits from several independent weak random sources. Previous constructions either required a large number of sources (polynomial in the input length), or required the entropy of each source to be large. Specifically, the best previous explicit construction using a constant number of n-bit sources required that at least one of the sources contains more than $n/2$ bits of (min-)entropy. In contrast, the optimal, non-explicit construction only requires the min-entropy to be more than $\log n$.

In this work, we manage to go beyond this $n/2$ "barrier" and give an explicit construction for extracting randomness from distributions with any constant entropy rate. The number of samples we require is a constant (depending polynomially on the rate). Our main tools are results from additive number theory and in particular a recent result by Bourgain, Katz and Tao and an improvement by Konyagin.

GUY KINDLER, **Breaking the $1/2$-barrier for bipartite Ramsey constructions and for linear source dispersers** (joint work with Boaz Barak, Ronen Shaltiel, Benny Sudakov, and Avi Wigderson)

The k-partite Ramsey construction problem with parameter δ, is to find explicit functions f, $f : [N]^k \to \{0, 1\}$, such that for every choice of k subsets $A_1, \ldots, A_k \subseteq [N]$ of size at least $[N]^\delta$ each, the restriction of

f to $A_1 \times \cdots \times A_k$ is non-constant. An alternative formalization would be to find a function $f : (\{0,1\}^n)^k \to \{0,1\}$, such that for every k sources $X_1, \ldots, X_k$ of size n-bits each and with min-entropy at least δn each, $f(X_1, \ldots, X_k)$ yields both 0 and 1 with positive probability.

So far, no bipartite Ramsey constructions were known for parameters $\delta < 1/2$. In this talk we present explicit constructions of Bipartite Ramsey graphs for all positive constant parameters δ (this trivially solves the k-partite problem for the same parameters for every $k > 2$). We also show 4-source extractors, that extract bits from four n-bit independent sources with min-entropy at least δn each. This answers a question of Barak, Impagliazzo, and Wigderson [Barak et al., *FOCS'04*].

Similar ideas lead also to explicit constructions of seedless condensers for Linear sources, namely explicit functions $f : \{0,1\}^n \to \{0,1\}$, which are non-constant on every affine linear subspace of $\{0,1\}^n$ of dimension at least δn. While no such construction was known for any $\delta < 1/2$, we can construct such functions for every positive constant δ.

PAVEL PUDLAK, **Pseudorandom sets and explicit constructions of Ramsey graphs** (joint work with Vojtech Rödl)

We shall show a polynomial time construction of a graph G on N vertices such that neither G nor $\overline{G}$ contains $K_{r,r}$, for $r = \sqrt{N}/2^{\sqrt{\log N}} = o(\sqrt{N})$. To this end we construct a subset $X \subset \mathbb{F}^m$ which has small intersections with all subspaces of dimension $m/2$.

Bounded Arithmetic and Proof Complexity

The framework of Bounded Arithmetic can be used to give machine-independent characterization of various complexity classes. Thus, complexity classes may be studied through the properties of logical theories of bounded arithmetic that "capture" these complexity classes. The overview of this approach was given in the talk by Stephen Cook. The logical theories for the classes NL and $PSPACE$ were presented by Antonina Kolokolova and Alan Skelley. Sam Buss and Tsuyoshi Morioka discussed the connections between systems of bounded arithmetic and propositional proof systems, and proved witnessing theorems for certain theories of bounded arithmetic.

STEPHEN COOK, **Making Sense of Bounded Arithmetic**

We present a unified treatment of logical theories for each of the major complexity classes between AC^0 and P, and give simple translations into the quantified propositional calculus.

SAM BUSS, **Bounded Arithmetic and Constant Depth Propositional Proofs**

We discuss the Paris-Wilkie translation from bounded arithmetic proofs to bounded depth propositional proofs. We describe normal forms for proofs in bounded arithmetic, and a definition of Σ'-depth for PK-proofs that makes the translation from bounded arithmetic to propositional logic particularly transparent. Using this, we give new proofs of the witnessing theorems for S^1_2 and T^1_2; namely, new proofs that the Σ^b_1-definable functions of S^1_2 are polynomial time computable and that those of T^1_2 are in Polynomial Local Search (PLS). Both proofs generalize to Σ^b_i-definable functions of S^i_2 and T^i_2.

ANTONINA KOLOKOLOVA, **A second-order theory for NL** (joint work with Stephen Cook)

We introduce a second-order theory V-Krom of bounded arithmetic for nondeterministic log space. This system is based on Grädel's characterization of NL by second-order Krom formulae with only universal first-order quantifiers, which in turn is motivated by the result that the decision problem for 2-CNF satisfiability is complete for $coNL$ (and hence for NL). This theory has the style of the authors' theory V_1-Horn [*APAL 124 (2003)*] for polynomial time. Both theories use Zambella's elegant second-order syntax, and are axiomatized by a set 2-BASIC of simple formulae, together with a comprehension scheme for either second-order Horn formulae (in the case of V_1-Horn), or second-order Krom (2-CNF) formulae (in the case of V-Krom). Our main result for V-Krom is a formalization of the Immerman-Szelepcenyi theorem that NL is closed under complementation. This formalization is necessary to show that the NL functions are Σ^B_1-definable in V-Krom. The only other theory for NL in the literature relies on the Immerman-Szelepcenyi's result rather than proving it.

TSUYOSHI MORIOKA, **The witnessing problems for Quantified Propositional Calculus** (joint work with Stephen Cook)

Let H be a proof system for the quantified propositional calculus (QPC). We define the Σ_j^q-witnessing problem for H to be: given a prenex Σ_j^q-formula A, an H-proof of A, and a truth assignment to the free variables in A, find a witness for the outermost existential quantifiers in A. We point out that the Σ_1^q witnessing problems for the systems G_1^* and G_1 are complete for polynomial time and PLS (polynomial local search), respectively. We introduce and study the systems G_0^* and G_0, in which cuts are restricted to quantifier-free formulas, and prove that the Σ_1^q-witnessing problem for each is complete for NC^1. Our proof involves proving a polynomial time version of Gentzen's midsequent theorem for G_0^* and proving that G_0-proofs are TC^0-recognizable. We also introduce QPC systems for TC^0 and prove witnessing theorems for them.

ALAN SKELLEY, **Theories and proof systems for PSPACE and beyond**

We present a new third-order theory W_1^1 for $PSPACE$ and discuss how Σ_1 theorems of it can be translated into polynomial-sized proofs in $BPLK$. $BPLK$ is a propositional proof system polynomially equivalent to G but using Boolean programs instead of quantified Boolean formulas. We then speculate as to how W_1^1 could be extended to obtain theories for the levels of the exponential-time hierarchy but, more interestingly, how $BPLK$ is uniquely amenable (unlike G) also to be extended in this direction.

Circuit Complexity, Probabilistic and Real Computation

Complexity theory studies the power of nonuniform (circuit-based) and uniform (Turing machine-based) models of computation. The talks by Ran Raz and Eric Allender discussed the computational power of restricted arithmetic and Boolean circuit models. Lance Fortnow presented the Probabilistic Time Hierarchy Theorem for Turing machines with constant amount of nonuniform advice. Finally, Mark Braverman discussed his results in the field of Real Computation.

RAN RAZ, **Multilinear formulas for Permanent and Determinant are of superpolynomial size** [Raz04]

An arithmetic formula is multilinear if the polynomial computed by each of its subformulas is multilinear. We prove that any multilinear arithmetic formula for the permanent or the determinant of an $n \times n$ matrix is of size superpolynomial in n. Previously, superpolynomial lower bounds were not known (for any explicit function) even for the special case of multilinear formulas of constant depth.

ERIC ALLENDER, **Toward a topology for $\mathbf{NC^1}$** (joint work with Samir Datta and Sambuddha Roy)

Hansen recently provided a characterization of ACC^0 as precisely the class of problems computable by constant-width PLANAR circuits of polynomial size (with AND and OR gates, with negation available at the inputs.) Barrington's theorem shows that, without the restriction of planarity, constant-width circuits characterize NC^1. We consider possible generalizations of Hansen's theorem, by considering circuits with small genus and thickness. Every problem in NC^1 is computed by a constant-width circuit of thickness two, and thus thickness does not seem to be a useful parameter for investigating the structure of NC^1. In contrast, we show that restricting constant-width circuits to have genus $O(1)$ again yields a characterization of ACC^0. It remains an intriguing open question if there are problems that are not believed to lie in ACC^0 that can be computed by constant-width, polynomial-size circuits of small (say, logarithmic) genus.

LANCE FORTNOW, **A hierarchy theorem for probabilistic polynomial time with one bit of advice** (joint work with Rahul Santhanam)

We show a hierarchy for probabilistic time with one bit of advice, specifically we show that for all real numbers $1 \leq \alpha < \beta$, $BPTIME(n^\alpha)/1 \subset BPTIME(n^\beta)/1$. This result builds on and improves an earlier hierarchy by Barak using $O(\log \log n)$ bits of advice. We build on Barak's idea by a careful application of the fact that there is a PSPACE-complete problem L such that worst case probabilistic algorithms for L take only slightly more time than average case algorithms.

MARK BRAVERMAN, **On the computability of Julia sets**

While the computer is a discrete device, it is often used to solve problems of a continuous nature. The field of Real Computation addresses the issues of computability in the continuous setting. We will discuss different models of computation for subsets of $\mathbb{R}^n$. The main definition we use has a computer graphics interpretation (in the case $n = 2$), as well as a deeper mathematical meaning. The Julia sets are particularly well studied sets arising from complex dynamics. In the talk we will present the basic facts about Julia sets and some computability results for them. Our computability results come in contrast to the Julia sets noncomputability results presented by Blum/Cucker/Shub/Smale. This discrepancy follows from the fact that we are using a different computability model.

Matrix Multiplication, Search Heuristics, Learning, and Quantum Computation

Determining the complexity of matrix multiplication is one of the most important questions in computer science. A very interesting new approach to this problem was described in the talk by Chris Umans. Josh Buresh-Oppenheim presented a formal model for the class of backtracking algorithms, and showed lower bounds on the power of algorithms in that model. Several new (both positive and negative) results on learnability were presented by Toniann Pitassi; some of these results exploited a connection between proof complexity and learning theory. Mario Szegedy showed a very general result on "speeding up" classical algorithms by quantum algorithms; he described the conditions on classical Markov-chain based algorithms that yield quadratic speedup in the quantum model of computation. Finally, Oded Regev presented an efficient lattice-based cryptographic system, whose security relies on the assumption of quantum (rather than classical) hardness of certain lattice problems.

CHRIS UMANS, **A group-theoretic approach to fast matrix multiplication** (joint work with Henry Cohn) [CU03]

How many operations are required to multiply two $n \times n$ matrices? The standard algorithm requires n^3 operations, but in 1969 Strassen showed that $O(n^{2.81})$ operations suffice. Over the next twenty years, a sequence of increasingly complex algorithms were devised, but since 1990 no one has been able to improve on the current best algorithm of Coppersmith and Winograd, that runs in time $O(n^{2.39})$. I'll describe work that develops a new (and self-contained) approach to the problem. In the new framework, one devises algorithms for matrix multiplication by constructing non-abelian groups with certain properties. The algorithms themselves are easy to describe, and they make critical use of the Discrete Fourier Transform over non-abelian groups. I'll outline some progress toward an improved algorithm using this new approach.

JOSH BURESH-OPPENHEIM, **Toward a model for Backtracking** (joint work with Allan Borodin, Russell Impagliazzo, Avner Magen, and Toniann Pitassi)

In this paper, we develop a hierarchy of models for backtracking algorithms (BT). Our model generalizes both the priority model of Borodin, Neilson and Rackoff, as well as the simple dynamic programming model due to Wögenger. We demonstrate the strength of our models by showing how well-known algorithms and algorithmic techniques can be simulated within our model, both those that are usually considered backtracking as well as a large family of greedy algorithms and dynamic programming algorithms. Finally we prove strong lower bounds on the capabilities of algorithms in this model, often essentially proving that the known algorithms are the best possible in the model.

After defining and discussing the BT family of models, we consider the following fundamental problems: interval scheduling with proportional profit, the knapsack problem, 2SAT, 3SAT, and vertex cover. Our main results are as follows: (1) For interval scheduling of n intervals on m machines with proportional profits, the optimal width of an adaptive BT algorithm is $\Theta(n^m)$. Further, for fixed-ordering BT, we obtain similar upper and lower bounds for approximating interval scheduling. (2) For knapsack, we prove an exponential lower bound in the adaptive BT model. (3) We prove that 2SAT has a linear size adaptive BT algorithm, but that any fixed-ordering BT algorithm requires exponential size. Further the lower also extends to show that neither 2SAT nor vertex cover can be approximated by subexponential size fixed-ordering BT programs. (4) For 3SAT we prove that any adaptive BT algorithm requires exponential size.

TONIANN PITASSI, **Learnability and automatizability** (joint work with Misha Alekhnovich, Mark Braverman, Vitaly Feldman, and Adam Klivans)

In this talk we prove new upper and lower bounds on the proper PAC learnability of decision trees, DNF formulas, and intersections of halfspaces. Several of our results were obtained by exploring a new connection between automatizability in proof complexity and learnability. After explaining this basic connection, we will prove the following new results: (1) We give new upper bounds for proper PAC learning of decision trees and DNF, based on similar known algorithms for automatizability of Resolution. (2) We show that it is not possible to PAC learn DNF by DNF in polynomial-time unless $NP \subseteq BPP$. We also prove the same negative result for proper PAC learning of intersections of halfspaces. (3) We show that decision trees cannot be proper PAC learned, under a different (less standard) complexity-theoretic assumption.

MARIO SZEGEDY, **Quantum speed-up of Markov chain based algorithms**
We develop a generic method for quantizing classical algorithms based on random walks. We show that under certain conditions, the quantum version gives rise to a quadratic speed-up. This is the case, in particular, when the Markov chain is ergodic and its transition matrix is symmetric. This generalizes the celebrated result of [Grover 1996] and a number of more recent results, including [Ambainis 2003] and [Ambainis, Kempe and Rivosh, 2004]. Among the consequences is a faster search for multiple marked items. We show that the quantum escape time, just like its classical version, depends on the spectral properties of the transition matrix with the marked rows and columns deleted.

ODED REGEV, **Lattice based cryptography, quantum and some learning theory**
We present strong and more efficient lattice based public key cryptographic schemes. In all previous systems, the encryption process increases the size of a message by a factor of n^2 where n is the hardness parameter. This is considered prohibitive since n has to be on the order of thousands in order to make the system secure. We reduce this blow-up to only n. This, we believe, makes our cryptographic scheme more practical. One curious feature of our construction is that it is based on the quantum hardness of lattice problems. All previous constructions were based on the classical hardness of lattice problems. The reason for this difference is the following: we present a quantum algorithm for a problem that we do not know how to solve classically.

List of Participants

Akavia, Adi (Massachusetts Institute of Technology)
Allender, Eric (Rutgers University)
Beame, Paul (University of Washington)
Ben-Sasson, Eli (Radcliffe Institute for Advanced Study)
Bogdanov, Andrej (University of California, Berkeley)
Braverman, Mark (University of Toronto)
Buresh-Oppenheim, Josh (University of Toronto)
Buss, Sam (University of California, San Diego)
Cleve, Richard (University of Calgary)
Cook, Stephen (University of Toronto)
Dinur, Irit (University of California, Berkeley)
Fortnow, Lance (University of Chicago)
Gal, Anna (University of Texas, Austin)
Gupta, Arvind (Simon Fraser University)
Guruswami, Venkatesan (University of Washington)
Impagliazzo, Russell (University of California, San Diego)
Kabanets, Valentine (Simon Fraser University)
Kapron, Bruce (University of Victoria)
Kindler, Guy (DIMACS, Rutgers University)
Kolokolova, Antonina (University of Toronto)
McKenzie, Pierre (University of Montreal)
Morioka, Tsuyoshi (University of Toronto)
Pitassi, Toni (University of Toronto)
Pudlak, Pavel (Institute of Mathematics, Prague)
Rackoff, Charles (University of Toronto)
Raz, Ran (Weizmann Institute)
Regev, Oded (Tel-Aviv University)
Reingold, Omer (Weizmann Institute)
Saks, Michael (Rutgers University)

Segerlind, Nathan (Institute for Advanced Study, Princeton)
Shaltiel, Ronen (Weizmann Institute)
Skelley, Alan (University of Toronto)
Srinivasan, Venkatesh (University of Victoria)
Sudan, Madhu (Massachusetts Institute of Technology)
Szegedy, Mario (Rutgers University)
Trevisan, Luca (University of California, Berkeley)
Umans, Chris (California Institute of Technology)
Watrous, John (University of Calgary)
Wigderson, Avi (Institute for Advanced Study, Princeton)
Zuckerman, David (University of Texas, Austin)
van Melkebeek, Dieter (University of Wisconsin)

Bibliography

[ALM$^+$98] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and the hardness of approximation problems. *Journal of the Association for Computing Machinery*, 45(3):501–555, 1998. (preliminary version in FOCS'92).

[AS98] S. Arora and S. Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the Association for Computing Machinery*, 45(1):70–122, 1998. (preliminary version in FOCS'92).

[BIW04] B. Barak, R. Impagliazzo, and A. Wigderson. Extracting randomness using few independent sources. In *Proceedings of the Forty-Fifth Annual IEEE Symposium on Foundations of Computer Science*, 2004 (to appear).

[CU03] H. Cohn and C. Umans. A group-theoretic approach to fast matrix multiplication. In *Proceedings of the Forty-Forth Annual IEEE Symposium on Foundations of Computer Science*, 2003.

[GRS04] A. Gabizon, R. Raz, and R. Shaltiel. Deterministic extractors for bit-fixing sources by obtaining an independent seed. In *Proceedings of the Forty-Fifth Annual IEEE Symposium on Foundations of Computer Science*, 2004 (to appear).

[Raz04] R. Raz. Multi-linear formulas for Permanent and Determinant are of super-polynomial size. In *Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing*, pages 633–641, 2004.

Chapter 18

Convex Geometric Analysis (04w5014)

July 10–15, 2004

Organizer(s): Nicole Tomczak-Jaegermann (University of Alberta), Vitali Milman (Tel Aviv University), Elisabeth Werner (Case Western Reserve University)

The main goal of the workshop was to bring researchers from different fields of Convex Geometric Analysis to exchange new ideas, to inform on new results and to consider new directions essential for further developments and applications. This goal was achieved and overachieved. We brought together senior experts and we ensured the participation of a significant number of young researchers – in fact, more than a half of all talks were given by people from this latter group. The subject was treated in a very broad sense, and some leading people from related fields (such as Classical Convexity and Asymptotic Combinatorics) were invited and contributed to the success of the meeting. By the request of some participants we also had an informal seminar lecture (see below) which attracted many participants and continued much more than an hour.

Below we collect together the abstracts of the talks, organized in the thematic groups, corresponding to the schedule of the workshop.

Franck Barthe: Orlicz Hypercontractive semigroups

This is a joint work with Patrick Cattiaux and Cyril Roberto.

The usual Ornstein Uhlenbeck semigroup is known to be hypercontractive (it is a contraction from L_2 into a smaller L_p space, and p increases with time). We study the analogue question for Heat semigroups of measures between exponential and Gaussian. An analogue of Gross theorem is presented, relating hypercontractivity in Orlicz spaces to general F-Sobolev inequalities. These Sobolev inequalities are analysed in connection with previous Sobolev type inequalities for these measures, as the ones of Latala and Oleszkiewicz. Applications to concentration and isoperimetric inequalities will be discussed too.

Bo'az Klartag: Approaches to the slicing problem

We will discuss some recent partial progress regarding the slicing problem. The slicing problem asks whether any n-dimensional convex body of volume one, has at least one hyperplane section, whose n-1 dimensional volume is larger than some positive, universal constant, independent of the dimension. This question is known to be equivalent to the question of the universal boundedness of the isotropic constant of centrally symmetric convex bodies. A few directions and possible tools to handle this problem will be described. We will focus on two main issues. The first is the use of geometric symmetrization techniques, and the second is the proof that any centrally symmetric n-dimensional convex body K, has a perturbation T such that the isotropic constant of T is bounded, and the Banach-Mazur distance between K and T is smaller than $c \log n$, where $c > 0$ is a numerical constant.

Vitali Milman: Explicit versus random

In this talk connections of the Asymptotic theory with Complexity Theory were exploited. The new notion of *Simplicity* was introduced which describes exactly the reverse direction to the standard notion of *Complexity*. Let a family of "simple" procedures be described and a family of "simple" objects are specified. Starting with some (supposedly complicated) object we would like to estimate the minimal number N of *simple* steps (i.e. steps from the family of simple procedures) that may be applied to our object in order to bring it to some other object that has been defined as a simple one. Then we say that N is a simplicity of our object. (Note that it is exactly the opposite direction of transformations which are used in defining the Complexity). So, in the process of constructing an algorithm which estimates Complexity we are starting with a simple object (system) and recover the original structure; but in estimating Simplicity, we "destroy" all specific information of our system to come to a simple one with very little specific information. A lot of recent results of the Asymptotic Theory of Convexity are directed to this goal: how quickly we may destroy all specification of a given (arbitrary, and *a priori* very complicated) object (normed space, or a convex body) and to derive some, say, isomorphic copy of an euclidean space (or an ellipsoid). There are really a number of breackthrough in this direction. For example, it is proved by Klartag that just $5n$ Minkowski symmetrizations are sufficient to bring any convex body in $\mathbb{R}^n$ to a body very close to an euclidean ball. Or, only $3n$ Steiner symmetrizations are enough to bring an arbitrary body to a neighbourhood of a euclidean ball (Klartag, Milman), and many others.

After describing this scheme and a number of examples, we moved to another complexity related subject. Standardly, we are describing some very interesting features of spaces and bodies (say, euclidean subspaces of very large dimensions, or euclidean quotients of subspaces of proportional dimension) through random selection of corresponding subspaces in a specified euclidean structure. To estimate the complexity of such features it would be right to demonstrate an explicit construction which leads to these properties. However, such explicit constructions are unknown. Then, we suggest to estimate a complexity through finding a small number of random steps which should be complemented by a number of explicit simple and short constructions. Then this number of (remaining) random steps will tell us what is the remaining complexity ("randomized complexity") of the feature we are studing. We consider a very famous example of the space ℓ_1^n: It is known that this space contains isomorphic copies of euclidean subspaces of any dimension proportional to n (with the isomorphic constant depending only on this proportion); this is so-called Kashin decomposition. We analys different ways for such an euclidean embedding, the problem which very recently attracted significant attention, including important talks on this conference, and we demonstrate some ways of reducing known "randomized complexity" by considering partially explicit steps using Walsh matrix (joint work with Artstein and Friedland).

Assaf Naor: Vertex expansion, edge expansion and the observable diameter.

Joint work with Yuval Rabani and Alistair Sinclair.

In this talk we will show that the edges of any n-point vertex expander can be replaced by new edges so that the resulting graph is an edge expander, and such that any two vertices that are joined by a new edge are at distance $O(\sqrt{\log n})$ in the original graph. This result is optimal, and is shown to have various geometric consequences. In particular, it is used to give a nearly optimal lower bound on the ratio between the observable diameter and the diameter of doubling metric measure spaces which are quasisymmetrically embeddable in Hilbert space.

Alexander Litvak: Behaviour of the smallest singular value of a random matrix and applications to geometry.

This is a report on the joint work with Alain Pajor, Mark Rudelson and Nicole Tomczak-Jaegermann.

We study behaviour of the smallest singular value of a rectangular random matrix, i.e., matrix whose entries are independent random variables satisfying some additional conditions. We prove a deviation inequality and show that such a matrix is a "good" isomorphism on its image. Then we show applications to geometry of random polytopes and to the problem of finding Euclidean subspaces of convex bodies.

Staszek Szarek: Saturation constructions in normed spaces

This is a joint work with Nicole Tomczak-Jaegermann.

Questions on how to detect possible regularities in the structure of a finite-dimensional normed space or improve and simplify this structure, by passing to its subspaces or quotients or, conversely, to what degree the structure of the entire space can be recovered from the knowledge of its subspaces or quotients, have constituted over the years one of the driving directions in the asymptotic theory of normed spaces. Many background results, starting with the fundamental Dvoretzky's theorem (especially in the form proved by Milman), through the Quotient of a Subspace theorem of Milman and its byproducts and relatives, show that one can achieve very considerable regularity for global invariants of a space by passing to a quotient or a subspace.

In this talk we present several results which clarify this circle of ideas. Here is a sample result: *Given finite dimensional normed space V there exists another space X with $\log \dim X = O(\log \dim V)$ and such that every subspace (or every quotient) of X, whose dimension is not "too small," contains a further subspace isometric to V. Moreover, some geometric properties of the space V can be "lifted" to X.* This sheds new light on the structure of such large subspaces or quotients (or, equivalently, of large sections or projections of convex bodies) and allows to solve several problems stated in the 1980s by V. Milman.

Semyon Alesker: The multiplicative structure on valuations.

We describe a canonical multiplicative structure on (a dense subspace of) continuous valuations. Then we discuss its properties and results from convexity and integral geometry staying behind these properties. If the time permits, we will discuss some applications.

Dario Cordero-Erausquin: On the convergence of Information in the Central Limit Theorem.

This is a joint work with Keith Ball.

The goal of the present work is to give uniform bounds for the converge in the Central Limit Theorem of quantities from information theory such as (Shannon) entropy and (Fisher) information. Only partial results were known, for instance under spectral gap assumptions. We show that under moment conditions (moment of order $2 + \varepsilon$ is enough)the information converges polynomially (with uniform rate). The proof uses the variational formulation for the information of the sum of two random variables discovered by Ball, Barthe and Naor. The different regimes that appear in the proof bring new light on the behavior of the information along the Central Limitprocess.

Roman Vershynin: Gromovs isoperimetry of waists and its use in asymptotic convex geometry

The isoperimetry of waists on the sphere is a recent result of Gromov. We will describe a simple way to use it in asymptotic convex geometry. For example, it implies the following "local versus global result. If two bodies K and L have nicely bounded sections, then the intersection of random rotations of K and L is nicely bounded. For $L =$ subspace, this yields a new "deterministic versus random phenomenon: if K has one nicely bounded section, then most sections of K are nicely bounded. The latter phenomenon was also independently discovered by Giannopoulos, Milman and Tsolomitis.

Shiri Artstein: On convexified packing and entropy duality

This is a joint work with Vitali Milman, Staszek Szarek and Nicole Tomczak-Jaegermann.

The notion of convexified packing (and convex separation) of convex sets will be introduced, and shown to satisfy a duality theorem: the convexified packings of a body by another body, and of its polars 9taken in the opposite direction) are comparable. A number of instances will be mentioned in which a relationship between convex separation and the usual separation can be established by geometric considerations. This will lead to a generalization of the recent duality result (by the first three authors [AMS]) from the case when one body is an ellipsoid, to the setting of two arbitrary bodies under only mild geometric assumptions about one of the underlying norms.

Robert McCann: A convex action principle giving steepest descent into a nonconvex landscape

This is a joint work with Nassif Ghoussoub.

Physical dynamics interpolate naturally between the dissipative and conservative extremes, in which friction either dominates or can be neglected. *Gradient flows* and *Hamiltonian systems* represent the archetypal examples of these two extremes. The orbits of a Hamiltonian system correspond to the critical paths of an *action* functional, but variational characterizations for the trajectories of a gradient flow are less familiar. For steepest descent into a *convex valley* such characterizations were formulated by Brezis-Ekeland and Auchmuty. Here we refine their approach, taking advantage of the Bolza self-duality introduced by Ghoussoub-Tzou, to formulate a *convex* variational principle for steepest descent into a valley which is merely semi-convex.

Van Vu: Random polytopes: High moments and beyond

Let K be a convex body with volume one in $\mathbb{R}^d$. Consider a selection of n random points in K (chosen with repsect to the unfirom distribution). The convex hull K_n of these points is a random polytope.

Basic parameters (such as the volume, number of vertices, number of facets etc) of random polytopes have been studied for many years. There is a huge amount of strong results about the expectation of these parameters. On the other hand, not too much has been obtained about the distribution. For instance, determining the higher moments is a major problem.

In this talk, we introduce a new method, which allows us to obtain useful information about the distribution. Using this, we can, among others, derive fairly accurate bounds for the high moments and prove limit theorems.

Mark Rudelson: Random processes via the combinatorial dimension

This is a report on a joint work with Roman Vershynin.

Let F be a class of real valued functions defined on a probability space X. For a given $t > 0$ we introduce a dimension $v(F, t)$ measuring the complexity of F in terms of the existence of specific patterns in F. More precisely, the combinatorial dimension $v(F, t)$ is the largest dimension of a structure in F, which is similar to a discrete cube of size t. This characteristic plays a crucial role in determining whether the class F satisfies the uniform Law of Large Numbers and the uniform Central Limit Theorem.

Combinatorial dimension provides a sharp estimate of the metric entropy of the function class. This allows to prove two basic combinatorial conjectures on random processes.

1. A class of functions satisfies the uniform Central Limit Theorem if the square root of its combinatorial dimension is integrable.

2. The uniform entropy is equivalent to the combinatorial dimension under minimal regularity.

Michael Krivelevich: Models of Random Graphs

In this survey talk I will discuss several, old and new, models of random graphs. The main aim of the talk is to familiarize the audience with the variety of models of random graphs, while stressing their differences and similarities and emphasizing common research approaches and methodology. Between the models I plan (or rather hope) to discuss are:

- binomial random graphs G(n,p) and the Erdos-Renyi model G(n,m);
- graph processes and hitting times;
- random regular graphs;
- network reliability model;
- adding random edges (smoothed analysis);
- random lifts;
- preferential attachment models.

Although no previous research experience with random graphs will be assumed, a genuine interest in the subject would be appreciated.

Gideon Schechtman: An observation regarding the dependence on ε in Dvoretzky's theorem

Recall that Dvoretzky's theorem says that there is a function $c(\varepsilon) > 0$ such that for all $n \geq 1$ and all $\varepsilon > 0$, every n-dimensional normed space contains a subspace $(1+\varepsilon)$-isomorphic to ℓ_2^k, for all $k < c(\varepsilon)\log n$. It was well-known that one may take $c(\varepsilon) \geq c\,\varepsilon^2$. In this talk it is shown that this estimate can be improved to

$$c(\varepsilon) > c\,\frac{\varepsilon}{(\log(1/\varepsilon))^2},$$

where $c > 0$ is an absolute constant.

Deane Yang: Moment-entropy Inequalities

This is a joint work with Erwin Lutwak and Gaoyong Zhang.

We establish a new link between the dual L^p Brunn-Minkowski theory and probability theory by using p-th moments to associate a star body to each $\mathbb{R}^n$-valued random variable and defining the dual mixed volume of a random variable with a star body. Using this, the authors generalized the fundamental dual Minkowski inequality for star bodies to an inequality of dual mixed volumes of star bodies and random variables. This in turn gives a fundamental inequality between the Renyi entropy of a random variable and its associated star body. Combining this with the L_p affine isoperimetric inequality of centroid bodies establishes a moment-entropy inequality of random variables that implies the well-known Blaschke-Santaló inequality of convex bodies.

Daniel Hug: On the L_p Minkowski problem

This is a joint work with Erwin Lutwak, Deane Yang, and Gaoyong Zhang.

A classical result of the Brunn-Minkowski theory is Minkowski's existence theorem. For a given centred and non-degenerate Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ it yields the existence of a unique (up to translation) convex body $K \subset \mathbb{R}^n$ such that the top order surface area measure of K equals μ. In the middle of the last century, Firey extended the Minkowski combination of convex bodies and thus laid the foundations of the Brunn-Minkowski-Firey (or L_p) theory. Subsequently, various elements of the classical theory such as Minkowski's inequality have been established in the L_p setting.

In this talk, we consider an L_p extension of Minkowski's existence theorem. A first proof making use of the machinery of PDE's is due to Chou and Wang. We describe two different elementary approaches to an L_p version of Minkowski's existence theorem. For this we first study polytopal solutions to the discrete-data L_p Minkowski problem.

Rolf Schneider: Size and limit shape of some random polytopes

This is a joint work with Daniel Hug. Familiar ways of generating a random polytope are either taking the convex hull of random points or the intersection of random halfspaces. In the first case, finitely many independent uniform points in a given convex body are an often studied setup. In the second case, which we consider here, an equally natural approach consists in taking a homogeneous Poisson hyperplane process and intersecting the halfspaces which are bounded by the hyperplanes of the process and contain a fixed point, say 0. Here, geometry comes in via the direction distribution of the hyperplane process. For the random polytope thus obtained, we study the existence of weak limits of the conditional shape distribution, given that the 'size' of the polytope is large. We show how the answer depends on the direction distribution and on the way how the 'size' is measured.

Olivier Guédon: Concentration of Mass on the Schatten Classes.

This is a report on a recent work with Grigoris Paouris.

Let $1 \leq p \leq \infty$ and $\widetilde{B(S_p^n)}$ be the unit ball of the Schatten trace class of matrices on $\mathbb{C}^n$ or on $\mathbb{R}^n$, normalized to have Lebesgue measure equal to one. We prove that

$$\lambda\left(\left\{T \in \widetilde{B(S_p^n)} : \frac{\|T\|_{HS}}{n} \geq c_1\, t\right\}\right) \leq \exp\left(-c_2\, t\, n^{k_p}\right)$$

for every $t \geq 1$, where $k_p = \min\{2, 1 + p/2\}$, $c_1, c_2 > 1$ are universal constants and λ is the Lebesgue measure. This concentration of mass inside a ball of radius proportional to n follows from an almost constant behaviour of the L_q norms (with respect to the Lebesgue measure on $\widetilde{B(S_p^n)}$) of the Hilbert-Schmidt operator norm of T.

Carsten Schuett: Approximation of Convex Bodies by Polytopes

In this joint work with Monika Ludwig and Elisabeth Werner, we study the approximation of convex bodies by polytopes. There are extensive investigations of this problem with the additional assumption that the polytope is either contained in the convex body or contains the convex body. Here we study the approximation without this additional assumption.

Monika Ludwig: A characterization of the intersection body transform

The intersection body transform associates with each convex body its intersection body. It is well know that it intertwines with the general linear group and that it is a valuation with respect to radial addition. We describe a classification of radial valuations on convex bodies that commute with the general linear group and obtain a characterization of the intersection body transform.

Hermann Koenig: Geometric inequalities for a class of exponential measures

This is a joint work with Nicole Tomczak-Jaegermann.

In this talk we consider versions of some geometric inequalities (of an isomorphic-type) for a natural class of exponential log-concave measures, replacing the usual volume in $\mathbb{R}^n$. These are versions of Milman's inverse Brunn-Minkowski inequality and Bourgain–Milman's inverse Santaló inequality, which have played an important role in the convex geometric analysis and the asymptotic theory of normed spaces during the last fifteen years. Both these inequalities can be viewed as a consequence of the existence, for any symmetric convex body in $\mathbb{R}^n$, of a special ellipsoid, called nowadays an M-ellipsoid, which, in a sense, reflects volumetric properties of the body. We show that the same ellipsoid also reflects, in an analogous way, properties of the body with respect to a large class of exponential (log-concave) measures on $\mathbb{R}^n$. This class contains in particular the Gaussian measure on $\mathbb{R}^n$.

An informal seminar was given by **Shiri Artstein** on her joint work with B. Klartag and V. Milman on the geometry of log-concave measures.

List of Participants

Alesker, Semyon (Tel Aviv University)
Anisca, Razvan (Texas A&M University)
Artstein, Shiri (Tel Aviv University)
Ball, Keith (University College London)
Barthe, Franck (Universite Toulouse III)
Colesanti, Andrea (University of Florence)
Cordero-Erausquin, Dario (Universite de Marne la Vallee)
Gordon, Yehoram (Israel Institute of Technology)
Guedon, Olivier (University Paris 6)
Hug, Daniel (Freiburg University)
Johnson, Bill (Texas A&M University)
Klartag, Bo'az (Tel Aviv University)
Koenig, Hermann (Kiel University)
Koldobsky, Alexander (University of Missouri)
Krivelevich, Michael (Tel Aviv University)
Latala, Rafal (Warsaw University)
Lindenstrauss, Joram (Hebrew University)
Litvak, Alexander (University of Alberta)
Ludwig, Monika (Technische Universitaet Wien)
McCann, Robert (University of Toronto)
Milman, Emanuel (Weizmann Institute)
Milman, Vitali (Tel Aviv University)
Naor, Assaf (Microsoft Research)
Oleszkiewicz, Krzysztof (Warsaw University)
Pajor, Alain (University Marne-la-Vallee)
Pelczynski, Aleksander (Institute of Mathematics, Polish Academy of Sciences)
Pisier, Gilles (Texas A&M University and Paris 6)
Pivovarov, Peter (University of alberta)
Reisner, Shlomo (University of Haifa)
Rudelson, Mark (University of Missouri)
Schechtman, Gideon (Weizmann Institute)

Schneider, Rolf (Freiburg University)
Schuett, Carsten (Kiel University)
Sodin, Alexander (Tel Aviv University)
Szarek, Stanislaw (Case Western Reserve University and Paris 6)
Tomczak-Jaegermann, Nicole (University of Alberta,)
Vershynin, Roman (University of California, Davis)
Vu, Van (University of Califormia at San Diego)
Yang, Deane (Polytechnic University, New York)
Zhang, Gaoyong (Polytechnic University, New York)

Chapter 19

Modelling Protein Flexibility and Motions (04w5017)

July 17–22, 2004

Organizer(s): Walter Whiteley (Mathematics and Statistics, York University), Michael Thorpe (Physics, Arizona State University), Leslie Kuhn (Biochemistry, Michigan State University)

Overview of the Subject

Following from work on the genome, the focus is shifting to protein structure and function. Much of the function of a protein is determined by its 3-D structure and motions (often in complexes of several molecules). The structure of many new proteins is being determined by x-ray crystallography and by nuclear magnetic resonance techniques. One can then study both local flexibility (adapting shape to fit with other molecules) and larger motions. One can also study the impact of other contacts such as ligands (drugs), or binding into complexes of proteins, DNA etc. in changing the shape and flexibility. An important area of current research in biochemistry, computational geometry and in applied mathematics is the computer modelling of such behaviour: which sections are rigid, under certain conditions; the possible motions; unfolding pathways; multiple configurations with different biological functions; and paths between these configurations.

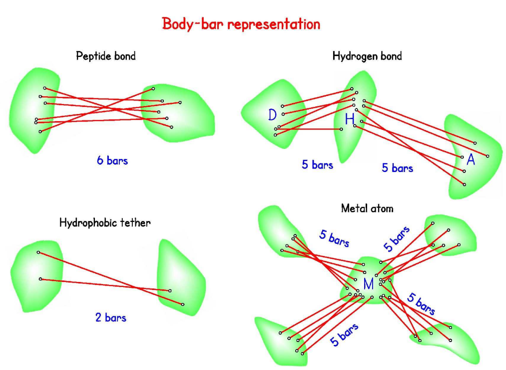


Figure 19.1: Showing the various elements in the body-bar representation of a folded protein structure that are used in many algorithms for rigidity including FIRST.

The mathematical theory of rigidity, and related techniques from geometric constraint theory (CAD, robotics), are one set of tools for such computer modelling. Applications of such techniques to protein flexibility have been expanding over the last few years, centred on the program FIRST. A short summary of the current state of the art for the combinatorics central to the rigidity methods and the robotics methods includes three factors:

1. the general problem of predicting whether a graph, build in 3-space as a bar and joint framework, will be rigid or flexible, for almost all realizations, is an long standing problem, going back at least to James Clerk Maxwell.

2. the general problem of predicting whether a graph built with vertices as rigid bodies, and edges as hinges, in 3-space, will be rigid or flexible for almost all choices of lines for the hinges, has a simple combinatorial solution and an efficient algorithm.

3. the general problem of frameworks extracted from covalent bonds of molecular structures (with fixed angles at the bonds) is conjectured to be covered by the algorithms of (2) (the Molecular Framework Conjecture) although it is a special class of frameworks under (1).

Algorithms have been implemented for certain models of proteins as frameworks within this mathematical theory. These models develop a graph based on the covalent bond network plus additional edges related to ionic bonding (salt bridges and hydrogen bonds, identified by proximity of these atoms in the 3-D structure) as well as graph edges for hydrophobic interactions, also identified by proximity of suitable heavy atoms in the 3-D structure. This graph yields a constraint matrix which will predict the first-order rigidity or flexibility of the corresponding model, and hopefully of the underlying molecules. However, for speed of computation (on works with up to 400,000 atoms), the rank is actually predicted from the combinatorics of the graph, using counting algorithms (often called the ‘pebble game’). The accuracy of these combinatorial results to the rank of the underlying matrix would follow from the ‘molecular conjectures’ of Tay and Whiteley, and there is significant experimental evidence, as well as partial results to support this correctness. These algorithms are fast enough to be used as preliminary screening in areas such as ligands as drugs. Other partial results have been obtained, and interesting comparisons have been made with measured biological data. Recent work has scaled up from single proteins to complexes such viral coats and RNA protein complexes but much work remains. The program FIRST [Floppy Inclusions and Rigid Substructure Topography] was discussed at some length during the workshop, and is available on the net at flexweb.asu.edu. The use of this web site was demonstrated during the workshop.

This rigidity/constraint based work has been extended from first-order predictions (in the rank of the matrix), using rigidity decompositions, and Monte-Carlo steps, to simulate larger motions, including pathways between known conformations of the same molecule. This is embedded in the program ROCK [Rigidity Optimized Conformational Kinetics], which was also presented during the workshop, and is also available at flexweb.asu.edu

Gaussian Network Models (GNM) represent another combinatorial and computation method for integrating ‘proximity’ constraints into linear algebra and predictions of motions of large bio-molecular structures. These models build a simple “incidence’ matrix for proximity of large atoms in the molecule, on a scale designed to ensure non-singular square matrices, then examine the dominant eigenvalues and vectors to predict significant overall motions. These methods were also presented at the workshop, along with some comparisons of predictions from GNM and FIRST and with known experimental measurements of forms of flexibility.

Recent work in computational has investigated the computational complexity of a variety of algorithms and questions around folding and unfolding chains, polygons and other simplified models which would relate to proteins. This includes the results in computational geometry (such as the Carpenter’s rule problem that combined computational geometry with results in rigidity theory. Other work on linkages in 3-space confirms that the 3-D problem is significantly harder, but also indicates that some results can be obtained. Work in robotics has also studied the kinematics of larger scale structures subject to geometric constraints. In particular, the Probabilistic Road Map method from robotic motion planning has been applied by several groups to generate possible folding pathways for proteins. One version of this was presented at the workshop, along with a brief introduction to their on-line service that was being mounted as the workshop progressed, at parasol.tamu.edu/foldingserver/.

A number of computational biochemists and biophysicists have generated a broader range of algorithms for predicting and simulating the shape and motion of proteins. Many of these include minimizing energy functions - a process is can, at critical points of the functions, have relationships to rigidity theory. The most intensive of these methods are the Molecular Dynamics Simulations (MDS), which work with all atom models and energy functions for many interactions, to both simulate local motions, and to examine larger scale motions, up to the level of protein unfolding and folding.

Other Biochemists are working at a more detailed level of the local geometric configurations and choices made in the placement of small sections of the backbone, and side chains, in creating the initial protein data bank (pdb) models from X-ray crystallography and NMR data. The quality of this data can be crucial as input to various modelling methods (above) and some of the modelling and computational methods above can, in turn, contribute to the quality of the pdb data. The interplay of data and modelling is an important feature of the state of the art these days, and all communities share an interest in this interplay.

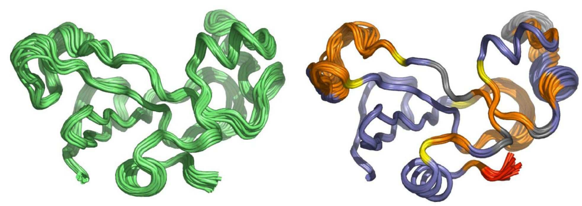


Figure 19.2: Showing the flexibility of the protein barnase. On the left is the structure as determined by NMR experiments. On the right additional conformers have been generated from the average X-ray structure using the programs FIRST and ROCK. [S. Menor, Ming Lei, M. Zavodszky and M.F. Thorpe, unpublished]

Each of these fields is in rapid evolution, due both to new theoretical results and to new experimental results that modify our assumptions and raise new questions. The work is increasingly interdisciplinary and the workshop reflected that reality.

Structure of the Workshop

There are a number of distinct communities working on computer (and mathematical) modelling of protein flexibility, rigidity, and folding, or on simplified and abstracted models with potential applications to these problems. This workshop brought together leading experts as well as current graduate students and post-docs from at least four of these communities: mathematicians working on the rigidity theory for structures (frameworks, molecular structures, tensegrity structures); computational geometers working on motions and paths of linkages, polygons, etc.; material scientists modelling rigidity in large molecular configurations; and biochemists modelling protein flexing, binding of molecules on proteins, detailed modelling of 3-D protein modelling and a variety of tools for predicting protein behaviour.

The workshop gathered together members of these communities of researchers to:

1. summarize the state of the art (as this time) for modelling protein flexibility and motions using models such as frameworks, linkages, Gaussian network models, robotics kinematics, etc.;

2. describe unsolved critical problems about current and potential models (mathematical, computational and biochemical), helping to sort the potential significance of various problems and potential results;

3. provide some grounding of mathematical and computational modelling efforts in biochemical data, to explore the effective use of this knowledge within modelling programs, and offer some reality checks on the meaning of that data for predictions of flexibility and folding.

4. Participate in working sessions to explore ways to clarify, resolve or solve these problems and propose priority problems and approaches.

Full hour talks the first day provided a survey of all four areas, with an emphasis on posing questions, conjectures, and directions for work that would connect the represented audiences. This was followed by an evening 'problem session' of issues worthy of follow up. These problems were immediately posted to the Web Comptes Rendu site (see below) and integrated into the ongoing discussions.

In advance of the workshop, participants were encouraged to post relevant papers, presentations and unsolved problems on a web site (see below). About 25 participants loaded materials, and many participants were able to prepare for the exchange by downloading and reading the materials. This easy ability to share materials played an essential role in building sufficient common ground to support strong exchanges among participants from diverse backgrounds, who had never met prior to the workshop.

Some 'problems needing to be solved' were posed in advance, and least three of these were solved before the workshop was over. Other problems posed were discussed during the workshop and additional problems were posed and posted during the workshop. Solutions or other follow up commentary continue to be posted at this time. We anticipate further follow up materials will be posted to the site over the next few months. This valuable site linking material on flexibility of proteins has now been linked from the flexweb.asu.edu web site. It has also been linked from the home pages of several of the participants, to become a resource for the wider community of researchers. In this way, it offers a basic source of information for new people to this area, including graduate students just moving into these areas. As such, this resource represents a clear outcome from the workshop that will continue to assist the building of a larger community with common goals, and building comparisons of results towards shared standards.

The program was deliberately flexible. Participants brought along PowerPoint or other presentations that were adapted overnight to address questions raised, or new approaches and issues that were relevant. All talks generated extensive discussion both during and after - confirming that we had achieved the desired engagement of people in interdisciplinary conversations. We also scheduled, from the second day on, some time for focused conversations with leadership from one, or several participants, and guidance from one (or both) of the organizers. give time for organized and informal working groups. As proposed, we also offered on site, software, and web site demonstrations, and access on a demonstration basis to software.

From there, the program evolved with

1. themed sections (Biochemistry, mathematics, computer science, biophysics with comparisons of methods (see below).
2. some wide ranging discussions with experts leading off an everyone pitching in;
3. substantial unscheduled time (noon to 3:30 most days) for informal conversations;
4. some evening discussions and software / web site demonstrations, running up to 10:30 at night;
5. discussions of shared concerns, including one session on the community responses to issues of patents, university intellectual property rules, sharing of code, etc.

Overall of the program elements there was active, spirited and informative discussion. No talk passed without engaging in extensive conversation about the methods and the results, and some sessions became extended conversations focused on themes from the problem sessions, or from debates which arose during previous discussions.

The patience of all speakers contributed to an atmosphere of respect and debate through which our different priorities, approaches, solved and unsolved questions were compared, and sometimes contrasted. The discussions on into the night engaged people from distinct communities in more detailed sharing of approaches, resources, and 'gold standards' for evaluating the quality of conclusions.

Since even the people within a single community had not previously gathered in one spot, there were also exchanges among people with similar backgrounds, and these exchanges among mathematicians, some computer scientists and some biophysicists were further reinforced in the follow-up Calgary Workshop on Rigidity (see below).

Working within Diverse Communities

There were some spirited, good natured and insightful exchanges about the priorities and contributions of various communities to the conversations. Here is how one graduate student (in computer science) presented her observations:

1. The mathematicians like theorems, but don't really care if the theorems can be used to compute results.
2. The physicists like computing results, but don't really care if the theory behind the results is correct.
3. The computer scientists like to compute results and have the theory be correct, but the mathematicians don't like our proofs, and the physicists don't like our results.

Here are a couple of other noted quotes, illustrating the good-humoured give and take:
"What we need is a good definition ... I am starting to think like a mathematician and the opposite was supposed to happen!" (A biophysicist)
"Mathematicians care about conjectures and things like that" (Post-doc).

Discussion and Outcomes

Everything said at the workshop supported the initial claim that flexibility of proteins (and other molecules) is an essential feature of the 3-D structure and its functioning. All of the speakers cast their work in terms of ways to explore this flexibility, to compare different measures and modelling methods and to predict flexibility and its impact on the interaction of molecules, both complexes of large molecules, and interactions of small drugs with large molecules.

Not surprisingly, everything said also supported the implicit theme that this modelling is hard. There were passing references to modelling protein folding (a very hard problem ab initio from the sequence) but this was not central to our discussions. The results, and the problems, addressed more modest goals, such as: - predicting, from a single 3-D structure, the regions of rigidity and flexibility within the molecular complex (e.g. FIRST); - predicting, from a single 3-D structure, the large scale patterns of the dominate modes of motion (e.g. Gaussian Network Models, ROCK) - searching for pathways between two known conformations of a molecule (e.g. ROCK PRM); - comparison of single molecule predictions with ensembles of structures generated by NMR methods, and increasingly generated by high quality X-ray crystallography methods (e.g. X-ray + ROCK = NMR), as illustrated in Figure 2. - ways of representing flexibility and motions of molecules, in ways that scale up and down for detail and overview; - providing solid mathematical foundations for the methods (above) and for comparisons among these methods. - improved models, computational techniques, and data base presentations, which will speed up work in many of these efforts; - incorporation of accurate biochemical information (e.g. rotomer data bases and improved Ramachandran plots) to improve the speed and accuracy of current algorithmic methods.

There was a good review of the successes (and limitations) of the current rigidity based algorithms. Discussion did confirm the advantages of switching from the former, 3-D bar and joint mathematical model to the molecular bar and hinge model. The advantages include: - a closer fit to kinematic models in use in robotics and in representations of molecular coordinates in terms of torsion angles; - simpler implementation of the combinatorial counting rules - more options for the value of constraints such as hydrophobic constraints, without resorting to 'pseudo-atom' insertions to trick to original algorithm - a more complete capture of 'stresses' and 'redundancy' in molecular models with small rings. - direct representation of possible 'collectively linked torsion angle changes' Other desirable features include: - equivalent matrix representations of first-order motions in the previous bar and joint models and the alternate models, - experimental equivalence with the prior algorithms for bar and joint models, - conjectures (with strong support) that the rigid region decompositions from the algorithms are identical; - close alignment with the mathematical conjectures and the broader mathematical model (body and hinge structures) for which the algorithms are known to be correct. An expository and research paper comparing the two models is being drafted by two of the mathematicians (Tay and Whiteley) to lay out the foundations of the molecular hinge model now being used, and to demonstrate the known equivalences and correspondences with the bar and joint model. This will support both those using these models for algorithmic work, and mathematicians investigating the conjectures about the algorithms for these models.

Extensive discussions, at Banff and the follow-up Calgary Workshop (see below) probed the mathematical complexities of the algorithms and the possible proofs of the molecular conjectures. A number of pieces were added to the puzzle, around connectivity and other features of 'rigid region decompositions' for general structures, and for molecular structures. As someone who has worked on these problems for over a decade, I was impressed by the new partial results, and new approaches which were discussed. Overall, the critical objective of engaging more mathematicians to work on the significant problems posed by the algorithms for modelling protein flexibility was achieved.

There were also discussions of how rigidity (FIRST) type results are being integrated into larger simulations, such as ROCK (flexweb.asu.edu/rock_index.html), and the newer Parasol Folding Server (parasol.tamu.edu/foldingserver/). In each case, grouping the atoms of a 'rigid cluster' together as a single moving body reduces the complexity of the computations.

Ring Closure: A number of computations involve steps which perturb the torsion angles along a bonds, and then relaxing or filling in missing values to ensure the closure of loops formed either by other bonds, such as ionic bonds, or implicity by fixing the position of more distant atoms. This ring closure is a classical problem that forms a bottleneck in terms of simulations. There was a morning session on algorithms for ring closure, and their applications in programs such a ROCK, the Parasol server, and the Richardson's work on 'protein chiropraxis' (snapping pieces of the backbone to alternative positions to generate more relaxed configurations). This sharing of techniques again opened the possibility of further collaborations to improve current algorithms and other examples to benchmark proposed methods with.

We had a wide-ranging introduction to the programs, viewers, and data-bases at the Richardson Lab (kinemage.biochem.duke.edu/). This provoked a lot of discussion about the quality of different sources of data, what errors to suspect, and what could be done to anticipate, and perhaps correct, the errors in the data which might impact the performance of the algorithms. As follow up to this discussion, some existing software will build in use of the improved systems For example, following discussions on possible collaborations, FIRST will soon provide Reduce placement of hydrogen atoms as an alternative to WhatIf placement now recommended. Other software will look at using the 'penultimate rotamer library', and other refined Ramachandron Plots to control which torsion angles are permitted in the simulations.

In general, there was an underlying theme that algorithms for protein flexibility can improved by better incorporation of accurate biochemical information during the initial processing or during selection of steps for simulation. This discussion was only possible because we had people from the multiple communities debating and exchanging over the full five days of the workshop.

A second complementary theme is adding computation expertise to current biochemistry (and perhaps biophysics) algorithms to improve their performance. The discussion of ring closure (above), and another discussion of the use of singular value decomposition for 'collective motions' were two examples of this theme. No definitive conclusions were reached, but possibilities were explored, and comparisons generated for further reflection.

There were a number of new ideas and approaches that were sparked for individuals and for small groups. Here is one illustration of how this worked.

There were two presentations about the flexibility of icosahedral virus capsids, using different techniques (engineering model building and FIRST analysis). Looking at the illustrations, the question was raised - can we find algorithms to detect only the motions respecting certain symmetries which are subgroups of the symmetries of the molecule? (This question can also be asked for much smaller structures, such as dimers of two protein chains with half-turn symmetry.) After some exploration, discussion during a coffee break generated a proposal (from an engineering participant) for extracting the combinatorial counts for matrices from the irreducible representations of the symmetry group. From these counts, the corresponding combinatorial pebble games were proposed (from a mathematician), and some simple examples computed and analyzed to ensure the overall results fit the larger patterns in cases we understand. The tentative conclusion is that we do have a 'program' for adapting known methods to symmetric motions of symmetric molecules, and a research / writing project is underway to tie up the details and share the results.

Finally, the most difficult outcome to document, but achievement, is that each participant came way with a broader perspective on what questions are significant, what resources are available for pursing our old questions (and some new ones) and what people might offer that additional insight which make carry us over from initial ideas to promising methods and new results. All the individual feedback received by the organizers has confirmed that participants, from graduate students to senior faculty, made new connections, saw new possibilities, and developed new respect for the contributions, and the difficulties of each of the participating communities.

We ended with one consensus conclusion - we should do this again in the future!

Special Features around this Workshop

Travel Funds. In addition to the BIRS funding, and the associated MSRI travel support, the NIH grant of one of the organizers (Walter Whiteley) included some funding for a workshop in the summer of 2004. This funding was used to provide support for a number of additional graduate students and post-doctoral fellows, as well as a few more senior researchers who would otherwise not have been able to attend. This funding was also used to partially subsidize the related Workshop on Rigidity (see below) in Calgary.

Web Comptes Rendus

biophysics.asu.edu/banf/list.php

This web site, hosted Arizona State University by one of the organizers, Michael Thorpe, permitted uploading of documents for sharing before, during and after the workshop. A number of people downloaded relevant articles and presentations to develop further background prior to the workshop. During the workshop, presentations, and related references were uploaded and accessed by people to further discussion. [The ready access to computers, wireless connections, and printers was a real assistance here.]

During the workshop, this web site became a ‘value added’ feature, as some presentations given in the morning became available to participants that evening. Overall, the easy access to the internet, via wireless and in-room connections, as well as the printers, was effectively mobilized to support the conversation, rather than distract from participation. When US visa problems forced a few cancellations, this web site also became a place to post their presentations, so these contributions could also be accessed.

University of Calgary Workshop on Rigidity July 22–24

This Banff Workshop gathered an important segment of the mathematical (and computational) research community on rigidity in one place. While the interdisciplinary workshop was an important source of problems and an impetus for future work, we also wanted a few days where the full range of current mathematical questions of rigidity could be shared and discussed. With the cooperation of the Department of Mathematics at the University of Calgary, in particular of the Canada Research Chair in Geometry, Karoly Bezdek, Robert Connelly, and Walter Whiteley organized a follow-up Workshop on Rigidity on Friday and Saturday, July 23-24. Funds from the University of Calgary, and some travel / housing funds from the NIH subcontract of Walter Whiteley, we were able to cover costs for housing, lunches, and transportation from Banff to the Calgary Hotel.

As a surprise (to the Rigidity Workshop Organizers), people from the computational and biophysics communities also stayed on for this workshop and contributed greatly to the discussions. As a result, we were able to have some follow up talks addressing issues raised in Banff, and several focused discussions on core conceptual and computational issues around ’collective motions’ and ’redundant constraints’, as well as follow up tasks from Banff, such as extending algorithmic work to describe symmetric patterns of stress and motions for symmetric structures such as viral capsids.

These discussions consolidated mathematical and computational developments from the Banff workshop, and have been followed up by active electronic discussions, several initiatives among the participants to write papers (both expository and with new results) and follow up collaborations engaging people from the workshops along with other collaborators who were not able to attend.

List of Participants

Amato, Nancy (Texas A&M University)
Bahar, Ivet (University of Pittsburgh)
Bereg, Sergey (University of Texas, Dallas)
Bezdek, Karoly (University of Calgary)
Borcea, Ciprian (Rider University)
Brock, Oliver (University of Massachusetts, Amherst)

Burkowski, Forbes J. (University of Waterloo)
Chubynsky, Mykyta (Arizona State University Tempe)
Connelly, Robert (Cornell University)
Gohlke, Holger (J.W. Goethe-Universitat)
Guest, Simon (University of Cambridge)
Hespenheide, Brandon (Arizona State University Tempe)
Jackson, Bill (Queen Mary College, University of London)
Jacobs, Don (California State University, Northridge)
Jiguo, Jiang (York University)
Jordan, Tibor (Eotvos University)
Kumar, Sanjay (Children's Hospital Boston, Harvard Medical School)
Lerner, Michael George (University of Michigan)
Mantler, Andrea (University of North Carolina, Chapel Hill)
Melnik, Roderick V.N. (Wilfred Laurier University)
Menor, Scott (Arizona State University)
Mousseau, Normand (Université de Montréal)
Rader, Andrew J. (University of Pittsburgh)
Richardson, David (Duke University)
Richardson, Jane (Duke University)
Ros, Lluis (Instutite of Robotics, Barcelona)
Rybnikov, Konstatin (University of Massachusetts)
Sanner, Michel (The Scripps Research Institute)
Servatius, Brigitte (Worcester Polytechnic Institute)
Servatius, Herman (Worcester Polytechnic University)
Sloughter, Maria (Cornell University)
Snoeyink, Jack (University of North Carolina, Chapel Hill)
Streinu, Ileana (Smith College)
Tay, Tiong-Seng (National University of Singapore)
Thomas, Shawna (Texas A&M University)
Thorpe, Michael (Arizona State University)
Whiteley, Walter (York University)
Whitesides, Sue (McGill University)
Zavodszky, Maria (Michigan State University)
Zuckerman, Daniel (University of Pittsburgh)

Chapter 20

Geometric Evolution Equations (04w5008)

July 24–29, 2004

Organizer(s): Christine Guenther (Pacific University), Jingyi Chen (University of British Columbia), Bennett Chow (University of California, San Diego), Klaus Ecker (Freie Universitaet Berlin)

In view of the continuing spectacular growth and advancement of the field of geometric evolution equations (GEEs for short), a 5-day workshop in the summer of 2004 at BIRS focused in this area would present a unique and timely opportunity to stimulate important new mathematical research and exposition on GEEs.

Methods and techniques which have been developed for one GEE often apply to numerous others. For example the method of P. Li and S.T. Yau for deriving differential Harnack inequalities for the heat equation has been further developed for many GEEs and applied to the analysis of singularity formation. Isoperimetric estimates are now known for many GEEs, both leading to proofs of global existence and convergence and finding applications in singularity theory. Pinching estimates hold for various GEEs; they are central to global existence and convergence theorems in the presence of convexity, and lead to proofs of the convexity of singularity models. Monotonicity formulas are common to the mean curvature flow (MCF) and the harmonic map flow, and yield important information on singularities, leading to the classification of singularities of Type I for the MCF. Entropy estimates are common to the 2-dimensional Ricci flow and the Gauss curvature flow, and have been generalized to many fully nonlinear GEEs. Maximum principle and gradient estimates are ubiquitous in the field. Because of this cross-fertilization of methodology, interaction between researchers on different GEEs is crucial, and often leads to new and important advances in the field.

Tentatively, a daily schedule of four 45-minute talks is planned for the 5-day workshop, with the possible exception of Tuesday, when we may organize an afternoon outing. A possible schedule consists of two talks in the morning and two in the afternoon: 9:00-9:45, 10:30 - 11:15, 1:30 - 2:15, 3:00 - 3:45. This would enable a good number of the 40 participants to present their recent research on GEEs. Additional lectures may be arranged as needed. The remainder of each afternoon will be devoted to problem sessions and/or informal discussion among the participants. To encourage the success of the problem sessions the organizers plan to contact a selection of participants who are leaders in the field to moderate the sessions, and ask that they prepare mini-topics for discussion. The setting of BIRS seems ideal for promoting informal interaction which has proved so fruitful in the past in the field of GEEs.

In recent years there has been increasing interaction among researchers on GEEs. In particular, a large percentage of papers on GEEs now appearing are joint papers. Important advances are happening in many parts of the world, and as the list of proposed participants of this workshop demonstrates, a goal of the organizers is to foster mathematical interaction between them.

The organizers would also like to encourage interaction between senior and junior mathematicians working in the area of GEEs. Such vertical integration at the research level is crucial to the continuing high level development of the field. The list of proposed participants includes a good mixture of both some of the most distinguished mathematicians in the world and many of the top young researchers in the field. We will also try to accomodate some advanced graduate students. We will ask each speaker to include open problems as a part of his or her talk to stimulate the research of junior participants in particular. It is the hope of the organizers that the combination of talks, problem sessions, informal discussions, and the special environment of BIRS will be the means of providing a wonderful educational opportunity for junior researchers.

Because of the geographic diversity of the researchers in the field of GEEs, bringing participants together for the 5-day workshop will facilitate the dissemination of the most recent research ideas and results, which otherwise might not be possible. The atmosphere of the workshop and its surroundings will hopefully lead to new collaborations during the workshop and especially in the years following the workshop.

In such a rapidly developing area as GEEs, it often important for researchers, both senior and junior, to have access to the most recent developments in the field. We plan to ask participants to send the organizers abstracts of their talks, and to bring recent papers for distribution during the workshop.

The combination of the breadth and the cohesiveness of the field of GEEs should make a 5-day workshop at BIRS have a significant impact on the field.

List of Participants

Buckland, John (Australian National University)
Butscher, Adrian (University of Toronto at Scarborough)
Cao, Xiaodong (Columbia University)
Chau, Albert (Harvard University)
Chen, Jingyi (University of British Columbia)
Chow, Bennett (University of California, San Diego)
Chu, Sun-Chin (National Chung Cheng University)
Ecker, Klaus (Freie Universitaet Berlin)
Guenther, Christine (Pacific University)
Gulliver, Robert (University of Minnesota)
Hong, Min-Chun (University of Queensland)
Huisken, Gerhard (Max Planck Institute for Gravit Phys)
Isenberg, Jim (University of Oregon)
Ivey , Thomas (College of Charleston)
Knopf, Dan (University of Texas, Austin)
Kuwert, Ernst (Albert-Ludwigs-Universitaet Freiburg)
McCoy, James (Australia National University)
Neves, Andre (Stanford University)
Ni, Lei (University of California, San Diego)
Sesum, Natasa (Massachusetts Institute of Technology)
Simon, Miles (Albert-Ludwigs-Universitaet Freiburg)
Sinestrari, Carlo (Universita di Roma)
Smoczyk, Knut (Max Planck Institute for Mathematics in the Sciences)
Struwe, Michael (Eidgen Technische Hochschule, Zentrum)
Tam, Luen-Fai (Chinese University of Hong Kong)
Tsai, Dong-Ho (National Tsing Hua University)
White, Brian (Stanford University)

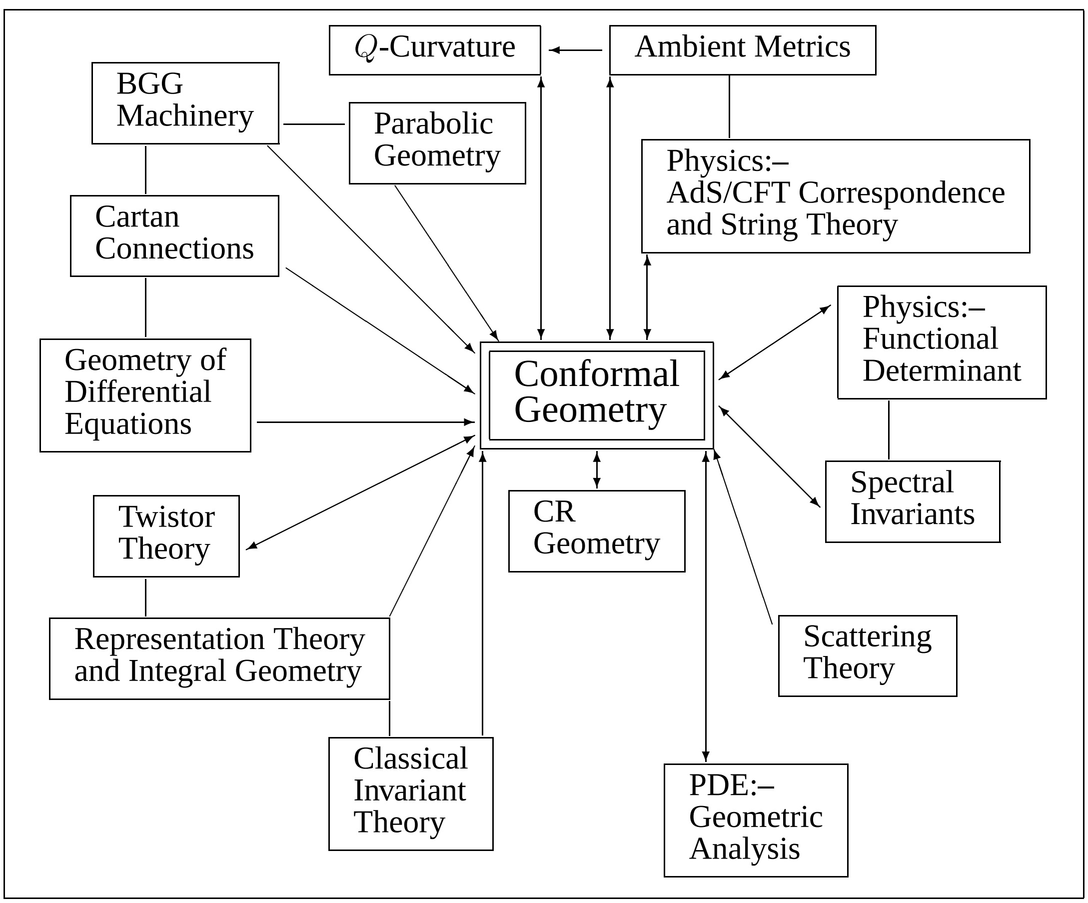
Chapter 21

Conformal Geometry (04w5006)

July 31–August 5, 2004

Organizer(s): Thomas Branson (University of Iowa), Michael Eastwood (University of Adelaide), McKenzie Wang (McMaster University)

The workshop organisers see conformal geometry as central in the following circle of ideas:–



Arrows in this diagram indicate input from one topic to another. Closely related topics are joined by lines. Conformal geometry is highly analogous to CR geometry, so their boxes are close together and arrows run in both directions. The left hand side of the diagram is largely algebraic. At the top of the diagram, Q-curvature and ambient metrics are specific aspects of conformal geometry, which are separated for special attention.

The right hand side of the diagram is more concerned with applications in geometric analysis and physics. The workshop touched on all aspects of this diagram and the discussion below will refer to topics in the diagram by underlining them.

In contrast with the more familiar Riemannian geometry, there are clear difficulties concerning the basic local geometry, symmetry, and invariance in conformal geometry. These difficulties may be traced to differences in the 'flat model': Riemannian geometry is a curved version of Euclidean geometry whereas conformal geometry is a curved version of the sphere S^n as a homogeneous space for $\mathrm{SO}(n+1,1)$. In particular, the isotropy subgroup in the Euclidean case is $\mathrm{SO}(n)$ and in Riemannian geometry there is a principal $\mathrm{SO}(n)$-connection on the frame bundle responsible for the usual Riemannian curvature. On the conformal sphere, however, the isotropy subgroup is a parabolic subgroup $P \leq \mathrm{SO}(n+1,1)$ whose algebraic structure and representation theory is much more subtle. There is a principal P-bundle induced on any conformal manifold but no principal P-connection. Instead one has only a 'Cartan connection' [18] (with values in $\mathfrak{so}(n+1,1)$).

Let us run now through the talks presented at the workshop and discuss how they fit in this general scheme of things. Some background is included. Also, some digressions are indulged in an effort to explain links between talks and to bring out some underlying themes. Speakers in appear in **bold type**.

SUNDAY Conformal differential geometry was a popular topic in the 1920s. It was then that Cartan introduced his conformal connection [18]. An equivalent formulation due to Thomas [51] followed shortly afterwards and is now understood as an important alternative viewpoint. But this was only the beginning of a theory of conformal differential geometry. At that stage, obvious questions remained unanswered. There was no classification, or even a means of writing down, conformal invariants or conformally invariant differential operators. A few intriguing examples were known: in addition to the Weyl tensor, there was the conformally invariant Bach tensor [3] in 4 dimensions and various differential equations from physics, including Maxwell's equations, were known to be conformally invariant. Little further progress was made until the mid 1980s although, with hindsight, hints of the ambient metric construction and AdS/CFT correspondence may be seen in articles of Thomas [52], Schouten-Haantjes [49], and Dirac [21]. Some renewed stimulus was also provided by Branson's finding [7] new conformally invariant operators with a more significant dependence on the Ricci tensor. There are some separate issues of geometric analysis concerning special Riemannian metrics and the Yamabe problem. We shall return to them in the discussion of Wednesday below. Laying them on one side for the moment, the real breakthrough in conformal geometry was presaged by developments in CR geometry. A real hypersurface in a complex manifold inherits a CR structure as a remnant of the ambient Cauchy-Riemann equations. There are clear parallels between CR geometry and conformal geometry. In particular, there is a Cartan connection in the CR case due to Chern-Moser [19] and Tanaka [50]. In 1979, Fefferman [26] developed the ambient metric construction for CR structures. It is a formal construction, which attempts to associate to every non-degenerate CR manifold a Kähler-Einstein manifold of higher dimension. As a consequence, CR invariants may be built from Lorentzian invariants of the ambient Kähler-Einstein manifold. Though the construction breaks down at a certain order, it builds CR invariants below a certain 'weight'. Fefferman also formulated a purely algebraic problem whose positive solution would show that all CR invariants below the critical weight arise in this way. This would be sufficient to predict the form of the coefficients of the asymptotic expansion up to the log term of the Bergman kernel of a strictly pseudo-convex domain. Since the isotropy subgroup $P \leq \mathrm{SU}(n+1,1)$ of the flat model (a hyper-quadric in $\mathbb{CP}_{n+1}$) is a parabolic subgroup and since the algebraic problems concerned the invariant theory of P, Fefferman dubbed these matters of 'parabolic invariant theory'. He was able to solve the algebraic problems subject to suitable restrictions on the degree and weight of the invariant.

The real breakthrough, alluded to above, was the ambient metric construction of Fefferman and Graham [28]. It is the (more difficult) conformal counterpart of Fefferman's ambient construction in the CR case [26]. Many of the recent developments in conformal differential geometry are based on this construction. It also lies at the heart of the spectacular AdS/CFT correspondence [44, 54] in the physics of string theory. Our first speaker on Sunday was **Robin Graham**. His talk concerned some new analytic aspects of the ambient metric construction. The most basic of questions concerning the ambient metric construction is whether the formal asymptotics detailed by Fefferman and Graham are attached to a genuine analytic construction. There has been a lot of recent work in this direction especially by Anderson [1] but this aspect was not seriously discussed at the workshop. Instead, a sufficient starting point for Graham was his older work with Lee [35], which shows that every conformal metric on the sphere sufficiently close to the round metric is the conformal infinity of a unique asymptotically hyperbolic Einstein metric on the ball. The usual

Dirichlet-to-Neumann mapping is obtained by taking a function on the sphere as the Dirichlet data for a harmonic function in the ball and then restricting the normal derivative back to the sphere. Graham considers a natural non-linear version of this construction starting with a deformation of the conformal metric on the sphere at infinity and picking off the conformally invariant part of the volume renormalisation [33]. This is an interesting construction even starting on the round sphere. In this case he uses an analysis of intertwining operators from representation theory [10] to see what is happening. What is happening more generally is clearly a very interesting but currently unanswered problem.

A little surprisingly, there was not a lot of discussion at this workshop concerning Branson's Q-curvature [8, 9]. A closely related entity, however, is the Fefferman-Graham obstruction tensor. This is a conformally invariant tensor defined in even dimensions generalising the Bach tensor [3] in four dimensions. Last year, Graham and Hirachi [36] showed that the metric variation of the integral of Q-curvature is the obstruction tensor. In their talks on Monday, **Rod Gover** and **Larry Peterson** showed how to obtain the obstruction tensor via certain conformally invariant differential operators constructed via 'tractor calculus'. Gover explained how some basic properties of the obstruction tensor could be seen from this point of view (e.g. that it vanishes for metrics conformal to Einstein) and Peterson explained how to use this method to teach a computer (using Lee's Ricci program [40]) to find explicit formulae for the obstruction tensor (in low dimensions). This should be compared with earlier work by the same two authors [31] on explicit formulae for Q via 'tractors' and computer assisted calculations. There now follows a brief digression on the tractor calculus.

As already mentioned, the natural connection in conformal geometry is a Cartan connection. It was first introduced in [18] as an $\mathfrak{so}(n+1,1)$-valued 1-form on the total space of a principal P-bundle naturally attached to any conformal manifold as a certain bundle of second order frames. Here, P is the stabiliser subgroup of $\mathrm{SO}(n+1,1)$ under its action on the n-sphere as conformal motions. A completely equivalent formulation of the Cartan connection arises as the induced connection on the vector bundle $\mathbb{T}$ arising from the defining representation of $\mathrm{SO}(n+1,1)$ on $\mathbb{R}^{n+2}$. In [51], Thomas independently and directly defined this bundle and its connection. Bearing in mind that this was before 'vector bundles' were standard mathematical objects, it was not so clear at the time that this construction was equivalent to Cartan's. A modern explanation of Thomas's construction is given in [4]. There are certainly some aspects to conformal geometry that are better pursued from the Thomas viewpoint. Thomas himself already understood [52] that the basic connection is an inadequate substitute for the Levi-Civita connection in Riemannian geometry. Roughly speaking, the Levi-Civita connection, as a connection on tensor bundles, can be iterated and so invariantly captures all the higher jets of a Riemannian metric. This is not the case for the Cartan or Thomas connections. It is this problem that the Fefferman-Graham ambient metric construction overcomes (to all orders in odd dimension and to some finite but obstructed order in even dimensions (see [30] for how to push past the obstruction in even dimensions)). The Thomas construction, or 'tractor calculus' as it is now known, is more closely related to the ambient metric construction [13] than is the Cartan connection. It seems to be more and more involved in progress in conformal geometry.

From the Cartan point of view, however, conformal differential geometry arises as a sort of 'curved version' of the flat model, which is the n-sphere under the action of $\mathrm{SO}(n+1,1)$ (the n-sphere is regarded as the space of generators of the null cone in $\mathbb{R}^{n+2}$ with its standard Lorentzian metric). It was pondered for some time whether there are 'curved versions' of all homogeneous spaces G/P where G is an arbitrary Lie groups and P is a parabolic subgroup. For some years, there were several known geometries (projective, conformal, CR, Cartan's five variables [17], $\ldots$) but no unified theory. The term parabolic geometry was already being employed before the general formulation, which now exists thanks to Morimoto [46] and Čap-Schichl [14]. The theory is now highly developed. **Andreas Čap** and **Jan Slovák** are writing a comprehensive text on the subject [15]. Čap gave a survey of the main points, especially Lie algebra cohomology and how it is used to normalise the Cartan connections and explore where lies the torsion and curvature of a normalised connection.

The final speaker on Monday was **Gestur Ólafsson** who talked about a certain integral geometric transform in representation theory. Though this was a little to one side of the main theme of the workshop, there are some links as follows. Twistor theory [38] is an alternative approach to basic physics introduced and developed by Roger Penrose over the past 30 years or so. A basic idea in twistor theory is that space-time should be view as a space of preferred 'cycles' in another, more fundamental, space: its 'twistor space'. All basic physics should then be seen as constructions on twistor space and the physical ramifications obtained by integral geometry. Usually, the twistor space has some complex structure (at least a CR structure) but the complex integral geometry is closely parallel to classical real integral geometry starting with the Radon and Funk transforms (now familiar in medical imaging). Ólafsson's talk was concerned with the real integral geometry of complex cycles in a complexified non-Riemannian symmetric space. There is a mathematical toolbox here that impinges of conformal geometry through the ideas of twistor theory and also because representation theory is finding its way more and more into conformal geometry.

MONDAY Conformal differential geometry is very much tied to physics. This is true on a classical level, where the most basic equations of physics (for example, the Dirac equation) are conformally invariant. It is also true on a quantum level and especially through the AdS/CFT correspondence [44, 54] linking string theory and supergravity on a 'bulk' Einstein manifold of negative scalar curvature and conformal field theory on its 'conformal infinity'. Monday's talks were largely devoted to physics.

For the benefit of mathematicians, **Louise Dolan** gave a very useful survey of her joint work [22] with Nappi and Witten from a couple of years ago. The idea is that there are equations that one can write down (due to Deser), in some sense between the usual massless and massive field equations on space-time. It turns out that these equations in the bulk have surprising consequences at infinity under the AdS/CFT correspondence, producing a known basic series of conformally invariant operators. These 'partially massless fields' are certainly intriguing but have not yet found their way into mathematics. Maybe this could be done now. There are further clues to a mathematical theory in that the inner product of the space of fields turns out to be sometimes negative (not physical).

Don Page talked about construction of Einstein metrics on certain compact manifolds (such as $S^2 \times S^n$) especially in odd dimensions. Like the classical Kerr metric, these metric admit many (conformal) symmetries. Finding explicit Einstein metrics is a difficult business. Nicely symmetrical examples (cf. [53]) are good for testing the behaviour of fields subject to invariant equations.

Lionel Mason's talk was very much based on the ideas of twistor theory. From the mathematical point of view he was talking about connections on $\mathbb{RP}_2$ with zero holonomy. He has worked out the Penrose-Ward transform for such connections and there are two parts to the twistor data. One is a field on $\mathbb{RP}_2^*$ much like the Funk-Radon transform would give. But the other part (to which the first part is coupled) takes the form on a holomorphic vector bundle on $\mathbb{CP}^*$. This is similar in spirit to Ólafsson's talk yesterday in that there is a unexpectedly useful complexification.

Spyros Alexakis gave an excellent review of how Q-curvature has recently been used (especially in 4 dimensions by Chang, Yang, and co-workers) in geometric analysis. Unfortunately, none of the researchers directly involved in this area were able to attend the workshop. Here is a digression on Q-curvature. It is a scalar Riemannian invariant canonically defined in even dimensions. In four dimensions $Q = \frac{1}{6}R^2 - \frac{1}{2}R^{ab}R_{ab} - \frac{1}{6}\Delta R$. Though there is a now a straightforward definition [29] in terms of the ambient metric, explicit formulae in higher dimensions are complicated [31] or unavailable. A characterisation of Q is also unavailable but one of its key properties is that, under conformal rescaling of the metric $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$ we have $\widehat{Q} = Q + P \log \Omega$, where P is a *linear* operator. Conformal invariance of P is forced. It is the Graham-Jenne-Mason-Sparling operator [34]. It also follows that $\int_M Q$, for M a compact manifold, is conformally invariant. It has been asserted in the physics literature that these properties force Q to have to form $Q = E + I + D$ where E is the Euler density or Pfaffian, I is a conformal invariant, and D is a divergence (and so does not contribute to $\int_M Q$). Alexakis's PhD thesis, currently in preparation, proves this assertion. It is not at all an easy argument and he was only able to sketch one ingredient (that if an identity on a Riemannian manifold holds formally (in the sense of Weyl's classical invariant theory), then it holds in all dimensions and so one can extract consequences by looking at the leading order terms as the dimension goes to infinity!).

Penrose's 'Weyl curvature hypothesis' states that the initial singularity in space-time (i.e. the 'big bang') is qualitatively different from final singularities (i.e. 'black holes' and the 'big crunch'). Specifically, at the initial singularity the conformal curvature should be smooth up to the boundary such as one might obtain by taking a smooth metric and rescaling it by a conformal factor that blows up at the boundary. But how can one recognise this phenomenon? There are no preferred coordinates and nor even a given boundary. One might choose bad coordinates and/or a bad conformal gauge. **Paul Tod** suggested various conformally invariant tests for the Weyl curvature hypothesis but complete recognition is still a problem.

Maciej Dunajski also spoke on material derived from twistor theory. Beginning with the Ward correspondence for instantons, there are twistor description of various integrable systems. All of them involve families of rational curves in an associated 'twistor space'. In higher dimensions, 'paraconformal' or 'almost

Grassmannian' structures seem to provide a suitable generalisation of conformal geometry in dimension four. Dunajski discussed a refinement of such structures, namely manifolds for which the tangent bundle has the form $S' \otimes \bigodot^k S$ where S and S' are rank 2 vector bundles.

TUESDAY A fundamental observation in even-dimensional conformal geometry is that the Hodge $\star$-operator is conformally invariant on forms of middle degree. For example, in 4 dimensions the 2-forms canonically split into self-dual and anti-self-dual types. Many of the special features of 4-dimensional geometry can be traced to this fact. Moreover, the general structure of conformally invariant differential operators follows similar patterns to the de Rham sequence. These are the Bernstein-Gelfand-Gelfand sequences (introduced in [24] in dimension 4). There is now an extensive theory, referred to as BGG machinery and largely due to Čap-Slovák-Souček [16] and Calderbank-Diemer [12]. Though the BGG sequence cannot usually be locally exact in the curved setting, it has been noticed that sometimes there are subcomplexes that have this property on special manifolds (for example, CR manifolds or self-dual manifolds in dimension 4). **Vladimír Souček** presented many examples of this phenomenon (found in joint work with Čap). Representation theory is involved, this time to see that curvature cannot possibly interfere with local exactness if the relevant representation is not present. These are exactly the sort of sequences that appear in trying to deform a given parabolic geometry (and in good circumstances this 'deformation complex' is elliptic).

As already mentioned, CR geometry of hypersurface type is closely analogous to conformal geometry. Remarkably, there are other examples of CR geometry that fit into this scheme, most notably the CR geometry of real codimension 2 submanifolds in $\mathbb{C}^4$ due to Schmalz and Slovák [48], which was only found via the general parabolic theory (and almost surely would have been impossible to find without the general theory). But is is far from clear whether parabolic geometry is the end of the road. More specifically, the basic construction in parabolic geometry is of a preferred Cartan connection and, until recently, the means for picking out a preferred connection was to insist that it be 'normal', a restriction best formulated with the BGG machinery. Earlier this year, Fox [27] found an example of a parabolic geometry (contact projective geometry), which possessed a canonical Cartan connection that was not necessarily normal. Instead, normality was an extra condition that showed up in the torsion of the Fox connection. In his talk on Tuesday, **Gerd Schmalz** considered the CR geometry of real codimension 2 submanifolds in $\mathbb{C}^3$. This geometry is far from parabolic: the 5-dimensional structure algebra is not even close to semisimple. Nevertheless, Schmalz reported on joint work with Beloshapka and Ezhov that constructs a canonical Cartan connection (and a normal form for the embedding analogous to Moser normal form [19] for hypersurfaces). It is interesting to note that the underlying real structure, of an 'Engel manifold' (a 4-dimensional real manifold M with 2-dimensional distribution D such that $[D, [D, D]] = TM$), carries no local information. In this sense it is quite close to the hypersurface case with underlying contact structure. However, the local structure algebra is quite different.

Kengo Hirachi continued the CR theme but back to hypersurface type. Of course, the prime example such geometry is found on the boundary of a strictly pseudo-convex domain in $\mathbb{C}^n$. Hirachi explained how to formulate the volume expansion with respect to the Einstein-Kähler or Bergmann metric. This is a parallel development to the conformal case [33] but there are new features: if the domain is taken to be a strictly pseudo-convex neighbourhood of the zero section of a sufficiently ample vector bundle, then the Chern classes of the bundle show up as CR invariants on the boundary.

There were two talks in the evening given by graduate students. Continuing Hirachi's volume renormalisation in the CR setting, his student **Neil Seshadri** talked about another aspect of the volume expansion: there are two interesting terms in the expansion and Seshadri talked about the coefficient of the 'log' term. As a digression, it is worth noting that there is a direct link between CR geometry and conformal geometry. Fefferman [25] (in the embedded case and Burns-Diederich-Shnider [11] intrinsically) found that a non-degenerate CR structure on a manifold M determines a canonical circle bundle over M, the total space of which is now known as the Fefferman space. This is naturally equipped with a conformal structure, which encodes the underlying CR structure. Conformally invariant constructions on the Fefferman space gives rise to CR invariant constructions on M. For example, the Chern-Moser chains on M arise as the images of null geodesics and CR Q-curvature pulls back to the conformal Q-curvature. However, the conformal structures that arise are special [32] and so it is wise to proceed independently. It is unknown in the CR case whether $\int_M Q$ can be non-zero.

The other evening talk was by **Doojin Hong**, a student of Branson. He talked about generating the spectrum for spinors over $S^1 \times S^n$ with Lorentzian metric. This should be compared with spectrum generating [10] for densities on S^n as an example of principal series representations and their intertwinors (and as used by Graham in the opening talk of this conference (and as may be used to understand the GJMS-operators operators on the sphere)).

WEDNESDAY The first talk was given by **Claude LeBrun** who discussed 'optimal metrics' on 4-manifolds, defined on an n-manifold M as a metric such that $\int_M |R|^{n/2}$ is absolutely minimised, where $|R|$ denotes the pointwise norm of the Riemann curvature tensor. The exponent $n/2$ is forced by requiring invariance under constant rescaling of the metric and already one can see that dimension 4 is special since then we are talking about the L^2 norm of the Riemann curvature. From the integral formulae for the Euler characteristic and signature, it follows immediately that there are two special ways of guaranteeing an optimal metric in 4 dimensions: Einstein metrics are always optimal and so are scalar-flat anti-self-dual metrics. LeBrun uses twistor theory to construct optimal metrics of this second type on the connected sum of 6 of more copies of the complex projective plane with its opposite orientation. Earlier results say that these manifolds are good test cases for the existence of optimal metrics. Twistor methods signal conformal geometry. The point is that for a metric to be anti-self-dual is a conformally invariant condition and these are exactly the manifolds with a twistor space. LeBrun uses the behaviour of the Green's function of the Yamabe operator to determine the sign of the Yamabe constant of a four manifold, this being reflected in the twistor space by dint of Atiyah's construction [2]. He uses this information to detect a change of sign in a family of conformal metrics and hence to find a find a metric with vanishing Yamabe constant. The details are in [39].

Helga Baum was concerned with Lorentzian metrics with special conformal properties. It is still unknown whether a compact Lorentzian manifold with a group of essential conformal transformations has to be constant curvature (where essential means that there is no metric in the conformal class for which the transformations are merely isometries). In the Riemannian case, there are no such exotic examples. The Lorentzian case is much more difficult. For example, there are compact Lorentzian manifolds with non-compact isometry group. These questions are very much connected with the existence of parallel spinors and holonomy in the Lorentzian case: see [41, 42]. In the conformal case, the existence of conformal Killing spinors and solutions to other overdetermined systems of partial differential equations (sometimes collectively known as twistor equations) is very much linked to tractor calculus. Baum explained how this link could be used. Solutions to these equations constrain the underlying geometry [5] and are the source of symmetries of basic equations such as the Dirac operator [6] and Laplacian [23].

Felipe Leitner continued with this theme, discussing the holonomy of conformal manifolds, meaning the holonomy of their Cartan connections. This is very much related to the existence of Einstein metrics in a given conformal class: a parallel standard tractor generically defines an Einstein scale. Unlike the Riemannian case, not much is known about the holonomy of conformal manifolds. With very symmetrical manifolds it may be calculated and this gives rise to the first real examples. Leitner finds [43], for example, the conformal holonomy of $\mathrm{SO}(4)$ (regarded as a Riemannian manifold via its bi-invariant metric) is $\mathrm{SO}(7)$ as a subgroup of $\mathrm{SO}(7,1)$. He also explained how special conformal holonomy gives rise to Einstein metrics in a given conformal class.

William Ugalde used the Wodzicki residue applied to a simple conformally invariant pseudo-differential operator to construct a natural conformally invariant differential pairing and an invariant differential operator, which looks very much like the critical GJMS-operator [34]. In dimension 4, Connes [20] obtained exactly this operator (sometimes called the Paneitz operator). In dimension 6, Ugalde has checked by direct calculation that one obtains the GJMS-operator. This is already a formidable task. It is extremely plausible that the GJMS-operators are indeed arising by Ugalde's construction. Perhaps the Wodzicki residue could be related to the scattering theory approach to the GJMS-operators [37]. Otherwise, we await a characterisation of these operators. Ugalde has checked many of the salient features of his operators but these features are insufficient precisely to pin them down.

Another popular geometry of the 1920s was projective differential geometry. Both Cartan and Thomas [4] developed the basic theory in parallel to conformal geometry. In particular, there is a canonically defined Cartan connection. **Vladimir Matveev** described his recent proof of the Lichnerowicz-Obata conjecture in projective geometry, namely that is a connected Lie group acts on a closed connected Riemannian manifold by transformations that preserve the geodesics (as unparameterised curves), then its acts by isometries or the manifold is covered by the round sphere. This is clearly parallel to the conformal Lichnerowicz conjecture (proved by Obata, Alekseevsky and Ferrand). Matveev converts the projective equivalence of two metrics into the existence of a tensor field, a self-adjoint endomorphism of the tangent bundle, satisfying a certain partial differential equation. He calls these tensors 'BM-structures'. He then embarks on a careful investigation of such structures bearing in mind how the eigenvalues of the endomorphism might coalesce. For details see [45].

After touching on Cartan connections in the geometry of differential equations [47] and reviewing the basic geometry of twistor theory, **George Sparling** described the the quantum Hall effect and how it has recently been viewed as a mathematical construction on the Riemann sphere under the action of $\mathrm{SU}(1,1)$. He speculated that this construction might be extended to a sort of 'cohomological twistor fluid' on the open orbits of $\mathrm{SU}(2,2)$ acting on $\mathbb{CP}_3$.

List of Participants

Alexakis, Spyridon (Princeton University)
Baum, Helga (Humboldt Universität zu Berlin)
Bland, John (University of Toronto)
Branson, Thomas (University of Iowa)
Cap, Andreas (University of Vienna)
Dolan, Louise (University of North Carolina)
Dunajski, Maciej (Cambridge University)
Derdzinski, Andrzej (Ohio State University)
Eastwood, Michael (University of Adelaide)
Gover, A. Rod (University of Auckland)
Graham, C. Robin (University of Washington)
Hirachi, Kengo (University of Tokyo)
Hong, Doojin (University of Iowa)
LeBrun, Claude (State University of New York, Stony Brook)
Leitner, Felipe (Humboldt University Berlin)
Maschler, Gideon (University of Toronto)
Mason, Lionel (Oxford University)
Matveev, Vladimir (University of Freiburg)
Olafsson, Gestur (Louisiana State University)
Page, Don (University of Alberta)
Peterson, Lawrence (University of North Dakota)
Raske, David (University of California, Santa Cruz)
Schmalz, Gerd (Universität Bonn)
Seshadri, Neil (University of Tokyo)
Slovak, Jan (Masaryk University)
Soucek, Vladimir (Charles University)
Sparling, George (University of Pittsburgh)
Tod, K. Paul (Oxford University)
Ugalde, William (Purdue University)
Villanueva, Alfredo (University of Iowa)
Wang, McKenzie (McMaster University)
Zhang, Genkai (Chalmers University of Technology, Sweden)

Bibliography

[1] M.T. Anderson, Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds, *Adv. Math.* **179** (2003), 205–249.

[2] M.F. Atiyah, Green's functions for self-dual four-manifolds. In *Mathematical Analysis and Applications*, Adv. Math. Suppl. Stud. **7**, 129–158, Academic Press, 1981.

[3] R. Bach, Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs, *Math. Zeit.* **9** (1921), 110-135.

[4] T.N. Bailey, M.G. Eastwood, and A.R. Gover, Thomas's structure bundle for conformal, projective and related structures, *Rocky Mtn. J. Math.* **24** (1994), 1–27.

[5] H. Baum, Twistor and Killing spinors in Lorentzian geometry. In *Global analysis and harmonic analysis (Marseille-Luminy, 1999)*, Sémin. Congr. **4**, 35–52, Soc. Math. France, 2000.

[6] I.M. Benn and J.M. Kress, First-order Dirac symmetry operators, *Class. Quant. Grav.* **21** (2004), 427–431.

[7] T.P. Branson, Conformally covariant equations on differential forms, *Commun. P.D.E.* **7** (1982), 393–431.

[8] T.P. Branson, Differential operators canonically associated to a conformal structure, *Math. Scand.* **57** (1985), 293–345.

[9] T.P. Branson, Sharp inequalities, the functional determinant, and the complementary series, *Trans. Amer. Math. Soc.* **347** (1995), 3671–3742.

[10] T.P. Branson, G. Ólafsson, and B. Ørsted, Spectrum generating operators and intertwining operators for representations induced from a maximal parabolic subgroup, *J. Funct. Anal.* **135** (1996), 163–205.

[11] D. Burns, K. Diederich, and S. Shnider, Distinguished curves in pseudoconvex boundaries, *Duke Math. J.* **44** (1977), 407–431.

[12] D.M.J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, *J. Reine Angew. Math.* **537** (2001), 67–103.

[13] A. Čap and A.R. Gover, Standard tractors and the conformal ambient metric construction, *Ann. Glob. Anal. Geom.* **24** (2003), 231–259.

[14] A. Čap and H. Schichl, Parabolic geometries and canonical Cartan connections, *Hokkaido Math. J.* **29** (2000), 453–505.

[15] A. Čap and J. Slovák, *Parabolic geometries*, Math. Surveys and Monographs, Amer. Math. Soc., to appear.

[16] A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, *Ann. Math.***154** (2001), 97–113.

[17] E. Cartan, Les systèmes de Pfaff á cinq variables et les équations aux derivées partielles du second ordre, *Ann. Ecole Norm. Sup.* **27** (1910), 109–192.

[18] E. Cartan, Les espaces à connexion conforme, *Ann. Soc. Pol. Math.* **2** (1923), 171–202.

[19] S.S. Chern and J.K Moser, Real hypersurfaces in complex manifolds, *Acta Math.* **133** (1974), 219–271.

[20] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[21] P.A.M. Dirac, The electron wave equation in de Sitter space, *Ann. Math.* **36** (1935), 657–669.

[22] L. Dolan, C.R. Nappi, and E. Witten, Conformal operators for partially massless states, *J. High Energy Phys.* **10** (2001) #16 (electronic).

[23] M.G. Eastwood, Higher symmetries of the Laplacian, *Ann. Math.*, to appear.

[24] M.G. Eastwood and J.W. Rice, Conformally invariant differential operators on Minkowski space and their curved analogues, *Commun. Math. Phys.* **109** (1987), 207–228, Erratum, *Commun. Math. Phys.* **144** (1992), 213.

[25] C. Fefferman, Monge-Ampere equations, the Bergman kernel, and geometry of pseudoconvex domains, *Ann. Math.* **103** (1976), 395–416.

[26] C. Fefferman, Parabolic invariant theory in complex analysis, *Adv. Math.* **31** (1979), 131–262.

[27] D.J.F. Fox, Contact projective structures, *arXiv*: math.DG/0402332.

[28] C. Fefferman and C.R. Graham, Conformal Invariants. In *Élie Cartan et les Mathématiques d'Aujourdui*, 95–116, Astérisque, 1985.

[29] C. Fefferman and K. Hirachi, Ambient metric construction of Q-curvature in conformal and CR geometries, *Math. Res. Lett.* **10** (2003), 819–831.

[30] A.R. Gover, Invariant theory and calculus for conformal geometries, *Adv. Math.* **163** (2001), 206–257.

[31] A.R. Gover and L.J. Peterson, Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus, *Commun. Math. Phys.* **235** (2003), 339–378.

[32] C.R. Graham, On Sparling's characterization of Fefferman metrics, *Amer. J. Math.* **109** (1987), 853–874.

[33] C.R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, *Suppl. Rend. Circ. Mat. Palermo* **63** (2000), 31–42.

[34] C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling, Conformally invariant powers of the Laplacian, I: Existence, *J. Lond. Math. Soc.* **46** (1992), 557–565.

[35] C.R. Graham and J.M. Lee, Einstein metrics with prescribed conformal infinity on the ball, *Adv. Math.* **87** (1991), 186-225.

[36] C.R. Graham and K. Hirachi, The ambient obstruction tensor and Q-curvature, *arXiv*: math.DG/0405068.

[37] C.R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Inv. Math.* **152** (2003), 89–118.

[38] S.A. Huggett and K.P. Tod, *An introduction to twistor theory*, Lond. Math. Soc. Student Texts vol. 4, Cambridge University Press, 1985.

[39] C.R. LeBrun, Curvature functionals, optimal metrics, and the differential topology of 4-manifolds, *arXiv*: math.DG/0404251.

[40] J.M. Lee, The ‘Ricci’ software package, http://www.math.washington.edu/ lee/Ricci.

[41] T. Leistner, Lorentzian manifolds with special holonomy and parallel spinors, *Suppl. Rend. Circ. Mat. Palermo* **69** (2002), 131–159.

[42] T. Leistner, Towards a classification of Lorentzian holonomy groups, in 2 parts, *arXiv*: math.DG/0305139 and *arXiv*: math.DG/0309274.

[43] F. Leitner, Conformal holonomy of bi-invariant metrics, *arXiv*: math.DG/0406299

[44] J. Maldecena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2** (1998), 231–252

[45] V. Matveev, Projective Lichnerowicz-Obata conjecture, *arXiv*: math.DG/0407337.

[46] T. Morimoto, Geometric structures on filtered manifolds, *Hokkaido Math. J.* **22** (1993), 263–347.

[47] P. Nurowski and G.A.J. Sparling, Three-dimensional Cauchy-Riemann structures and second-order ordinary differential equations, *Class. Quant. Grav.* **20** (2003), 4995–5016.

[48] G. Schmalz and J. Slovák, The geometry of hyperbolic and elliptic CR-manifolds of codimension two, *Asian J. Math.* **4** (2000), 565–598.

[49] J.A. Schouten and J. Haantjes, Beiträge zur allgemeinen (gekrümmten) konformen Differentialgeometrie I, II, *Math. Ann.* **112** (1936), 594–629 and **113** (1936), 568–583.

[50] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, *Japan. J. Math.* **2** (1976), 131–190.

[51] T.Y. Thomas, On conformal geometry, *Proc. Nat. Acad. Sci.* **12** (1926), 352–359.

[52] T.Y. Thomas, Conformal tensors I, *Proc. Nat. Acad. Sci.* **18** (1932), 103–112.

[53] M.Y. Wang, Einstein metrics from symmetry and bundle constructions. In *Essays on Einstein manifolds*, Surv. Diff. Geom. **VI**, 287–325, Int. Press, 1999.

[54] E. Witten, Anti de Sitter space and holography, *Adv. Theor. Math. Phys.* **2** (1998), 253–291.

Chapter 22

Stochastic Processes in Evolutionary and Disease Genetics (04w5015)

August 7–12, 2004

Organizer(s): Ellen Baake (Universität Bielefeld), Don Dawson (Carleton University), Warren Ewens (University of Pennsylvania, Philadelphia), Bruce Rannala (University of Alberta)

General Overview of the Field and its Developments

The broad subject area of the meeting was that of *mathematical population genetics*. It is concerned with the analysis of the generation, nature, and maintenance of genetic variation within and between biological populations. In its evolutionary aspects it describes the change in the genetic composition of populations under the influence of various evolutionary forces, the most important of which are mutation, selection, recombination, migration and random genetic drift. The latter is a consequence of the fact that even without fitness differences, some individuals may, just by chance, have more offspring than others, so that the offspring of one genotype may displace another one in a finite population. Thus there is a significant element of randomness in genetic systems. From the point of view of disease genetics, many diseases are caused by deleterious mutant genes, and the analysis of the variation in a population for the disease and the normal gene is a significant component of this area of research.

These two components of the theory have hitherto been somewhat separate. However, recent trends in evolutionary genetics theory have brought them together, and one of the aims of this meeting was to further this fusion of two important areas of population genetics.

Three new developments are shaping the area at present: a change in biological thinking, the emergence of new data, and new mathematic(ian)s; these are, of course, all interrelated. Let us explain this in some more detail.

The basic processes of evolution are known in principle, along with fundamental equations which describe the effects of interactions between genes. Indeed, of the biological sciences, genetics is the one with the most clearly defined mathematical models. The evolutionary behavior of a population may be described by a stochastic model of gene frequency change, which is similar to corresponding models in interacting particle systems. These models are well understood if mutation and drift are the only forces present, or if selection is also present but acts on the genes at one or a small number of gene loci. In particular, the pattern of genetic variation generated under these scenarios is quite well known.

But for several decades, the area suffered from a lack of data to support - or reject - the hypotheses about the evolutionary process and the genetic basis of various diseases. It was therefore often criticized as “l’art pour l’art”. This situation has suddenly been reversed due to the wealth of molecular data flowing in during the past few years. The data come from the various genome projects, and from studies aimed at the genetic basis of human diseases. The most valuable data derive from samples from a population (population sequence data), as opposed to single individuals. For the first time in the history of the field, the theory is now lagging behind the data, and the lack of analytical results translates into a lack of statistical methods for data analysis. The immediate need of data evaluation methods is often satisfied by heuristic techniques of a preliminary nature. But in the long run, there is a real need for methods which rest on a solid foundation with respect to the underlying genetic stochastic processes.

Evolutionary genetics theory has thus moved in large part to an analysis of the corresponding inverse problem, namely the reconstruction of evolutionary history from the observed patterns of genetic variation. A particularly challenging problem is the detection of selection at the molecular level. Selective forces are not easy to analyze since their effects must be discerned against a background of stochastic effects. Thus the analysis of the genetic data used to assess the effects of selection presents particularly difficult statistical problems, which have no entirely satisfactory answer even today.

The theoretical foundations of such analyses have been laid by the change in direction in population genetics from the classical prospective theory, considering the evolution of a population forwards in time, to the retrospective theory, which considers the past history of the currently-observed population. Mathematically, the backward view corresponds to the dual of the forward process. Coalescent theory, the most frequently used area of the retrospective theory, is concerned specifically with properties of the ancestry of a sample of genes as they trace back to a common ancestor. If, for example, a disease mutation occurs only once, two or more disease genes in a contemporary sample have an ancestry that traces back to a most recent common ancestor disease gene.

Problems beyond those listed above are far more complex, have not been solved, and will require significant mathematical analysis for their resolution. This is particularly so if processes like recombination are included, or if selection acts in a complicated way. These are exactly the problems encountered in disease genetics. Significant properties of the disease locus itself are in practice often unknown, including its location - indeed a major aim of the theory is to attempt to locate it. Inferences about its location are made by using genes at known marker loci. This leads to the problem that the coalescent process of the disease gene is different from that of the markers, because of recombination between disease and marker loci. The situation is further complicated by the fact that diseases are often polygenic (and possibly under selection, which may be of complicated type due to interaction between loci). Such diseases are called complex diseases, and their study forms the center of current genetic investigation.

Another important development is the increasing interest of the mathematical community in theoretical biology in general, and genetics in particular. Many professional probabilists have recently moved into the area, with powerful modern methods at their fingertips, which has helped to turn the mathematics of biological evolution into an active and growing field. This has resulted in a productive interplay in which the problems of population biology have stimulated new mathematics which in turn has provided powerful new analytical tools to address the emerging problems of biological evolution. In particular there have been important developments in the theory of Markov processes stimulated by population biology - these include the introduction of important families of interacting particle systems and the class of measure-valued Fleming-Viot processes including the infinitely many types and infinitely many sites processes that now play an important role in population genetics. A number of effective mathematical tools for the analysis of these systems have been developed. Other mathematical tools will be described below, where appropriate.

After this general overview, let us now give a more detailed description of the various matters described above, as discussed at the meeting. It had six main topics, each led by a key speaker: Particle systems (Rick Durrett), The Coalescent (John Wakeley), Evolutionary population genetics (John Gillespie), Branching processes (Peter Jagers), Human genetics (Robert Elston), and Haplotype blocks (Peter Donnelly). We will proceed from the more theoretical to the more applied, and put special emphasis on the connections between the topics.

Particle Systems and Coalescent Process

A fundamental model class in population genetics is defined by the Moran model and its relatives (and the closely related Wright-Fisher model with its variants). This is best described as an interacting particle system with a fixed number of N individuals, each of which is assigned a type; individuals reproduce and mutate independently, in discrete or continuous time. Every time an individual reproduces, the offspring is assigned a type (according to a Markov chain that describes mutation), and replaces another individual that is randomly chosen to die, thus keeping population size constant. If N gets large, the system is described by a diffusion limit, known as the Fleming-Viot measure-valued process. A very general particle representation that also remains valid in this limit is provided by the so-called look-down process, which yields a joint representation of particles and their genealogies [4].

From an evolutionary perspective, it is of particular interest to consider these particle systems *backward* in time. Given a sample from the present population, one aims at finding its genealogy. Here, a reproduction event forward in time corresponds to merging of individual lineages to a common ancestor backward in time, that is, a coalescence event. Since its invention by Kingman [22], the coalescence process has revolutionized population genetic thinking and data analysis.

This coalescent process is tractable and has been much studied in the case of neutral evolution, that is, all types of individuals have the same reproduction rate (this is the 'vanilla-flavoured' coalescent). The emphasis of current research is on the extension of the underlying ideas and methods to more complex systems involving selection, recombination, migration, and variable population size.

The mathematical description of the process becomes a great challenge when types have different reproduction rates, that is, if selection is involved. This situation is particularly relevant for many questions in molecular evolution, in particular, when one wants to infer the (most likely) evolutionary history from a sample of individuals of a present-day population, and pinpoint selective events that have happened in the past.

One major step has been the construction of Neuhauser and Krone [24, 26], which uses, forward in time, two different reproduction events: definitive ones that will be used by every individual regardless of its type, and potential ones that may only be used by 'fit' individuals. Backward in time, this now induces a coalescing/branching structure, where the branching events correspond to unresolved birth events, meaning that the ancestry here may only be decided in a second step, when the types of the ancestors have been resolved. This process is rather complex, but some explicit results may be obtained, with considerable technical effort, about the time to the most recent common ancestor, for example.

The process becomes much simpler if, rather than full geneologies, only the ancestral lines of single individuals are considered. This seems to have been overlooked for quite some time; some explicit results (for two types) have recently appeared [13].

If more than two types are considered, explicit analytical results seem out of reach at present, and even simulation of the backward coalescent is a challenge. The 'first generation' simulation algorithms require sampling from the stationary distribution, which is, however, known only for the unrealistic case of parent-independent mutation. Recently, however, some progress could be made through *exact sampling* algorithms [10]. They do not require explicit knowledge of the stationary distribution and are, therefore, more generally applicable.

A second important direction concerns the inclusion of spatial structure into the Moran model and the resulting coalescent. A popular model here is the *stepping-stone model*, where individuals perform a random walk on a one- or two-dimensional lattice (or torus, in a mathematical idealization). Genetic structure may then be analyzed through the homozygosity as a function of the separation of the colonies, and a genetic distance known as FST (fixation index of subpopulations relative to the total population). Various limits, depending on the scaling of migration rate, subpopulation size and number of subpopulations, must be considered. Recent results include the logarithmic growth of FST with the number of colonies, the identification of parameter regimes where the stepping-stone model is effectively panmictic, the structure of genealogies, the effect of migration on the mutation patterns expected under the infinite-sites model, and the additional effect of selection [2, 5, 33, 32].

Another important line of development concerns the assumption of constant population size, which is a severe limitation. Traditionally, variable population size has been treated with the help of the concept of *effective population size*; but a certain confusion has been associated with this notion. Recently, this has been analyzed within the framework of the coalescent [30]: Having identified conditions under which a model with stochastic demography converges to the coalescent with a linear change in time scale, Sjodin et al. [30] have argued that this is a necessary condition for the existence of a meaningful effective population size. Such a linear time scale change is obtained when demographic fluctuations and coalescence events occur on different time scales.

Branching Processes

Branching processes have a long history in population genetics theory. They were first used by Wright to determine the fate (fixation or loss) of a rare mutant within a finite population (for review, see [9, p. 27ff]); for this purpose, a single-type Galton-Watson process is relevant. Recently, multi-type branching processes have been used in the context of mutation-selection models for large populations [18]; here, as in the coalescent process, the view back in time has become important, and earlier results by Jagers et al. [21] can now be used to investigate the relationship between the forward and the backward process, and the present and ancestral distribution of types, respectively.

But coalescent and branching processes have more in common than the backward view along single lines. Motivated by an earlier meeting on mathematical population genetics, Geiger [17] has recently started to investigate an analogue to the coalescent for branching processes. If k individuals are sampled uniformly at random from one generation of a large Galton-Watson population that has persisted for n generations, then the shape of the subtrees spanned by the sampled vertices and the root depends essentially on the tail of the offspring distribution: While in the finite variance case the subtrees are asymptotically binary (as $n \to \infty$), multiple branch-points do persist in the limit if the variance of the offspring distribution is infinite.

Apart from concrete questions like this one, branching processes and particle systems can also be subsumed under the general framework of particle systems and look-down processes mentioned earlier [4].

Evolutionary Genetics

An important topic in modern evolutionary genetics is the identification of selective events in the history of a sample from the patterns of genetic variation observed in a present-day population. This is often done by means of the so-called *hitchhiking effect*, namely the fact that the fixation of a strongly selected beneficial mutation is accompanied by the increase of variants at other loci linked with the beneficial mutation. This effect leaves numerous signatures of diversity in DNA sequences, both within and between species, and affects the frequency spectrum of alleles, as well as linkage disequilibria and codon bias. Depending on whether there has been a single (recent) hitchhiking event or several repeated ones, the effects may be local or over a broader range. By comparing theoretical predictions with actual sequence data, one can infer the rate and strength of beneficial mutations in nature (among the many references available, see [23] for a recent example).

The hitchhiking effect has recently entered the level of large-scale analysis of SNP data. SNP's, '*single nucleotide polymorphisms*', are single nucleotide sites that are polymorphic in a population. Much effort is devoted to the problem of detecting selective sweeps using large SNP data sets from genomic scans. However, special care must be taken to overcome the ascertainment problem: Most population genetical methods do not correctly accommodate the special discovery process used to identify SNPs, which results in biased allele frequency distributions that must be corrected for [27].

Last but not least, our traditional understanding of the interplay of selection and genetic drift is challenged by the *pseudohithhiking model* proposed by Gillespie [19]. Strongly selected substitutions at one locus can induce stochastic dynamics that resemble genetic drift at a closely linked neutral locus. The pseudohitchhiking model is a one-locus model that approximates these effects and can be used to describe the major consequences of linked selection. The coalescent of the pseudohitchhiking model has a random number of branches at each node, which leads to a frequency spectrum that is different from that of the equilibrium neutral model. If *genetic draft*, the name given to these induced stochastic effects, is a more important stochastic force than genetic drift, then a number of paradoxes that have plagued population genetics disappear – but, at the same time, the estimation procedures commonly employed in genetic analyses may be estimating parameters other than those that are assumed.

Apart from its impact on population genetics, this approach is also having a significant impact on mathematical research. Since the model relies on *strongly* selected mutants, the usual diffusion limit and associated coalescence theory is not applicable. Durrett and Schweinsberg [6] have approximated it with the help of random partitions created by a stick-breaking process, and Etheridge, Pfaffelhuber and Wakolbinger have modelled the ancestry at the neutral locus by means of a structured coalescent in a random background, and derived a corresponding sampling formula [8].

Recombination and Haplotype Blocks

Recombination is the formation of a chromosome passed on by a parent to an offspring by physical exchange between the two parental chromosomes, so that the transmitted chromosome consists of parts of each of the two parental chromosomes. There has been much recent speculation (based on patterns of genetic variation), and occasionally experimental confirmation (via sperm typing), that rates of recombination across the human genome vary on a fine scale. In particular, some regions of the genome appear to contain *recombination hotspots*, where recombination occurs at rates several times higher than the background average rate. Aside from inherent interest, an understanding of this local variation is essential for the appropriate design and analysis of many studies aimed at elucidating the genetic basis of common diseases or of human population histories. Standard pedigree-based approaches do not have the fine scale (< 0.1 cM) resolution that is needed to address this issue, because thousands of meioses are needed per recombination event. In contrast, samples of DNA sequences from unrelated chromosomes in the population carry relevant information, as there are a large number of meioses in the history of a sample of population data. But inference from such data is extremely challenging in several respects: the underlying stochastic model (the coalescent with recombination, a process that is practically intractable in the full-fledged version required here), the statistical analysis, and the computational requirements.

Although there has been much recent interest in the development of full likelihood inference methods for estimating local recombination rates from such data, they are not currently practicable for data sets of the size being generated by modern experimental techniques. Fearnhead and Donnelly [11, 12] introduced and studied two approximate likelihood methods. The first, a marginal likelihood, ignores some of the data. For larger sequences, they introduced a *composite likelihood*, which approximates the model of interest by ignoring certain long-range dependencies. With a combination of both methods, data from the lipoprotein lipase gene have been analyzed. A different approach was pursued by Li and Stephens [25], who have related the patterns of genetic variation to the underlying recombination process through their PAC model (product of approximate conditionals). This method has already been applied to two problems: determining whether a recombination hotspot identified in human males via sperm typing is also present in chimps; and quantifying the frequency of recombination hotspots in genes in the human genome.

Closely related to the local variation of recombination rates is the concept of haplotype blocks. A haplotype block is a region of a chromosome that tends to be passed on intact, without recombination, from parent to offspring. Partly as a result of this, the region of a chromosome corresponding to a block tends to exhibit only a few haplotypes in the entire population. Identification of haplotype blocks is a way of examining the extent of linkage disequilibrium in the genome, which generally provides useful information for the planning of association studies in human genetics (see the next Section). The aim is to identify a minimal subset of SNPs that can characterize the most common haplotypes. No uniform definition of a haplotype block has yet been agreed upon; however, various operational definitions are in use, see, e.g., Daly et al. [3]. The *Hap-Map project* (http://www.hapmap.org/index.html.en) describes haplotype blocks in the human genome. Particular interest is in the question whether there is similar haplotype block-structure between and within populations (Nigeria/Yoruba, Asia, African Americans, Europeans), see, e.g., [16].

Human Genetics

In human genetics, finding genes underlying heritable traits has been a long-standing problem. In recent years attention has shifted from ‘Mendelian’ disorders (that is, diseases caused by one defective gene, such as Huntington’s disease or cystic fibrosis) to so-called complex traits, which are thought to be influenced by multiple genes possibly interacting with each other and with environmental risk factors. Many of these are common diseases, like diabetes.

As mentioned above, inference relies on the association with known marker genes, i.e., on linkage disequilibrium; this association is complete if there is no recombination between disease and marker locus, and decreases with distance (i.e. recombination rate) between them, thus giving a method of estimating this distance.

On a finer scale, the coalescent-based methods of the previous Section are the methods of choice; but for larger distances, pedigree-based methods are more appropriate. Here, one takes advantage of one basic difference between general population genetics and human genetics, which uses family (or pedigree) data rather than population data. Observations are made on a collection of markers (usually SNP's) transmitted from parents to affected (and sometimes unaffected) offspring, and as a result an assessment can be made about which SNP's the disease gene is close to. Since the locations of the SNP's are known, inferences can be made about the location of the disease gene. This area of research is known as *linkage analysis*, is based on probability models and parametric inference, and has a very long tradition; for review, see [28]. Over the years, each major development in parametric statistical inference has been adopted by the developers of linkage analysis methods, and questions of genetic analysis have prompted new statistical developments, from the work of Fisher [15] onwards. In many ways, statistical inference and genetic analysis have developed in parallel over the last 100 years.

Currently, the field is moving from a situation in which marker typing was hard and expensive to an era where this is relatively cheap, fast and easy, and the major cost of a study of a complex trait is now in the family collection and trait phenotyping. Recent progress in sequence analysis has made available the joint analysis of thousands and even hundreds of thousands of SNP's, thus making possible genome-wide screens for disease genes. Indeed, many researchers are already taking advantage of this fact. The statistical challenge is now how to deal with the vastly larger number of variables than observations: The enormous number of genotype configurations leads to a *curse of dimensionality*. This situation is analogous to that in microarray expression analysis, where expression levels of large numbers of genes are measured on a comparatively very small number of individuals. In both cases, false positives are the main problem. This is now an area of intensive statistical research; some recent approaches are discussed in [20].

List of Participants

Baake, Ellen (University of Bielefeld)
Baake, Michael (University of Bielefeld)
Birkner, Matthias (Weierstrass-Institut fuer Angewandte Analysis und Stochastik)
Bull, Shelley (University of Toronto, Samuel Lunenfeld Research Institute)
Chang, Joseph (Department of Statistics, Yale University)
Clark, Andrew (Cornell University)
Clerget-Darpoux, Francoise (INSERM U535, Hopital Paul Brousse)
Cox , Ted (Syracuse University)
Daly, Mark (Whitehead Institute)
Dawson, Don (Carleton University)
De, Arkendra (Cornell University)
Donnelly, Peter (Oxford University)
Durrett, Rick (Cornell University)
Elston, Robert (Case Western Reserve University)
Ewens, Warren (University of Pennsylvania, Philadelphia)
Fearnhead, Paul (Lancaster University)
Geiger, Jochen (Technische Universität Kaiserslautern)
Gillespie, John (University of California, Davis)
Greenwood, Priscilla (Arizona State University)
Griffiths, Robert (University of Oxford)
Jagers, Peter (Chalmers University of Technology)
Joyce, Paul (University of Idaho)
Kim, Yuseob (University of Rochester)
Krone, Steve (University of Idaho)
Kurtz, Tom (University of Wisconsin - Madison)
Limic, Vlada (University of British Columbia)
Luo, Yuqun (Case Western Reserve University)
Möhle, Martin (University of Tübingen)
Nielsen, Rasmus (Cornell University)
Ott, Jurg (Rockefeller University)

Patterson, Nick (Broad Institute/MIT)
Popovic, Lea (University of Minnesota)
Rannala, Bruce (University of Alberta)
Siegmund, David (Stanford University)
Stephens, Matthew (University of Washington)
Stewart, William (University of Washington)
Sturm, Anja (University of British Columbia)
Taylor, Jessie (Institute for Cell, Animal and Population Biology, University of Edinburgh)
Thompson, Elizabeth (University of Washington)
Wakeley, John (Harvard University)
Wakolbinger, Anton (Universität Frankfurt am Main)
Wiuf, Carsten (University of Aarhus)
Yu, Feng (University of British Columbia)

Bibliography

[1] J.D. Balding, M. Bishop, C. Cannings (eds.), Handbook of Statistical Genetics, 2nd ed., Wiley, Chichester, 2003.

[2] T. Cox and R. Durrett, The Stepping Stone Model: New Formulas Expose Old Myths, *Ann. Appl. Prob.* **12** (2002), 1348-1377.

[3] M.J. Daly, J.D. Rioux, S.F. Schaffner, T.J. Hudson, and E.S. Lander, High-resolution haplotype structure in the human genome, *Nature Genetics* **29** (2001), 229–232.

[4] P. Donnelly and T.G. Kurtz, Genealogical processes for Fleming-Viot models with selection and recombination, *Ann. Appl. Prob.* **9** (1999), 1091–1148.

[5] R. Durrett, Probability Models for DNA Sequence Evolution, Springer 2002.

[6] R. Durrett and J. Schweinsberg, Approximating selective sweeps, *Theor. Pop. Biol.* **66** (2004), 129–138.

[7] A. M. Etheridge, Introduction to Superprocesses, American Mathematical Society, Providence, RI, 2000.

[8] A.M. Etheridge, P. Pfaffelhuber, and A. Wakolbinger, An approximate sampling formula under genetic hitchhiking, submitted. math.PR/0503485

[9] W.J. Ewens, Mathematical Population Genetics, I. Theoretical Introduction, 2nd ed., Springer 2004.

[10] P. Fearnhead, Perfect simulation from population genetic models with selection, *Theor. Pop. Biol.* **59** (2001), 263–279.

[11] P. Fearnhead and P. Donnelly, Estimating recombination rates from population genetic data, Genetics **159** (2001), 1299-1318

[12] P. Fearnhead and P. Donnelly, Approximate likelihood methods for estimating local recombination rates, J. Roy. Stat. Soc. B. **64** (2002), 657-680.

[13] P. Fearnhead, The common ancestor at a non-neutral locus, *J. Appl. Prob.* **39** (2002), 38-54.

[14] P. Fearnhead, Ancestral processes for non-neutral models of complex diseases, *Theor. Pop. Biol.* **63** (2003), 115-130.

[15] R.A. Fisher, The systematic location of genes by means of crossover observations, *Amer. Nat.* **56** (1922), 406–411.

[16] S.B. Gabriel et al., The structure of haplotype blocks in the human genome. *Science* **296** (2002), 2225-2229.

[17] J. Geiger, The tree structure of a k-sample from a large Galton-Watson generation, in preparation.

[18] H.-O. Georgii and E. Baake, Supercritical multitype branching processes: the ancestral types of typical individuals, *Adv. Appl. Prob.* **35** (2003), 83–114; q-bio.PE/0311020.

[19] J.H. Gillespie, Genetic drift in an infinite population: The pseudohitchhiking model, *Genetics* **155** (2000), 909-919.

[20] J. Hoh and J. Ott, Mathematical multi-locus approaches to localizing complex human trait genes, *Nat. Rev. Genet.* **4** (2003), 701-709.

[21] P. Jagers, Stabilities and instabilities in population dynamics, *J. Appl. Prob.* **29** (1992), 770–780.

[22] J.F.C. Kingman, The coalescent, *Stoch. Proc. Appl.* **13** (1082), 235–248.

[23] Y. Kim and R. Nielsen, Linkage disequilibrium as a signature of selective sweeps, *Genetics* **167** (2004), 1513-1524.

[24] S.M. Krone and C. Neuhauser, Ancestral processes with selection, *Theor. Pop. Biol.* **51** (1997), 210-237.

[25] N. Li and M. Stephens, (2003) Modelling linkage disequilibrium and identifying recombination hotspots using single nucleotide polymorphisms data, Genetics **165** (2003), 2213-2233.

[26] C. Neuhauser and S.M. Krone, The genealogy of samples in models with selection, *Genetics* **145** (1997), 519–534.

[27] R. Nielsen et al., A scan for positively selected genes in the genomes of humans and chimpanzees. *PLoS Biology* (2005), in press.

[28] J. Ott, *Analysis of human genetic linkage*, 3rd ed., Johns Hopkins, 1999.

[29] J. Schweinsberg and R. Durrett, Random partitions approximating the coalescence of lineages during a selective sweep, submitted.

[30] P. Sjödin, I. Kai, S. Krone, M. Lascoux and M. Nordborg, On the meaning and existence of an effective population size, *Genetics* **165** (2005), 1061-1070.

[31] M. Stephens and P. Donnelly, Ancestral inference in population genetics models with selection, *Austral. New Zeal. J. Stat.* **51** (2003), 901–931,

[32] J. Wakeley and T. Takahashi, The many-demes limit for selection and drift in a subdivided population, *Theor. Pop. Biol.* **66** (2004), 83–95.

[33] I. Zähle, T. Cox and R. Durrett, The stepping stone model, II: Genealogies and the infinite sites model, submitted.

Chapter 23

Statistical Science for Genome Biology (04w5519)

August 14–19, 2004

Organizer(s): Jennifer Bryan (University of British Columbia), Sandrine Dudoit (University of California, Berkeley), Mark van der Laan (University of California, Berkeley)

The main objective for this workshop is to facilitate the development and dissemination of statistical methods relevant to genome-scale biology, by bringing together leading researchers working at the interface between the biological and mathematical sciences. An important goal, that is reflected in our list of proposed invitees, is to include statisticians working on different aspects of statistics but all related to genomics. We have intentionally targeted areas ranging from classical statistical genetics, such as the genetic mapping of complex traits, to the emerging area of high throughput gene and protein expression analysis. It is becoming increasingly important for statisticians working in genomics to be aware of the quantitative problems and solutions related to diverse experimental platforms. Biological investigators are taking advantage of opportunities to study a system or process from several angles simultaneously and there is a growing need for quantitative methods to handle disparate data types seamlessly, for example, a holistic analysis based on sequence, expression, and molecular interaction data. An aim of this workshop is to devise appropriate statistical formulations for such analyses, which will expedite the creation and application of sound and powerful statistical methodologies.

The utility of sophisticated statistical methods in modern biology is well-established and has been addressed in more detail elsewhere in this proposal. Promoting genomics as a source of interesting and important problems is also of great benefit to the statistics research community. The fascinating developments in the biological sciences have generated an unprecedented enthusiasm in the computational sciences generally (statistics, mathematics, and computer science) by raising novel and challenging methodological questions. However, there can be significant 'barriers to entry and expansion' in this type of interdisciplinary work. This workshop would enable researching statisticians, with documented expertise and interest in genomics, to broaden their knowledge of current research and important open problems.

The question of timeliness is quite easy to address in this case. The pace at which genomic data are accumulating far exceeds the scientific community's ability to interpret it. Likewise, the emerging subspecialty in statistics – statistical genetics and genomics – is growing and changing rapidly and a workshop specifically aimed at this area is sorely needed.

List of Participants

Allison, David (University of Alabama, Birmingham)
Birkner, Merrill (University of California, Berkeley)
Brumm, Jochen (University of British Columbia)

Bryan, Jennifer (University of British Columbia)
Buhlmann, Peter (ETH Zurich)
Bull, Shelley (University of Toronto)
Bureau, Alexandre (University of Lethbridge)
Carey, Vince (Harvard University)
Dudoit, Sandrine (University of California, Berkeley)
Durbin, Blythe (University of California, Berkeley)
Fridlyand, Jane (University of California, San Francisco)
Gentleman, Robert (Harvard University/Dana Farber)
Hansen, Kasper Daniel (University of Copenhagen)
Holmes, Susan (Stanford University)
Hubbard, Alan (University of California, Berkeley)
Joe, Harry (University of British Columbia)
Jornsten, Rebecka (Rutgers University)
Keich, Uri (Cornell University)
Keles, Sunduz (University of Wisconsin, Madison)
Kendziorski, Christina (University of Wisconsin, Madison)
Kopciuk, Karen (Alberta Cancer Board/University of Calgary)
Kustra, Rafal (University Toronto)
Larkin, Jennie (The Institute for Genomic Research)
Lumley, Thomas (University of Washington)
Marin, Michael (University of British Columbia)
Molinaro, Annette (University of California, Berkeley/NCI)
Newton, Michael (University of Wisconsin)
Pachter, Lior (University of California, Berkeley)
Pollard, Katherine S. (University of California, Santa Cruz)
Purdom, Elizabeth (Stanford University)
Quackenbush, John (The Institute for Genomic Research)
Rocke, David (University of California, Davis)
Rossini, Anthony (University of Washington)
Ruczinski, Ingo (Johns Hopkins University)
Sinisi, Sandra (University of California at Berkeley)
Temple Lang, Duncan (University of California, Davis)
Trivedi, Prinal (University of Alabama, Birmingham)
Yang, Yee Hwa (University of California, San Francisco)
van der Laan, Mark (University of California, Berkeley)

Chapter 24

Dynamics, Control and Computation in Biochemical Networks (04w5550)

August 21–26, 2004

Organizer(s): Brian Ingalls (University of Waterloo), Leon Glass (McGill University), John Reinitz (State University of New York, Stony Brook), Eduardo Sontag (Rutgers University), Erik Winfree (California Institute of Technology)

This workshop brought together a diverse group of people working in interdisciplinary areas touching on molecular biology, computer science, nonlinear mathematics, control theory, and computer and biomedical engineering. Since few meetings attract individuals from all these groups, this meeting was the first opportunity that many of the participants had to meet one another. This provided for a lively and intellectually stimulating environment and enabled the forging of several new friendships and collaborations.

Cells and organisms have evolved elaborate mechanisms to carry out their basic functions. Networks of biochemical reactions are responsible for processing environmental signals, inducing the appropriate cellular responses, and sequencing internal events. The overall molecular algorithms carried out by such networks are as yet poorly understood. Recent years have witnessed remarkable advances in elucidating the components of these networks due to technological achievements. Prominent among these achievements are the means for rapid sequencing of genomes, the means for simultaneously determining the expression levels of thousands of different genes, and recombinant DNA techniques to isolate, identify, manipulate, and synthesize genetic and metabolic networks. These advances have confronted the biological sciences with massive amounts of data that require huge computational resources backed by flexible software easily used by the non-expert. All cellular and molecular biologists now use sophisticated computer-based algorithms to identify and analyze DNA and protein sequences

The workshop was designed to address a range of questions that go beyond the development of algorithms for the searching and analysis of genomic and protein data bases. This report summarizes some of the main themes and questions emerging from the meeting. Further details and selected reprints are available. Since the main themes of the meeting are inextricably woven together, it is hard to generate subheadings that do not overlap each other. In recognition of this, the talks of the meeting were not organized into specific themes in any given session, but presented in a “quasi-random” order. This presentations however does identify main themes – but it will be clear that these overlap and several of the presentations are relevant to many of the themes.

(i) DNA computers and nano-mechanical devices (Seeman, Yurke,Winfree). The function of DNA as the primary means for storing genetic information is well known. In order to accomplish the biological tasks, there are a vast range of molecules and organelles that have evolved to carry out the requisite biological functions. In recent years, this growing knowledge about DNA function and chemistry has been exploited in a variety of ways. An early paper by Adleman demonstrated how to use DNA recombinant technology to solve combinatorial problems such as the Hamiltonian path problem. At this conference, there were demonstrations of the further and growing power of DNA technology. DNA is providing a new basis for nanotechnology. By controlling the base sequence in DNA and the sequences of DNA bases to generate sticky ends of molecules, DNA can assemble in a huge variety of geometrical shapes and patterns in 2 and 3 dimensions. For example, Winfree showed how it is possible to build the fractal shape of the Sierpinski gasket from the self-organization of specially constructed DNA, Seeman generated polytopes out of DNA molecules, Seeman and Yurke build nanorobotic devices that such as walkers and tweezers. Theoretical studies by Winfree and others have demonstrated that tiling can be used to build a Turing machine. Now the first steps are being taken to use tiling to carry out rudimentary computations such as XOR computations (Seemans) or counting using binary logic (Winfree). DNA might also be used to build scaffolds that could then be used to house molecules for further analysis. Crystals with dimensions up to a millimetre ordered to 10 angstrom resolution (as determined by X-ray diffraction) can be produced. Seeman proposed that this should be able to produce high resolution crystals of DNA host lattices with heterologous guests, leading to well-ordered bio-macromolecular systems amenable to diffraction analysis. Fundamental questions involve developing ways to design the DNA molecules to carry out desired tasks in a robust fashion, developing an understanding of the way to program tiling algorithms to carry out specified computational tasks, and determining ways to correct non-desired dynamic behaviour on a molecular scale.

(ii) Genetic networks. Gene expression is partially regulated by transcription factors binding to control regions of the genome. The logic of the control of expression is difficult to work out. The problem of genetic control was addressed from several perspectives. Highly idealized models composed of Boolean devices that update at discrete times were used to construct classes of networks whose dynamic properties could be studied and analyzed (Kauffman, Peterson). There is evidence that real biological systems are only restricted to some subclasses of the possible boolean functions. By studying the dynamics properties of ensembles of networks composed of such devices Kauffman is defining new classes of observations that could potentially be tested experimentally in real systems using emerging technologies. For example, it would be of interest to test the modification of the global gene expression induced by the knockout or knocking of a single gene. Such results would reflect the global structure of the genome that might well be captured by very simple models. The notion that real genes may be modelled by simple rules was a concept that ran through a number of the talks. Here are several examples. Jaeger and Kozlov described methods used to determine the structure of gene networks using equations based on neural networks in Drosophila and demonstrated that these networks could generate observed patterns of gene expression. Drosophila was also addressed by Louis, who presented a model of the genetic control circuitry responsible for sex-determination. Stojanovic applied modular design to construct molecular logic gates with oligonucleotides as both inputs and outputs. These gates could be arranged in circuits that autonomously perform complex Boolean calculations. Edwards demonstrated general mathematical properties of differential equations modelling genetic circuits with switch-like control but operating in continuous time, and Glass demonstrated how electronic circuits (modelling the equations discussed by Edwards) could evolve novel dynamics. General issues of computability were addressed more abstractly by Maass, while Prusinkiewicz presented an algorithmic framework for simulating growth driven by genetic control. Oudenaarden and Bolouri emphasized the roles of functional modules and design motifs that are the building blocks for the entire cellular network. Oudenaarden showed how positive feedback loops can be used to store information and generate steep switches, and presented both theoretical models and experiments on natural and synthetic genetic networks in the bacterium Escherichia coli and the budding yeast Saccharomyces cerevisiae. Finally, Bolouri described efforts being made to develop software to help analyze and simulate real networks. Underlying the talks about the genetic networks are questions relating to the relative roles of simplified models versus realistic models. Though simplified models have transparency that can in some cases lead to strong theoretical results, as in the presentations by Kauffman and Edwards, the relevance of these results to real networks is completely open. It is necessary to figure out ways to analyze and deconstruct real networks that have some transparency, but which are sufficiently complex that they capture the relevant properties and dynamics of the real networks.

(iii) Engineered Genetic Circuits. Our understanding of genetic networks has progressed to the point where it is possible to design and construct circuits which serve a particular purpose. Successful designs have included a switch and an oscillator. Much current work is focused on reliable implementation of interconnections of such components. Weiss designed and built simple genetic circuits, implemented in bacteria, that could generate a pulse when the appropriate stimulus was given. Issues of computing with minimal genetic circuits were addressed by Kim. Swain and Elowitz presented work on characterizing the sources of randomness in genetic networks, with the aim of improving design procedures for engineered genetic circuits. Improved design was also dealt with on an experimental level by Surette, who discussed mechanisms of genetic regulation. Endy presented a novel approach to biological design. Work in his laboratory aims to modify existing organisms to make them more amenable to study and control.

(iv) Signal Transduction Networks. Cells receive information about their external environment through proteins situated in the cell membrane. This information is then processed by signal transduction networks which are responsible for integrating the incoming signals and eliciting the appropriate cellular response. These networks function in a highly dynamic manner, constantly updating to represent the current environment. Analysis of such systems can only be carried out through the use of mathematical tools designed for addressing dynamic behaviour. Such analyses were presented by Levchenko, who presented a model of cross-talk among various signal transduction pathways and by Chaves, who demonstrated a model for the receptor-ligand binding which sets such pathways in motion. Elston presented an analysis of the mitogen-activated protein kinase cascade, a ubiquitous module in signal transduction architecture. He addressed the importance of stochasiticty in the behaviour of these networks. Ferrell discussed bistability and other phenomena in cell cycle models. Othmer chose signal transduction networks to illustrate work on robustness in biochemical networks. This is a central issue and was touched upon in a number of presentations.

(v) Control-Theoretic Analysis of Biochemical Networks. The regulation of cellular activity is achieved through a complex network of interacting components. The reverse-engineering of these self-regulating systems can be aided by application of the theory that has been developed for the design and synthesis of automatic feedback systems, namely the theory of systems and control. A number of the participants presented work in applying expertise in control to problems of biochemical regulation. Some of this work addresses particular systems – identifying the architecture of key regulatory components and elucidating their interconnections. Such work was presented by Iglesias (chemotaxis of the slime mold Dictyostelium discoidium) and Khammash (heat shock response in E. Coli). Other work addresses theoretical tools which can be used in the study of such systems. Doyle presented a sensitivity analysis of oscillating systems using sophisticated mathematical tools, while Ingalls demonstrated a control-theoretic interpretation of a biochemically-inspired sensitivity analysis. Work on identification and interpretation of stochasticity in biochemical systems was presented by El-Samad. Sepulchre presented results on stability of coupled oscillators which has application to interconnected biochemical systems.

(vi) Monotone Systems. The theory of monotone systems provides one of the foundations of dynamics in mathematical biology, and Gedeon's talk described applications to gene regulation models. A recent extension of the theory allows for the consideration of "open" systems with inputs and outputs, and the talks by Sontag and Enciso focused in these extensions, which allow one to reconstitute the behaviour of certain large scale systems by decomposing them into smaller and easier to analyze input/output components. These two talks covered small-gain and multistability theorems, as well as several applications to gene and protein network models.

List of Participants

Bolouri, Hamid (Institute for Systems Biology)
Chaves, Madalena (Aventis)
Davidson, Carla (University of Calgary)
Dawes , Adriana (University of British Columbia)
Doyle, Frank (University of California, Santa Barbara)
Edwards, Roderick (University of Victoria)
El-Samad, Hana (University of California, Santa Barbara)
Elowitz, Michael (California Institute of Technology)
Elston, Timothy (University of North Carolina)
Enciso, German (Rutgers University)
Endy, Drew (Massachusetts Institute of Technology)
Ferrell, James (Stanford University)
Gedeon, Tomas (Montana State University)
Glass, Leon (McGill University)

Heuett, Will (University of Washington)
Iglesias, Pablo (Johns Hopkins University)
Ingalls, Brian (University of Waterloo)
Jaeger, Johannes (State University of New York, Stony Brook)
Kauffman, Stuart (Santa Fe Institute)
Khammash, Mustafa (University of California, Santa Barbara)
Kim, Jongmin (California Institute of Technology)
Kozlov, Konstantin (St. Petersburg State Polytechnical University)
Levchenko, Andre (Johns Hopkins University)
Louis, Matthieu (Rockefeller University)
Maass, Wolfgang (Technische Universität Graz)
Mincheva, Maya (University of Lethbridge)
Narang, Atul (University of Florida)
Othmer , Hans (University of Minnesota)
Peterson , Carsten (Lund University)
Prusinkiewicz, Przemyslaw (University of Calgary)
Seeman, Nadrian (New York University)
Sepulchre , Rodolphe (Université de Liège)
Sontag, Eduardo (Rutgers University)
Stojanovic, Milan (Columbia University)
Surette, Michael (University of Calgary)
Swain, Peter (McGill University)
Van Oudenaarden, Alexander (Massachusetts Institute of Technology)
Weiss, Ron (Princeton University)
Winfree, Erik (California Institute of Technology)
Yurke, Bernard (Bell Labs/Lucent)

Chapter 25

Combinatorial Hopf Algebras (04w5011)

August 28–September 2, 2004

Organizer(s): Frank Sottile (Texas A&M University), Nantel Bergeron (York University), Louis Billera (Cornell University), Stephanie van Willigenburg (University of British Columbia)

Recent developments have linked heretofore distinct subjects within combinatorics, algebra, geometry, and theoretical physics thereby uncovering exciting new avenues for research. Our workshop would bring together experts in these newly linked subjects, many of whom have not previously interacted, to focus attention on these topics. We expect to follow a leisurely pace at the workshop, reserving time for consultation among the participants. The talks would concentrate on new developments and open problems, serving to define this area and its future directions.

An old theme in algebraic combinatorics initiated by Rota is that many combinatorial objects possess natural product and coproduct structures. Enumeration and classification of these structures often give rise to an associated graded Hopf algebra. This theme has matured with recent work of Ehrenborg, Aguiar, and others, who give a natural Hopf morphism from these Hopf algebras of combinatorial objects to the Hopf algebra of quasi-symmetric functions Qsym. This morphism arises from a universal property of the Hopf algebra Qsym as a terminal object in the category of graded Hopf algebras equipped with a zeta-function. In particular, many enumerative combinatorial invariants (among them flag f-vectors, Littlewood-Richardson coefficients, and chromatic symmetric polynomials) are obtained from this universal property. In light of this, the Hopf algebra Qsym and its sub-structures are bound to play a very important role in algebraic combinatorics.

The Hopf algebra Qsym of quasi-symmetric functions was introduced by Gessel as a source of generating functions for Stanley's P-partitions. Since then, quasi-symmetric functions have appeared in many combinatorial contexts. For example, Gessel showed that multiplication in Qsymm is given by a shuffle product, giving showing that the descent sets of all shuffles of two sequences depends only on the decent sets of the sequences being shuffled. The relation of Qsym to the ring of symmetric functions was first clarified by Malvenuto and Reutenauer via graded Hopf duality to the Solomon descent algebras, and then Gelfand, et al., defined the graded Hopf algebra NC of non- commutative symmetric functions and identified it with the Solomon descent algebra. Recently, there is a growing interest in symmetric and quasi- symmetric functions and their non-commutative analogs.

The dual pair NC and Qsym of Hopf algebras are only the first of a growing number of combinatorial Hopf algebras which possess interesting structures and are related to NC and Qsym. For example, Malvenuto and Reutenauer studied a non-commutative graded Hopf algebra whose basis is all permutations having a natural map to Qsym, and which contains NC as a subalgebra. This map factors through a Hopf algebra of planar binary trees studied by Loday and Ronco, and others. Loday and Ronco discovered the notion of dendriform algebra studying this Hopf algebra. Other combinatorial Hopf algebras possess interesting operadic structures, some of which have been elucidated by Loday and Chapoton. Brouder and Frabetti studied a related (actually isomorphic) Hopf algebra of planar binary trees, linking it to renormalization of Quantum Electrodynamics. This is similar to, but different from, the Hopf algebra of rooted trees of Connes and Kreimer which encodes some renormalization in Quantum Field Theory. Relations between these algebras have yet to be fully studied, and we believe that special structures of these algebras will have relevance to the original combinatorics we began with.

To give one important example, the connections uncovered by the work of Aguiar suggest a number of exciting possibilities. This includes a unified approach to positivity questions in different realms of enumerative combinatorics or using these results as a bridge to transfer ideas and techniques between these heretofore different realms. For example, both the Schubert calculus and polytope/poset theory have important open positivity questions and well-developed techniques to study these (Schensted insertion and geometry/representation theory for the Schubert calculus and shelling/homology of posets for polytopes and posets). It would be very fruitful to connect these techniques or to apply them to other areas.

For example, Billera and Liu considered elements of the algebra NC as flag- enumeration functionals on all graded posets, and they defined a quotient E of NC consisting of all such functionals on Eulerian posets. Bergeron, Mykytiuk, Sottile and van Willigenburg showed that the algebra E is dual to Stembridge's algebra Pi of peak quasi-symmetric functions. More precisely, they showed that both algebras have natural coproducts that make them into Hopf algebras, and that these Hopf algebras are, in fact, dual. This duality links the study of the enumerative properties of Eulerian posets, including associated geometric objects such as spheres, convex polytopes and hyperplane arrangements, with that of Stembridge's enriched P-partitions and related questions having to do with peaks and shuffles in permutations. More recently, Billera, Hsiao, and van Willigenburg showed a direct connection between the cd-index for Eulerian posets and the standard basis for the algebra Pi, giving a direct link between natural non-negativity questions in each field. This line of study also shows an interesting connection between the map relating "descents" to "peaks" and the classical theory of Zaslavsky relating enumeration in geometric lattices to that of the regions in an associated arrangement of hyperplanes. This is yet to be completely understood. They also showed this map to define a random walk on the collection of all peak sets whose stationary distribution is the distribution of peak sets in the symmetric group. This is a direct outgrowth of a shuffle interpretation for multiplication in Pi.

Another exciting line of study is the investigation of Qsym (in finitely many variables) considered as a subalgebra of the polynomial algebra. In recent work, Aval and Bergeron and Bergeron have shown that the quotient of the polynomial ring over the ideal generated by Qsym is linked to Catalan numbers. This new investigation promises a wealth of research as it is related to the analogous quotients in invariant theory involving symmetric polynomials. The later are intensively studied in geometry and representation theory as they encode cohomology of various (partial) flag manifolds. Its generalization leads to the famous n!-conjecture of Garsia and Haiman, in relation to the positivity conjecture of the Macdonald symmetric functions. Haiman recently proved these conjectures. It would be extremely stimulating to construct an analogous theory for the quasi-symmetric functions.

Because the work connecting these areas is quite recent, its implications for future research have only just begun to be disseminated within the research community. Our workshop would foster deeper connections and enable future collaborations between researchers in these various areas.

List of Participants

Aguiar, Marcelo (Texas A&M University)
Aval, Jean-Christophe (Université de Bordeaux I)
Bayer, Margaret (University of Kansas)
Bergeron, Francois (Université du Quebec a Montréal)
Bergeron, Nantel (York University)
Billera, Louis (Cornell University)
Cartier, Pierre (Institut Mathematique de Jussieu)
Chapoton, Frédéric (Institut Girard Desargues, Lyon 1)
Ebrahimi-Fard, Kurusch (Institut des Hautes Études Scientifiques)
Ehrenborg, Richard (University of Kentucky)
Frabetti, Alessandra (Institut Girard Desargues, Lyon 1)
Garsia, Adriano (University of California, San Diego)

Gessel, Ira (Brandeis University)
Grossman, Robert (University of Illinois, Chicago)
Guo, Li (Rutgers University)
Hazewinkel, Michiel (Centrum voor Wiskunde en Informatica)
Hetyei, Gabor (University of North Carolina, Charlotte)
Hivert, Florent (Université de Marne-la-Vallée)
Hohlweg, Christophe (The Fields Institute)
Holtkamp, Ralf (Ruhr-Universität Bochum)
Hsiao, Samuel (University of Michigan)
Kapetanovic, Selma (York University)
Lauve, Aaron (Rutgers University)
Loday, Jean-Louis (Institut de Recherche Mathématique Avancée)
Moreira, Walter (Texas A&M University)
Novelli, Jean-Christophe (Marne la valle)
Nyman, Kathryn (Texas A&M University)
Pereira, Mariana (MSRI)
Readdy, Margaret (University of Kentucky)
Reading, Nathan (University of Michigan)
Reiner, Vic (University of Minnesota)
Ronco, Maria (Universidad de Buenos Aires)
Schmitt, William (George Washington University)
Schocker, Manfred (Oxford University)
Sottile, Frank (Texas A&M University)
Stembridge, John (University of Michigan)
Taskin, Muge (University of Minnesota)
Thomas, Hugh (University of New Brunswick)
Zabrocki, Mike (York University)
van Willigenburg, Stephanie (University of British Columbia)

Chapter 26

Pluripotential Theory and its Applications (04w5035)

September 4–9, 2004

Organizer(s): Len Bos (University of Calgary), Eric Bedford (Indiana University), Alex Brudnyi (University of Calgary), Al Taylor (University of Michigan)

As a relatively new field, the methods and results of Pluripotential Theory are not well known to researchers in other parts of mathematics. In the proposed workshop, it is our intention to bring together experts in Pluripotential Theory with those who might best benefit from exposure to its methods or have used similar tools in their work. Conversely, pluripotential theorists would benefit immensely through the learning of possible new applications and problems that arise from other fields. It is our hope and expectation that such a workshop will lead to fruitful cooperation across areas. The topics of the workshop will be General PSH Theory, PSH in Complex Dynamics, PSH Functions on Varieties, Multivariate Potential Theory and Multivariate Polynomial Inequalities.

List of Participants

Alan, Muhammed Ali (Indiana University)
Baran, Miroslaw (Jagiellonian University)
Bedford, Eric (Indiana University)
Blocki, Zbigniew (Jagiellonian University)
Bloom, Thomas (University of Toronto)
Bos, Len (University of Calgary)
Branker, Maritza (University of Toronto)
Brudnyi, Alex (University of Calgary)
Cegrell, Urban (Mid Sweden University)
Coman, Dan (Syracuse University)
DeMarco, Laura (University of Chicago)
Diller, Jeffrey (University of Notre Dame)
Gogus, Nihat (Syracuse University)
Guedj, Vincent (Université Paul Sabatier)
Kolodziej, Slawomir (Jagiellonian University)
Larusson, Finnur (University of Western Ontario)
Levenberg, Norm (University of Auckland)
Mau, Sione (University of Auckland)
Meise, Reinhold (Universität Düsseldorf)

Milman, Pierre (University of Toronto)
Poletsky, Evgeny (Syracuse University)
Ransford, Thomas (Universite Laval)
Rumely, Robert (University of Georgia)
Révész, Szilárd (Alfréd Rényi Institute of Mathematics)
Saff, Ed (Vanderbilt University)
Sigurdsson, Ragnar (University of Iceland)
Stawiska, Malgorzata (Purdue University)
Taylor, Al (University of Michigan)
Totik, Vilmos (University of South Florida and University of Szeged)
Vogt, Dietmar (Bergische Universitat Wuppertal)
Wiegerinck, Jan (University of Amsterdam)
Wikstrom, Frank (University of Michigan)
Yomdin, Yosef (The Weizmann Institute of Science)
Zaharyuta, Vyacheslav P. (Sabanci University)

Chapter 27

Commutative Algebra: Homological and Birational Theory (04w5027)

September 11–16, 2004

Organizer(s): Ragnar-Olaf Buchweitz (University of Toronto), Paul Roberts (University of Utah), Bernd Ulrich (Purdue University)

In this part of the proposal we outline the topics listed above and explain in more detail the current state of research and why a workshop in this area would be very useful at the present time.

1. Problems in positive and mixed characteristic

For rings of positive characteristic, the Frobenius map, which sends an element of the ring to its pth power, where p is the characteristic of the ring, is an extremely powerful tool. Many conjectures on the homological theory of Noetherian rings have been proven using this technique, and more recently these ideas have been extended in the theory of tight closure. For rings of mixed characteristic, an important line of research has been to prove these conjectures and to extend these ideas to that case.

Positive characteristic and tight closure

The theory of tight closure was introduced by Hochster and Huneke and is currently a very active area. Connections have been proven to exist between the concepts which arise naturally in this theory and classical properties of singularities by Karen Smith, Watanabe, and others, and there is now a considerable amount of interest in relations with deep properties of local cohomology. Among the current topics of research in this area are test ideals, preservation under base change, and relations with local cohomology, including work by Smith, Lyubeznik, and Singh. Much of this work is related to one of the fundamental questions, whether tight closure commutes with localization. One of the new developments which we intend to include in our workshop is the connection with mutliplier ideals, a topic we discuss in more detail in the third section.

Mixed characteristic and the homological conjectures

During the last year a major step forward was made in the homological conjectures for rings of mixed characteristic with Heitmann's proof of the Direct Summand Conjecture in dimension three. This case of the conjecture had been open and had been the subject of intense investigations for over thirty years. In addition to solving an important problem, the ideas used in the proof are related to methods using the Frobenius map and tight closure which have been shown to be very useful in studying problems in the case of equal characteristic. While Heitmann's results do not quite extend this theory to the mixed characteristic case, they do show that many of the ideas carry over and can be used successfully.

The new methods also have counterparts from an unexpected source, that of Arithmetic Geometry. The results outlined in the previous section have also been studied by Faltings, Gabber, Ramero, and others in their work on almost etale extensions with different aims but with some very similar results, including a result on the vanishing of local cohomology similar to that of Heitmann. One of the aims of this workshop is to invite mathematicians from these different areas to combine the expertise of these groups working on related questions.

Hilbert-Kunz multiplicities

The third topic is Hilbert-Kunz functions and multiplicities. These invariants are an analogue of traditional multiplicities in Algebraic Geometry and record the action of the Frobenius homomorphism on a ring in positive characteristic. They are related to local Chern classes and other arithmetic invariants. Recent results by Fakruddin and Trevedi use vanishing theorems by Haboush and Andersen to study cones over elliptic curves in positive characteristic. In addition, Monsky and his student Teixeira have found an exciting description in terms of dynamical systems on curves over finite fields that yields rationality of the Hilbert-Kunz multiplicity, and, even better, recursive formulae for the whole Hilbert-Kunz series in terms of ideal class groups. There are also striking similarities to the Hasse-Weil Zeta function in arithmetic geometry that are not yet understood.

2. Homological methods

In addition to the homological questions for rings of positive and mixed characteristic, we intend to include more general homological topics. We will concentrate upon the study of free resolutions, including the tremendous amount of work being done on resolutions of special classes of ideals, and how the structure of the resolution of the coordinate ring of a projective variety reflects the geometry or arithmetic of the variety.

There are in particular two areas of current research activity that we intend to address.

Commutative Algebra and Exterior Algebra

The first of these areas concerns the interplay between classical commutative algebra and exterior algebra. Although the underlying Bernstein-Gelfand-Gelfand correspondence, which establishes a correspondence between the derived categories of modules over a polynomial ring and a dual exterior algebra, has been known for 25 years, it is only now that the dictionary is sufficiently well understood to bear fruit on classical problems such as resolutions of ideals of points in projective space, syzygies of Veronese or Segre embeddings, Chow forms and resultants. Among the major players in this area are Avramov, Eisenbud, Herzog, S. Popescu, and Schreyer.

An exciting very recent aspect is the appearance of the Koszul complex in the study of D-branes in String Theory, and it is reasonable to expect that in the time leading up to this workshop the connections to classical theory will become clearer and provide for in depth understanding of rather surprising insights into Algebraic Geometry originating from Physics.

Vanishing results for Ext and Tor functors

A consistent thread in the development of the homological theory of commutative rings has been conditions for Ext and Tor functions to vanish. Although these topics have been studied extensively, it came as a major surprise recently that vanishing of Ext over complete intersections is symmetric in its arguments, and there are strong indications that similar results may hold over general Gorenstein rings. This active area of research, pursued by Avramov, Buchweitz, Huneke, Claudia Miller and their collaborators, is remarkable in that it exposes basic questions about the structure of ubiquitous homological functors. Some of these fundamental problems were raised a long time ago by Auslander and Reiten in connection with the so-called Generalized Nakayama Conjecture, which states that the vanishing of certain Ext modules forces a module to be projective. Recent work in representation theory suggests that deeper understanding will require the study of Hochschild cohomology and concepts related to the fine structure of derived categories and their equivalences.

3. Integral dependence and integral closures

The concepts of integral extensions and integral closures of rings are central to much of Commutative Algebra. In this part of the proposal we discuss a generalization of these concepts to ideals. This defines a closure operation for which the closure is in general larger than the tight closure discussed in the first section. Integral closure of ideals is closely related to singularity theory, the study of Rees algebras, and multiplier ideals.

Multiplier ideals and cores

Multiplier ideals are integrally closed ideals that have been defined in complex algebraic geometry using log resolutions. The theory of multiplier ideals has surprising applications in local commutative algebra, as shown by Ein, Lazarsfeld and Smith. On the other hand, Hara, Watanabe, and their coauthors have established a connection to test ideals, a characteristic p notion arising in tight closure theory. The proposed workshop will help in bringing together the different points of view in this rich subject. Multiplier ideals are also related to cores of ideals, a topic receiving a great deal of attention lately. A better understanding of cores would lead to improved versions of the celebrated Briancon-Skoda theorem and solve a conjecture of Kawamata on the non-vanishing of sections of line bundles. This surprising connection was recently discovered by Smith and her coauthors, and one can expect further exciting developments in the near future.

Rees algebras and singularities

There has been an abundance of new results about Rees algebras and their structure over the past ten years. Rees algebras are the rings in which integral dependence of ideals can be studied and they are the algebraic objects that appear in the process of resolution of singularities. Rees algebras have been used in Kawasaki's celebrated proof of the existence of Macaulifications, a weak form of resolution of singularities. Kawasaki's work, which builds on research of Goto and his school, has not been widely disseminated. The proposed workshop will serve as a forum for discussion of his ideas and techniques. Another important contribution in singularity theory is Cutkosky's work on Abhyankar's conjecture about local factorization of birational maps between nonsingular varieties. We plan to bring together many of the leaders in this subject, including Teissier who has made substantial progress on the problem of desingularization by using toric methods.

Multiplicities and computational problems

The concept of integral dependence of ideals and modules is essential in intersection theory and the theory of multiplicities. It plays an important role in equisingularity theory as well, through the work of Gaffney, Kleiman and Teissier. On the other hand there is still no efficient algorithm for computing integral closures of ideals: this computational problem has been a focus of work by Vasconcelos and his coauthors.

List of Participants

Avramov, Luchezar (University of Nebraska at Lincoln)
Bruns, Winfried (University of Osnabrueck)
Buchweitz, Ragnar-Olaf (University of Toronto)
Chardin, Marc (Centre National de la Recherche Scientifique)
Corso, Alberto (University of Kentucky)
Cutkosky, Steven Dale (University of Missouri-Columbia)
Ein, Lawrence (University of Illinois at Chigago)
Faridi, Sara (LaCIM Universite du Quebec a Montreal)
Flenner, Hubert (University of Bochum)
Goto, Shiro (Meiji University, Tokyo)
Hara, Nobuo (Tohoku University)
Hochster, Mel (University of Michigan)
Huneke, Craig (University of Kansas)
Iyengar, Srikanth (Univeristy of Nebraska-Lincoln and University of Missouri-Columbia)

Jorgensen, David (University of Texas, Arlington)
Katzman, Moty (University of Sheffield)
Leuschke, Graham (University of Toronto)
Lipman, Joseph (Purdue University)
McDermott, Moira (Gustavus Adolphus College, Minnesota)
Miller, Claudia (Syracuse University)
Monsky, Paul (Brandeis University)
Polini, Claudia (University of Notre Dame)
Roberts, Paul (University of Utah)
Sather-Wagstaff, Sean (University of Nebraska-Lincoln)
Sega, Liana (Michigan State University)
Sharp, Rodney (University of Sheffield)
Singh, Anurag (Georgia Institute of Technology)
Srinivas, Vasudevan (Tata Institue of Fundamental Research)
Ulrich, Bernd (Purdue University)
Watanabe, Keiichi (Nihon Umiversity, Tokyo)

Chapter 28

Quantum Computation and Information Theory (04w5041)

September 18–23, 2004

Organizer(s): John Watrous (University of Calgary), Richard Cleve (University of Calgary), Umesh Vazirani (University of California, Berkeley)

Various sub-fields of quantum information processing have developed, including the sub-fields of quantum information theory and quantum algorithms and complexity. The objective of the workshop is to bring together outstanding researchers from these sub-fields in order to interact with one another, to share problems and recent discoveries, to collaborate on problems these areas have in common, and to explore other connections between these areas. BIRS offers an excellent opportunity to do this.

List of Participants

Aaronson, Scott (University of California, Berkeley)
Aharonov, Dorit (Hebrew University)
Ambainis, Andris (University of Waterloo)
Bacon, Dave (Santa Fe Institute)
Cheung, Donny (University of Waterloo)
Childs, Andrew (California Institute of Technology)
Cleve, Richard (University of Calgary)
Fenner, Stephen (University of South Carolina)
Fuchs, Christopher (Bell Labs)
Gavinsky, Dmitry (University of Calgary)
Gerhardt, Heath (University of Calgary)
Gottesman, Daniel (Perimeter Institute)
Hallgren, Sean (NEC Labs)
Hoyer, Peter (University of Calgary)
Kerenidis, Iordanis (University of California, Berkeley)
Klauck, Hartmut (University of Frankfurt)
Kobayashi, Hirotada (Japan Science and Technology Agency)
Leung, Debbie (California Institute of Technology)
Moore, Cris (University of New Mexico)
Nayak, Ashwin (University of Waterloo)
Perez, Carlos (University of Waterloo)
Preda, Daniel (University of California, Berkeley)

Reichardt, Ben (University of California, Berkeley)
Roehrig, Hein (University of Calgary)
Russell, Alex (University of Connecticut)
Sanders, Barry (University of Calgary)
Schulman, Leonard (California Institute of Technology)
Scott, Andrew (University of Calgary)
Selinger, Peter (University of Ottawa)
Sen, Pranab (University of Waterloo)
Shi, Yaoyun (University of Michigan)
Smolin, John (IBM)
Szegedy, Mario (Rutgers University)
Toner, Ben (California Institute of Technology)
Vazirani, Umesh (University of California, Berkeley)
Watrous, John (University of Calgary)

Interaction of Finite Dimensional Algebras with other areas of Mathematics

04w5501Vlastimil Dlab (Carleton University), Claus Ringel (Universitaet Bielefeld), Leonard Scott (University of Virginia)September 25–30, 2004

The Workshop will concentrate on several topics reflecting a close relationship between the theory of finite dimensional associative algebras and other areas of Mathematics. It will deal, in particular, with relations between Lie theory and the representation theory of quivers. Methods concerning quivers and their representations have been used in the past 30 years extensively in order to describe the structure of length categories (Abelian categories where every object has a finite composition series) which arise very frequently not only in algebra, but also in geometry and analysis. They enable a better understanding of the indecomposable objects and allow often a definite presentation of the category by generators and relations. There are several quite surprising relations to Lie theory: first of all, several length categories play a prominent role in Lie theory such as the category $\mathcal{O}$ of categories of Harish-Chandra modules and they can be investigated successfully using the representation theory of finite dimensional algebras. And second, one may use the representation theory of special finite dimensional algebras in order to construct Lie algebras and quantum groups.

The Workshop will also reflect the latest work on quasi-hereditary algebras and their generalizations. These algebras may be viewed as generalizations of quiver algebras more appropriate for understanding the standard module/irreducible module relationship in the category $\mathcal{O}$, perverse sheaves, and categories of representations of algebraic groups in positive characteristic. They were introduced by Cline-Parshall-Scott in their efforts to understand Kazhdan-Lusztig theory, and notably developed further by work of Dlab and Ringel, both jointly and individually. Some of the most interesting generalizations of quasi-hereditary algebras, called stratified algebras, have been defined in independent (and slightly different) ways by Agoston-Dlab-Lukács and Cline-Parshall-Scott. In each case, the stratification involved occurs categorically at the derived category level and internally in terms of particular ideals. Applications include nondescribing characteristic representations of finite groups of Lie type (Du-Parshall-Scott), to pure finite group representation theory (Webb), and to new generalizations, involving non-highest weight irreducible modules, of characteristic 0 Lie algebra representation theory (Futorny-Koenig-Mazorchuk).

In other recent developments, Soergel and others have continued to push the model of finite dimensional modules of cohomology rings as a replacement for relevant perverse sheaf categories, and Ginzburg et al have announced a re-working of the Kazhdan-Lusztig + Kashiwara - Tanisaki understanding of quantum group representations at an lth root of 1 in these terms. Tilting modules, an older development originating in quiver theory, now impact many areas of representation theory, especially characteristic p, through work of Ringel, Donkin, Soergel and Andersen, for example, and have even proved useful for the study of maximal subgroups of finite groups of Lie type (Seitz, Saxl).

Also, Schur-Weyl duality that was at the heart of the introduction of quantum groups by Drinfeld and Jimbo is now understood completely in a characteristic-free quantum context (Donkin, Du- Parshall-Scott) with tilting modules as a major tool. Vesserot, generalizing work of Erdmann, has used this and Ariki's work mentioned below to provide a complete equivalence of the problems of understanding representations of Hecke Algebras representations in type A and corresponding representations of q-Schur algebras. The latter, introduced by Jimbo for physics and, independently, introduced by Dipper-James for finite groups of Lie type in nondescribing characteristic, control finite rank representations of quantum enveloping algebras.

The workshop will also focus the attention to the use of the representation theory of special finite dimensional algebras in order to construct Lie algebras and quantum groups. The exciting development in the representation theory of finite dimensional algebras in the last 30 years was based on the use of very intricate combinatorial methods (quivers, root systems, posets, integral quadratic forms). The combinatorial approach has an algebraic counterpart which leads to Hall polynomials and quantum groups. Presently, the investigations use machinery of perverse sheaves (Lusztig), methods of differential geometry (Nakajima) and Hall algebras (Ringel).

An exciting development has been the realization (by Jie Xiao and others) of non-affine Kac-Moody Lie algebras in natural terms, not just using abstract generators and relations (an unsolved problem for many years). In the affine case, especially, through the Fock space representation in type A, many new connections with known finite dimensional or finite rank Hecke algebras have been discovered by Ariki, who also proved a conjecture of Lascoux-Leclerc-Thibon in type A, and in other classical types (B and C, extended to type

D by Jun Hu) successfully parameterized positive characteristic irreducible representations.Older work of Kleshchev on the modular representation of the symmetric group, proving a conjecture of Millineux, is now seen as having broader significance in Jun Hu's work.

The main interest at present lies in an extension of these investigations to Lie algebras defined by intersection matrices or even more general data. It has been outlined already that some of the elliptical Lie algebras studied by Saito (those of type D_4, E_6, E_7, E_8) can be obtained using tubular algebras and it will be of interest to deal with those Lie algebras which arise from a general canonical algebra (or, equivalently, a weighted projective line).

All the references to "representation theory" above should be interpreted in the broadest current sense, which includes homological as well as geometrical considerations. Note that representations of finite dimensional algebras, through translation to quiver or other problems, often enter into other geometric issues beyond those strictly related to Lie representation theory. Though it is not a principal focus of our Workshop, the connections of representation theory as specific to Lie theory and Lie groups with the general theory of representations of finite dimensional algebras and of finite groups is always present in our thinking. The unification of as much as possible of Mathematics in the areas of these disciplines is an additional goal.

Thus, the scope of the applications of finite dimensional algebras will extend to algebraic geometry, automorphic forms, finite group representations and mathematical physics along the lines initiated at the 1992 Annual Canadian Mathematical Seminar at Carleton University and published by Kluwer Academic Publishers as Volume 424 of Series C: Mathematical and Physical Sciences. It should be pointed out that extensive work is in progress and that significant fresh developments are expected in these areas by 2004.

The timeliness of our proposal should be clear from the above developments. It is easy to summarize some of the remaining objectives: 1) Understanding in the broadest infinite-dimensional terms, but through finite dimensional algebras, the representations of Lie algebras in characteristic 0, and related geometric structures, such as perverse sheaves. 2) Understanding representations finite groups of Lie type in characteristic p, and related theory for the symmetric groups and other Coxeter groups important in finite groups and Lie theory. 3) The use of the representation theory of quivers and species for getting insight into the structure of Kac-Moody Lie algebras which are not of finite or affine type. 4) The use of tubular algebras in order to get insight into the structure of the so-called elliptical Lie algebras.

In addition to people mentioned explicitly above, the following names should be mentioned as potential participants - speakers. Mathas, Geck and Hiss to report on Hecke algebras and algebraic groups. Possibly also Rouquier, Broué and Rickard, as well as Putcha and Renner whose work could have been mentioned above. The same applies to Buchweitz, Crawley-Boevey and Reiten who should present the latest developments in the representation theory of quivers and its interaction with geometry. In addition, there is a quite large number of more junior mathematicians (Gruber, Brundan, Rui, Hodge, Francis to name a few) who work in the area of the Workshop and will be interested in participating; a more detailed list of them may be compiled later.

List of Participants

Agoston, István (Eötvös University)
Ariki, Sumusu (Kyoto University)
Avramov, Luchezar L. (University of Nebraska)
Berman, Stephen (University of Saskatchewan)
Brüstle, Thomas (Bishop's University)
Chen , Xueqing (University of Ottawa)
Dlab, Vlastimil (Carleton University)
Du, Jie (University of New South Wales)
Erdmann, Karin (University of Oxford)
Ginzburg, Victor (University of Chicago)
Holtmann, Angela (Universität Bielefeld)
Iyama, Osamu (Kyoto University)
Kessar, Radha (Ohio State University)
Kleshchev, Alexander S. (University of Oregon)

Koenig, Steffen (University of Leicester)
Kumar, Shrawan (University of North Carolina, Chapel Hill)
Leclerc, Bernard (Université de Caen)
Lin, Zongzhu (Kansas State University)
Linckelmann, Markus (Ohio State University)
Liu, Shiping (Université de Sherbrooke)
Malkin, Anton (Massachusetts Institute of Technology)
Mathas, Andrew (University of Sydney)
Nakano, Daniel K. (University of Georgia)
Neher , Erhard (University of Ottawa)
Parshall, Brian (University of Virginia)
Phillips, Aaron M. (University of Virginia)
Pospichal, Tomas (Carleton University)
Pucinskaite, Daiva (Universität Bielefeld)
Ringel, Claus Michael (Universität Bielefeld)
Rouquier, Raphaël (CNRS - Université Paris 7)
Schiffmann, Olivier (École Normale Supérieure)
Schmidmeier, Markus (Florida Atlantic University)
Schröer, Jan (University of Leeds)
Scott, Leonard L. (University of Virginia)
Stroppel, Catharina (University of Glasgow)
Tanisaki, Toshiyuki (Osaka City University)
Turner, Will (University of Bristol)
Wang, Weiqiang (University of Virginia)
Webb, Peter (University of Minnesota)
Xiao, Jie (Tsinghua University)

Chapter 29

Self-Stabilizing Distributed Systems (04w5531)

October 2–7, 2004

Organizer(s): Lisa Higham (University of Calgary), Anish Arora (Ohio State University), Faith Fich (University of Toronto), Maurice Herlihy (Brown University), Ted Herman (University of Iowa)

Fundamental synchronization and coordination primitives lie at the heart of distributed computer systems. These systems rely on algorithms for synchronization and coordination problems such as mutual exclusion, dining philosophers, leader election, token-circulation and identifier assignment to manage the use of resources and to control communication. However, as the number of components in these systems grows the likelihood of some component failure also increases, causing the traditional solutions to these problems to fail. Hence, algorithm designers seek to make systems more reliable by building fault-tolerance into their distributed protocols.

Self-stabilization is a particularly robust and appealing model of fault-tolerance for distributed computation. A distributed system is self-stabilizing if, when started from an arbitrary initial configuration, it is guaranteed to reach a legitimate configuration as execution progresses, and once a legitimate configuration is achieved, all subsequent configurations remain legitimate. Thus a self-stabilizing system will recover from a burst of transient faults (moving the system to some arbitrary configuration) without the need for manual intervention, as long as no further faults occur. The definition of self-stabilizing systems implies that they need not be initialized. This is an additional advantage particularly for systems that are physically widely dispersed. Furthermore, frequently (but not always) the techniques used to make the system converge apply even when the system can change dynamically. In this case systems need not be reset when a processing node or communication channel is added, reconfigured or removed.

The possibility of self-stabilizing distributed computation was first pointed out and explored by Edsger Dijkstra in a paper in 1974 when he asked the question: would it be possible for a set of machines to stabilize their collective behaviour in spite of unpredictable initial conditions and distributed control? This brief paper went largely unnoticed until Leslie Lamport's invited talk at PODC (Principles of Distributed Computing) 1983, where he said: "I regard this as Dijkstra's most brilliant work — at least, his most brilliant published paper ... a milestone in work on fault tolerance ... I regard self-stabilization to be a very important concept in fault tolerance, and to be a very fertile field for research." Subsequently, there has been a concerted research effort producing innovative theoretical results that can be used in practice. Some of the tasks that have been discussed in these result includes topology update, clock synchronization, multiprocessor synchronization, network flow control, leader election, and many different graph algorithms.

The main challenge associated with self-stabilization is complexity. It is difficult to design and prove correct protocols for asynchronous distributed systems. It is even more challenging to provide self-stabilizing solutions. One direction has been to rely on familiar paradigms of distributed programming — mutual exclusion, leader election, logical time, snapshots, wave algorithms, and broadcast — all of which now have stabilizing counterparts that can be combined for application to system design. Another research direction explores complexity in terms of resource requirements, including memory needed to satisfy particular task requirements, time needed for stabilization, and size requirements on messages. A third research direction attacks complexity by automating system design, and efforts along this line include composition methods, program transformers, program refinement, and systematic proof techniques for self-stabilizing programs.

Considerable progress has been made along all three of directions discussed above. However, there is still much that is not well understood, the set of known techniques for design of self-stabilizing algorithms is limited, and there are few practical applications of self-stabilization to existing systems. The BIRS workshop would focus on progress in these three directions.

A more formal and unified theory for self-stabilization would contribute to a deeper general understanding of the major issues. The existing self-stabilizing algorithms are designed for a plethora of models. Both shared-memory and message-passing models have been considered. Some shared-memory solutions assume that a processor can atomically read the entire state of all its neighbours and update its own state in one un-interruptible step. Others assume that an atomic step consists of just one read or one write to one shared variable. Design is easier under the first assumption but the second is more realistic. Some solutions assume that the computation proceeds under the control of a centralized scheduler, which selects exactly one processor to execute its next action at each step; others, more realistically, assume the scheduler can select any non-empty subset of processors to execute in each step. In either case, the scheduler may be constrained by any one of several possible fairness assumptions. Computation may be synchronous or asynchronous; processors may have distinct identifiers or may be indistinguishable; the system configuration may be static or subject to dynamic changes; protocols may be deterministic or may exploit randomization. A self-stabilizing solution to a specific problem may exist under one set of assumption but not under another. Some work has addressed the relationships between some of these models. Also, there are now some results that compile a solution that is correct assuming a strong model to a solution for a weaker model that makes fewer assumptions about the underlying system. But the complete picture of how all of these model properties are related for self-stabilizing systems is far from complete and frequently imprecise.

The workshop would first focus on constructing a framework capable of capturing the various assumptions and using the framework to precisely define the various models. Work would proceed to investigate the relationships between models. The goal is, for a set of model assumptions A and B, to develop a general transformation that converts an arbitrary self-stabilizing algorithm for strong model A into a self-stabilizing algorithm for the corresponding problem for weak model B, or to prove that no such transformation exits. Proving impossibility or lower bounds on the costs of desired transformations will serve to highlight the essential differences between models.

A surprising proportion of the existing self-stabilizing algorithms rely on a small collection of ad hoc techniques or small variants of these techniques. Recently, however, there has been some initial work that attempts to connect self-stabilization to other well-established research areas. The hope is that we can exploit the long history of research and the more highly developed insights in these domains to develop more sophisticated techniques for self-stabilization. Control theory appears to be one possibility. So far, however, the only progress has been to recast some existing algorithms into the framework of control theory and use this reformulation to produce a new proof of correctness. The acid test is to find self-stabilizing solutions to unsolved problems with control theory techniques. As a second example, self-stabilization is closely related to attractors. One paper has explored the relationship, but the connection has yet to been exploited for gain. Randomization is a tool that is now used widely in distributed systems. Typically, when a system has some costly situations, which a deterministic algorithm may not be able avoid, randomization is used to make these bad cases highly unlikely. Thus the expected performance can be quantified and controlled. Randomization has been used much less by the self-stabilization community, perhaps because its adds another level of difficulty onto an already highly complex situation. We would seek participation at BIRS from experts in all these research domains as an efficient way to get up to speed in these potentially valuable related research areas.

Finally, there has not yet been enough progress in moving the existing self-stabilization research into practice. Now, however, interest in robust systems that have built-in fault tolerance is high. So new activity is emerging that seeks to add fault-tolerance to several applications. Particularly obvious targets include embedded systems and Internet applications. To profitably apply self-stabilization to these areas, we need to combine self-stabilization with other practical goals. For example, it may be necessary to provide conditional safety properties during the period of convergence before a system stabilizes. It may be desirable to relate the cost of convergence to the scope of transient faults, by showing how self-stabilizing systems can confine repair to greater locality in time (faster repair) and space (fewer processes) for initial system states that represent only a “small” fault. In cases where self-stabilization is too costly, there may be practical ways to weaken the self-stabilization requirement to a realistic and achievable compromise. In other cases, it may be necessary to combine self-stabilization with other notions of fault-tolerance for even more robustness. Our list of proposed attendees contains a good balance between researchers with theoretical expertise and expert practitioners to ensure the best climate for progress toward this final goal.

List of Participants

Abraham, Uri (Ben-Gurion University of the Negev)
Arora, Anish (Ohio State University)
Bejan, Alina (University of Iowa)
Boulinier, Christian (LaRIA, Centre National de la Recherche Scientifique)
Brukman, Olga (Ben-Gurion University of the Negev)
Cobb, Jorge (The University of Texas at Dallas)
Dolev, Shlomi (Ben-Gurion University of the Negev)
Fich, Faith Ellen (University of Toronto)
Ghosh, Sukumar (University of Iowa)
Gouda, Mohamed (University of Texas, Austin)
Gradinariu, Maria (IRISA, Université Rennes)
Herman, Ted (University of Iowa)
Higham, Lisa (University of Calgary)
Huang, Shing-Tsaan (National Central University Taiwan)
Ioannidou, Kleoni (University of Toronto)
Johnen, Colette (Université de Paris Sud)
Kamaluddeen, Nuha (University of Calgary)
Karaata, Mehmet (Kuweit University)
Katayama, Yoshiaki (Nagoya Institute of Technology)
Manne , Fredrik (University of Bergen, Norway)
Masuzawa, Toshimitsu (Osaka University)
Nesterenko, Mikhail (Kent State University)
Schiller, Elad (Ben-Gurion University of the Negev)
Srimani, Pradip (Clemson University)
Tixeuil, Sébastien (Université de Paris Sud)
Yamashita, Masafumi (Kyushu University)
Zaks, Shmuel (Technion - Israel Institute of Technology)
Zhang, Chen (University of Iowa)
Zhang, Hongwei (The Ohio State University)

Chapter 30

Free Probability Theory (04w5028)

October 9–14, 2004

Organizer(s): Alexandru Nica (University of Waterloo), Roland Speicher (Queen's University), Dan Voiculescu (University of California, Berkeley)

Introduction

Free probability theory is a line of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out from attempts to solve some longstanding problems about von Neumann algebras of free groups. In the twenty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability and the theory of large deviations, algebraic combinatorics. Free probability also has connections with some mathematical models in theoretical physics.

The BIRS workshop on free probability brought together a very strong group of mathematicians representing the current directions of development in the area. This continued a sequence of very successful 5-day workshops organized on these lines, like the ones at the Fields Institute in March 1995, at CIRM Luminy in January 1998, and at MSRI in January 2001.

In this report we look in more detail at what are the current directions of development in free probability, with an emphasis on how they were represented in the BIRS workshop.

Developments related to operator algebras and random matrices

Free probability has far-reaching connections both with the field of operator algebras (where the subject is originally coming from) and with the one of random matrices. Since the interactions between free probability and these two fields are closely related to each other, we will discuss them together.

Background and general overview

Let $L(G)$ denote the von Neumann algebra (that is, the weakly closed subalgebra of $B(\,l^2(G)\,))$ generated by the left regular representation of the discrete group G. The so-called "isomorphism question" for von Neumann algebras of free groups asks: denoting by F and F' the free groups with 2 and respectively 3 generators, is it true or not that $L(F)$ is isomorphic to $L(F')$? This outstanding problem (still open today) was the original motivation for the birth of free probability. We should mention here that several other questions raised in the 60's about the von Neumann algebras of free groups were still open when free probability was born — e.g. questions about the so-called fundamental groups of these algebras, or the question of whether they admit (a von Neumann algebra version of) Cartan subalgebras.

Substantial progress was brought in these old problems by the development of free probability and of its connection with random matrices. The first such connection appeared with the "random matrix model for freeness" established in [53]; this model very quickly was found to have groundbreaking applications concerning the fundamental groups of the free group von Neumann algebras (see e.g. [42]). Also, random matrix model techniques showed that, with F and F' as above, the von Neumann algebras $L(F)$ and $L(F')$ do become isomorphic when the natural stabilization operation of tensoring with $B(H)$ is applied to them. (This was seeming to suggest that $L(F)$ and $L(F')$ might be, after all, isomorphic to each other!)

On the other hand there was another, very penetrating, line of attack, which came with the appearance of the concept of free entropy for n-tuples of non-commutative random variables. The developments related to free entropy are gathering arguments in the support of the idea that (with F and F' as above) $L(F)$ and $L(F')$ are not isomorphic. At the current stage, free entropy arguments limit the possibilities on how a hypothetical isomorphism between $L(F)$ and $L(F')$ could go; further technical progress in free entropy may completely rule out the existence of such an isomorphism. Free entropy is discussed in the next subsection.

It is worth pointing out that, beyond the drive for solving the celebrated isomorphism question, free probability has built a solid theory of the free group von Neumann algebras, which parallels the one going on in the hyperfinite case. Among von Neumann algebras, the hyperfinite ones are by far the ones with the richest theory, and the free group von Neumann algebras are considered as the "best" among the non-hyperfinite ones. It has become clear that the parallelism between the hyperfinite and and the free case goes deep, and it includes, for example, the development of a type III theory (for infinite von Neumann algebras), see e.g. [22], [46].

Moreover, free probability has also found outstanding applications in C^*-algebra theory, and in relation to the invariant subspace problem. This was done in work by Haagerup and collaborators ([24],[25], [28]–[31]), and was an important topic of the workshop – cf. the discussion in the subsections 2.3 and 2.4 below.

Finally, the subsections 2.5 and 2.6 of this section discuss two other operator algebra topics that were addressed in the workshop. One is in connection to the Connes embedding problem, and the other is in connection to a q-deformation of the von Neumann algebra of a free group.

Free entropy

Free entropy is an invariant for n-tuples of non-commutative random variables. There are in fact two versions (both well-motivated) of this concept: the "microstates" free entropy, and the "non-microstates" one.

The microstates free entropy was introduced in [54]. The key idea for this concept is to look at sets of n-tuples of matrices of large size (the "matricial microstates") which approximate in distribution a given n-tuple of non-commutative random variables. Very soon after being introduced, the microstates free entropy found powerful applications to free group von Neumann algebras. One such application was the proof (in [55]) of the fact that these von Neumann algebras do not have Cartan subalgebras. Another application was the so-called primality of the free group von Neumann algebras (in the sense that they cannot be decomposed as tensor products in a non-trivial way), see [26].

The non-microstates free entropy was introduced in [56], and uses an approach where one first considers the free analogue for the concept of Fisher information. There are a number of particular cases when the two approaches to free entropy (via matricial microstates and via the free Fisher information) are known to coincide; it is in fact believed that this should be true in general. Finding bridges between the two approaches to free entropy is at present one of the central problems in free probability. A notable progress on this line was made in [10]; by using large deviation techniques for n-tuples of matrices, it was shown there that one always has an inequality between the two free entropies (the free entropy defined with microstates can never exceed the one defined via free Fisher information).

An invariant related to free entropy is the free entropy dimension. In the microstate approach this is, very roughly, a normalized dimension of Minkowski type for the sets of matricial microstates. There now exist several versions of the the free entropy dimension; in particular we should mention that a surprising facet of this invariant has recently occurred in a von Neumann homological context, in the work of Connes and Shlyakhtenko [21].

For a survey on free entropy and free entropy dimension, see [58].

Free entropy was a central topic at the BIRS workshop. In the opening talk of the workshop, Dima Shlyakhtenko presented his work on estimates for free entropy dimension (cf. [21], [47]). The talk by Kenley

Jung addressed the issue of free entropy dimension inequalities for subfactors (cf. [35]). The talk by Fumio Hiai was devoted to presenting his work in [28] on the free analogue of pressure, a concept which is dual via a Legendre transform to a free entropy type of invariant. Related to that, the talk by Denes Petz presented the work in [34] on free transportation cost inequalities via random matrix approximation.

The invariant subspace problem

Let $A \subset B(H)$ be a von Neumann factor (i.e. a von Neumann algebra such that the centre of A is reduced to $\mathbb{C}1_A$). The invariant subspace problem for A asks if every element $a \in A$ has an invariant subspace of the form Ran(p) with $p = p^* = p^2 \in A$. In the case when $A = B(H)$, this is the "classical" invariant subspace problem; but the problem is also very interesting (and still open) for other von Neumann factors – in particular for those of type II_1 (a class of von Neumann factors which includes the von Neumann algebras of free groups).

Significant progress in the invariant subspace problem in the type II_1 case was made in the last few years, by using the concept of Brown measure for an element of a II_1 factor; this was introduced by L. Brown [16], and is a generalization (defined via subharmonic function theory) for the notion of spectral distribution for normal elements. The very recent groundbreaking work of Haagerup and Schultz [29] shows that an element of a II_1 factor has invariant subspaces whenever the support of its Brown measure is not reduced to a single point. This is the the culmination of intense work building in this direction, by Haagerup [28] (proving the same result under a restriction related to the Connes embedding problem mentioned in Section 2.5), and by several authors ([24], [25], [49]) who studied Brown measures and invariant subspaces in examples occurring naturally in free probability.

The recent progress in the invariant subspace problem was one of the highlights of the BIRS workshop, and was reflected in the combined talks given by Uffe Haagerup and Hanne Schultz.

Applications to C*-algebra theory

Free probability considerations can also be made in a C^*-algebra (rather than von Neumann algebra) framework. In the C^*-algebra universe, the counterpart of $L(G)$ is $C^*_r(G)$, the reduced C*-algebra of the group G.

In the recent paper [31], Haagerup and Thorbjornsen obtain a sweeping generalization to several matrices for a number of results concerning the largest eigenvalue of a Gaussian random matrix. This has strong consequences for the reduced C^*-algebras of free groups. In particular, they obtain that, for G a free group on 2 or more generators, the Ext semigroup of $C^*_r(G)$ is not a group (this has been one of the most popular open questions in C^*-algebra theory since the late 70's).

In the BIRS workshop, Steen Thorbjornsen presented another application of random matrices to reduced C^*-algebras of free groups – a new proof for the fact that these C^*-algebras have no non-trivial projections (cf. [30]).

Connes embedding problem

The Connes embedding problem [20] asks whether every type II_1 factor can be embedded into an ultraproduct of the hyperfinite II_1 factor. The problem has several equivalent reformulations, one of them being that every n-tuple of elements of a II_1 factor has matricial microstates.

The BIRS workshop had a talk on this topic, given by Florin Radulescu. He presented his work in [43] showing that the Connes embedding problem is equivalent to a statement on matrix trace inequalities which is, in a certain sense, an analytic version of Hilbert's 17th problem (formulated in a von Neumann algebra framework).

q-deformations of the free group factors

A basic way of producing examples of free families of random variables goes by using creation and annihilation operators on the full Fock space. The most important such example is the "free semicircular system" of Voiculescu, which is the analogue in free probability for a family of independent Gaussian random variables.

The von Neumann algebra generated by a semicircular system with n elements is isomorphic to $L(F_n)$, where F_n is the free group on n generators.

In [14], Bozejko and Speicher put into evidence a deformation, depending on a parameter $q \in (-1, 1)$, for the creation and annihilation operators on the full Fock space. The corresponding q-deformation of a free semicircular system is called "q-Gaussian family". In the case when $q = 0$ one has the full Fock space and the semicircular families, while in the limit $q \to 1$ one gets back to the classical objects, independent families of Gaussians. The von Neumann algebra generated by a q-Gaussian family was given quite a bit of attention in the last few years, but remains fairly mysterious. In fact even the basic question of whether this algebra has trivial centre was only partly solved until the recent work of Ricard [45]. In his talk at the workshop, Eric Ricard presented this work, showing the factoriality of the von Neumann algebra generated by a q-Gaussian family with n elements, for all $n \geq 2$ and all $q \in (-1, 1)$.

Relations to probability theory

An important insight brought by free probability is that the concept of freeness for a family of non-commuting random variables in a von Neumann algebra should be treated as an analogue of the notion of independence from classical probability. Acting on this line, one is prompted to start a programme of developing free counterparts for fundamental theorems of classical probability.

It is remarkable how far this programme can go. There exist now quite a few deep theorems about the internal structure of free probability, inspired (though certainly not following from!) developments in classical probability.

For example, there exists a notion of free convolution of distributions, and a well-developed analytic machinery (replacing the classical machinery of the Fourier transform) for dealing effectively with this new type of convolution. We have characterizations for freely infinitely divisible distributions, also for freely stable distributions and their domains of attraction [8]. On the other hand there exists a well-developed theory of stochastic integration and of stochastic analysis for the free Brownian motion [12]. Quite a few talks at the BIRS workshop addressed free analogues of the classical situation.

Analytic properties of free convolution

Free convolution is a binary operation on probability measures on the real line which corresponds to the sum of two free random variables (in the same way as the usual convolution corresponds to the sum of two independent random variables). There exists also a multiplicative version of the free convolution, which goes with the product of free random variables. Many theorems about classical convolution have free counterparts, see e.g. [57], [59]. The analytic treatment of such questions relies on a good understanding of the Cauchy transforms of the convolved probability measures, and of a couple of other transforms specifically used by free probability, the R-transform and the S-transform. Thus on one hand many questions about free convolution result in interesting new statements about analytic functions, and on the other hand complex analysis is an important tool for investigating properties of the free convolution. It should be pointed out that although free convolution and classical convolution have many properties in common, there are also differences – in particular, free convolution has better regularity properties than classical convolution. Whereas the basic analytic theory of free convolution is by now a well-established theory, there are many questions in this context which are still open, giving rise to interesting investigations.

The BIRS workshop had two talks about recent developments in this direction: by Hari Bercovici, on Hincin's theorem for multiplicative free convolution and by Serban Belinschi, on the regularity properties of the free convolution. It is a remarkable fact (first noticed in [40]) that any probability measure μ on the real line belongs to a partial semigroup μ_t ($t \geq 1$) relative to the free additive convolution. In their work, Belinschi and Bercovici proved similar results for free multiplicative convolution of measures supported on the positive half-line and (in a slightly less general context) on the unit circle. They also investigated regularity properties of measures in these semigroups, and connections with additive and multiplicative Boolean convolutions of probability measures. See [5]–[7].

Free extreme values, free transport inequalities, and free de Finetti's theorem

As mentioned above there are quite a few parts of classical probability theory which have a free counterpart – well established are by now, e.g., the theory of free convolution (as addressed in 3.1) or the basic theory of free stochastic processes and free stochastic integration. The BIRS workshop had several talks about new analogues of classical theories.

• Gerard Ben-Arous talked about free extreme values; in recent joint work [4] with Dan Voiculescu, they obtained free probability analogues of the basics of extreme value theory, based on Ando's spectral order. This includes classification of freely max-stable laws and their domains of attraction, using free extremal convolutions on the distributions.

• A recent development in free probability theory is to look for analogues of cost of transportation inequalities in the free setting. A free Wasserstein distance was introduced and investigated by Biane and Voiculescu [13]. In particular, they proved a free version of Talagrand inequality. In her talk titled "About optimal transport for non-commutative variables", Alice Guionnet reported on her ongoing joint work with Cedric Villani on how to develop this program even further.

• Franz Lehner reported on his free version of de Finetti's theorem, characterising amalgamated free products as noncrossing exchangeability systems which satisfy a so-called weak singleton condition, see [36].

Free probabilistic aspects of random matrices

Whereas the above developments show that free probability really deals with a beautiful and rich structure, it happens at the same time that the framework created by free probability may indicate new conceptual approaches to problems from other fields. This was in fact the main theme in the section 2 above, where the other field was the one of operator algebras. Another outstanding illustration of how this can happen is provided by work of Ben-Arous, Guionnet, and Cabanal-Duvillard (see e.g. [3], [17], [57]), who showed that free entropy is useful in the study of the rate functions of large deviation principles.

In the BIRS workshop, James Mingo talked about fluctuations of random matrices and second order freeness. He reported his joint work with Speicher [37] and with Sniady and Speicher [38], where they extend the relation between random matrices and free probability theory from the level of expectations to the level of fluctuations by introducing the new concept of second order freeness.

Another connection between special random matrix ensembles and concepts from free probability was made in the talk of Benoit Collins. He considered the product of two independently randomly rotated projectors, the square of whose radial part turns out to be distributed as a Jacobi ensemble, and studied its global and local properties in the large dimension scaling relevant to free probability theory, see [19].

Other versions of non-commutative probability theory

Free probability theory is a prominent example of a non-commutative probability theory, where one tries to extend notions and ideas from classical probability theory to a non-commutative setting (by replacing the commutative algebra of random variables by a non-commutative algebra). Free probability theory is that part of non-commutative probability theory where the notion of freeness (as a replacement for independence) plays a crucial role. However, there are also some other possibilities for non-commutative versions of independence; in particular, we have the theories of conditional freeness (which is a generalization of freeness), of boolean independence, and of monotonic independence.

At the BIRS workshop, Marek Bozejko talked about a relation between conditional freeness and some classes of free Levy processes. In joint work with W. Bryc, they have shown a relation with free pairs of random variables which have linear regressions and quadratic conditional variances when conditioned with respect to their sum, see [15].

There exist also more general axiomatic notions for non-commutative independence. The investigation of processes with independent increments in such a general frame was presented in the talk of Claus Koestler. In joint work with J. Hellmich and B. Kümmerer [32], they introduced a non-commutative extension of

Tsirelson-Vershik noises, which they call continuous Bernoulli shifts, and they established a bijective correspondence between additive and unital shift cocycles in this setting.

Combinatorial aspects of free probability

Cumulants and R-transforms

Even when restricted to a purely algebraic framework, free random variables have an interesting combinatorics, which stems from their non-commuting character. The combinatorics of free probability turns out to be governed by the theory of Moebius inversion in lattices (as developed by Rota and collaborators), applied to lattices of non-crossing partitions. The specific way how the lattices of non-crossing partitions show up in this framework is via the non-crossing cumulant functionals associated to a non-commutative probability space – see e.g. the survey paper [51].

The combinatorial study of freeness had an important impact in the development of the R-transform, which is the free probabilistic counterpart of the Fourier transform from classical probability. The R-transform for one bounded selfadjoint random variable has both an analytic and a combinatorial incarnation, and each of these two incarnations generated its own direction of research. A success of the combinatorial approach to the R-transform was that it could be extended (by using non-crossing partitions) to the case when joint R-transforms for several non-commuting random variables are considered. Moreover, the pursuit of the multivariable R-transform lead to a whole collection of combinatorial tools, which can often streamline and make transparent complicated algebraic computations involving freeness. A nice illustration of the power of the combinatorial method is provided for instance by the solution to the problem of computing the distribution of a free commutator, see [41].

The basic theory of non-crossing cumulants has inspired a number of variations. For instance: by starting from the fact that the lattices of non-crossing partitions have analogues of type B (cf [44]), one can work on developing an analogue of type B for the theory of non-crossing cumulants – see [11]. Another very interesting variation was presented at the workshop by Mireille Capitaine; in a very recent joint work with Muriel Casalis [18], they have introduced a concept of "cumulants for $N \times N$ random matrices", where letting $N \to \infty$ gives back the non-crossing cumulants used in free probability.

Last but not least let us mention that a substantial part of the theory of the non-crossing cumulants and R-transforms can be extended to an "operator-valued" version, where we talk about free independence with amalgamation over a subalgebra B (rather than over $\mathbb{C}$). For a survey of non-crossing cumulants in this more general framework, see the memoir [52]. At the BIRS workshop, the operator-valued R-transform was the main tool used by Ken Dykema in his talk on nearest neighbour random walks on amalgamated free product groups (cf [23]).

Asymptotic representation theory for symmetric groups

The combinatorial machinery of free probability has found powerful applications in the asymptotic representation theory of symmetric groups. The study of limit shapes of Young diagrams goes back to the 70's, and was well-developed in a sequence of papers by S. Kerov in the 90's. The connection with free probability came with the paper [9], where P. Biane showed how the non-crossing cumulants of the transition measure of a limit diagram can be used to study asymptotics for characters and for natural operations with representations. Research on this topic was presented at the BIRS workshop in a survey talk given by Aikihito Hora, and in a talk by Piotr Sniady, presenting his work in [48].

Other combinatorial connections

There were several other talks at the BIRS workshop which touched on combinatorial aspects pertinent to developments in free probability.

• Michael Anshelevich presented his work [1] where he puts the basis for a theory of orthogonal polynomials in several non-commuting indeterminates, and where some non-trivial examples are obtained by looking at free analogues for the classical Meixner systems.

• A new possible direction of development was addressed in the talk by Ed Effros, on the relations between non-crossing cumulants and combinatorial Hopf algebra theory.

• The talk by Ian Goulden discussed the well-known formula of Harer and Zagier (which is directly related to the moments of a GUE random matrix), and presented a direct combinatorial way of deriving this formula, via tree enumeration (cf. [27]).

• The talk by Michael Neagu discussed another occurrence of free independence in connection to the symmetric groups – random permutation matrices are asymptotically free from GUE or Wishart matrices (cf. [39]).

Reaching out to new directions

The organizers of the BIRS workshop found it important to invite also some participants who do not work in free probability, but are interested in it and whose areas of interest indicate possible new connection points and new directions of development for free probability. The talks of those participants did not report on genuine free probability results, but gave some kind of introduction to questions and results in their respective areas.

• Robert Bauer gave a survey talk on conformal invariance and stochastic Loewner equations. In [2], he showed, using concepts of noncommutative probability, that Loewner's evolution equation can be viewed as providing a map from paths of measures to paths of probability measures, whose fixed point is the convolution semigroup of the semicircle law.

• Michael Pimsner gave a talk about graded groups in KK-theory, with emphasis on an application to computing the K-theory groups for a crossed-product by an action of a symmetric group.

• Alexander Soshnikov talked about Poisson statistics for the largest eigenvalues of Wigner and Wishart random matrices with heavy tails. He considered large Wigner random matrices in the case when the marginal distributions of matrix entries have heavy tails and proved that the largest eigenvalues of such matrices have Poisson statistics, see [50].

List of Participants

Anshelevich, Michael (University of California, Riverside)
Bauer, Robert (University of Illinois, Urbana)
Belinschi, Serban (Indiana University)
Ben Arous, Gerard (New York University)
Bercovici, Hari (Indiana University)
Bozejko, Marek (University of Wroclaw)
Cabanal-Duvillard, Thierry (Universite Rene Descartes)
Capitaine, Mireille (Universite Paul Sabatier)
Collins, Benoit (University of Kyoto)
Dykema, Ken (Texas A&M University)
Effros, Edward (University of California, Los Angeles)
Ghosh, Shamindra (University of New Hampshire)
Goodman, Frederick (University of Iowa)
Goulden, Ian (University of Waterloo)
Guionnet, Alice (Ecole Normale Superieure de Lyon)
Haagerup, Uffe (University of Southern Denmark)
Hiai, Fumio (Tohoku University)
Hora, Akihito (Okayama University)
Jung, Kenley (University of California, Berkeley)
Koestler, Claus (Queen's University)
Lehner, Franz (Graz University of Technology)
Mingo, James (Queen's University)

Neagu, Michael (University of Waterloo)
Nica, Alexandru (University of Waterloo)
Petz, Denes (Budapest University of Technology and Economics)
Pimsner, Michael (University of Pennsylvania)
Radulescu, Florin (University of Iowa)
Ricard, Eric (Universite de Besancon)
Savu, Ana Maria (Queen’s University)
Schultz, Hanne (University of Southern Denmark)
Shlyakhtenko, Dimitri (University of California, Los Angeles)
Sniady , Piotr (University of Wroclaw)
Soshnikov, Alexander (University of California, Davis)
Speicher, Roland (Queen’s University)
Thorbjornsen, Steen (University of Southern Denmark)
Ueda, Yoshimichi (Kyushu University)
Voiculescu, Dan (University of California, Berkeley)

Bibliography

[1] M. Anshelevich, Orthogonal polynomials with a resolvent-type generating function, preprint 2004 (math-archive 0410482).

[2] R. Bauer, Loewner's equation from a noncommutative perspective, *J. Theoret. Prob.* **17** (2004), 435-456.

[3] G. Ben-Arous, A. Guionnet, Large deviations for Wigner's law and Voiculescu's non-commutative entropy, *Probability Theory and Related Fields* **108** (1997), 517-542.

[4] G. Ben Arous, D. Voiculescu, Free extreme values, preprint 2005 (math-archiv 0501274).

[5] S. Belinschi, H. Bercovici, Atoms and regularity for measures ina partially defined free convolution semigroup, *Math. Zeitschrift* **248** (2004), 665-674.

[6] S. Belinschi, H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, to appear in *International Math. Res. Notes*.

[7] S. Belinschi, H. Bercovici, Hincin's theorem for multiplicative free convolution, preprint 2004.

[8] H. Bercovici, V. Pata (with an appendix by P. Biane), Stable laws and domains of attraction in free probability theory, *Annals of Math.* **149** (1999), 1023-1060.

[9] P. Biane, Representations of symmetric groups and free probability, *Advances in Math.* **138** (1998), 126-181.

[10] P. Biane, M. Capitaine, A. Guionnet, Large deviation bounds for matrix Brownian motion, *Invent. Math.* **152** (2003), 433-459.

[11] P. Biane, F. Goodman, A. Nica, Non-crossing cumulants of type B, *Transactions of the Amer. Math. Soc.* **355** (2003), 2263-2303.

[12] P. Biane, R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, *Probability Theory and Related Fields* **112** (1998), 373-409.

[13] P. Biane, D. Voiculescu, A free probability analogue for the Wasserstein metric on the trace-state space, *Geometric and Functional Analysis* **11** (2001), 1125-1138.

[14] M. Bozejko, R. Speicher, An example of a generalized Brownian motion, *Communications in Mathematical Physics* **137** (1991), 519-531.

[15] M. Bozejko, W. Bryc, On a class of free Levy laws related to a regression problem, preprint 2004 (math-archiv 0410601).

[16] L.G. Brown, Lidskii's theorem in the type II case. In *Geometric methods in operator algebras (H. Araki and E.Effros, eds)*, Pitman Research Notes in Mathematics **123** (1986), 1-35.

[17] T. Cabanal-Duvillard, A. Guionnet, Large deviations upper bounds for the laws of matrix-valued processes and non commutative entropies, *Annals of Probability* **29** (2001), 1205-1261.

[18] M. Capitaine, M. Casalis, Cumulants for random matrices as convolutions on the symmetric group, preprint 2004.

[19] B. Collins, Products of random projections, Jacobi ensembles and universality problems arising from free probability, preprint 2004 (math-archive 0406560).

[20] A. Connes, Classification of injective factors. Cases II_1, III_λ, $\lambda \neq 1$, *Annals of Math.* **104** (1976), 73-115.

[21] A. Connes, D. Shlyakhtenko. L^2–homology for von Neumann algebras, preprint 2003 (math-archive 0309343).

[22] K. Dykema, Factoriality and Connes' invariant T(M) for free products of von Neumann algebras. *J. Reine und Angewandte Math.* **429** (1994), 159-180.

[23] K. Dykema, Symmetric random walks on certain amalgamated free product groups, preprint 2004 (math-archive 0409595).

[24] K. Dykema, U. Haagerup, Invariant subspaces of Voiculescu's circular operator, *Geometric and Functional Analysis* **11** (2001), 693-741.

[25] K. Dykema, U. Haagerup, DT-operators and decomposability of Voiculescu's circular operator, *Amer. J. Math.* **126** (2004), 121-189.

[26] L. Ge, Applications of free entropy to finite von Neumann algebras, *Annals of Math.* **147** (1998), 143-157.

[27] I. Goulden, A. Nica, A direct bijection for the Harer-Zagier formula, preprint 2004, to appear in *J. Comb. Theory Series A.*

[28] U. Haagerup, Spectral decomposition of all operators in a II_1-factor which is embeddable in R^ω, handwritten notes distributed at MSRI, 2001.

[29] U. Haagerup, H. Schultz, Invariant subspaces for operators in a general II_1-factor, preprint 2005.

[30] U. Haagerup, H. Schultz, S. Thorbjornsen, A random matrix approach to the lack of projections in $C^*_{red}(F_2)$, preprint 2004 (math-archive 0412545).

[31] U. Haagerup, S. Thorbjornsen, A new application of random matrices: Ext$(\,C^*_{red}(F_2)\,)$ is not a group, preprint 2002, to appear in *Annals of Math.*

[32] J. Hellmich, C. Köstler, B. Kümmerer, Noncommutative continuous Bernoulli shifts, preprint 2004 (math-archiv 0411565).

[33] F. Hiai, Free analog of pressure and its Legendre transform, preprint 2004 (math-archive 0403210).

[34] F. Hiai, D. Petz, Y. Ueda, Free transportation cost inequalities via random matrix approximation, to appear in *Probability Theory and Related Fields*.

[35] K. Jung, Some free entropy dimension inequalities for subfactors, preprint 2004 (math-archiv OA/0410594).

[36] F. Lehner, Cumulants in noncommutative probability theory IV. De Finetti's Theorem, L^p-Inequalities and Brillinger's formula, preprint 2004 (math-archiv 0409025).

[37] J. Mingo, R. Speicher, Second Order Freeness and Fluctuations of Random Matrices: I. Gaussian and Wishart matrices and Cyclic Fock spaces, preprint 2004 (math-archiv 0405191).

[38] J. Mingo, P. Sniady, R. Speicher, Second Order Freeness and Fluctuations of Random Matrices: II. Unitary Random Matrices, preprint 2004 (math-archiv 0405258).

[39] M. Neagu, Asymptotic freeness of random permutation matrices from Gaussian matrices, preprint 2004 (math-archiv 0410028).

[40] A. Nica, R. Speicher (with an appendix by D. Voiculescu), On the multiplication of free n-tuples of noncommutative random variables, *American J. Math.* **118** (1996), 799-837.

[41] A. Nica, R. Speicher, Commutators of free random variables, *Duke Math. J.* **92** (1998), 553-592.

[42] F. Radulescu, The fundamental group of $L(F_\infty)$ is $R_+ \setminus \{0\}$, *Journal of the Amer. Math. Soc.* **5** (1992), 517-532.

[43] F. Radulescu, A non-commutative, analytic version of Hilbert's 17th problem in type II_1 von Neumann algebras, preprint 2004 (math-archive 0404458).

[44] V. Reiner, Non-crossing partitions for classical reflection groups, *Discrete Mathematics* **177** (1997), 195-222.

[45] E. Ricard. Factoriality of q-Gaussian von Neumann algebras, preprint 2003 (math-archive 0311413).

[46] D. Shlyakhtenko: Free quasi-free states, *Pacific J. Math.* **177** (1997), 329-368.

[47] D. Shlyakhtenko: Some estimates for non-microstates free entropy dimension, with application to q-semicircular families, *International Math. Res. Notices* **51** (2004), 2757-2772.

[48] P. Sniady, Asymptotics of characters of symmetric groups, Gaussian fluctuations of Young diagrams and genus expansion, preprint 2004 (math-archive 0411647).

[49] P. Sniady, R. Speicher, Continuous family of invariant subspaces for R-diagonal operators, *Inventiones Math.* **146** (2001), 329-363.

[50] A. Soshnikov, Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails, *Electron. Comm. Probab.* **9** (2004), 82-91 (electronic).

[51] R. Speicher, Free probability and non-crossing partitions, *Sem. Lothar. Combin.* **39** (1997), Art. B39c (electronic).

[52] R. Speicher, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, *Memoirs of the American Math. Society* **132** (1998), x+88.

[53] D. Voiculescu, Limit laws for random matrices and free products. *Inventiones Math.* **104** (1991), 201-220.

[54] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory II, *Inventiones Math.* **118** (1994), 411-440.

[55] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory III: the absence of Cartan subalgebras, *Geometric and Functional Analysis* **6** (1996), 172-199.

[56] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory V: non-commutative Hilbert transforms, *Inventiones Math.* **132** (1998), 182-227.

[57] D. Voiculescu, Lectures on free probability theory. In *Lectures on probability theory and statistics*, St. Flour **28** (1998), Springer Lecture Notes in Mathematics vol. 1738, 279-349.

[58] D. Voiculescu, Free entropy. *Bull. London Math. Soc.* **34** (2002), 257-278.

[59] D. Voiculescu, K. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series, Vol. 1, American Math. Society, 1992.

Chapter 31

Braid Groups and Applications (04w5526)

October 16-21, 2004

Organizer(s): Joan Birman (Columbia University), Patrick Dehornoy (University of Caen), Dale Rolfsen (University of British Columbia), Roger Fenn (University of Sussex), Vaughan Jones (University of California, Berkeley)

The braid groups B_n were introduced by E. Artin in 1926 [1] (see also [2]). They have been of importance in many fields – algebra, analysis, cryptography, dynamics, topology, representation theory, mathematical physics – and many of these aspects were represented in the BIRS workshop. This workshop involved not only leading experts in the field, but also, importantly, a number of young researchers, postdoctoral fellows and several graduate students. This made for an exciting and informative mix of ideas on the subject.

The importance of the braid groups is based, in part, on the many ways in which they can be defined. This is outlined in the following introductory section.

Six Definitions of the Braid Groups

Definition 1: Braids as particle dances. Consider n particles located at distinct points in a plane. To be definite, suppose they begin at the integer points $\{1, \ldots, n\}$ in the complex plane $\mathbb{C}$. Now let them move around in trajectories

$$\beta(t) = (\beta_1(t), \ldots \beta_n(t)), \quad \beta_i(t) \in \mathbb{C}, \quad 0 \leq t \leq 1.$$

A *braid* is then such a time history with the proviso that the particles are noncolliding:

$$\beta_i(t) \neq \beta_j(t) \quad \text{if} \quad i \neq j$$

and end at the spots they began, but possibly permuted:

$$\beta_i(0) = i, \quad \beta_i(1) \in \{1, \ldots, n\}, \quad i = 1, \ldots, n.$$

If one braid can be deformed continuously into another (through the class of braids), the two are considered equivalent – we will say equal.

Braids α and β can be multiplied: one dance following the other, each at double speed. The product is associative but not in general commutative. The identity dance is to stand still, and each dance has an inverse; doing the dance in reverse time. These (deformation classes of) dances form the group B_n.

A braid β defines a permutation $i \to \beta_i(1)$ which is a well-defined element of the permutation group Σ_n. This is a homomorphism with kernel, by definition, the subgroup P_n of *pure* braids. P_n is sometimes called the *coloured* braid group, as the particles can be regarded as having identities, or colours. P_n is of course normal in B_n, of index $n!$, and there is an exact sequence

$$1 \to P_n \to B_n \to \Sigma_n \to 1.$$

Definition 2: Braids as strings in 3-D. This is the usual and visually appealing picture. A braid can be viewed as the graph, or timeline, of a braid as in the first definition, drawn in real x, y, t-space, monotone in the t direction. The complex part is described as usual by $x + y\sqrt{-1}$. The product is then a concatenation of braided strings.

This viewpoint provides the connection with knots. A braid β defines a knot or link $\hat{\beta}$, its closure, by connecting the endpoints in a standard way so that no new crossings are introduced. J. W. Alexander showed that all knots arise as the closure of some braid and by a theorem of Markov (see [4] for a discussion and proof) two braids close to equivalent knots if and only if they are related by a finite sequence of moves and their inverses: conjugation in the braid group and a stabilization, which increases the number of strings.

Definition 3: B_n as a fundamental group. In complex n-space $\mathbb{C}^n$ consider the big diagonal

$$\Delta = \{(z_1, \ldots, z_n); \quad z_i = z_j, \quad \text{some} \quad i < j\} \subset \mathbb{C}^n.$$

Using the basepoint $(1, 2, \ldots, n)$, we see that

$$P_n = \pi_1(\mathbb{C}^n \setminus \Delta).$$

In other words, pure braid groups are fundamental groups of complements of a special sort of complex *hyperplane arrangement*, itself a deep and complicated subject.

To get the full braid group we need to take the fundamental group of the *configuration space*, of orbits of the obvious action of Σ_n upon $\mathbb{C}^n \setminus \Delta$. Thus

$$B_n = \pi_1((\mathbb{C}^n \setminus \Delta)/\Sigma_n).$$

Notice that since the singularities have been removed, the projection

$$\mathbb{C}^n \setminus \Delta \longrightarrow (\mathbb{C}^n \setminus \Delta)/\Sigma_n$$

is actually a covering map. As is well-known, covering maps induce injective homomorphisms at the π_1 level, so this is another way to think of the inclusion $P_n \subset B_n$.

Finally, we note that the space $(\mathbb{C}^n \setminus \Delta)/\Sigma_n$ can be identified with the space of all complex polynomials of degree n which are monic and have n distinct roots

$$p(z) = (z - r_1) \cdots (z - r_n).$$

This is one way in which the braid groups play a role in classical algebraic geometry, as fundamental group of the space of such polynomials.

Definition 4: The algebraic braid group. B_n can be regarded algebraically as the group presented with generators $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the braid with one crossing, with the string at level i crossing over the one at level $i + 1$ and the other strings going straight across.

These generators are subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1,$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad |i - j| = 1.$$

We can take a whole countable set of generators $\sigma_1, \sigma_2, \ldots$ subject to the above relations, to define the infinite braid group B_∞. If we consider the (non-normal) subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$, these algebraically define B_n. Notice that this convention gives "natural" inclusions $B_n \subset B_{n+1}$ and $P_n \subset P_{n+1}$.

Definition 5: B_n as a mapping class group. Going back to the first definition, imagine the particles are in a sort of planar jello and pull their surroundings with them as they dance about. Topologically speaking, the motion of the particles extends to a continuous family of homeomorphisms of the plane (or of a disk, fixed on the boundary). This describes an equivalence between B_n and the mapping class of D_n, the disk D

with n punctures (marked points). That is, B_n can be considered as the group of homeomorphisms of D_n fixing ∂D and permuting the punctures, modulo isotopy fixing $\partial D \cup \{1, \ldots, n\}$.

Definition 6: B_n as a group of automorphisms. A mapping class $[h]$, where $h : D_n \to D_n$, gives rise to an automorphism $h_* : F_n \to F_n$ of free groups, because F_n is the fundamental group of the punctured disk. Using the interpretation of braids as mapping classes, this defines a homomorphism

$$B_n \to Aut(F_n),$$

which Artin showed to be faithful, i. e. injective.

The generator σ_i acts as

$$x_i \to x_i x_{i+1} x_i^{-1}; \quad x_{i+1} \to x_i; \quad x_j \to x_j, \quad j \neq i, i+1.$$

Thus B_n may be considered a group of automorphisms of $Aut(F_n)$ satisfying a condition made precise by Artin.

Representations of Braid Groups

One of the most active aspects of braid theory is the study of linear representations. A major breakthrough has been the proof in 2000 by S. Bigelow [3] and D. Krammer [12] of the long-standing conjecture that Artin's braid groups B_n are linear groups. That is, there exists a faithful representation of B_n in a finite-dimensional linear group. The Lawrence–Krammer representation that provides a linear representation of B_n has dimension $n(n-1)/2$. After the result was established, considerable efforts have been made to better understand the algebraic underlying socle on which the representations arise. The general question is to identify the non-trivial finite-dimensional quotients of the group algebra $\mathbb{C}B_n$, on the shape of the Iwahori–Hecke algebra investigated in the past decades. The general philosophy is: the bigger the quotient algebra, the better the results. Until recently, the biggest known algebra was the Birman–Murakami–Wenzl algebra [6].

An exciting development presented during the workshop is the description by Stephen Bigelow of a new family of finite-dimensional quotients of the algebra $\mathbb{C}B_n$ that naturally extends the Iwahori–Hecke and the Birman–Murakami–Wenzl algebras. The latter are just the first two steps in the new family. The new algebras, called "Zipper algebras" and denoted $Z_n(q, r)$, depend on two nonzero complex parameters, and they are defined using a diagrammatic approach. The principle is to introduce an additional generator Box_k visualized by a box with $k+1$ input and $k+1$ output strands, and to extend the usual skein relation declaring that a q-twisted combination of opposite crossings is 0 (case of Hecke algebra), or is the 2-2-tangle (case of BMW) into the relation declaring the q-twisted combination of opposite crossings is the new free generator Box_2. Then, inductively, one adds a similar skein relation relating the diagrams with a k-box and the two possible positions of an additional strand with a $k+1$-box. What Bigelow proves so far is that the algebras $Z_n(q, r)$ have finite dimension, and make a proper extension of the BMW algebra. What remains open is the exact dimension of the Zipper algebra, as well as the degeneracy at $q = 1$.

A graduate student from China, Hao Zheng, also discussed a topological approach to representations of B_n, much in the spirit of Bigelow's earlier work.

One of the best-known (but unfaithful) representations of braid groups is the Burau representation. Jones noted that this representation can be interpreted in terms of probabilities related to a particle jumping from one string to another at a crossing in a braid picture. X. S. Lin expanded on this basic idea to produce a new representation of B_n, using ideas related to probability, which is closely related to the coloured HOMFLY polynomial.

The student Holly Hauschild presented a concrete tangle-theoretic approach to the Birman–Murakami–Wenzl algebra. Morton and Wasserman had shown that the BMW algebra is isomorphic to a Kauffman tangle algebra. Haushchild described an extension of this isomorphism to give a similar correspondence between the affine BMW algebra and a corresponding algebra of tangles in the solid torus.

Taking a more abstract approach, Hans Wenzl described how representations of braid groups can be used to construct and classify certain braided tensor categories which are useful in low dimensional topology, physics and operator theory. In particular, he has classified all representations of B_3 up to dimension five.

The talk of Ivan Marin also considered representation theory of braid groups and their generalizations. He discussed representations obtained in a systematic way from the representations of “infinitesimal braids.” This approach sheds new light on the decompositions of tensor products and the unitarisability properties of braid representations, as well as the actions of the universal Galois group involved in this setting.

Thus it is fair to say that great strides were made in the BIRS workshop toward the understanding of the representation theory of braid groups and the many applications of these ideas. Of course, much remains to be understood in this important subject.

Applications to Knot Theory and Topology

The Jones Polynomial

The most obvious applications of braid theory are to the study of knots. About two decades ago, work of V. Jones [11] established a new powerful knot invariant via representations of B_n. This work led to exciting and unsuspected connections with operator theory, statistical mechanics and other aspects of mathematical physics. It was also generalized to the so-called HOMFLY polynomial, the Kauffman polynomial and a plethora of other knot invariants.

An outstanding open question is whether the Jones polynomial detects the unknot. In other words, if the Jones polynomial $V_K(t)$ of a knot K is trivial, does it imply that K is unknotted? The corresponding question for links of two or more components was settled very recently by Eliahou, Kauffman and Thistlethwaite [9], who displayed infinite families of links with the same Jones polynomial as the unlink, but which are nontrivially linked.

It is also well-known that there are many examples of distinct knots with the same Jones (and HOMFLY) polynomial, using various techniques: Conway mutation, a construction of Kanenobu (producing an infinite family with common Jones polynomial), etc. A new technique was discussed at the workshop by the student Liam Watson, which employs the idea of a braid group action on Conway tangles in a knot diagram to produce distinct knots with the same Jones polynomial, which nevertheless are not Conway mutants. A consequence of his work is that, given any Conway tangle, there exist distinct knots containing that tangle as part of their diagrams, and having the same Jones polynomial. Watson’s techniques (unlike Conway mutation) have the possibility of settling the question of whether the Jones polynomial detects the unknot.

Hitoshi Murakami gave a fascinating lecture on the current state of the art of the so-called volume conjecture, which relates the volume of the complement of a hyperbolic knot K with limits of values of the coloured Jones polynomial. Originally posed by Kashaev, and following work of J. Murakami and H. Murakami, this conjecture can be made precise:

$$Vol(S^3 \setminus K) = 2\pi \lim_{N\to\infty} \log |J_N(K; \exp(\frac{2\pi\sqrt{-1}}{N})|,$$

where $J_N(K;t)$ denotes the N^{th} coloured Jones polynomial of the knot or link K.

This conjecture has been verified for various special cases – the knots 4_1 and 5_2, the Whitehead link and the Borromean rings – by various authors, but remains open in general and is the focus of considerable attention by topologists. Murakami also discussed a complexified version of the formula, in which the absolute value signs in the above equation are removed, and one has the imaginary part of the left-hand side expressed as the Chern-Simons invariant of the complement.

Three-dimensional Manifolds and TQFT’s

One of the most important new tools in the study of 3-manifolds is the Casson invariant $\lambda(M)$, defined by A. Casson for any integral homology 3-sphere M. The original definition by Casson in 1984 involved counting $SU(2)$ representations of the fundamental group of M. Greg Kuperberg and Dylan Thurston showed, in 1999, how to express $\lambda(M)$ as a configuration space integral. A very interesting new approach was explained in the BIRS workshop by Christine Lescop. She showed that $6\lambda(M)$ is the algebraic intersection of three codimension 2 manifolds in the 6-dimensional space of two-point configurations of M, for any integral homology sphere M. Lescop went on to show it extends to the Walker generalisation of the Casson invariant to rational homology spheres, giving a topological characterisation of the Walker invariant.

Partly inspired Jones' lead in connecting braid theory with mathematical physics, and subsequent work by Atiyah, Witten and many others, topological quantum field theories have become an important new field of study. Several of the lectures in the BIRS workshop concentrated on aspects of TQFT's and their application to 3-manifolds.

Gregor Masbaum discussed joint work with P. Gilmer regarding naturally defined lattices in the vector spaces associated to surfaces, by the SO(3) TQFT at an odd prime. These lattices, whose existence comes from the fact that the associated quantum invariants of 3-manifolds are algebraic integers, form an "Integral TQFT" in an appropriate sense. Masbaum defined an explicit basis for this lattice.

In a talk entitled "Braids and hypergeometric integrals," Toshitake Kohno discussed two approaches to braid group representations: the homological approach of Laqwrence, Krammer, Bigelow, et. al. and a more physically motivated approach involving monodromy of flat connections. The latter involves, in particular, solutions to the Khizhnik-Zamolodchikov equations and conformal blocks, as well as the Drinfel'd approach using quantum groups. In his lecture, Kohno related the theory of conformal blocks to certain hypergeometric integrals. This deep subject promises to enrich both the theory of braid representations, as well as questions of interest to mathematical physicists.

Related to the above, the postdoctoral fellow Alissa Crans discussed new methods for finding solutions to the Zamolodchikov tetrahedral equations. In particular, after a discussion of 2-categories, she showed that, just as any Lie algebra gives a solution of the Yang-Baxter equation, any Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation.

Braids and Homotopy Theory

Fred Cohen spoke on some striking connections between braid theory and deep questions of homotopy theory. He related questions some elementary constructions in the pure braid groups, such as string doubling and forgetting strands, with open questions in homotopy theory. As an example, the homotopy groups $\pi_N(S^2)$ have not been calculated for high values of N, and settling questions regarding the constructions on pure braids would determine those groups.

Braids, Combinatorics and Algorithms

A very active area which was well-represented at the conference concerns ideas surrounding Garside's 1969 solution to the word and conjugacy problems in the braid groups [10]. Three talks (by Gonzalez-Meneses, by Gebhardt and by Krammer) related directly to this circle of ideas, with Gonzalez-Meneses and Gebhardt focussing on ways to understand and simplify the combinatorics, while Krammer's efforts were directed toward extending it to surface mapping class groups. The discussions that followed these talks were broadly based, because at least some number of the other participants (e.g. Dehornoy, Paris, Michel, Birman and Brendle) had themselves made important contributions to what have become known as "Garside structures", so that the workshop was a major event for workers in the area.

Another very exciting development was presented by Daan Krammer. Building on the seminal work by Garside, many authors have developed a general theory of Garside groups, which are groups of fractions of monoids in which divisibility has a lattice structure. The braid groups have several Garside structures, namely (at least) the one originally defined by Garside, and the one associated with the recent Birman–Ko–Lee monoid. Krammer proposes new developments that seem to go far beyond the previous attempts. The point is to weaken the condition that the group is the group of fractions of a lattice into the weaker one that the group acts on a lattice, by an action that need not be transitive.

An equivalent way of describing the framework is to introduce the notion of a Garside groupoid (small catgeory where all arrows are invertible). Technically, an extended Garside structure is specified by axiomatizing the intervals $[a, a\Delta]$ of a Garside monoid, where Δ is a Garside element. The main interest of this extended framework is to make it possible to define completely new Garside structures on braid groups — and, possibly, on more general mapping class groups, but this remains a conjecture. The construction starts with considering the braid group B_n as acting on a disk with n punctures, as in Definition 5 above.

Now, the new ingredient is to add q marked points on the boundary circle. By considering certain cell decompositions of such "bi-punctured" disks (punctures in the interior and on the boundary) up to isotopy, one obtains a lattice and, under a convenient version of Dehn's half-twist in which the boundary punctures are shifted, one obtains an action of the braid group B_n on that lattice. In the case $q = 2$ (only the North and the South poles of the disk are marked), the action is simply transitive, and one obtains the standard Garside structure of B_n. For $q \geq 3$, the action is not transitive, and one obtains a completely new structure. In particular, for $q = 3$ (3 punctures on the boundary disk), the lattice can be described explicitly, and, surprisingly enough, the famous MacLane pentagon shows up, and, more generally, the intervals $[a, a\Delta]$ are closely related with the Stasheff associahedra. This opens a new, fascinating connection between Artin's braid group and Richard Thompson's groups, and certainly much more is still to come.

The word and conjugacy problems in the braid groups have importance for their role in public key cryptography. It is well known that the complexity of the word problem in the braid group B_n is $(|W|^2 n)$, where $|W|$ is word length and n is braid index, whereas all solutions to the conjugacy problem known at this time are exponential. Codes have been designed which are based on the assumption that the conjugacy problem is fundamentally exponential, so a polynomial solution to the conjugacy problem would be of major importance.

A new idea was to apply the partial solutions to the same problems by Thurston, by treating braids which are"reducible, finite order and pseudo-Anosov" separately. This proved to be very fruitful as regards the combinatorics of Garside's work in the braid groups. Since Thurston's ideas apply to all mapping class groups, not just to the braid groups, it was then very interesting when Daan Krammer presented his fascinating talk, which aimed to go the other way and introduce Garside-like combinatorics into the study of surface mapping class groups.

It can be mentioned that a different connection between Artin's braid group and Richard Thompson's groups was discussed in Dehornoy's talk in the workshop, devoted to Bar Natan's parenthesized braids. The latter can be made into a group which contains both the braid groups and the Thompson groups, and some new results about self-distributive operations on that new group are quite intriguing. This group also enjoys a left-invariant ordering extending the well-known ordering of B_n.

Another connection with combinatorics was given by Christian Kassel. In joint work with Christophe Reutenauer, they considered the classical idea of Sturmian sequences of two symbols, which occur in fields such as number theory, ergodic theory, dynamical systems, computer science and crystallography. They show that the class of special Sturmian sequences (a submonoid of $Aut(F_2)$) can be realized naturally as a submonoid of the four strand braid group B_4. As an application, this leads to a new criterion for determining when two words form a basis for the free group F_2.

The Markov theorem without stabilization (MTWS) of J. Birman and W. Menasco established a calculus of braid isotopies that can be used to move between closed braid representatives of a given oriented link type without having to increase the braid index by stabilization. Although the calculus is extensive there are three key isotopies that were identified and analyzed— destabilization, exchange moves and braid preserving flypes. One of the critical open problems left in the wake of the MTWS is the *recognition problem*— determining when a given closed n-braid admits a specified move of the calculus. Bill Menasco described an algorithmic solution to the recognition problem for three isotopies of the MTWS calculus—destabilization, exchange moves and braid preserving flypes. The algorithm is "directed" by a complexity measure that can be *monotonic simplified* by the application of *elementary moves* on a modified braid presentation.

Generalizations of the Braid Groups

Because of the many definitions of the braid groups, there are various natural ways to generalize them, some of which have far-reaching applications. Several such generalizations were considered in the BIRS workshop, namely Artin groups (an algebraic generalization), mapping class groups (also known as modular groups), configuration spaces and their algebraic properties

Artin Groups

Deligne [7] and Brieskorn-Saito [5], introduced a family now referred to as Artin groups, which generalizes the braid groups and is also closely related to the so-called Coxeter groups which arise in the study of Lie groups and symmetries of Euclidean space. For a fixed positive integer n, consider an n by n matrix $M = \{m_{ij}\}$, where m_{ij} is a positive integer or ∞, with the assumption that $m_{ij} = m_{ji} \geq 2$ and $m_{ii} = 1$. The corresponding Artin group has a presentation with generators $x_1, \ldots, x_n$ and, for each pair i, j there is a relation:

$$x_i x_j x_i \cdots = x_j x_i x_j \cdots$$

where the product on each side has length m_{ij} ($m_{ij} = \infty$ indicates no relation is present). If one adjoins relations $x_i^2 = 1$, the result is the so-called Coxeter group corresponding to the given matrix.

In this context, the $n+1$ by $n+1$ matrix with entries equal to 3 just above and below the diagonal, and 2 in entries farther from the diagonal, corresponds exactly to the braid group B_n; in this case the Coxeter group is the symmetric group Σ_n. The Artin groups for which the corresponding Coxeter group is finite are an important subclass, referred to as "spherical." As with the braid groups, Artin groups of spherical type correspond to fundamental groups configuration spaces associated to hyperplane arrangements.

Fundamental to the understanding of semisimple Lie groups is the well-known classification of finite Coxeter groups into several infinite families and certain sporadic types E_6, E_7, E_8, F_4, etc. These Coxeter groups are well known to be distinct, but the corresponding question for the associated Artin groups had been open until now. This question was finally settled by L. Paris, as announced in the BIRS workshop. He used various group-theoretic invariants to establish that the spherical Artin groups of (apparently) different type really are non-isomorphic.

In a different approach to the subject, Dan Margalit discussed embeddings of three infinite families of Artin groups (modulo their centres) as finite index subgroups of the mapping class group of a punctured sphere. As a corollary Margalit, in joint work with Bob Bell, was able to classify all injections of these Artin groups into each other.

Reflection Groups

The finite Coxeter groups can be considered as groups of reflections of R^n, acting on configuration spaces, as described in Definition 3 for the case of the braid groups. Several talks focussed on this aspect, as well as natural generalizations to complex reflection groups

The lecture of Gus Lehrer dealt with the cohomology of these configuration spaces, with local coefficients. For the case of the braid groups, this calculation was accomplished by Arnol'd in 1969, with further progress made by Brieskorn, F. Cohen, Orlik-Solomon and others. In particular, the rank of the cohomology in various dimensions is encoded in a Poincaré polynomial, a sort of generating function. Lehrer's lecture gave a method of calculating these polynomials using Z-functions, defined using centralizers, and related this work to varieties defined over number fields.

In the lecture "Hurwicz action on euclidean reflections," Jean Michel discussed a theorem of Dibrovin and Mazocco that, if the Hurwicz action of the braid group on a triple of Euclidean reflections in R^3 has a finite orbit, then the group generated by these reflections is finite. Michel extended this result to the case of R^n, correcting an erroneous proof which had appeared recently in the literature and simplifying the Dibrovin and Mazocco proof as well.

Mapping Class Groups

The mapping class group $Mod(S)$ of an orientable surface S is well-known to be generated by Dehn twists about simple closed curves in S. An important subgroup of this is the Torelli subgroup, consisting of (classes of) homeomorphisms which induce the identity on the homology of S. In particular, the subgroup K of $Mod(S)$ generated by twists along separating curves of S, called the Johnson kernel, lies in the Torelli subgroup. Tara Brendle, in joint work with Dan Margalit, outlined a proof that the abstract commensurator of K satisfies $Comm(K) = Aut(K) = Mod(S)$, thus verifying a conjecture of Benson Farb.

In the interpretation of braid groups as mapping class groups of a punctured disk, one notes that the homeomorphisms involved may be taken to be smooth and area-preserving. Thus B_n is related to the group G of area-preserving diffeomorphisms of the disk. The study of G is also important in understanding flows related to magnetic fields in the solid torus. This was the subject of a fascinating talk by Elena Kudryavtseva, which concentrated on the so-called Calabi invariant, the averaged linking number for pairs of orbits of the magnetic flow in the solid torus. Her main result is that any C^1-smooth function on G is, in fact, a function of the Calabi invariant. This has the consequence that higher-order knot and braid invariants cannot be generalized to invariants of magnetic fields in the solid torus.

Thus we have three classes of groups: the braid groups, mapping class groups of more general surfaces and Artin groups, and while it has been known since the early 1970's that they are interrelated, the full richness of the interrelationship is just now beginning to be made clear. The last word on this fascinating subject does not appear to have been said.

List of Participants

Bigelow, Stephen (University of California, Santa Barbara)
Birman, Joan (Barnard College, Columbia University)
Brendle, Tara (Cornell University)
Clay, Adam (University of British Columbia)
Cohen, Frederick (University of Rochester)
Crans, Alissa (Loyola Marymount University)
Dehornoy, Patrick (University of Caen)
Digne, Francois (Université de Picardie Jules-Verne)
Erlijman, Juliana (University of Regina)
Gebhardt, Volker (University of Sydney)
Ghosh, Shamindra (University of New Hampshire)
Godelle, Eddy (Universite de Caen)
Gonzalez-Meneses, Juan (University of Seville)
Hao, Zheng (Peking University)
Hauschild, Holly (University of Iowa)
Kassel, Christian (CNRS et Université Louis Pasteur)
Kawamuro, Keiko (Columbia University)
Kohno, Toshitake (University of Tokyo)
Krammer, Daan (University of Warwick)
Kudryavtseva, Elena (University of Calgary)
Lehrer, Gus (University of Sydney)
Lescop, Christine (Université Joseph Fourier, Grenoble)
Lin, Xiao-Song (University of California, Riverside)
Margalit, Dan (University of Utah)
Marin, Ivan (Universite d'Evry)
Masbaum, Gregor (Universite Paris 7)
Menasco, William (University at Buffalo)
Michel, Jean (Universite Paris 7)
Morrison, Scott (University of California, Berkeley)
Murakami, Hitoshi (Tokyo Institute of Technology)
Paris, Luis (Université de Bourgogne)
Rolfsen, Dale (University of British Columbia)
Shpilrain, Vladimir (City College of New York)
Watson, Liam (University of British Columbia)
Wenzl, Hans (University of California at San Diego)
Xicoténcatl, Miguel (Cinvestav Inst. Polytechnico Nacional)
Yurasovskaya, Katya (University of British Columbia)

Bibliography

[1] Artin, E. ,Theorie der Zöpfe, Abh. Math. Sem. Hamburg. Univ. 4(1926), 47 – 72.

[2] Artin, E. , Theory of braids, Annals of Math. (2) 48(1947), 101 – 126.

[3] Bigelow, S. J., Braid groups are linear. J. Amer. Math. Soc. 14 (2001), no. 2, 471–486 (electronic).

[4] Birman, J., *Braids, links and mapping class groups*, Annals of Math. Studies 82, Princeton Univ. Press, 1974.

[5] Brieskorn, E. and Saito, K., Artin-Gruppen und Coxeter-Gruppen, Invent. Math. 17 (1972), 245–271.

[6] Birman, J. and Wenzl, H., Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc. 313(1989), 249 – 273.

[7] Deligne, P. Les immeubles des groupes de tresses généralisés. Invent. Math. 17 (1972), 273–302.

[8] Dubrovin, B. and Mazzocco, M., Monodromy of certain Painlevé-VI transcendents and reflection groups. Invent. Math. 141 (2000), no. 1, 55–147.

[9] Eliahou, Shalom; Kauffman, Louis H. and Thistlethwaite, Morwen B., Infinite families of links with trivial Jones polynomial. Topology 42 (2003), no. 1, 155–169.

[10] Garside, F. A. , The braid group and other groups, Quart. J. Math. Oxford Ser. (2) 20 1969 235–254.

[11] Jones, V. F. R., A polynomial invariant for knots via Von Neumann algebras, *Bull. Amer. Math. Soc.* 12(1985), 103–111.

[12] Krammer, D, Braid groups are linear. Ann. of Math. (2) 155 (2002), no. 1, 131–156.

Chapter 32

Mathematical Image Analysis and Processing (04w5512)

October 23–28, 2004

Organizer(s): Selim Esedoḡlu (UCLA), Sung-Ha Kang (University of Kentucky), Mary Pugh (University of Toronto), Jackie (Jianhong) Shen (University of Minnesota)

Scientific Statement of the Workshop

Thanks to technological breakthroughs in the past few decades, mankind is now able to see images of worlds without (distant planets, galaxies, and the surface terrain of the Earth) and worlds within (human organs, geological imaging, and atomic and molecular structures at the nanoscale level). See Figure 32.1.

As the state-of-the-art imaging technologies became more and more advanced, yielding scientific data at unprecedented detail and volume, the need to process and interpret all the data has made image processing and computer vision also increasingly important. Sources of data that have to be routinely dealt with today include video transmission, wireless communication, automatic fingerprint processing, from massive databanks, non-weary and accurate automatic airport screening (e.g., the USA Federal government's experiment of retina-image based automatic screening at the Minneapolis-St. Paul International Airport), robust night vision for civilian rescue workers or battlefield soldiers, and vision repair for patients with vision defects.

Input from mathematicians has had a fundamental impact on many scientific disciplines. When accurate, robust, stable, and efficient tools were required in more traditional areas of science and technology, mathematics often played a role in helping to supply them. No doubt the same will be true in the case of imaging and vision sciences.

The workshop *Mathematical Image Analysis and Processing* was motivated by both the imminence of

Figure 32.1: (a) Images from Mars' explorer, (b) Brain MRI image.

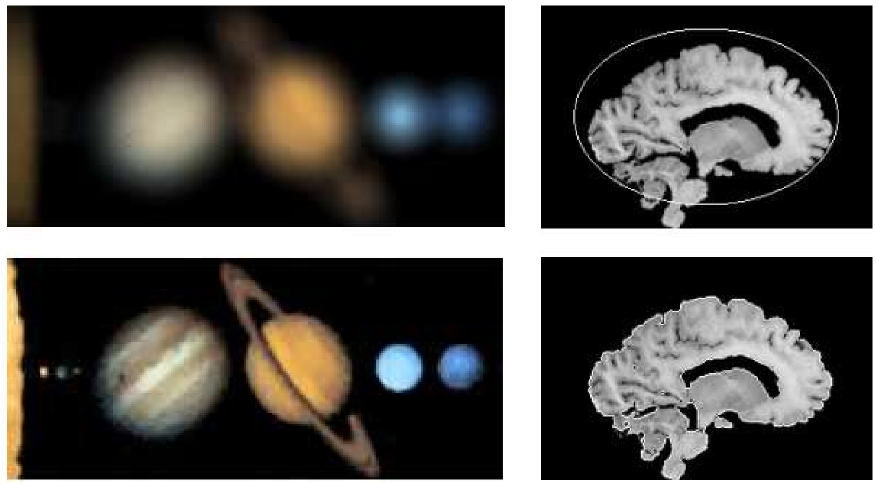


Figure 32.2: Example of image restoration (a) deblurring, and (b) image segmentation

vision sciences, and this principle about the role of mathematics. With the support of BIRS, it provided a solid platform for enthusiastic mathematicians to further their research in imaging and vision sciences, and start collaborations that will lead to new discoveries.

Mathematical Statement of the Workshop

Mathematical Image Processing is a rapidly growing field. As such, there are many different approaches for addressing similar questions. The main task of our workshop has been to concentrate on a few mathematically intriguing problems, and to allow researchers to present the state-of-the-art in their approaches to these problems. In this way, the workshop has achieved two goals. First, it has provided a forum where results from different approaches can be systematically compared as well as integrated. Second, by making sure a variety of mathematical areas (from pure theoretical analysis to practical computational techniques) are represented, the workshop will encourage more mathematicians to work on imaging and vision problems.

The scientific structure of the workshop was based upon the following intrinsic principle: *from image analysis to image processing*. Designing a successful processing technique relies on having a successful model for images themselves. Image analysis mainly focuses on image spaces and efficient ways to represent images, such as spectral analysis, wavelets, statistics, level-sets and PDEs. Image processing involves modifying the original images in order to improve the visual qualities or to extract valuable information for further higher-level processing. Some familiar processing tasks are image restoration, compression, segmentation, shape analysis, and texture extraction (see Figure 32.2).

The three crucial ingredients of image analysis and processing are: modelling, analysis, and computational implementation. In the past couple of decades, mathematicians have been able to make substantial contributions in all these areas. Our workshop was instrumental in helping both active and newly interested mathematicians to refine and further existing ideas, as well as to highlight and concentrate on challenging problems.

Participants

The glowing reputation of BIRS allowed us to attract a highly diverse group of participants for our workshop.

Disciplinary Diversity. The themes and missions of the workshop successfully attracted not only mathematicians, but also scientists from closely connected areas of electrical engineering, computer science, information sciences, biomedical engineering, industrial engineering, psychology, as well as from national research labs. Such multidisciplinary interaction and collaboration is a characteristic of mathematical image

Figure 32.3: Some of our participants.

analysis and processing as a growing field in applied mathematics, and is a solid foundation for long-term healthy development.

Global Diversity. The majority of the thirty-eight participants came from Canada, Mexico, and the USA. In addition, there was representation from Austria, Finland, France, Germany, Israel, Italy, and Norway. Such global collaboration will make it easier for researchers to communicate, coordinate, and optimize their scientific and academic resources (e.g., funding or holding future conferences on mathematical image processing, training and exchanging research assistants or postdocs, and collaborating on and unifying research projects).

Visible Minorities. There were eight female participants and eleven participants from racial groups that are under-represented in North American science.

Level Diversity. There was a good mixture of participants from all levels: graduate students, post-doctoral researchers, tenure-track professors, and senior scientists. Senior scientists who participated are: Bill Allard, Andrea Bertozzi, Tony Chan, Gerardo E. García Almeida, Jim Little, Brad Lucier, Riccardo March, Dimitris Metaxas, Mila Nikolova, John Oliensis, Mary Pugh, Martin Rumpf, Fadil Santosa, Otmar Scherzer, Volker Schmidt, Jayant Shah, Kaleem Siddiqi, Xue-Chang Tai, Baba Vemuri, Luminita Vese, Curt Vogel, and Ross Whitaker. Post-docs and tenure-track assistant professors who participated are: Mimi Boutin, Selim Esedoglu, John Greer, Sinan Gunturk, Sung-Ha Kang, Stacey Levine, Kirsi Majava, Francois Malgouyres, Jackie Shen, Richard Tsai, Kevin Vixie, Lior Wolf, and Haomin Zhou. There were three advanced graduate students present: Toni Buades-Capo, Fred Park and Alon Spira. This range of levels demonstrates the strength of mathematical image analysis and processing in both contemporary scientific arenas and its potential in future ones.

Scientific Overview of the Workshop:

The workshop covered a variety of topics and methodologies in contemporary image and vision analysis.
In terms of scientific areas, the workshop touched upon:

(a) computer vision, especially shape analysis and scene reconstruction;

(b) theoretical image analysis;

(c) general image processing including edge detection, denoising, deblurring, inpainting, registration, and segmentation;

(d) biomedical image processing including diffusion tensor imaging, tumour detection, and retinal movement quantification.

(e) industrial image processing, including the automobile and printer industries.

(f) information and communication sciences, including efficient image data compression, coding, and error concealment.

(a) (b)

Figure 32.4: (a) Dr. Martin Rumpf and (b) Dr. Tony Chan giving talks at the workshop.

The following mathematical methodologies were represented at the workshop:

(1) variational optimization, especially on non-quadratic, non-convex, geometry-oriented, and data adapted "energy" functionals;

(2) inverse problems for ill-posed problems and regularization;

(3) nonlinear partial differential equations (PDE), to model, simulate, or achieve equivalent physical actions such as diffusion, convection, free-boundary interface motions, and phase transitions;

(4) differential geometry, for processing data information on general Riemannian manifolds and embedded surfaces, as well as for processing generic geometric information such as total curvatures and principle curvatures;

(5) Lie groups and invariant theory, for studying affine or projective invariants of image acquisition from camera motions and different views;

(6) quantum information theory, especially using the quantum probabilistic view to develop novel approaches to mathematical learning theory;

(7) statistical and information theory, to process data features or patterns that are beyond the effective description of deterministic models such as PDEs and variational energies;

(8) harmonic analysis, on analyzing wavelets compressing and coding schemes, as well as investigating the interaction between such atomic decompositions and variational/PDE methods;

(9) real and functional analysis, such as using distributions and oscillatory functional spaces (e.g., BMO) to model textures;

(10) computational logic, e.g., how to make sound Boolean decisions for noisy or multi-channel data;

(11) numerical analysis, including the stability and accuracy of high-order nonlinear PDE schemes;

(12) scientific computation, including the level-set method, numerical PDE, Γ-convergence regularization, thresholding dynamics, iterative algorithms, as well as multiphase computation.

The applications of theories and methodologies of the workshop have been found in:

(i) medicine and the health sciences, including tumour detection and robust nerve fibre tracing in the brain;

(ii) industrial engineering, such as designing an automatic feed-back control (vision) system in the automobile industry and improving the quality of inkjet printers;

(iii) astronomy, such as enhancing and improving the quality of telescope observations by adaptive denoising, deblurring, and repairing;

(iv) communication technologies, including efficient data coding and error concealment for noisy or lossy channels;

(v) artificial intelligence, including parameter-free learning processes;

(vi) movie and art restoration, and computer graphics;

(vii) surveillance video and airport security;

(viii) robot vision system, including object and scene perception;

(ix) military applications, such as for autopiloting planes to automatically track enemy vehicles and moving information.

Such a broad scope and range made Mathematical Image Analysis and Processing'04 a unique workshop. The participants found themselves in an ideal environment where they could freely communicate their research ideas as well as be nourished with fresh ideas from other participants.

Highlights of Presentations

The following are not merely summaries of the presentations, but also highlight how the researchers have made novel contributions as well as what new important trends they represent. (The summaries are given in presentation order.)

Dr. Baba Vemuri, from the University of Florida, proposed novel metrics for measuring the distances between tensors, as motivated by the application of diffusion tensor imaging in medical MRI. The talk showed the power of combining knowledge and tools from different areas, such as the probabilistic view of diffusion tensors (e.g., Gaussian distributions, the Kullback-Leibler distances between distributions, etc.), variational/PDE methods like region based active contour models, and restoration of Riemannian features (e.g., unit vectors on the spheres or orthonormal frames). The talk also demonstrated the growing realization that any imaging model or computational scheme often crucially depends upon using the proper metrics or measures.

Dr. Stacey Levine, from Duquesne University, presented her work on applying non-standard growth functionals for image denoising and image decomposition. Inspired by the previous works of Chambolle and Lions (1997), You and Kaveh (2000), and Chan et al. (1998), Dr. Levine studied the following functional on a given image domain Ω:

$$\int_\Omega \phi(x, \nabla u)dx,$$

where, unlike the conventional uniform Sobolev norm $\phi(x, \mathbf{p}) = |\mathbf{p}|^2$ or the Total Variation Radon measure $\phi(x, \mathbf{p}) = |\mathbf{p}|$, a non-standard growth exponent $q(x)$ is incorporated by

$$\phi(x, \mathbf{p}) = \frac{1}{q(x)}|\mathbf{p}|^{q(x)}, \quad |\mathbf{p}| < 1; \qquad |\mathbf{p}| - 1 + \frac{1}{q(x)}, \quad \text{otherwise.}$$

The spatially dependent growth exponent $q(x)$ is assumed between $1+\epsilon$ and 2 in order to ensure convexity. Dr. Levine studied both the mathematical theory (e.g., existence and uniqueness) and the computational performance of such models. The experiments showed noticeable improvement over conventional models for denoising and texture extraction.

Dr. Otmar Scherzer, from the University of Innsbruck, talked about studying illposed inverse problems using general nonlinear, non-differentiable, or non-convex regularizers. One particular problem is to restore an image whose pixels have been randomly (and blindly) switched. When the random displacement is local, a Taylor expansion inspired the following degenerate, nonlinear, and non-convex functional for solving such an ill-posed inverse problem:

$$\min_u \int_\Omega |Du| + \int_\Omega \lambda(x,u,\nabla u)(u-u_0)^2 dx,$$

where $\lambda(x,u,\nabla u) = \lambda + |\nabla u|^{-2}$, with the constant λ also handling potential intensity noises. The non-convex and singular dependence of $\lambda(x,u,\nabla u)$ on ∇u leads to analytical difficulties. Dr. Scherzer showed how to resolve these issues.

Alon Spira, from the Technion, discussed a general framework for evolving images and curves on parametrized Riemannian manifolds. Of particular interest for image processing and vision tasks are the Eikonal equations and Beltrami flows on Riemannian manifolds. The computational efforts presented in the work are representative of the literature for robustly and efficiently handling geometry while performing certain image processing tasks.

Dr. Martin Rumpf, from the Gerhard Mercator Universität-Gesamthochschule, focused on his work on high-order geometric flows for anisotropic images and surface processing, as motivated by Willmore-type energies and anisotropic surface functionals, e.g.,

$$A_\gamma(M) = \int_M \gamma(n)da,$$

where M denotes an embedded surface in $\mathbb{R}^3$, n its unit normal, and da the surface element. The anisotropy comes from the choice of γ. If γ has the general form of $\gamma(x,n,H,K)$ with mean curvature H and Gaussian curvature K, then the resulting Euler-Lagrange equations are inevitably of high order (e.g., 4th order). Dr. Rumpf also presented efficient numerical methods for handling such high order geometric flows.

Dr. Lior Wolf, from MIT's Center for Biological and Computational Learning, presented his recent work on a novel learning theory inspired by the Born Rule from Quantum Mechanics. Let Ψ be a pure quantum state (represented by a column vector for simplicity) and $\rho = \Psi\Psi^T$ be its associated projection. Let M be a rank-one observable, i.e., $M = aa^T$ for some column vector a. Then the Born Rule claims that the expected value of the observable M is given by

$$\mathrm{trace}(M\rho) = \Psi^T M \Psi = |\langle a, \Psi\rangle|^2.$$

Dr. Wolf explained how one can design novel learning algorithms based on the Born Rule, especially for data clustering (either two-class or multiple-class).

Dr. Mila Nikolova, from the Ecole Normale Superieure de Cachan, presented her work on edge recovery via nonconvex regularized least-squares. The properties of restored images and signals were carefully investigated based on the shape information of the regularization function. One remarkable result revealed a major difference between the edge-preserving convex functionals and non-convex regularizers: in the latter case, given a (local) minimizer the (intensity) difference between adjacent pixels are either suppressed or enhanced.

Toni Buades-Capo, from the Ecole Normale Superieure de Cachan, presented his work on rigorously comparing the performance of different image denoising algorithms and models. Work of this nature is of paramount importance to the imaging and vision literature, since for any given task, different research groups often develop different models and algorithms. Image denoising is a well-known example. Popular denoising methods include spectral methods, Wiener filtering, wavelet thresholding, adaptive filtering, anisotropic diffusions, etc. The key issue in developing rigorous comparative criteria is how to properly define performance measures, and how to properly interpret them from both vision and information theoretic points of view. That is; how to move beyond the "eyeball metric" in which the researcher visually judges the quality of the method, risking the possibility of bias.

Dr. Brad Lucier, from Purdue, investigated the variational smoothing of a class of Besov images $B^1_\infty(L^1)$ based on both a wavelet representation and on direct pixel-domain smoothing. This work reflects a growing need and trend for hybridizing different methodologies in image processing. Lucier's earlier works with Chambolle and DeVore in 1998 initiated the interaction between the variational formulations and wavelet algorithms. The present work further expands this direction by comparing the performance using different (but equivalent) formulations of the Besov semi-norms.

Dr. Ross Whitaker, from the University of Utah, presented impressive computational results for image denoising and other applications, based on an information-theoretic approach for adaptive filtering. The results appear superior to those in the variational literature which often implicitly assume that the underlying ideal images belong to the space of functions of bounded variation, $\mathrm{BV}(\Omega)$, or to some Sobolev space $W^{\alpha,p}(\Omega)$. The information-theoretic and statistical nature of Dr. Whitaker's new approach enables more robust and general prediction of individual pixel values based on local information as well as on suitable entropy measures. The work reflected the benefit of applying information-theoretic tools (often developed for the information and communication sciences) to image and vision analysis.

Dr. Luminita Vese, from UCLA, explored tools from real and functional analysis to study image decomposition. The primary goal of image decomposition is to separate a given image u_0 into visually or functionally meaningful components. A popular framework, initiated by Yves Meyer, is to decompose u_0 into $u + v + w$, where $u \in \mathrm{BV}$, w is Gaussian white noise, and v is an oscillatory pattern in $\mathrm{div}(L^\infty)$. The goal of solving such a model is to perform three functions at once — to denoise the image (find w), to segment the image (find u), and to find the textures of the various surfaces (find v). Dr. Vese presented her work on how to compute her new model using $\mathrm{div}(\mathrm{BMO})$ to model v, where BMO denotes all the locally summable functions with bounded mean oscillations.

Dr. Xue-Cheng Tai, from the University of Bergen, discussed how to develop computational and numerical analysis on piecewise constant level-set methods. Unlike the level-set setting of Osher and Sethian (1987) for which the level-set functions are Lipschitz continuous, Dr. Tai proposed employing piecewise constant level-set functions. The range of the level set function is preassigned, e.g., a finite set of integers, or merely binary values of -1 and $+1$. Dr. Tai also presented the successful applications of his methods to digital image processing as well as to solving ill-posed inverse problems.

Dr. Bill Allard from Duke presented his new results on the regularity of level sets of minimizers to total variation based image denoising models. These results, which draw on tools from geometric measure theory, apply to a wide range of models that have been considered in the image processing literature. Dr. Allard's results thus elucidate common features of these models, especially in regard to the bias that these models introduce (because of their regularization terms) into image reconstructions. Dr. Allard's results also allowed him to construct relevant and nontrivial exact solutions to several of these total variation based models. Such exact solutions are valuable in that they allow us to rigorously contrast the behaviour of competing variational models. They also constitute test cases for numerical algorithms, so that the algorithms' performance can be scientifically measured instead of judging them solely on whether their results are visually pleasing or not.

Dr. Kevin Vixie, from Los Alamos National Lab, discussed his work investigating exact solutions to a variational denoising model based on the TV regularizer with L^1 fidelity. When the given target image is an indicator function of some set, under certain conditions the exact solutions can be characterized in terms of disks with appropriate radii.

Dr. Jayant Shah, from Northeastern University, a pioneer in mathematical image and vision analysis, presented some new developments on processing the skeleton information of generic shapes. The skeleton of a given closed shape with Lipschitz boundary is the locus along which the distance function of shape boundary has first order singularities, like the ridge of a mountain range. In computer vision, skeleton information has been used for object recognition, detection, and differentiation. The advance in Dr. Shah's work is to study gray skeletons instead of conventional binary ones.

Dr. Riccardo March, from the Istituto per le Applicazioni del Calcolo, presented his work on the curvature-dependent nonlinear and nonconvex functionals emerging from recent works in image processing. In order to more faithfully process edge information in models, Dr. March proposed to incorporate the information about endpoints, length, and curvature into the variational formulation. A Γ-convergence theory is then developed for such geometric functionals. It allows approximating the original functional by ones that are numerically much more convenient. Dr. March then explained the main ideas of the proof that establishes the convergence of approximate functionals to the original one. The work presented was an example of how rigorous mathematics helped solve some very practical problems encountered in image processing and computer vision.

Fred Park, from UCLA, presented his work on a total variation based model for simultaneous image inpainting and blind deconvolution. Blind deconvolution is used for recovering information from telescope images such as that in Figure 32.2; image inpainting is how scratches are "removed" from old photographs. He demonstrated that these tasks are best treated as a coupled image processing task. This allows him to take boundary conditions for the deconvolution that are naturally generated by the inpainting; thus reducing ringing effects that can arise from a poor choice of boundary conditions.

Dr. Haomin Zhou, from Georgia Tech, spoke about his efforts to combine PDE ideas and methods with wavelet techniques. A new class of singularity-adapted and more efficient wavelet transforms are invented by using shock-capturing schemes from computational fluid dynamics. Dr. Zhou also discussed how to use variational approaches and well-chosen functionals in the pixel domain to help improve the quality of wavelet compression and interpolation. Such work represents the exciting and necessary trend of combining the variational/PDE methodology with harmonic analysis.

Dr. Richard Tsai, from the UT Austin, presented a fast algorithm for constructing minimizing sequences for the Mumford-Shah segmentation functional. Inspired by earlier works of Merriman, Bence, and Osher on threshold dynamics for geometric curve evolution, Dr. Tsai discussed how the new fast algorithm was discovered and should be understood, and presented numerical results illustrating its effectiveness.

Dr. Kaleem Siddiqi, from McGill University, discussed a novel approach to shape analysis. Given the distance function of a shape, one computes its gradient. Loosely speaking, Dr. Siddiqi then views this gradient as a flux, reducing the problem of finding the Blum skeleton of a shape to finding sources in the flux.

Dr. Curt Vogel, from Montana State University, presented a new image processing problem which is markedly different from most. His goal is to robustly estimate and track the motion of the retina, based on high resolution scan data produced by laser devices. Retinal tracking is a crucial step in understanding the early human visual system. The major challenge is to distinguish irrelevant eye motion from the intrinsic motion of the retina, and to properly determine the optical properties. This work signifies the importance of image and vision processing in biological data probing and analysis.

Dr. Tony Chan, from UCLA, presented a new framework of logical segmentation for multichannel images as well as logical tracking for video images. This work improves the applicability of the conventional Mumford-Shah segmentation model by incorporating Boolean algebra. The work is especially relevant when information from multiple sources (such as different colour channels, or even different imaging modalities) needs to be combined for the segmentation task.

Dr. Dimitris Metaxas, from Rutgers University, presented work in deforming shapes and interior textures (Metamorphs) as applied to medical imaging of the human heart. He also presented work in computer animation of breaking water waves. A major advantage of his algorithms was their computational efficiency.

Dr. Kirsi Majava, from the University of Jyväskylä, discussed her work on applying the active-set algorithm for nonsmooth variational optimization problems like

$$\min_u \int_\Omega |u-u_0|^s dx + \beta \int_\Omega |\nabla u|^r dx,$$

commonly seen in image denoising and restoration. She presented computational results suggesting this new algorithm is very promising. Potential connections to discretized active contour algorithms were suggested by the audience and further investigation into this novel algorithm is a hot topic.

Dr. Francois Malgouyres, from the Universite Paris 13, presented his work on achieving good image compression schemes through projections on polyhedral sets, extending his earlier work on image restoration.

Dr. Fadil Santosa, from the University of Minnesota, presented an interesting inverse problem arising from the imaging of spotwelds. A typical car has more than 20,000 spotwelds holding together metal sheets. The evaluation and monitoring of their quality is achieved by noninvasive thermal imaging devices. Dr. Santosa has been able to develop models, analyze the resulting illposed inverse problems, as well as computationally simulate the models based on regularization techniques.

Dr. Sinan Gunturk, from New York University, explored the mathematics behind a well-known class of analog-to-digital converters in signal processing ($\Sigma\Delta$-transition) as well as a similar type of converter in modern inkjet printers, called digital halftoning algorithms. Dr. Gunturk also discussed how to use modern multiscale ideas to develop novel and more accurate halftoning algorithms.

Dr. Andrea Bertozzi from UCLA, and Dr. John Greer, from New York University, both presented interesting work applying high-order PDE to image denoising, regularization, and image inpainting. High order PDE have emerged in the recent literature of image processing, such as the inpainting model of Bertalmio et. al. (2000):

$$u_t + (\nabla u)^\perp \cdot \nabla(\Delta u) = 0$$

and the LCIS (low-curvature image simplifier) equation:

$$u_t + \nabla \cdot (g(|\Delta u|)\nabla \Delta u) = 0.$$

Rigorous study of the existence, uniqueness, and proper boundary conditions is highly challenging and is needed in the image processing literature. The works of Dr. Bertozzi and Dr. Greer shed fresh light on both the analytical and algorithmic structures on these equations.

Dr. Mimi Boutin, from Purdue, presented work on applying invariant theory and the moving-frame method to the reconstruction of 3-D scenes from 2-D image projections. These tools from Lie Groups can extract valuable information parameters while discarding superfluous unknowns, and thus lead to a simple and robust scene reconstruction theory.

Dr. Gerardo Garcia Almeida, from the University Antonoma de Yucatan, presented his work on integral equations of the first kind. He presented anisotropic generalizations of the Tikhonov regularization and then sought the optimal value for the regularization parameter. The choice of the image space is particularly important for this analysis. He considers anisotropic Nikol'ski-Besov space to give more possibilities for applications of this method. His methods are from harmonic analysis.

Dr. Volker Schmidt, from the Universität Ulm, presented a new method for the statistical analysis of binary features. By using tools such as convex and stochastic geometry, the morphological image characteristics are robustly estimated statistically. This was a lovely piece of work in which he was able to extract topological features of all degrees (points, edges, facets, etc.)

List of Participants

Allard, William K. (Duke University)
Bertozzi, Andrea (Duke University/University of California, Los Angeles)
Boutin, Mireille (Purdue University)
Buades Capo, Toni (Centre de Mathematiques et de Leurs Applications)
Chan, Tony (University of California, Los Angeles)
Esedoglu, Selim (University of California, Los Angeles)
Garcia Almeida, Gerardo E. (Universidad Autónoma de Yucatán)
Greer, John B. (Courant Institute of Mathematical Sciences)
Gunturk, Sinan (New York University)
Kang, Sung Ha (University of Kentucky)
Levine, Stacey (Duquesne University)
Little, James (University of British Columbia)
Lucier, Bradley (Purdue University)
Majava, Kirsi (University of Jyvaskyla)
Malgouyres, Francois (IMaitre de conference a l'universite Paris 13)
March, Riccardo (Istituto per le Applicazioni del Calcolo, Consiglio Nazionale delle Ricerche)
Metaxas, Dimitris (Rutgers University)
Nikolova, Mila (Ecole Normale Supérieure de Cachan)
Oliensis, John (Stevens Institute of Technology)
Park, Fred (University of California, Los Angeles)
Pugh, Mary (University of Toronto)
Rumpf, Martin (Universität Duisburg)
Santosa, Fadil (University of Minnesota)
Scherzer, Otmar (University of Innsbruck)
Schmidt, Volker (University of Ulm)
Shah, Jayant (Northeastern University)
Shen, Jackie (University of Minnesota)
Siddiqi, Kaleem (McGill University)
Spira, Alon (Technion - Israel Institute of Technology)
Tai, Xue-Cheng (University of Bergen)
Tsai, Richard (University of Texas, Austin)
Vemuri, Baba (University of Florida)
Vese, Luminita (University of California, Los Angeles)
Vixie, Kevin (Los Alamos National Laboratory)

Vogel, Curtis (Montana State University)
Whitaker, Ross (University of Utah)
Wolf, Lior (Massachusetts Institute of Technology)
Zhou, Hao Min (Georgia Institute of Technology)

Chapter 33

The Structure of Amenable Systems (04w5045)

October 30–November 4, 2004

Organizer(s): George Elliott (University of Toronto), Andrew Dean (Lakehead University), Thierry Giordano (University of Ottawa), Guihua Gong (University of Puerto Rico), Huaxin Lin (University of Oregon), N. Christopher Phillips (University of Oregon)

The subject matter of the recent BIRS workshop on amenable systems could be roughly divided into four broad categories: classification of amenable C^*-algebras and related topics, C^*-algebras associated to directed graphs and related objects, commutative dynamical systems and C^*-algebras, and non-commutative dynamical systems. The state of research in amenable systems and the results presented at the workshop are discussed below under these headings.

1. Classification of Amenable C*-algebras and Related Topics

Classification of amenable $C*$-algebras was only a dream some 16 years ago, a dream that started with Elliott's classification of AF-algebras. The Elliott program could be simply described as classification of amenable C^*-algebras by a K-theoretical invariant (the Elliott invariant). Today, the Elliott program of classification of amenable C^*-algebras has become a very successful and continuing story. To name a few break-through results in the program we mention: the Elliott-Gong theorem, which classified simple AH-algebra of real rank zero with local spectra of dimension at most three; the Kirchberg-Phillips theorem on classification of separable, amenable, purely infinite, simple, C^*-algebras which satisfy the Universal Coefficient Theorem; and the Elliott-Gong-Li classification theorem for simple AH-algebras with no dimension growth.

On the other hand, Villadesen's amazing construction of simple AH-algebras with higher stable rank opened a whole new horizon, as well as indicated new difficulties in the Elliott program. During the workshop, A. Toms exploited Villadesen's construction further. He reported that one can construct a class of simple AH-algebras whose isomorphic invariant set must include something other than the conventional Elliott invariant. This mystery injects new excitement into the Elliott program.

Z. Niu demonstrated possibilities of attacking general simple ASH-algebras which are not simple AH-algebras.

Interesting results on classification of non-simple C^*-algebras were also given in the work shop, for example, by Dadalart and Pasnicu.

Closely related to dynamical systems, H. Lin and N. C. Phillips reported that simple crossed products arising from minimal dynamical systems on finite dimensional compact metric spaces have zero tracial rank and therefore are classifiable if the ranges of their K_0-groups are dense in the affine functions on their tracial spaces. This result, together with Lin's work on amenable simple C^*-algebras with lower tracial rank, demonstrated that the above mentioned classification theorem of Elliott-Gong (as well as the result of Elliott-Gong-Li) can be applied to many naturally arising C^*-algebras, in particular, as the title of this work shop suggests, those C^*-algebras arising from amenable dynamical systems.

Other related topics were discussed during the work shop.

A C^*-algebra is said to be self-absorbing if $A \otimes A$ is isomorphic to itself. W. Winter reported that there are only a few such amenable simple C^*-algebras.

D. Kucerovski and P-W. Ng reported a number of absorbing theorems which are closely related to the classification of amenable C^*-algebras.

M. Dadarlat reported a new development regarding the Universal Coefficient Theorem. He revisited the topology on the Kasparov groups and showed that for two separable amenable C^*-algebras, KL-equivalence is the same as KK-equivalence. One may hope that all separable amenable C^*-algebras satisfy the UCT.

2. C*-algebras Associated to Directed Graphs and Related Objects

A directed graph is a combinatorial object consisting of vertices and oriented edges joining pairs of vertices. We can represent such a graph by operators on a Hilbert space $\mathcal{H}$: the vertices are represented by mutually orthogonal closed subspaces, or more precisely the projections onto these subspaces, and the edges by operators between the appropriate subspaces. The graph algebra is, loosely speaking, the C^*-algebra generated by these operators.

When the graph is finite and highly connected, the graph algebras coincide with a family of C^*-algebras first studied by Cuntz and Krieger in 1980 [3]. The Cuntz-Krieger algebras were quickly recognised to be a rich supply of examples for operator algebraists, and also cropped up in unexpected places [13], [16]. In the past 10 years there has been a great deal of interest in graph C^*-algebras associated to infinite graphs, and these have arisen in new contexts: in non-abelian duality [12], [5], as deformations of commutative algebras [17],[7], in non-commutative geometry [4], [15], and as models for the clasification of simple C^*-algebras [8].

Graph algebras have an attractive structure theory, in which algebraic properties of the algebra are related to combinatorial properties of paths in the directed graph. The fundamental theorems of the subject are analogues of those proved by Cuntz and Krieger, and include uniqueness theorems and a description of the ideals in graph algebras. But we know so much more: just about any C^*-algebraic property a graph algebra might have can be determined by looking at the underlying graph.

Higher-rank graphs are, as the name suggests, higher dimensional analogues of directed graphs. They were introduced by Kumjian and Pask [11], and have recently been attracting a good deal of attention. Uniqueness theorems have been proved, and though they are significantly more complicated than graph algebras, we are finding out more about them every day. Recently there have been some partial results on their K-theory [1], [6] and there are some recent results by Raeburn, Sims, and Pask which show that a large class of simple AT algebras can be realised as two dimensional graph algebras. The future may hold many more intriguing results.

Other generalisations of graph algebras that have been studied include the ultragraph C^*-algebras introduced by Tomforde [18] and the labelled graph C^*-algebras introduced by Bates and Pask [2]. An ultragraph is a generalisation of a directed graph in which the edges have a set-valued range. To form labelled graphs, the edges of a directed graph are given labels coming from some alphabet. At this time the basic uniqueness and simplicity results have been proved for these algebras, and theorems have been proved which show that some of their structural properties can be determined by looking at the underlying graph and its labelling. Katsura [9] has done a vast amount of work in describing C^*-algebras associated to topological graphs. There is also a substantial group of mathematicians working on non self-adjoint operator algebras associated to directed graphs (see for example, [10]. Other practitioners include Muhly, Solel and Hopenwasser). The results here are remarkable: the directed graph itself is the invariant for classification of these operator algebras.

At the workshop, Teresa Bates presented some preliminary results on labelled graph C^*-algebras, Toke Carlsen and Alex Kumjian discussed higher rank graph C^*-algebras, and Takeshi Katsura presented some results related to topological graph C^*-algebras. David Kribs presented some new results on weighted graph C^*-algebras.

Bibliography

[1] Allen, S , Pask, D and Sims, A *A dual graph construction for higher rank graphs, and K-theory*, Proc. Amer. Math. Soc., to appear.

[2] Bates, T and Pask, D *The C^*-algebras of labelled graphs*, in preparation.

[3] Cuntz, J and Krieger, W *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

[4] Consani, C and Marcolli, M *Spectral triples from mumford curves* Interant. Math. Res. Not **36** (2003), 1945–1972.

[5] Deicke, K, Pask, D and Raeburn, I *Coverings of directed graphs and crossed products of C^*-algebras by coactions of homogeneous spaces*, Internat. J. Math. **14** (2003), 773–789.

[6] Evans, G *K-theory for the C^*-algebras of higher rank graphs*, preprint, University of Copenhagen.

[7] Hong, J-H and Szymanski,W *Quantum spheres and projective spaces as graph algebras* Comm. Math. Phys. **232** (2002), 157–188.

[8] Hong, J-H and Szymanski, W *Purely infinite Cuntz-Krieger algebras of directed graphs*. Bull. London Math. Soc. **35** (2003), 689–696.

[9] Katsura, T. *A class of C^*-algebras generalising both graph algebras and homemomorphism algebras. I Fundamental results*, Trans. Amer. math. Soc. **356** (2004), 4287–4322.

[10] Kribs, D. W. and Powers, S.C *Free semigroupoid algebras*, J. Ramanujan Math. Soc. **19** (2004), 117–159.

[11] Kumjian, A and Pask, D *Higher rank graph C^*-algebras*, New York J. Math **6** (2000), 1–20.

[12] Kumjian, A, Pask, D and Raeburn, I *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.

[13] Pask, D and Raeburn, I *On the K-theory of Cuntz-Krieger algebras*, Publ. Res. Inst. Math. Sci. **32** (1996), 415–443.

[14] Pask, D, Raeburn, I and Sims, A *AT k-graph C^*-algebras*, in preparation.

[15] Pask, D and Rennie, A *Building spectral triples on graph C^*-algebras*, in preparation.

[16] Speilberg, J *Free product groups, Cuntz-Krieger algebras and covariant maps*, Internat. J. Math. **2** (1991), 457–476.

[17] Szymanski, W *Quantum lens spaces and graph algebras*, Pacific J. Math. **211** (2003), 249–263.

[18] Tomforde, M *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs*, J. Operator Theory **50** (2003), 345–368.

3. Commutative Dynamical Systems and C*-algebras

Already in Murray and von Neumann's first papers, the links between the theories of dynamical systems and operator algebras have been very important. Thanks to Connes' classification of injective von Neumann factors (the type III_1 case having been settled by Haagerup), Krieger's theorem, and the Connes-Feldman-Weiss characterization of amenable measurable actions; there is a bijective correspondence between amenable, ergodic, non-singular actions up to orbit equivalence, and injective von Neumann factors up to isomorphism.

In this report, we will present some of the known results in the interplay between topological dynamics and C*-algebras. Many of the new results were presented at this BIRS workshop. We begin by reviewing the transformation group C*-algebras of minimal homeomorphisms of compact metric spaces. For example, both the C*-crossed products associated to Cantor minimal systems and the irrational rotation algebras are AT-algebras (direct limits of circle algebras) with real rank zero and therefore belong to the class of algebras classifiable by K-theoretical invariants. The first result was proved by Putnam and the second one by Elliott and Evans. In a very recent preprint, H. Lin and N.C. Phillips proved the following remarkable result:

> *Let (X, ϕ) be a minimal dynamical system where X is an infinite compact metric space with finite covering dimension. Let $A = C^*(X, \phi)$ be the associated crossed product, and Aff$(T(A))$ be the space of real valued affine continuous functions on $T(A)$, the compact convex set of tracial states of A. If the natural map from $K_0(A)$ to Aff$(T(A))$ has dense range, then A is a simple unital AH algebra with rank zero and therefore is classifiable.*

In the smooth case, let us recall that Q. Lin and N.C. Phillips showed that the C*-crossed product associated to a minimal diffeomorphism of a compact smooth manifold is also classifiable, being a direct limit, with no dimension growth, of recursive subhomogeneous C*-algebras.

For a general minimal dynamical system (X, ϕ), no Krieger type theorem has yet been proved. Only for two classes of dynamical system have dynamical characterizations of isomorphism of the associated C*-crossed products been given. Before describing them, let us notice first of all that, due to an old result of Sierpinski, two (topologically) orbit equivalent minimal homeomorphisms on a connected compact metric space are flip conjugate.

For minimal homeomorphisms of the circle, the isomorphism of the C*-crossed product implies flip conjugacy (this follows from the following two facts: every minimal homeomorphism of S^1 is conjugate to an irrational rotation, and $C^*(S^1, R_\alpha) \cong C^*(S^1, R_\beta)$ iff α has the same image as $\pm\beta$ in $\mathbb{R}/\mathbb{Z}$).

For Cantor minimal systems, Giordano, Putnam, and Skau introduced the slightly technical notion of strong orbit equivalence (SOE) and proved that two Cantor minimal systems are SOE iff the associated C*-crossed products are isomorphic.

Using the Bratteli-Vershik model of Cantor minimal systems created by Herman, Putnam and Skau, H. Dahl has characterized the (finite dimensional) Choquet simplices of probability measures on the Cantor set which are the set of invariant measures of a Cantor minimal system. This generalizes a result of E. Akin.

Recently H. Lin has proposed the study of different versions of approximate conjugacy for minimal dynamical systems. The first results appear in three preprints by Lin, Lin and Matui, and Matui. For Cantor minimal systems, the approximate conjugate relation is closely related to orbit equivalence and strong orbit equivalence.

For minimal actions of groups other than $\mathbb{Z}$, the situation is more complicated. Itzá-Ortiz has recently established a correspondence between the group of the eigenvalues of a minimal suspension dynamical flow (whose ceiling function is not necessarily constant) and a multiplicative subgroup of the K_0-group associated to the base transformation of this flow. For minimal actions of $\mathbb{Z}^n$, it is not yet known if the corresponding C*-crossed-product is classifiable in the Elliott sense. The best result up to now has been that obtained by N.C. Phillips, who showed that the C*-crossed-product associated to a minimal, free $\mathbb{Z}^n$-action on the Cantor set has stable rank one, real rank zero, and cancellation of projections, and that the order on its K_0-group is determined by traces.

On the dynamical side, Giordano, Putnam, and Skau studied the so-called affable equivalence relations and proved that a "small extension" of an AF-equivalence relation is still (orbit equivalent to) an AF-equivalence relation. This gives a new topological dynamic proof that any Cantor minimal system is orbit equivalent to an AF-equivalence relation. In a recent preprint, they introduce a cohomological condition on minimal $\mathbb{Z}_2$-actions on the Cantor set, give two large classes of actions satisfying it and show that such minimal $\mathbb{Z}_2$-actions are orbit equivalent to AF-equivalence relations, using the extension result mentioned above.

4. Noncommutative Dynamical Systems

The classification program for C*-algebras has had the most success with purely infinite, simple C*-algebras (see, for example, [5] and [8]), with simple C*-algebras with tracial rank zero as introduced in [6] (see, for example, [7]), and especially with various classes of C*-algebras obtained as direct limits of special kinds of type I C*-algebras (see the discussion in Section 1. of this report). The classification program is currently interacting with noncommutative dynamics in two important ways. First, C*-algebraists tend to be more interested in crossed product C*-algebras than in most of the classes just mentioned. Work on the classification of crossed products has generally taken the form of proving that certain crossed products belong to one of the classes already covered by other classification theorems, or, less satisfactorily, at least proving structural properties of crossed products which suggest that they should belong to one of these classes. Work on crossed products by groups acting on compact spaces is discussed in Section 3. of this report, but some results for actions on noncommutative C*-algebras were presented at the workshop. Secondly, having classified algebras, it is natural to try to classify group actions on algebras.

Recent work on classifiability of crossed products of noncommutative C*-algebras has relied on the tracial Rokhlin property. This property is a weakening of the Rokhlin property [3] that Izumi uses in his classification work for automorphisms. The Rokhlin property is a rather rigid condition: K-theoretic obstructions (some obvious, some less so; see [4]) show that many purely infinite simple C*-algebras admit no actions of finite groups with the Rokhlin property. The tracial Rokhlin property for actions of finite cyclic groups first appeared in [9], where it was proved that if A is a simple separable unital C*-algebra with tracial rank zero, and if $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite cyclic group with the tracial Rokhlin property, then $C^*(G, A, \alpha)$ again has tracial rank zero. The applications there were to C*-algebras on which no nontrivial action of a finite group can have the full Rokhlin property.

Hiroyuki Osaka talked about actions of $\mathbb{Z}$ with the tracial Rokhlin property. For $\mathbb{Z}$, there are no known K-theoretic obstructions which prevent an action from having the Rokhlin property while allowing it to have the tracial Rokhlin property. However, there are a number of interesting actions of $\mathbb{Z}$ which are known to have the tracial Rokhlin property but not known to have the Rokhlin property. Osaka described two results, strongly suggestive but still incomplete. Let A be a simple separable stably finite unital C*-algebra, and let $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ be an action with the tracial Rokhlin property. If A has real rank zero and stable rank one, and if the order on projections over A is determined by traces, then $C^*(G, \mathbb{Z}, \alpha)$ again has these properties. If A has tracial rank zero, and if α_n is approximately inner for some nonzero n, then $C^*(\mathbb{Z}, A, \alpha)$ again has tracial rank zero.

As seen above, in some ways crossed products by finite groups are more accessible than crossed products by $\mathbb{Z}$. Their K-theory, however, is much harder to compute. For example, there is an action of $\mathbb{Z}/2\mathbb{Z}$ on a contractible C*-algebra such that the K-theory of the crossed product is nonzero, which rules out anything resembling the Pimsner-Voiculescu exact sequences for crossed products by $\mathbb{Z}$ and by free groups. There are standard actions of $\mathbb{Z}/n\mathbb{Z}$ on the irrational rotation algebras A_θ, for $n = 2, 3, 4, 6$, which are among the actions of finite groups which have attracted the most attention. Computations of K-theory in the rational case (when the crossed products are type I and can be described explicitly) have led to the conjecture that, in the irrational case, all the crossed products are AF algebras. This has been known for some time for $\mathbb{Z}/2\mathbb{Z}$ (the proof relies on a fortuitous coincidence), and has been proved by Walters for $\mathbb{Z}/4\mathbb{Z}$ and "most" θ. It is shown in [9] that, for θ irrational, all the crossed products are AH algebras with slow dimension growth and real rank zero. Thus, the remaining step is to compute the K-theory. Julian Buck talked about work in this direction with Walters for $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ (where the least is known). It depends on cyclic cohomology in an essential way.

In the second direction, Masaki Izumi has previously proved some classification results for actions of finite groups with the Rokhlin property on Kirchberg algebras [3], [4]. In his talk at the conference, he described results for quasifree actions of finite groups on $\mathcal{O}_\infty$. These actions do not have the Rokhlin property; in fact, as follows from Izumi's earlier work, there are no nontrivial actions of finite groups on $\mathcal{O}_\infty$ which have the Rokhlin property. However, Izumi proved that quasifree actions are locally representable, which in a certain sense is dual to the Rokhlin property. (For an action α of a finite *abelian* group, α is locally representable if and only if $\widehat{\alpha}$ has the Rokhlin property.) One should note that this theory is really only just beginning; as with the classification of C*-algebras, the purely infinite simple case is the place to start.

Andrew Dean talked about classification for actions on AF algebras which are explicitly given as direct limit actions, but where the group is not compact. Such actions ("locally representable" in a sense stronger than that used by Izumi) for compact groups were considered long ago by Handelman and Rossmann [1], [2], as well as others. While keeping the direct limit structure (in contrast to Izumi), Dean has obtained results for certain specific kinds of actions of noncompact groups. In previous work, he has considered actions of $\mathbb{R}$, and in his talk at this conference he examined actions of two relatively elementary groups which have infinite dimensional irreducible representations, and in particular are neither compact nor abelian, namely $\mathrm{SL}_2(\mathbb{R})$ and the group of Euclidean motions of the plane. The direct limits are set up so as to allow these representations to appear in at least a limited way, and thus allow infinite dimensional algebras (copies of the compact operators) in the system. They can only appear in a limited way because the partial maps must have finite multiplicity; otherwise, the direct limit will not be AF.

Two talks at the conference described work on noncommutative dynamics farther afield from the classification program. Rui Okayasu presented work relating the entropy of certain subshifts to the values of a a numerical invariant introduced some time ago by Voiculescu for the purpose of measuring the obstruction to the existence of a quasicentral approximate identity relative to the Macaev ideal for a finite set of operators. Specifically, the set of operators should be the creation operators which appear in Matsumoto's construction of the C*-algebra of the shift. Okayasu has also computed this invariant for the images of generating sets of certain groups under the regular representation.

Ilan Hirshberg talked about finding certain kinds of representations of C*-correspondences (bimodules which are Hilbert modules on one side). A C*-correspondence can be thought of as a generalization of an automorphism of the algebra (also, simultaneously, as a generalization of some other things), and some of the associated C*-algebras (Cuntz-Pimsner algebras [10]) have attracted considerable interest recently. These algebras generalize not only crossed products but also Cuntz-Krieger algebras and graph algebras. From the point of view of dynamics, a representations of a C*-correspondence is a generalization of a covariant representation of $(\mathbb{Z}, A)$. Hirshberg's situation was of course much more complicated than just finding covariant representations.

Bibliography

[1] D. Handelman and W. Rossmann, *Product type actions of finite and compact groups*, Indiana Univ. Math. J. **33**(1984), 479–509.

[2] D. Handelman and W. Rossmann, *Actions of compact groups on AF C*-algebras*, Illinois J. Math. **29**(1985), 51–95.

[3] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*, Duke Math. J. **122**(2004), 233–280.

[4] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. II*, Adv. Math. **184**(2004), 119–160.

[5] E. Kirchberg, *The classification of purely infinite C*-algebras using Kasparov's theory*, preliminary preprint (3rd draft).

[6] H. Lin, *Tracially AF C*-algebras*, Trans. Amer. Math. Soc. **353**(2001), 693–722.

[7] H. Lin, *Classification of simple C*-algebras with tracial topological rank zero*, Duke Math. J. **125**(2005), 91–119.

[8] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*, Documenta Math. **5**(2000), 49–114 (electronic).

[9] N. C. Phillips, *Crossed products by finite cyclic group actions with the tracial Rokhlin property*, preprint.

[10] M. Pimsner, *A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* $\mathbb{Z}$, pages 189–212 in: *Free Probability Theory (Waterloo, ON, 1995)*, Fields Inst. Commun. vol. 12, Amer. Math. Soc., Providence, RI, 1997.

List of Participants

Bates, Teresa (University of Victoria)
Blackadar, Bruce (University of Nevada-Reno)
Blanchard, Etienne (Centre National des Recherches Scientifiques, Jussieu)
Boca, Florin (University of Illinois)
Brenken, Berndt (University of Calgary)
Brown, Nathanial (Pennsylvania State University)
Buck, Julian (University of Northern British Columbia)
Carlsen, Toke (Norwegian University of Science & Technology)
Dadarlat, Marius (Purdue University)
Dean, Andrew (Lakehead University)
Eilers, Soren (University of Copenhagen)
Elliott, George (University of Toronto)
Giordano, Thierry (University of Ottawa)
Gong, Guihua (University of Puerto Rico)
Hirshberg, Ilan (University of Southern Denmark)
Itza-Ortiz, Benjamin (University of Ottawa)
Ivanescu , Cristian (University of Toronto)
Izumi, Masaki (Kyoto University)
Katsura, Takeshi (Hokkaido University)
Kribs, David (Univeristy of Guelph)
Kucerovsky, Dan (University of New Brunswick)
Kumjian, Alex (University of Nevada, Reno)
Li, Liangqing (University of Puerto Rico)
Lin, Huaxin (University of Oregon)
Matui, Hiroki (Chiba University)
Neufang, Matthias (Carleton University)
Ng, Ping Wong (University of New Brunswick)
Niu, Zhuang (University of Toronto)
Okayasu, Rui (Osaka Kyoiku University)
Osaka, Hiroyuki (Ritsumeikan University, Kyoto)
Ozawa, Narutaka (University of Tokyo)
Pasnicu , Cornel (University of Puerto Rico)
Phillips, N. Christopher (University of Oregon)
Putnam, Ian (University of Victoria)
Robert, Leonel (University of Toronto)
Savu, Ana (Queen's University)
Skau, Christian (University of Trondheim)
Toms, Andrew (University of New Brunswick)
Wassermann, Simon (University of Glasgow)
Winter, Wilhelm (University of Muenster)

Chapter 34

Functional Differential Equations (04w5026)

November 6–11, 2004

Organizer(s): Jianhong Wu (York University), Hans-otto Walther (University of Giessen), John Mallet-paret (Brown University)

The main purpose of this proposed workshop is to bring together international leaders and active researchers working in the theory and applications of functional differential equations for communication of new ideas and results, for review and summary of topics of current interest, for discussions of future research directions and for initiation of further collaborations.

The workshop shall have three (two-hours) featured lectures in each of the following areas: mixed functional differential equations; delay differential systems with state-dependent lags; delayed reaction-diffusion equations with non-local effects.

There will be multiple half-hour lectures, in addition to special sessions on specific topics.

Training of the junior researchers, postdoctoral fellows and graduate students will be an important part of this workshop. We are planning to invite approximately 10 young researchers to the workshop and we will encourage the speakers of the three featured lectures to make their lecture notes available to the general public.

Workshops of this type have been previously held in many major international centres of mathematical research, such as the ICM2002 satellite conference "Differential and Functional Differential Equations" (Moscow, Aug. 11-17, 2002), the CRM workshop "Memory, Delays and Multistability in Neural Systems" (October 11-15, 2000), International Conference on Functional Differential and Difference Equations (Lisboa, Portugal, July 26-30, 1999), Oberwolfach workshop "Global and Geometric Theory of Delay Differential Equations" (January 11-17, 1998), IMA program in Mathematical Physiology and Differential-Delay Equations (March 19 - April 13, 1990). There has been either a special session or minisymposium in the area of functional differential equations associated with almost every major international event of dynamical systems sponsored by AMS, CAIMS, CMS, SIAM and EQUADIFF.

Canada, Germany and USA have strong groups of researchers, and have been playing leading role in both theoretical research and applications of functional differential equations. Germany previously hosted such a workshop in Oberwolfach in 1998, IMA organized a special program in 1990, and thus it is natural and desirable that the next international workshop in the area be held in Canada, at Banff station. Canada has active researchers from coast to coast (see listed possible participants below), a workshop at Banff will further stimulate the nationwide collaboration.

List of Participants

Braverman, Elena (University of Calgary)
Buono, Pietro-Luciano (Universite de Montreal)
Campbell , Sue Ann (University of Waterloo)
Chen, Yuming (Wilfrid Laurier University)
Chow, Shui-Nee (Georgia Institute of Technology)
Diekmann, Odo (Utrecht University)
Eichmann, Markus (Universitat Giessen)
Erneux, Thomas (Universite Libre de Bruxelles)
Faria, Teresa (Universidade de Lisboa)
Gedeon, Thomas (Montana State University)
Hadeler, Karl-Peter (Universitaet Tuebingen)
Humphries, Tony (McGill University)
Krisztin, Tibor (University of Szeged)
Kyrychko , Yuliya (University of Surrey)
Lani-Wayda, Bernhard (Universitaet Giessen)
Litsyn, Elena (Ben-Gurion University of the Negev)
Mallet-paret, John (Brown University)
Mincheva, Maya (University of Lethbridge)
Ruess, Wolfgang (Universitat Essen)
Sell, George (University of Minnesota)
Sieber, Jan (University of Bristol)
Skubachevskiy, Alexander L. (Moscow State Aviation Institute)
Solonukha, Oles (Central Economical Mathematical Institute of Russian Academy of Science, Moscow)
Verduyn Lunel, Sjoerd (Universiteit Leiden)
Walther, Hans-Otto (University of Giessen)
Wu, Jianhong (York University)
Zou, Xingfu (University of Western Ontario)

Chapter 35

New Techniques in Lorentz Manifolds (04w5521)

November 6–11, 2004

Organizer(s): Virginie Charette (University of Manitoba), Todd A. Drumm (University of Pennsylvania), William M. Goldman (University of Maryland)

Recently discovered examples of Lorentz manifolds have renewed interest in the field among group theorists, differential geometers, topologists and dynamicists. The purpose of the November 6 BIRS workshop was to assemble specialists in these fields to discuss these new discoveries.

A *Lorentz manifold* is a manifold with an indefinite metric of index 1. Such structures arise naturally in relativity theory and, more recently, string theory.

Unlike the considerably more familiar *Riemannian manifolds* (with metric tensors of index 0), Lorentzian manifolds are poorly understood. Basic global questions remain unanswered, even for Lorentzian manifolds of constant curvature.

The simplest example is *Minkowski space* $\mathbb{R}^n_1$, a real affine space of dimension n, with a nondegenerate inner product of index 1. Although its compact quotients have been classified [17], its noncompact quotients, and more generally manifolds locally isometric or conformal to it are still mysterious. Closely related are the model constant curvature Lorentz manifolds, namely de Sitter space $\mathbb{S}^n_1$ and anti-de Sitter space $\mathbb{AS}^n_1$. Constant curvature *Riemannian* manifolds are also Lorentz manifolds.

Some of the topics discussed during the workshop included:

- Foliations of Lorentz manifolds and globally hyperbolic spacetimes;
- Global hyperbolicity in constant curvature manifolds;
- Conformal Lorentzian dynamics;
- Fundamental domains in anti-de Sitter space;
- Spinors on Lorentz manifolds;
- Topology of the future causal boundary of a spacetime.

We expand here on topics that generated discussion in the "open problems" session, and possible new research directions. The workshop facilitated many discussions which led to several new results.

Affine Spaces; Margulis Spacetimes

In 1977 Milnor [23] asked whether a nonabelian free group acts properly by affine transformations of $\mathbb{R}^n$. He suggested taking a discrete free subgroup Γ_0 of $SO(2,1)$ (for example a Schottky group) and "adding

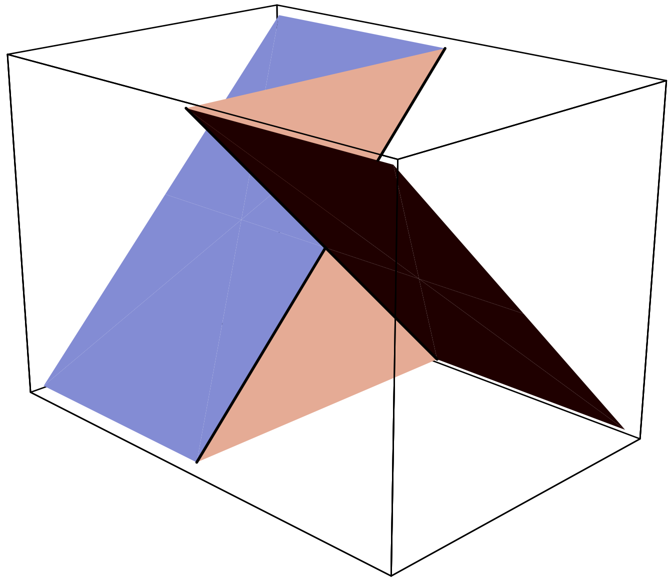


Figure 35.1: A crooked plane

translational components” (that is, an *affine deformation*) to make the group act properly. In 1983 Fried-Goldman [15] reduced the classification of complete affine 3-manifolds to Milnor’s question. Also in 1983 Margulis [20] constructed proper free actions of nonabelian free groups, answering Milnor’s question. Margulis’s examples were startling and unexpected.

In his 1990 doctoral thesis, Drumm [11] gave examples by constructing fundamental domains for such actions, using polyhedra called *crooked planes*. A crooked plane is depicted in Figure 35.1 and the intersections of a tiling of $\mathbb{R}^2_1$ by crooked planes by a horizontal plane are depicted in Figures 35.2 and 35.3.

Hence the interest in flat Lorentz 3-dimensional space forms, or *Margulis spacetimes*. Margulis found a criterion for a group Γ to *not* act properly. The Margulis invariant of an affine hyperbolic transformation measures signed Lorentzian displacement along an invariant spacelike line. When Γ acts properly and contains no parabolics, the quotient spacetime $M = \mathbb{R}^{2,1}/\Gamma$ enjoys the property that every essential loop is freely homotopic to a unique closed geodesic (necessarily spacelike). The absolute value of the Margulis invariant is the *signed Lorentzian length spectrum* of M.

Margulis showed that in order for Γ to act properly, the sign of the Margulis invariant must be constant over the group. It was conjectured that this is a sufficient condition; it was even hoped that we could find some sort of condition involving only a finite set of elements of Γ.

In the case where Γ is a free group on two generators, this conjecture has already led to surprising findings. If Γ is the holonomy of a three-holed sphere, it does act properly if and only if the values of the Margulis invariants for a certain “generating triple” (see below) all carry the same sign [19]. This is equivalent, via a beautiful interpretation of the signed Lorentzian length, to a result by Thurston: all closed geodesics of a hyperbolic three-holed sphere are shortened (resp., lengthened) if the three bounding closed geodesics are shortened (resp., lengthened) [33].

In the case of the punctured torus, the conjecture was answered in the negative by showing that there is no hope to ensure properness of an action by a “same sign” condition on a finite number of elements of the group [6].

In each case, Γ is a free rank two subgroup. Thus the moduli space of affine deformations depends on three parameters, namely, the values of the Margulis invariant for a pair of generators and their product – call these a *generating triple*. In fact, since an affine deformation may be considered up to rescaling without loss of generality (this corresponds to rescaling Minkowski space), the real projective plane is the moduli space of affine deformations of Γ, and by Margulis’ result, the proper deformations are bounded by the triangle with homogeneous coordinates (1,0,0), (0,1,0) and (0,0,1).

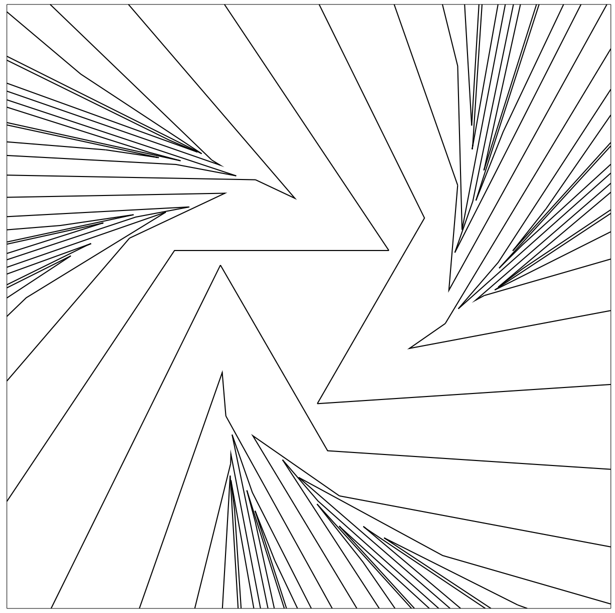


Figure 35.2: Cross-section of a crooked tiling

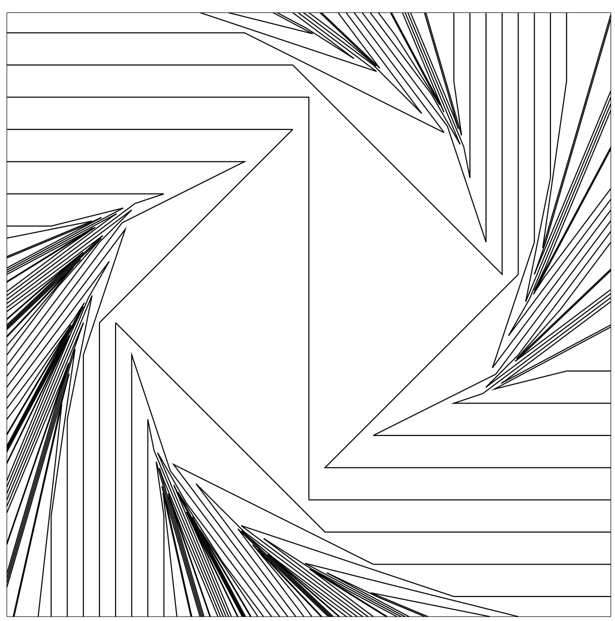


Figure 35.3: Proper affine deformation of the modular group

The contrast between the two cases is evident in Figures 35.4 and 35.5. Each line corresponds to a word in the abstract group associated to Γ: it is the space of deformations whose Margulis invariant for that word is zero. In the case of the three-holed sphere (Figure 35.4), the triangle of positive values for the generating triple appears to be contained in the intersection of the positive half-planes. However, in the case of the punctured torus, there are subsets for which every element of the generating triple admits a positive Margulis invariant, but some word in the group does not.

Crooked planes were discussed in Drumm's talk. He described the conjectural relationship between crooked planes and the Margulis invariant. The finite determination of the Margulis invariant was discussed in Charette's lecture [8].

As for the original conjecture, it is now believed to be false and that instead, one must consider an extension of the Margulis invariant, which we outline here.

Set E to be the affine space modelled on $\mathbb{R}^3_1$ and let Γ be a free rank two group of isometries of E, such that its linear part $\dot{\Gamma}$ is a convex cocompact subgroup of $SO(2,1)$ – thus $\dot{\Gamma}$ is discrete and finitely generated, and $\Sigma = H^2/\dot{\Gamma}$ has no cusps.

Consider the flat Minkowski bundle $\tilde{E} \to U\Sigma$. The affine deformation Γ corresponds to a cocycle class in $H^1(\dot{\Gamma}, E)$, which in the de Rham interpretation corresponds to a class $\omega \in H^1(\tilde{E})$. The bundle $\tilde{E} \to U\Sigma$ admits a preferred spacelike section ν, that is an extension of e_γ, the preferred unit-spacelike eigenvector of γ which appears in the definition of the Margulis invariant. The following function is not uniquely defined:

$$\begin{aligned} f : U\Sigma &\longrightarrow \mathbb{R} \\ (x,u) &\longmapsto \langle \omega(X), \nu \rangle, \end{aligned}$$

where X is the generator of the geodesic flow. However, given a probability measure invariant by the geodesic flow, λ, the following only depends on the cohomology class of ω:

$$\mu(\lambda) = \int_{U\Sigma} f \, d\lambda.$$

Goldman, Labourie and Margulis have shown that Γ acts properly on E if and only if the sign of $\mu(\lambda)$ is constant over all λ [16]. Here are some open problems remaining around this question. (See also Section 35.)

- Is the Goldman-Labourie-Margulis theorem the sharpest possible? It is believed to be so, that is, that Γ may not act properly on E, even though the sign of the Margulis invariant is constant over the group.
- Extend the result to the case when Σ has cusps, i.e. when Γ admits parabolic elements.

Margulis's original definition of the signed Lorentzian length was extended to include parabolic elements by Charette and Drumm [9].

Surfaces in Lorentz Space-forms

This was the subject of Schlenker's talk, as well as Pratoussevitch's talk. While Schlenker discussed the extension of Aleksandrov's theorem to Minkowski space, Pratoussevitch described a surprising construction of fundamental polyhedra for $\mathbb{AS}^3_1$-structures on Seifert 3-manifolds.

A theorem of Aleksandrov states that any metric on the two-sphere S^2 with curvature $K > -1$ is induced on a unique convex surface in H^3, three-dimensional hyperbolic space. Schlenker and Labourie have worked on the analogous problem in de Sitter space S^3_1. In particular, the same result holds in S^3_1, except that the curvature is now bounded above by one and the closed geodesics must have length greater than 2π. Let $\Sigma \subset H^3$ be a smooth, strictly convex surface; denote by I the induced metric. We define the third fundamental form on Σ to be:

$$III(X,Y) = I(\nabla_X N, \nabla_Y N),$$

where N is the unit normal vector. Then there is a dual statement to Aleksandrov's theorem: any metric h on S^2 with curvature less than one and whose closed geodesics have length greater than 2π is the third fundamental form of a unique convex surface in H^3. This follows from the duality between surfaces in H^3 and surfaces in S^3_1, which we will outline here.

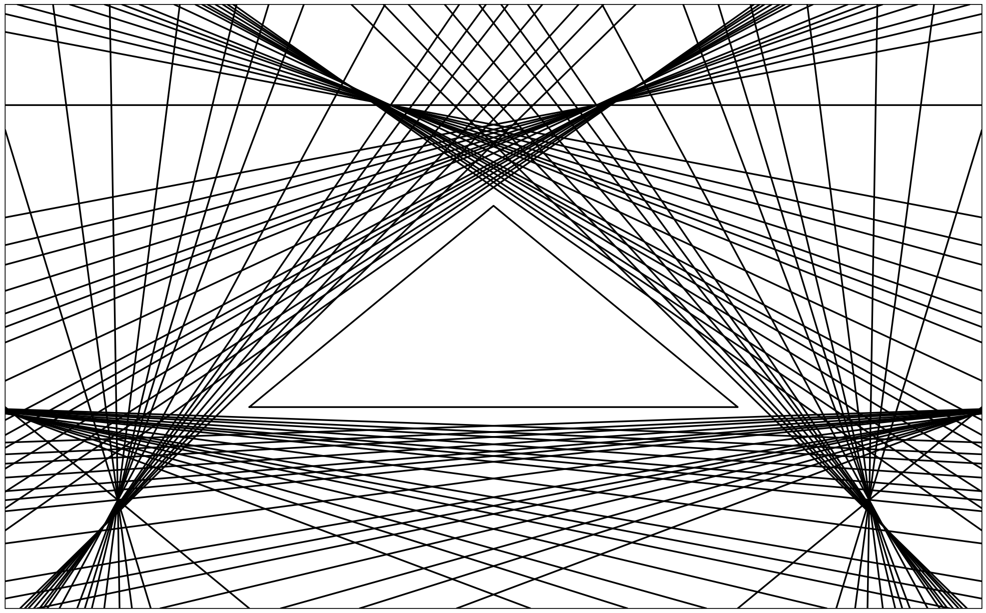


Figure 35.4: The moduli space of proper affine deformations of a hyperbolic 3-holed sphere

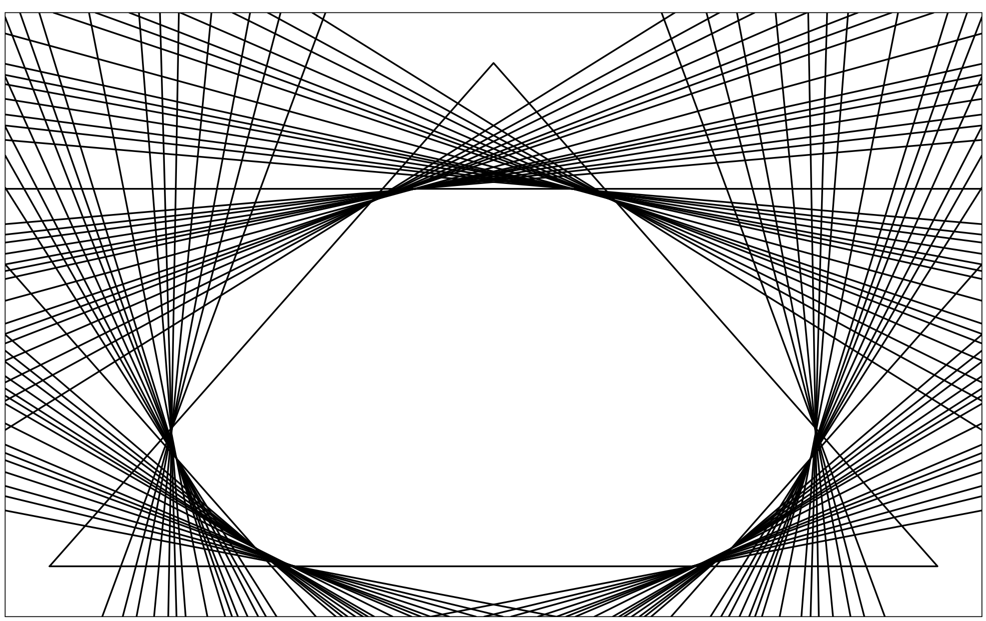


Figure 35.5: The moduli space of proper affine deformations of a hyperbolic 1-holed torus

Let $\Sigma \subset H^3$; for $p \in \Sigma$, the tangent space at p corresponds to a plane in H^3, which in turn admits a polar point $p^* \in S^3_1$. The induced metric I^* on the polar surface turns out to be III.

Thus any statement concerning convex surfaces in H^3 translates to a dual statement in S^3_1.

Let us define a *Fuchsian equivariant embedding* of a surface. Given a surface Σ of genus $g \geq 2$, an *equivariant embedding* of Σ in H^3 (resp. S^3_1, H^3_1) is a pair (ϕ, ρ), where:

- ϕ is an embedding of $\tilde{\Sigma}$ into H^3 (resp. S^3_1, H^3_1);

- ρ is a monomorphism of $\pi_1(\Sigma)$ into the isometry group of H^3 (resp. S^3_1, H^3_1) such that, for every $x \in \Sigma$ and $\gamma \in \pi_1(\Sigma)$,
$$\phi(\gamma x) = \rho(\gamma)\phi(x).$$

An equivariant embedding (ϕ, ρ) is *Fuchsian* if it fixes a totally geodesic plane in H^3 (resp. a point in S^3_1, H^3_1).

In this context, Aleksandrov's theorem is stated as follows: a convex surface Σ with curvature $K > -1$ admits a unique Fuchsian equivariant embedding into H^3, such that $I = h$. Dually, if $K < 1$ and every closed geodesic has length greater than 2π, Σ admits a unique Fuchsian equivariant embedding into H^3 such that $III = h$.

In the anti-de Sitter world, analogous statements hold. Namely, a surface Σ with metric h, whose curvature is bounded *above* by -1, admits a unique equivariant embedding into H^3_1 such that $I = h$. Dually, the same result holds with $h = III$ instead of I.

Now, Aleksandrov's theorem is a special case of a statement about hyperbolic three-manifolds with convex boundary. Let $h_\pm$ be metrics on a convex surface Σ with curvature $K > -1$. Then there exists a unique hyperbolic metric g on $\Sigma \times [-1, 1]$ such that the induced metric on each component of the boundary is given by $h_+,\ h_-$, respectively. Dually, the same statement holds for $K < 1$, as long as the lengths of the closed geodesics are greater than 2π, substituting the third fundamental form for I.

In the anti-de Sitter world, all evidence points to the existence of an analogous statement; but this remains conjectural.

Causality

Perhaps the most salient feature of a Lorentzian structure is its underlying causality structure. Unlike Riemannian manifolds, geodesics (and more generally, smooth curves) come in several flavors, depending on the restriction of the metric tensor to these curves. Steve Harris described the notion of the ideal causal boundary on Lorentz manifolds [18].

In a sequence of talks, Thierry Barbot and François Begun described their joint work [3] with Zeghib, on foliating globally hyperbolic 3-dimensional spacetimes by constant mean curvature surfaces. The principal result is that every maximal such spatially compact Lorentzian manifold admits a *time function*, that is, a function which increases along future-directed timelike curves.

Lorentzian Foliations and Group Actions

The subject of Riemannian foliations (that is, foliations whose holonomy groupoid preserves a transverse Riemannian metric) was developed in the 1970's and 1980's. Pierre Mounoud presented his recent work [24, 25, 26] on Lorentzian foliations at the workshop.

Frances's lecture dealt with the extension of Obata's theorem to Lorentz manifolds. Obata proved that the only Riemannian manifolds which admit noncompact conformal automorphism groups are Euclidean space $\mathbb{R}^n$ and the Euclidean sphere $\mathbb{S}^{n-1}$. Frances gave surprising examples of compact conformally flat Lorentz manifolds whose automorphism groups are noncompact. Furthermore he discussed which 3-manifolds support such *essential* flat Lorentzian conformal structures. This was the topic of his recent doctoral thesis [12, 13, 14].

In a different direction, Karin Melnick discussed her recent results, concerning which groups can act by Lorentzian isometries on compact manifolds. Building on earlier work of Zeghib [34, 35] and Adams-Stuck [2], she showed that the possible connected isometry groups of a compact connected Lorentz manifold have the form $K \times R^m \times S$, where K is compact and S is locally isomorphic to one of the following:

- $PSL_2(R)$;
- a Heisenberg group H_n;
- one of a countable family of solvable extensions isomorphic to $S^1 \ltimes H$, where H is a Heisenberg group.

She went on to describe which manifolds admit an action of the Heisenberg group, particularly one of codimension one–where the dimension of the Heisenberg group is one less than the dimension of the manifold. This work, recently posted to the archives [21], may be part of her forthcoming doctoral thesis.

Closely related to Killing vector fields are *Killing spinor fields* which generalize to *conformal Killing fields.* In her talk, Helga Baum showed how conformal Killing spinor fields lead to new examples of manifolds with essential Lorentzian conformal structures. In particular if the associated vector field is lightlike, then the manifold is one of a few special types (for example, a strictly pseudoconvex boundary of a domain (*a Fefferman space,* or a circle bundle over a Kähler manifold). The proof [4] involves a careful analysis of the zero-set of a conformal Killing spinor field.

Low-dimensional Topology and Other Topics

The workshop benefited from several lectures which were not exactly on the topic of the conference, but nonetheless closely related. Suhyoung Choi presented his solution [10] of Marden's Tameness Conjecture for hyperbolic 3-manifolds (proved independently by Agol and Calegari-Gabai).

Dave Morris lectured on which arithmetic groups can act on the line.

Kevin Scannell discussed deformations of hyperbolic 3-manifolds, which through work begun in his thesis [30, 31, 32], closely relate to $\mathbb{R}^3_1$-manifolds.

Problem Session

The items outlined above represent just a sample of the topics discussed at the workshop. On the last day a problem session was held. Here is a list of some of the problems which were suggested:

1. (Labourie) Mess shows that compact oriented orthochronous $2 + 1$ AdS spacetime with non-empty spacelike boundary S is a product $S \times [0, 1]$ and embeds in a domain of dependence. Is it possible to construct a singular AdS manifold with more than two ends, say by branching on a spacelike geodesic in a domain of dependence?

2. (Scannell) Generalize the "no topology change" theorem of Mess noted above to all constant curvature $3 + 1$ spacetimes. Or (even better) characterize when a constant curvature $3 + 1$ maximal domain of dependence embeds in a larger constant curvature spacetime.

3. (Schlenker) Let M be a compact AdS cone manifold with m singular curves. Given real numbers $\dot{\alpha_1}, \ldots \dot{\alpha_m}$, is there a first order deformation of the AdS structure inducing these derivatives of the cone angles? This is related to the following problem, posed by Mess.

4. (Mess) Let $\rho = (\rho_L, \rho_R)$ be the representation of the fundamental group of a closed surface into $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ corresponding to an AdS domain of dependence.

   (a) Is ρ determined by the two measured laminations on the boundary of the "convex hull"?

   (b) Is ρ determined by the hyperbolic structure on the future boundary of the convex hull together with the measured lamination on the past boundary?

(c) Is ρ determined by ρ_L together with the hyperbolic structure on one of the boundary components of the convex hull?

These are analogous to well-known questions about the parameterization of quasi-Fuchsian space by the pair of conformal structures at infinity, and how these relate to the bending laminations and hyperbolic structures on the convex hull boundary.

5. (Barbot) Let M_0 be a globally hyperbolic static AdS spacetime with closed spacelike slices and consider $v_0 = vol(M_0)$. Is the volume of a non-static AdS spacetime of the same topological type less than or equal to v_0?

6. (Schlenker) Is the volume of the convex core of a $2 + 1$ AdS domain of dependence strictly concave as the bending lamination varies? This question, and Barbot's question above, can be thought of as refinements of the following question posed by Mess in his preprint:

7. (Mess) For a $2 + 1$ AdS domain of dependence, the volume of the maximal domain of dependence and of the convex core are invariants on $Teich \times Teich$. How do they behave? Are they related, perhaps asymptotically, to invariants of quasi-Fuchsian space, such as the volume of the convex core and the Hausdorff dimension of the limit set?

8. (Harris) A static complete spacetime is conformal to $(\mathbb{L}^1 \times M)/G = U$ with $G \subset Isom(M)$ for a Riemannian manifold M. Here $\mu : G \to \mathbb{R}$ is a homomorphism and G acts on $\mathbb{L}^1 \times M$ by $g(t, x) = (t + \mu(g), g \cdot x)$. Does $\hat{\partial}(U)$ depend on μ?

9. (Goldman) Let M be a complete flat $2 + 1$ spacetime.

   (a) Does M have a fundamental domain bounded by crooked planes?

   (b) Is the interior of M diffeomorphic to a solid handlebody?

   (c) Do there exist natural smooth approximations of crooked planes?

   (d) (Properness conjecture). It is known that if an affine deformation of a Fuchsian group acts properly, then the value of the Margulis invariant is everywhere positive or everywhere negative. Is the converse true?

10. (Goldman) Extend crooked planes to surfaces in AdS space. Are there conformally invariant surfaces that could be used as boundaries of fundamental domains of AdS spacetimes?

11. (Abels)

    (a) Auslander Conjecture: Is every affine crystallographic group (i.e. discrete, cocompact subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ acting properly) virtually solvable?

    (b) Are there properly discontinuous affine groups (not necessarily cocompact) that are neither virtually polycyclic nor virtually free?

12. (Scannell) Characterize closed hyperbolic 3-manifolds which admit affine deformations into $Isom(\mathbb{R}^4)$. Do they always admit quasi-Fuchsian deformations into $Isom(H^4)$?

Many of the talks were influenced by Mess's unpublished preprint [22].

During the problem session it was decided to undertake the project to annotate the preprint (in order to update the results) and eventually publish it.

The organizers solicited papers based on the workshop, possibly including the updated annotated version of Mess's paper. Since one of the organizers of the workshop (Goldman) is editor-in-chief of the journal *Geometriae Dedicata*, that journal seems a particularly appropriate for such a volume.

Kevin Scannell's workshop website `http://borel.slu.edu/lorentz/index.html` facilitates communication between the participants following the workshop. In particular, the summaries of the discussions (and soon the papers arising from the workshop) will be posted there.

List of Participants

Abels, Herbert (Bielefeld University)
Bader, Uri (Technion - Israel Institute of Technology)
Barbot, Thierry (École Normale Supérieure de Lyon)
Baum, Helga (Humboldt University)
Beguin, Francois (Universite de Paris-Sud)
Brenner, Eliot (Yale University)
Brill, Dieter (University of Maryland)
Charette, Virginie (University of Manitoba)
Choi, Suhyoung (Korea Advanced Institute of Science and Technology)
Drumm, Todd (University of Pennsylvania)
Dumas, David (Rice University)
Frances, Charles (École Normale Supérieure de Lyon)
Goldman, William (University of Maryland)
Harris, Steven (Saint Louis University)
Labourie, Francois (Universite de Paris-Sud)
Melnick, Karin (University of Chicago)
Morris, Dave (University of Lethbridge)
Mounoud, Pierre (Max Planck Institut fur Mathematik)
Pratoussevitch, Anna (University of Bonn)
Scannell, Kevin (Saint-Louis University)
Schlenker, Jean-Marc (Universite Paul Sabatier)

Bibliography

[1] H. Abels, Properly discontinuous groups of affine transformations: A survey, *Geometriae Dedicata* **87** (2001), 309–333.

[2] S. Adams and G.Stuck, The isometry group of a compact Lorentzt manifold I and II, *Inv. Math.* **129**, 1997 no 2, 239-287)

[3] T. Barbot, F. Begun, and A. Zeghib, Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3, *Geometriae Dedicata* (submitted).

[4] H. Baum, Twistor spinors on Lorentzian manifolds, CR-Geometry and Fefferman spaces, *Proc. of the International Conference on Differential Geometry and its Applications*, Brno 10.-14.8.1998, Brno 1999, 29-38.

[5] D. Brill, V. Charette and T. Drumm, Closed Timelike Curves in Flat Lorentz Spacetimes *J. Geom. Phys.* **46** (2003) 394-408

[6] V. Charette, Non-proper affine actions of the holonomy group of a punctured torus, *Forum Mathematicum* (to appear).

[7] V. Charette, T. Drumm, W. Goldman and M. Morrill, Complete flat affine and Lorentzian manifolds, *Geometriae Dedicata* **97** (2003), 187–198.

[8] V. Charette, T. Drumm, Strong marked isospectrality of affine Lorentzian groups, *J. Diff. Geom.* **66** (2004), 437–452.

[9] V. Charette, T. Drumm, The Margulis invariant for parabolic transformations, *Proc. AMS*, (to appear).

[10] S. Choi, Drilling cores of hyperbolic 3-manifolds to prove tameness, arXiv:math/0410381.

[11] T. Drumm, Fundamental polyhedra for Margulis space-times, *Topology* **31** (1992), 677–683.

[12] C. Frances and C. Tarquini, Autour du théoréme de Ferrand-Obata, *Annals of Global Analysis and Geometry* **21** (2002), no. 1, 51–62.

[13] C. Frances, *Des contre-exemples au théoréme de Ferrand-Obata en géométrie Lorentzienne conforme*, *Mathematische Annalen* (to appear).

[14] C. Frances, The conformal boundary of Margulis spacetimes, *C. R. Acad. Sci. Paris Ser. I* **336** (2003), 751-756.

[15] D. Fried and W. Goldman, Three-dimensional affine crystallographic Groups, *Advances in Mathematics* **47**, (1983), 1-49.

[16] W. Goldman, F. Labourie and G. Margulis, Proper affine actions and geodesic flows of hyperbolic surfaces. math.DG/0406247. (submitted)

[17] W. Goldman, Y. Kamishima, The fundamental group of a compact flat Lorentz space form is virtually polycyclic, *J. Diff. Geom.***19** (1984), 233–240.

[18] S. Harris, Topology of the future chronological boundary: universality for spacelike boundaries, *Class. Quantum Gravity* **17** (2000), 551-603.

[19] C. Jones, doctoral dissertation, University of Maryland, 2003.

[20] G. Margulis, Free properly discontinuous groups of affine transformations, *Dokl. Akad. Nauk SSSR* **272** (1983), 937–940.

[21] K. Melnick, Doctoral dissertation, University of Chicago (expected 2006).

[22] G. Mess, Lorentz spacetimes of constant curvature, IHES preprint, 1990.

[23] J. W. Milnor, On fundamental groups of complete affinely flat manifolds, *Advances in Mathematics* **25** (1977), 178–187.

[24] Some topological and metric properties of the space of Lorentzian metrics, *Differential Geom. Appl.* **15** (2001), no. 1, 47–57.

[25] Some dynamical properties of the space of Lorentzian metrics, *Comment. Math. Helv.* **78** (2003) 463-485.

[26] P. Mounoud Complétude et flots nul-géodésibles (completeness and nul-geodesible flows), *Bulletin de la Societe Mathematique de France* (to appear).

[27] C. Boubel, P. Mounoud and C. Tarquini Foliations admitting a transverse connexion (submitted).

[28] A. Pratoussevitch, Uber Fundamentalbereiche diskreter Untergruppen von SU(1,1) Diploma Thesis, University of Bonn, 1998.

[29] A. Pratoussevitch, Fundamental Domains in Lorentzian Geometry, math.DG/0308279, 2003, submitted.

[30] K. Scannell Flat conformal structures and the classification of de Sitter manifolds, *Comm. Anal. Geom.* **7** (1999), no. 2, 325-345.

[31] K. Scannell Infinitesimal deformations of some SO(3,1) lattices, *Pacific J. Math.* **194** (2000), no. 2, 455-464.

[32] K. Scannell 3-manifolds which are spacelike slices of flat spacetimes, *Classical Quantum Gravity* **18** (2001), no. 9, 1691-1701.

[33] W. Thurston, Minimal stretch maps between hyperbolic surfaces, math.GT/9801039, 1986.

[34] A. Zeghib, Sur les espaces-temps homogenes, The Epstein Birthday Schrift, *Geom. Topol. Publ.*, Coventry, 1998.

[35] A. Zeghib, Sur les espaces-temps homogenes, The identity component of the isometry group of a compact Lorentz manifold, *Duke Math. J.* **92**, 1998 no 2, 321-333.

Chapter 36

Explicit Methods in Number Theory (04w5502)

November 13–18, 2004

Organizer(s): Peter Borwein (Simon Fraser University), Hendrik W. Lenstra (University of California, Berkeley), Peter Stevenhagen (Universiteit Leiden), Hugh Williams (University of Calgary)

With this workshop, we intend to provide an opportunity for the participants to communicate recent developments in the various participating disciplines to experts in the same and in neighbouring areas. Furthermore, the workshop will facilitate and promote new and existing collaborations by giving an opportunity for participants to meet their colleagues in a relatively small, informal and intensive environment.

Developments in the participating areas are vast and quick. Many collaborations between physically distant researchers are ongoing and new results in one area often spark off new collaborations with researchers in other areas. The proposed meeting will give the participants an excellent platform for disseminating their results to a relevant audience and will give them a chance to absorb results by others. Recent meetings at MSRI and Oberwolfach have shown that the subject area is very much in flux and that there is a clear demand for more opportunities for dissemination and collaboration in this field.

Information technology industries have shown serious interest in computational number theory. Many number theoretic constructions find an application in cryptography or coding theory. Furthermore, the computational challenges offered by number theory give an excellent incentive and clear benchmarks for the computing industry to enhance hardware and the constant quest for faster algorithms enhances computational tools in general.

By organising a meeting on a smaller scale at BIRS rather than a large conference, better conditions are created to work out correspondences and relevant applications of presented results informally afterwards. We expect that the scientific spin-off of an intensive, informal and small-scale meeting will be more significant than that of a formal, big conference. The formula that BIRS offers matches with the intent of the proposed meeting and we would be delighted to be enabled to organise the proposed meeting at BIRS.

List of Participants

Bauer, Mark (University of Calgary)
Bernstein, Daniel (University of Illinois, Chicago)
Bhargava, Manjul (Princeton University)
Borwein, Peter (Simon Fraser University)
Bosma, Wieb (Katholieke Universiteit Nijmegen)
Boyd, David (University of British Columbia)
Bruin, Nils (Simon Fraser University)

Calegari, Frank (Harvard University)
Cohen, Henri (Université Bordeaux)
Cremona, John (University of Nottingham)
Dembele, Lassina (University of Calgary)
Ferguson, Ron (Simon Fraser University)
Girard, Martine (University of Sydney)
Gunnells, Paul (University of Massachusetts)
Hare, Kevin (University of Waterloo)
Hess, Florian (Technische Universität Berlin)
Klüners, Jürgen (Universität Kassel)
Kohel, David (University of Sydney)
Krutelevich, Sergei (University of Ottawa)
Lauter, Kristin (Microsoft Research)
Lenstra, Hendrik W. (Universiteit Leiden)
Mercer, Idris (Simon Fraser University)
Mossinghoff, Michael (Davidson College)
Mueller, Siguna (University of Calgary)
Mukunda, Keshav (Simon Fraser University)
O'Neil, Catherine (Massachusetts Institute of Technology)
Pinch, Richard (HMG, Cheltenham)
Schaefer, Ed (Santa Clara University)
Scheidler, Renate (University of Calgary)
Schoof, Rene (University of Rome II)
Stein, William (Harvard University)
Stevenhagen, Peter (Universiteit Leiden)
Stoll, Michael (International University Bremen)
Teske, Edlyn (University of Waterloo)
Watkins, Mark (Institut Henri Poincare)
Wetherell, Joseph L. (Center for Communications Research, La Jolla)
Wiese, Gabor (Universiteit Leiden)
Williams, Hugh (University of Calgary)
de Smit, Bart (Universiteit Leiden)
van Luijk, Ronald (University of California, Berkeley)

Chapter 37

Diophantine Approximation and Analytic Number Theory (04w5507)

November 20–25, 2004

Organizer(s): Michael Bennett (University of British Columbia), Greg Martin (University of British Columbia), John Friedlander (University of Toronto), Andrew Granville (Université de Montréal), Cameron Stewart (University of Waterloo), Trevor Wooley (University of Michigan)

Analytic number theory and Diophantine approximation are two major areas in the field of number theory. Historically, they have both been strongly represented in the North American community and remain so to this day. Furthermore, the frequency with which methods and advances in one area have fuelled research efforts in the other is increasing, as deeper and more pervasive connections between the two areas are constantly discovered.

Some of the major tools of analytic number theory involve the theory of meromorphic functions (which was in large part commenced by the study of the Riemann zeta function in connection with the distribution of prime numbers), the evaluation and estimation of exponential sums, sieve methods, and many techniques from the fields of harmonic analysis, probability, and random matrix theory. In Diophantine approximation, common techniques involve versions of the subspace theorem, lower bounds for the sizes of linear forms in logarithms, the hypergeometric method of Thue and Siegel, and the circle method, which itself combines elements of harmonic analysis and exponential sums.

The primary problems to which these tools are applied include, on the analytic side, the distribution of prime numbers and of the prime factors of integers, special values of zeta functions (including multiple zeta values) and L-functions, and uniform distribution of arithmetic sequences; and on the Diophantine side, determining the transcendentality of natural constants and of values of modular functions, irrationality measures for these values and for algebraic numbers, and applications to rational points on algebraic varieties and solutions of Diophantine equations. Indeed, the difficulty in placing any of these topics firmly within one category or the other makes the interrelations between these two areas even more evident.

The objective of the workshop on Diophantine approximation and analytic number theory was to gather together researchers with expertise in both Diophantine approximation and analytic number theory in an environment that fosters the presentation and sharing of the latest ideas in both fields. The participants were chosen either as experts in analytic number theory whose work involves problems in Diophantine approximation, or as experts in Diophantine approximation whose methods also lend themselves to the resolution of open questions in analytic number theory. The workshop was also intended to provide a significant learning experience and exposure to current research for number theorists in these two areas in the early stages of their careers.

A focal point of the workshop was that of rational and K-rational points on curves and surfaces (see [3]), an area where the aforementioned interaction between the fields is particularly pronounced. By way of example, Jordan Ellenberg (Princeton) spoke about his research into insolubility of the equation $x^2+y^4=z^n$ and the relation between possible solutions and the modularity of Q-curves, extending the now famous Frey curve approach first taken to success by A. Wiles. His techniques include bounds for the average central values of the L-functions in question; in practice, he only needed information on a single L-function central value. Kannan Soundararajan and Trevor Wooley (Michigan) were thus able to recommend various weighting and mollification techniques from the analytic number theory arsenal the purpose of which is precisely to retain the analyzability of an averaged sum while giving more information about particular values.

Regarding surfaces, Pietro Corvaja (Udine) spoke on his continuing work with Zannier involving a wide range of applications of explicit versions of the Subspace Theorem to counting integer points on surfaces. His interaction with David McKinnon (Waterloo) during the informal discussion sessions allowed them to trade perspectives on some of McKinnon's conjectures on the bounds for such integer points. In addition, this subject is closely related to the work of Ellenberg, Venkatesh and Helfgott (all present at the workshop) on counting rational points on varieties.

On the subject of Diophantine equations, one of the most significant results of the past decade was Preda Mihailescu's proof of Catalan's Conjecture. At the workshop, Mihailescu described his partial progress towards a generalization of his work, to the Fermat–Catalan equation $x^p+y^p=z^q$. These results rely upon classical cyclotomic theory, and complement current work on similar equations of Bennett and Ellenberg, who apply extensions of the methods of Wiles et al. involving Frey elliptic curves. In this setting, the modularity of representations of dimension greater than 1 becomes paramount; computational aspects of Hilbert modular forms were discussed by Lassina Dembele (University of Calgary).

A central theme in current research in Analytic Number Theory is the conjectural connection between random matrix theory and distribution questions regarding the zeros of L-functions (see [1]). At the workshop, Chantal David (Concordia University) presented the results of her evaluation of the predictions that random matrix theory made to the question of how often twists of L-functions corresponding to elliptic curves by characters of fixed order vanish at the centre of the critical strip. She gave the surprising answer that for twists by characters of order 3 or 5 the number of such vanishing central values is predicted to be infinite, while for characters of higher prime order the number of such vanishing central values is predicted to be finite. Martin pointed out that David's techniques might well be able to predict that the number of vanishing central values should be finite, even when the characters of all prime orders greater than 5 are taken together. Among further results on L-functions, Valentin Blomer (Göttingen) spoke on his contributions and improvements to the literature of bounds for values of automorphic L-functions within the critical strip. Subconvexity bounds for the growth of such functions are often crucial in obtaining nontrivial estimations for technical sums that arise in problems in multiplicative and additive analytic number theory. For instance, these results are important for current research of Paula Cohen (Texas A&M University) on hyperbolic equidistribution problems (generalizing prior work of W. Duke on the equidistribution of Heegner points). Cohen's results apply more generally to Hilbert modular varieties of arbitrary dimension.

Further themes represented in the workshop were the Hardy-Littlewood circle method (Wooley, Lucier, etc, see [2]), a major tool in analytic number theory now finding application to questions of rational points on varieties, and the hypergeometric method of Thue and Siegel, traditionally the province of transcendence theory and now, as demonstrated by Michael Filaseta (University of South Carolina), important for a variety of questions in classical analytic theory. Classical transcendence theory [4] was also well-represented, including a stimulating talk by Yann Bugeaud (University of Strasbourg) on the transcendence of real numbers whose decimal or base-b expansions are expressible by automata.

Number theory is unique among the major fields of mathematics in that it combines problems and questions of incredible simplicity and accessibility with truly deep and technical tools and methods for addressing these questions. A reduction of a problem in one area of number theory (and indeed in many other mathematical fields as well) often involves a very simply stated question in the other area, which can seem difficult to resolve if one is not well-versed in the techniques of the second area. Often, contact and communication between Diophantine approximation researchers and analytic number theorists is the greatest obstacle to overcome on the way to significant advances on both sides. This accessibility that number theory possesses is another reason that involving young researchers in the workshop was so profitable.

List of Participants

Bauer, Mark (University of Calgary)
Bennett, Michael (University of British Columbia)
Blomer, Valentin (University of Toronto)
Bugeaud, Yann (Universite Louis Pasteur)
Choi, Stephen (Simon Fraser University)
Cohen, Paula (Texas A&M University)
Corvaja, Pietro (University of Udine)
David, Chantal (Concordia University)
Ellenberg, Jordan (Princeton University)
Evertse, Jan-Hendrik (University of Leiden)
Ferguson, Ron (Simon Fraser University)
Filaseta, Michael (University of South Carolina)
Ford, Kevin (University of Illinois)
Foster, Chris (University of Calgary)
Helfgott, Harald (Yale University)
Kadiri, Habiba (Universite de Montreal)
Liu, Yu-Ru (University of Waterloo)
Lucier, Jason (University of Waterloo)
Martin, Greg (University of British Columbia)
McKinnon, David (University of Waterloo)
Mihailescu, Preda (Universität Paderborn)
Mulholland, Jamie (University of British Columbia)
Ng, Nathan (Universite de Montreal)
Roy, Damien (University of Ottawa)
Soundararajan, Kannan (University of Michigan)
Stewart, Cameron (University of Waterloo)
Thunder, Jeff (Northern Illinois University)
Venkatesh, Akshay (Massachusetts Institute of Technology)
Waldschmidt, Michel (Universite Pierre et Marie Curie (Paris VI))
Walling , Lynne (University of Colorado, Boulder)
Walsh, Gary (University of Ottawa)
Wooley, Trevor (University of Michigan)
van der Poorten, Alf (Macquarie University)

Bibliography

[1] J. B. Conrey, L-functions and random matrices. In *Mathematics unlimited—2001 and beyond*, 331–352, Springer, Berlin, 2001.

[2] R. Heath-Brown, Rational points and analytic number theory. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, 37–42, Progr. Math., 226, BirkhŁuser Boston, Boston, MA, 2004.

[3] P. Swinnerton-Dyer, Diophantine equations: progress and problems. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, 3–35, Progr. Math., 226, BirkhŁuser Boston, Boston, MA, 2004.

[4] M. Waldschmidt, Open Diophantine problems. *Mosc. Math. J.* **4** (2004), no. 1, 245–305, 312.

Chapter 38

Mathematical Models for Biological Invasions (04w5539)

November 27–December 2, 2004

Organizer(s): Mark Kot (University of Washington), Mark Lewis (University of Alberta), Pauline van den Driessche (University of Victoria)

Background

The spread of introduced species is one of the most important applied problems in ecology. In North America, invasive exotic species are widespread, ranging from zebra mussels to Africanized honey bees to weedy plants. Although some invaders are thought of as beneficial, many become pests, and the associated costs are immense, exceeding $100 billion US per year.

Early models for invasive species were nonlinear reaction-diffusion equations such as Fisher's equation [8], which describes quadratic growth coupled to Brownian motion. Here the analysis of travelling waves and of the convergence of initial data to wave solutions has been a fruitful area of classical mathematical research [9]. The travelling wave speed, interpreted biologically as the rate of spread of the introduced population, has successfully predicted spread rates of many introduced species, but has failed dramatically with others.

From a scientific perspective, the field of invasion biology has matured greatly in the last few years as ecologists have tried to come to grips with the risks, damages, and spatial spread of introduced species. This is evidenced by new journals (eg, 'Biological Invasions'), large sections of meetings devoted to the subject of biological invasions (eg, Ecological Society of America annual meeting), and many new books and new text books on the subject. At the same time, quantitative biologists and mathematical modellers have become increasingly aware of the limitations inherent in the early quantitative models.

Ingredients missing in early models include: rare, long-distance dispersal events which cannot be described by classical diffusion, age- and stage-structured population dynamics, interspecific interactions and nonlinear stochastic effects. It is possible to include such ingredients in systems of coupled nonlinear reaction-diffusion equations, systems of integral-based equations, such as integro-difference (discrete-time, continuous space) equations, or as stochastic, interacting particle models.

Analysis of the resulting mathematical systems is a daunting task, and provides a modern-day challenge for applied mathematicians. Some progress has been made on such analysis of these systems, although, to date, results have not always been communicated widely. Moreover, a broad scientific impact requires a multidisciplinary effort which includes mathematicians, biologists and modellers.

The purpose of this meeting was to bring together a group of expert mathematicians and quantitative biologists with the following goals: (i) communicate recent advances in the mathematical analysis of invasion problems, and advances in the application of these results to real ecosystems (ii) propose future directions for research in the mathematics of biological invasions with a view to developing areas where the interaction between models and science is strong. Because the field of biological invasions is immense, we focused of four subareas where the interaction between models and science is already promising.

1. How do invader life-history details affect spread, and are there particular stages that are most sensitive to control measures?

2. How do secondary ecological interactions with other species impact spread, and what is the impact of the invader on these other species?

3. How can model inputs (such as dispersal kernels) be measured under practical field conditions, and how can model predictions (such as spreading speeds) be tested against field data?

4. What is the impact of rare, long-distance dispersal events on the rate of spread, and the precision of spread rate estimates?

Some results of these efforts give us a detailed understanding of biological invasions, including the spatial spread of disease, new methods to predict the response of vegetation to climate change, the spread of weed species through ecosystems, and new methods for spatial biocontrol of pest species.

Our workshop was roughly focused around the above four themes, and involved a range of participants, ranging from mathematicians to quantitative biologists. The synergistic interaction between mathematics and biology lead to advances in both fields.

Mathematical Theory

Several large classes of models for the growth and spread of multiple species can be unified into a simple recursion model of the form $u_{n+1} = Q[u_n]$. Here the operator Q takes the set of densities of the species at an initial time into the values at time later. This provides a generalization of the early partial differential equation (PDE) models of Fisher [8] , Skellam [21] and others, to include the possibility of non-Gaussian dispersal and discrete-time dynamics.

Hans Weinberger presented a survey of the qualitative spreading properties of solutions of such models in which all the species cooperate [11, 23]. The main results are that there are, in general, a slowest spreading speed such that no species spreads at a speed less than this number and at least one species spreads no faster, and a fastest spreading speed such that no species spreads more quickly and at least one species spreads no more slowly. These results were illustrated with some simple invasion models, one of which showed the development of 'stacked waves' of mutualistic species, moving at different speeds, and another that treated two-species competition models. More recent work on the existence of travelling wave solutions was discussed. Here the existence of a family of travelling wave solutions was shown, with the spreading speed characterized as the slowest speed of the family of travelling waves [1]. This recent work builds on the earlier theory developed by Roger Lui for recursion models [13, 14].

The effects of quiescent states on ecological systems were discussed by Karl Hadeler. Quiescent states, with random switching in and out of these states, damp oscillations locally and can suppress periodic orbits. This was illustrated by the introduction of a quiescent state for the prey into the MacArthur-Rosenzweig model. Coupled reaction-diffusion equations with a quiescent state [6] can be analyzed by the methods in [11, 23] to yield spread rates and travelling fronts. Here the impact of the quiescent states can be dramatic on spreading speeds, often reducing the speed to a fraction of what it would be without the quiescent state. The mathematical methods in Hadeler's research [6] have been recently applied to model the spread of West Nile Virus across North America [10]

Xiao-Qiang Zhao gave a historical survey of results on travelling waves and spread speeds for different population models. He summarized studies of monostable and bistable waves for a variety of different formulations (reaction-diffusion equations, integro-differential equations, etc.) and unified these with an integral equation approach. He gave rigorous results on the asymptotic spread speed and travelling wave speed for symmetric kernels [22]. This general method was illustrated by examples from the literature in which the spread speed and minimum wave speed were equal, and this value was estimated. Numerical simulations [24] of spreading speeds were presented.

Invasions of diseases into new territory is a worldwide problem, which traditionally has been modelled with reaction-diffusion equations. If dispersal is nonlocal, these equations can greatly underestimate speeds of invasion. Integro-differential models can incorporate nonlocal dispersal. Jan Medlock showed how to use knowledge about the dispersal of either disease propagules (distributed contacts) or infected hosts (distributed infectives) to model disease spread with integro-differential equations [16]. Both models have travelling wave solutions and the wave speed can be computed in terms of the moment generating function of the contact distribution or dispersal kernel. The magnitude of the force of infection determines which dispersal mechanism gives rise to faster wave speeds. A perturbation scheme can be used to approximate the wave shape. Integro-differential equations seem more flexible than reaction-diffusion equations for continuous time scenarios; they will clearly be the focus of much future work.

Development of the Interface Between Model and Data

The theoretical work of Lui [13, 14], cited above, was popularized and applied in an ecological context to stage-structured (matrix) models with dispersal by Neubert and Caswell [17]. Here the combination of stage-specific information on demography and dispersal makes it possible to predict invasion wave speeds. However, such predictions are not the only, or even the most interesting, results of the model. For example, analysis of sensitivity and elasticity of the speed to model parameters makes it possible for managers to determine where invasive species are most susceptible to control measures. These issues were discussed in detail by Mike Neubert and Hal Caswell. They also presented a large number of examples of successful application of the theory to biological invasions across biological taxa. Some of the work they presented was the output of a US NSF-funded "National Center for Ecological Analysis and Synthesis" working group.

Beneficial 'invasions' may be the goal when managing endangered species. Variation in the rate of spread of a population is of fundamental importance for managing the species of conservation concern, for which spatial spread is beneficial [20]. James Bullock presented case studies in which this approach was used to model and understand constraints on spread for a range of conservation questions: how we facilitate habitat restoration; how we speed up species re-introduction; what role do mutualisms have in population persistence and spread; and how do we predict risks from Genetically Modified Organisms? The methods used for the analysis in these studies was based on the Neubert and Caswell modelling approach given above [17].

When rare, long-distance dispersal events occur, spread rates of populations are very sensitive to the so-called 'tails' of the dispersal kernels (probability density functions for dispersal distance). Here, the rare, long-distance dispersal events are the ones that cause rapid spread of an invading population. At the same time, the spreading speed becomes highly variable, as it is uncertain precisely when the rare, long-distance dispersal will occur [3]. In this context fecundity (number of viable offspring produced) makes a strong contribution to invasion speed [4]. The importance of fecundity has been largely overlooked, because traditional models of diffusion are weakly influenced by net reproductive rate (R_0) and, thus, seed production. By contrast, fat-tailed dispersal kernels effectively translate small differences in fecundity over large distances [2]. Among the challenges for predicting invasion speed is the estimation of fecundity and of recruitment success in new landscapes. Together, these components of population success far from the resident population control the capacity to spread. Jim Clark discussed the components of R_0 that must be inferred or predicted in order to anticipate invasion speed, and provided perspectives on those components we can expect to predict well and those that will remain uncertain for the foreseeable future [5].

Although recent studies have highlighted the importance of detailed dispersal data for the accurate prediction of spread rates, there are few, if any, standardized methods for the measurement of dispersal. Katriona Shea reported on recent efforts to standardize dispersal study designs. These include simulation models to investigate the efficiency of different trap layouts; to assess the importance of trap areas, source strengths, and dispersal geometry; and to compare the effectiveness of trapping (Eulerian) and tracking (Lagrangian) approaches. For thin-tailed dispersal data, transects were especially effective, but for fat-tailed data sector sampling was more effective. Under constant environmental conditions tracking of seeds often required smaller sample sizes than trapping for reasonable goodness of fit. At the same time, tracking data, which is often of limited duration, is more susceptible to error from autocorrelation in the environment. Dispersal models based on limited samples should be used with caution in population dynamics.

Mountain pine beetle attacks on lodgepole pine are a major problem for forests in the western United

States and, more recently, in Alberta and British Columbia. James Powell began by describing the ecology and phenology of pine beetles [12]. The development of mountain pine beetles is under direct thermal control, and success of attack depends upon the beetles' ability to emerge simultaneously at an appropriate time of year [19]. Before 1995, data on outbreaks in Sawtooth Valley in Idaho showed a declining period-two oscillation, but since 1995 data show exponential growth in the area of infestation. To describe these outbreaks, Powell developed a discrete-time model, the Red-Top Model, in which the pines are divided into three age classes. Two key parameters were estimated from the data. The presence of an Allee effect makes the calculation of the invasion speed difficult, even with good dispersal data. Predicted spread speeds match data for Sawtooth Valley, but are too low for current British Columbia outbreaks. Other factors such as wind dispersal and global warming may account for this discrepancy.

The most common method of harvesting forests is clearcutting, which presents a challenge to species that live in the forest, for example, tree squirrels in the dry interior forests of British Columbia. Rebecca Tyson presented a model for recolonization that includes a habitat quality depending on time since clearcutting. The model includes migration between patches and a patch selection function. Tyson applied the model to tree squirrels in both a two-patch system (mature and second growth forest) and a four-patch model that includes edge effects. In the latter case recolonization can take more than twice as long as forest regeneration. If the recutting schedule in based only on forest regeneration, then it is quite possible that even small mammal populations living in the forest are still a long way from recolonization.

Development of New Models and Their Analysis

Are generalist predators effective biological control agents for invasive species? Chris Cosner described a model for an invasive leaf miner and a generalist parasitoid that attacks the leaf miner but that can survive without the leaf miner. Each species has its own carrying capacity, but there is a Holling type-II predator-prey term that links the dynamics of the two species. In addition, both species diffuse. The resulting reaction-diffusion model predicts a number of possible outcomes, depending on the parameters of the model. In some cases, there are pulled waves of leaf miner invasion. For other parameters, the predator induces an Allee effect in the prey and the leaf miner invades by means of pushed waves. Finally, the predator may prevent invasion by the leaf miner altogether. Cosner used this model to focus attention on the factors that lead to effective biological control by generalist predators.

William Fagan continued the theme of predator-prey interactions in ecological invasions by summarizing recent experimental and theoretical work on native herbivorous insects that attack invasive lupine plants at Mount St. Helens, Washington, USA [7]. Detailed data on the life history and interaction of the lupine and its herbivores have been used to parameterize a system of stage-structured integrodifference equations for the recolonization of the volcano's primary successional landscape. A key ingredient of these models is the presence of inverse density-dependent herbivory: herbivores that attack high-density patches of lupine encounter low nutrient quality and high toxicity. A preliminary analysis of this model suggests that Allee effects in the predator play a pivotal role and that too much plant "invasion momentum" prevents the herbivore from reversing the plant invasion. Fagan described the implications of this work for successional dynamics and the biological control of invasive species.

Species persistence in river ecosystems is a subject of ongoing concern, especially as these ecosystems are affected by human disturbance. Individuals in rivers and streams are subject to downstream-advection in their environment. The somewhat surprising observation that species can persist in such environments even though the individuals cannot actively move against the advection has been termed the "drift paradox" in the ecological literature. Mathematical models for populations in environments with unidirectional flow, such as rivers and streams, can be used to analyze conditions under which species can persist. In particular the models allow us to analyze the consequences of movement behaviour of individuals with respect to invasion speed and critical domain size. As shown by Lutscher and coworkers in a series of papers [15, 18], it turns out that these two ecological quantities are related as follows: If the advection speed is so large that the critical domain size approaches infinity, then the population cannot invade upstream, and vice versa. As shown by Lutscher, it is possible to extend one simple model to include spatial heterogeneity, given by a "pool-and-riffle" environment in a river, and study the model with respect to persistence and travelling periodic waves.

Collaborative Research, Interchange and Open Questions

The workshop was the ideal venue for discussion and collaborative interactions. For every 50 minutes of lecture there was at least 30 minutes of formal discussion time. This was supplemented by more informal discussion in the afternoons. The afternoons were also used for informal "breakout sessions" in which groups discussed subjects such as: how to estimate observed population spread rates from data, and the formulation and analysis of stochastic models for population spread.

Collaborative research groups tackled specific applied problems where biological questions and mathematical theory came together. For example, two separate groups started work on deriving a simplified model for plant-insect recolonization interactions on the Pumice Plains region of Mt. St. Helens. The ideas for these groups followed on from the ideas presented in the talks of Bill Fagan and Steve Cantrell (above).

The organizers asked that workshop participants submit informal "open problems" as a basis for the final discussion for the workshop. The list appended shows the breadth and depth of the issues addressed at the meeting.

Discussion Questions

We invited workshop participants to submit open questions in the area of biological invasions. These open questions were the foundation for our discussion on the final day. A list is given below.

Mathematical Theory

1. Can we come up with a general theorem on the existence of spread rates, even when systems are not cooperative? (Hans Weinberger)

2. Just as there are well-known precisely defined kinds of stability, applicable to classes of dynamical systems, someone (if they haven't already done so) ought to come up with general well-defined notions of invadability of a given ecological community by another given community, which are applicable to broad classes of evolutionary population dynamics models. In the case of models of spatially distributed populations, some but not all of these definitions would involve propagative ideas (well represented in this conference). These definitions of invadability would be mathematical, but of course clearly tied to concrete events. Example: systems with travelling fronts connecting two "stable" constant states—the state being invaded could be considered invadable according to one definition, but not if something like the hair trigger effect is required. (Paul Fife)

3. Find a framework in which to study travelling waves for non-monotonic evolutions. Maybe take a clue from physics; there has been a lot of work, usually not rigorous, on non-monotonic invasions, sometimes invasions of or by a spatially or temporally oscillatory state. (Paul Fife)

4. I wish we would have a clear definition of an accelerating front. (Karl Hadeler)

5. What error does one commit (with respect to critical domain size (leading eigenvalue) or speed of fronts if one replaces a kernel by its diffusion approximation? (Karl Hadeler)

6. Is there any scaling (cf. Paul Fife's comment on Barenblatt: long time, but not too long) of time and/or space that would make a fat kernel biological meaningful. Remember: you try to investigate the behaviour of a front for large time at distant space positions and at the same time you assume that the fat kernel reaches to infinity with high probability. No wonder there is singular behaviour. (Karl Hadeler)

7. How quickly do invasions reach the wave speed c*, what role do transients play, and how do demographic and dispersal contributions shift as an invasion progresses? (Kat Shea)

8. What to do about overcompensation? (Mike Neubert)

9. What is the biological interpretation of the condition for invasion in the bistable equation? (I.e. $\int f(x)dx > 0$?) (Mike Neubert)

10. Why hasn't any mathematician written a paper, in plain language, that lays out exactly what we know about rates of spread in population models, exactly what we don't know, and and exactly what we wish we knew? (This last part will be different for mathematicians and biologists.) (Mike Neubert)

11. What happen if we don't have a cooperative system? Can we still define an invasion speed? Mathematically, this is very challenging as Prof. Weinberger pointed out. The phenomenon of accelerating waves that Jan talked about is also very interesting. (Roger Lui)

12. Difference integral equations can formally be written as double integral equations using the Dirac measure in time concentrated at one. Jan Medlock's epidemic model with movement via an integral kernel can formally be transformed to a double integral equation with a Dirac measure in space concentrated at 0. (Horst Thieme)

13. Can the double integral equation theory be extended from Lebesgue densities to more general measures such that these models are not only formally covered, but become actual special cases? (Horst Thieme)

14. If a multi-species or multi-stage system is cooperative, certain useful results follow. If it is not, no one knows quite what to expect. Faced with this situation, some students of invasion (like me, for instance) go ahead and calculate properties of wave speed anyway. This reliance on faith seems touching, but dangerous. So, are there any calculations (hopefully simple) that would support one's belief in the existence of a wave, with a speed, in the absence of a demonstration of cooperativity? An obvious possibility would be a numerical simulation for a specific set of parameters, but does this really provide much comfort? Are there other, less obvious, calculations? If so, what are they? (Hal Caswell)

15. Is it true that the flows of integrodifferential equations do not compactify (although they may contract with respect to measures of non-compactness) and hence their qualitative analysis is technically more difficult than that of reaction diffusion equations ? (Karl Hadeler)

Development of the Interface Between Model and Data

1. Should we be paying attention to correspondence between projected spread rates and observed spread (from field data)? When an for what kinds of questions? Under what conditions should we de-emphasize this comparison? What can we learn from it? (Ingrid Parker)

2. Can some of the complicated behaviour seen in plant-herbivore invasions on Mt. St. Helens be duplicated in simple models with a couple of PDEs or integro-difference equations? (Chris Cosner)

3. Patchy spread with more than one species motivated by the Mount St Helens system. This would involve something like random draws from a dispersal kernel in which a dispersed individual of species X starts a patch where it lands, which then starts to grow radially. The same type of process then occurs for species Y, but Y only succeeds if it lands on one of the patches of species X. Once a Y landed on a patch of Xs there would have to be some submodel for local growth of Y plus rules for how the newly founded patches of X and Y started sending off their own dispersers (or not). (Bill Fagan)

4. It seems that we lack a proper framework for 2D (and 3D) phenomena. Not only are the problems solved in 1D, but even questions are asked in a 1D shape (speed of invasion). The issue is first to know whether things essentially stay the same in 2D or not, and if not, in which shape can we ask the relevant ecological questions in 2D ? (Amaury Lambert).

5. There seem to be several open questions surrounding the choice of dispersal kernels to use in a model. These are of a more technical nature and not grand theoretical questions. What kernels are appropriate for a given dispersal mechanism(s). What are the appropriate statistics to use to estimate parameters in kernels. Most of what I am aware of in this area treats the kernels as mere statistical models and doesn't take into account the mechanisms. Surely there is a way to combine experiments on individuals (tracking) with experiments on populations (tracking). What is to be done with outliers, or extreme events. Should they be used in estimating a dispersal kernel, or treated differently in a model. Eg, an integral equation model with either rare events as initial conditions or as a stochastic (rare) forcing. (James Watmough)

6. Given that long distance dispersal events are hard to quantify and measure, due to their stochastic nature, and that the effect of this data on the resulting dispersal kernel: little on parametric kernels; and large on spline type kernels, is unsatisfactory (Clark). How can this be resolved with the importance of LDD events on spread rate and wave speed (Neubert, Caswell etc)? Following Shea and Bullock's optimization method, can this be extended to determine the amount of effort that is worth expending on collecting LDD? Or should mechanistic models be investigated more fully? (Caroline Bampfylde)

7. Should we be paying attention to correspondence between projected c* and observed spread? When and for what kinds of questions? When should we de-emphasize this comparison? What can we learn from it? (Ingrid Parker)

Development of New Models and Their Analysis

1. "How much is enough?" There is a spectrum from simple mathematical toy models to highly complex simulation models, and similarly there seem to be simple field experiments and highly complicated ones. How much of all this information do we really need to understand and to make informed decisions? (Frithjof Lutscher)

2. Is there a simple model that can be used to determine conditions under which the intrinsic stochasticity in dispersal and environmental conditions make it impossible to estimate a spread rate with reasonable certainty? (Mark Lewis)

3. What happens in coinvasions or successive invasions involving prey and generalist predators or suites of predators? (Chris Cosner)

4. One of the main features of the Mt.St. Helens lupine-moth interaction is that moth feeding success is much higher on small lupine patches than it is on larger patches.In part the reason is that the nutrient content of larger patches is lower; however,the larger patch lupines appear to generate a substance that is toxic to the moth. It seems to me that one possible way to model such a scenario is to think of the interaction as predation at low resource levels and as competition at high levels. What are the implications for such a formulation regarding the invasion of consumer and resource across a spatial landscape? (Steve Cantrell)

5. I gather Okubo did something on invasion into a competitor, which I still have to find and read, but I'd be interested in how the degree of similarity between invaders and natives mediates invasion wave speed. Also, how the presence of different types of natural enemies in such competitive systems (generalists, specialists on the invader or on the native) would alter the dynamics. (Kat Shea)

6. Amaury Lambert and I talked about the following branching process problem, again based on the Mount St Helens system. The branching process considers how patches give rise to new patches (at rate B) and how patches are terminated by the herbivores (at rate D) There are old analytical results governing the probability of failure of the whole colony if B and D are constants. But consider the following scenario: After a patch is founded it continues to grow locally in size in addition to having the possibility of founding new patches. One way of treating this idea is to have B be some increasing function of patch age (the time since that patch was founded). Patch termination rate D would also have to be an increasing function of patch age to counter the increasing rate B and maintain the possibility for process failure. The interesting scenarios are ones in which B and D have different shapes, including the case where D is hump shaped (increasing faster than B for young ages but then below B for long ages [to capture the decline in lupine tissue quality]). This would be complicated analytically so one possibility would be to rephrase the problem as a hierarchy of branching processes where at the fine scale individuals give birth to other individuals (at a constant rate B) but the D term operates against whole "limbs" (i.e., patches) of the process. There is a relevant book on this type of system called "Modelling extinction" which deals with theoretical treatments of extinctions in the fossil record trying to find ways of studying mass-extinction processes like those found in the geological record using mathematics. (Bill Fagan)

7. Analytically treat the case of co-invasions of predator and prey in the integrodifference case even if it is not tractable for PDEs and thus falls outside of the Weinberger-Li-Lewis general theory. (Bill Fagan)

8. Treat analytically cases where species interactions manifest through changes in each others movement terms rather than through the growth terms. So what is the effect of a mutualist like a seed disperser or a natural enemy like a seed predator on invasion success or wave speed. (Bill Fagan)

9. The focus is overwhelmingly on deterministic models (even if they sometimes have to be interpreted as integrated stochastic models). However, the role of stochasticity (demography but also dispersion) is ubiquitous in ecology and has not to be missed, at least because you might ask what is the probability that an invasion will succeed. The recent interest of some of the audience into branching random walks instead of PDE or IDE (Kot, Medlock, Reluga, Neubert, Caswell...) suggests that the community is mature for dealing more often with stochastic versions of known models, but also hopefully with truly new stochastic models. (Amaury Lambert)

10. Stochastic models - especially random environment and how they affect invasion speed. Prof. Hadeler mentioned in his talk that a lot of work has been done in this area but I am unaware of any papers on this subject. Is this related to LDD that Prof. Clark talked about? (I am not talking about nonhomogeneous environment.) (Roger Lui)

11. A more complete theory of the functional response is needed that takes into account: use of a spatially distributed resource, and the effects of multiple predator and prey species, and does not assume dispersal, reproduction and predation are independent. (James Watmough)

12. I'm wondering how much we can learn about predator invasions from what is known about epidemiology. For example, we know that a susceptible population at K but spreading outwards can be invaded by a pathogen in the interior and the wave of infecteds will follow the wave of susceptibles, etc. We also know that for the disease, R0¿1 can invade and that immunization can reduce the density of susceptibles below the level necessary for an epidemic. (Kevin Drury)

13. How can integro-differential equation models for disease spread best be used to model disease control strategies? (Pauline van den Driessche)

14. If we consider a population where females and males disperse differently, is the spread speed of the travelling wave determined by the parameters of the less-readily dispersing sex? (Rebecca Tyson)

15. What proportion of "attempted" invasions fail, and how does invasion failure fit into the framework of wave speeds via the linear conjecture? Is it simply that failure is due to an overall negative growth rate, or should we be looking for threshold effects, which do not allow the use of the linear conjecture? There was some discussion to the effect that many invasions do fail, or require multiple initiating events, so this would appear to be a common feature. (Markus Owen)

16. What do our models tell us about types of intervention that may be desirable? E.g. control measures or conservation measures. (Markus Owen)

17. One thing I was thinking about was types of models other than PDEs. Alan Hastings briefly mentioned cellular automata and interacting particle systems in his 1996 Ecology paper. I think that there is some analytical theory concerning these approaches - it's not just all simulation.

    Also these approaches might be more amenable to incorporating detailed landscape data into a model. With all the GIS data that there must be now, this might be worthwhile. (Andy Edwards)

18. My question has to do with can we identify the extent to which dispersal makes a species a better invader. Most ecologists normally think that dispersal is a plus but (1) during establishment, dispersal may amplify the effects of an Allee effect and decrease establishment rates (2) If range expansions functions via formation of coalescing colonies (e.g. Shigesada et al) then dispersal may decrease the rate of colony establishment (again by magnifying Allee effects) and thus lead to lower rates of spread. I guess the big question is when does dispersal have these negative effects. (Sandy Liebhold)

19. Some plants produce fruits with a surplus of nutrient in order to attract spreaders (apple), others produce seeds with just the right amount for germination but they produce much more seeds than they can ever be expect to be dispersed in order to attract spreaders by the mere amount (acorn). In such cases most seeds fall very close to the parent plant and the actual dispersal happens later. Dow we know how to average over positions of located seeds to get the true dispersal pattern? Does standard sampling underestimate dispersal rates ? (Karl Hadeler)

20. An invader that is rather similar to existing species (example leaf miner) can be sure that the per capita mortality caused by a generalist predator does not exceed that of its competitors. Is it true that its invasion success will largely depend on exploitation of resources and reproduction and not so much on predation? In other words, can it hide amongst its competitors? (Karl Hadeler)

21. Is it true that Europeans should not bother about invaders since their flora (and perhaps fauna) is relatively poor in comparison to North America? (Karl Hadeler)

List of Participants

Atapour, Mahshid (University of Saskatchewan)
Bampfylde, Caroline (University of Alberta)
Bullock, James (NERC Centre for Ecology and Hydrology, Winfrith)
Cantrell, Steve (University of Miami)
Caswell, Hal (Woods Hole Oceanographic Institution)
Clark, Jim (Duke University)
Cosner, Chris (University of Miami)
Desjardins, Sylvie (Okanagan University College)
Drury, Kevin (University of Notre Dame)
Edwards, Andrew (Dalhousie University)
Fagan, Bill (University of Maryland)
Fife, Paul (University of Utah)
Hadeler, Karl (Universität Tübingen)
He, Fangliang (University of Alberta)
Hosono, Yuzo (Kyoto Sangyo University)
Hu, Xinsheng (University of Alberta)
Hunter, Christine (Woods Hole Oceanographic Institution)
Jerde, Chris (University of Alberta)
Kot, Mark (University of Washington)
Lambert, Amaury (Ecole Normale Supérieure)
Lewis, Mark (University of Alberta)
Li, Bingtuan (University of Louisville)
Liebhold, Andrew (USDA Forest Service)
Lui, Roger (Worcester Polytechnic Institute)
Lutscher, Frithjof (University of Alberta)
Medlock, Jan (Yale University)
Neubert, Mike (Woods Hole Oceanographic Institute)
Owen, Markus (University of Nottingham)
Parker, Ingrid (University of California, Santa Cruz)
Potapov, Alex (University of Alberta)
Powell, James (Utah State University)
Reluga, Timothy (Yale University)
Robbins, Tom (University of Utah)
Shea, Katriona (Pennsylvania State University)
Thieme, Horst (Arizona State University)
Tyson, Rebecca (Okanagan University College)
Watmough, James (University of New Brunswick)

Weinberger, Hans (University of Minnesota)
Zhao, Xiao-Qiang (Memorial University of Newfoundland)
Zou, Xingfu (University of Western Ontario)
de-Camino-Beck, Tomas (University of Alberta)
van den Driessche, Pauline (University of Victoria)

Bibliography

[1] B. Li, H.F. Weinberger, and M.A. Lewis, Existence of travelling waves for discrete and continuous time cooperative systems (submitted to *Math. Biosci.*).

[2] J.S. Clark, M.A. Lewis, and L. Horvath, Invasion by extremes: Population spread with variation in dispersal and reproduction *American Naturalist* **157** (2001) 537-554.

[3] J.S. Clark, M.A. Lewis, J.S. McLachlan, and J. Hille Ris Lambers, Estimating population spread: what can we forecast and how well? *Ecology* **84** (2003) 1979-1988.

[4] J.S. Clark, S. LaDeau, and I. Ibanez, Fecundity of trees and the colonization-competition hypothesis *Ecological Monographs* **74** (2004) 415-442.

[5] J.S. Clark, Why environmental scientists are becoming Bayesians. *Ecology Letters* (2004) (in press).

[6] K.P. Hadeler and M.A. Lewis, Spatial dynamics of the diffusive logistic equation with sedentary component. *Canadian Applied Mathematics Quarterly* **10** (2002) 473–500.

[7] W.F. Fagan, J.G. Bishop, Trophic interactions during primary succession: Herbivores slow a plant reinvasion at Mount St. Helens. *Am. Nat.* **155** (2000) 238–251.

[8] R.A. Fisher, The wave of advance of advantageous genes *Ann. Eugen. London* **37** (1937) 355–369.

[9] A. Kolmogorov, Petrovsky and N.I. Piscounov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application a un problème biologique *Bull. Moscow Univ. Math. Mech.* **1** (1937) 1–26.

[10] M.A. Lewis, J. Renclawowicz, and P. van den Driessche, Traveling waves and spread rates for a West Nile virus model (submitted to *Bull. Math. Biol.*).

[11] M.A. Lewis, B. Li, and H.F. Weinberger, Spreading speed and the linear conjecture for two-species competition models *J Math. Biol.* **45** (2002), 219–233.

[12] J.A. Logan, J. Regniere and J.A. Powell, Assessing the Impacts of Global Warming on Forest Pest Dynamics, *Frontiers in Ecology and the Environment* **1** (2003) 130–137.

[13] R. Lui, Biological growth and spread modeled by systems of recursions. I Mathematical theory, *Mathematical Biosciences* **93** (1989), 269–295.

[14] R. Lui, Biological growth and spread modeled by systems of recursions. II Biological theory, *Mathematical Biosciences* **93** (1989), 297–312.

[15] F. Lutscher, E. Pachepsky, and M.A. Lewis, The effect of dispersal patterns on stream populations (*SIAM J. Appl. Math.*, in press).

[16] J. Medlock and M. Kot, Spreading disease: integro-differential equations old and new *Math. Biosci.* **184** (2003), 201–222.

[17] M.G. Neubert and H. Caswell, Demography and dispersal: calculation and sensitivity analysis of invasion speed for stage-structured populations *Ecology* **81** (2000) 1613–1628.

[18] L. Pachepsky, F. Lutscher, R. Nisbet, and M.A. Lewis, Persistence, spread, and the drift paradox (*Theor. Pop. Biol.*, in press).

[19] J. Powell, J. Jenkins, J. Logan and B. Bentz, Seasonal Temperature Alone Can Synchronize Life Cycles *Bulletin Math. Biol.* **62** (2000) 977–998.

[20] R.F. Pywell, J.M. Bullock, D.B. Roy, L. Warman, K.J. Walker and P. Rothery, Plant traits as predictors of performance in ecological restoration *Journal of Applied Ecology* **40** (2003) 65.

[21] J. G. Skellam, Random dispersal in theoretical populations *Biometrika* **38** (1951) 196–218.

[22] H.R. Thieme and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models *J. Diff. Eq.* **195** (2003), 430–470.

[23] H.F. Weinberger, M.A. Lewis, and B. Li, Analysis of the linear conjecture for spread in cooperative models *J Math. Biol.* **45** (2002), 183–218.

[24] D. Xu and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for a nonlocal epidemic model. (2004) (preprint).

Chapter 39

Generalizations of de Bruijn Cycles and Gray Codes (04w5039)

December 4–9, 2004

Organizer(s): Brett Stevens (Carleton University), Joe Buhler (Reed College), Persi Diaconis (Stanford University), Fan Chung Graham (University of California, San Diego), Ronald Graham (University of California, San Diego), Frank Ruskey (University of Victoria)

Introduction

The Banff International Research Station workshop on "Generalizations of de Bruijn Cycles and Gray Codes" was held the 5-9 Dec. 2004. We had many wonderful talks, open problem sessions, informal discussions. This workshop was a wonderful opportunity for many researchers in the breadth of this area to meet and exchange ideas. What follows is an outline of the talks, their abstracts, some summary of the informal discussions and a summary of the open problems discussed.

Sunday 5 Dec. 2004

Frank Ruskey, University of Victoria
Title: "Gray Codes, Polyominoes and Distributive Lattices"
Joint work with Stirling Chow.
Abstract: A polyomino is a configuration of unit squares in the plane. Squares are joined along their edges and the configuration may not contain holes. In this talk we consider two topics on polyominoes: Gray codes for polyominoes and Venn diagrams of minimum area constructed from polyominoes.

The Gray codes that we consider are for column-convex polyominoes, which are those whose intersections with vertical lines are connected. In the code successive polyominoes differ by the movement of one square. These Gray codes have interesting connections with certain classical distributive lattices studied in algebraic combinatorics, particularly with regard to questions of rank unimodality.

A Venn diagram is a collection of n simple closed curves (e.g., the outlines of polyominoes) whose 2^n possible interior intersections are all non-empty and connected. A Venn diagram whose curves are polyominoes has minimum area if every intersection of curve interiors is a unit square, with the exception of the empty set. Previously they were known to exist only for $n <= 4$ polyomino curves; we give examples showing that they exist up to $n = 7$ polyomino curves, and give an approximation algorithm to construct them with close to minimum area for all n.

Aaron Williams, University of Victoria
Title: "A Gray Code that Knuth Missed"
Abstract: While studying a pre-fascicle of Donald E. Knuth's new volume of "The Art of Computer Programming" the authors discovered a new Gray Code for generating combinations. The Gray Code is remarkably simple to describe, has several interesting properties, can be implemented with an efficient loop-free algorithm, and will ultimately appear in Volume 4 of the "Art of Computer Programming

Robert Johnson, Queen Mary University of London
Title: "Long Cycles in the Middle Two Layers of the Cube"
Abstract: Let B_r be the graph formed by the middle two layers of the discrete cube of dimension $(2r+1)$. That is the vertices of B_r are the r-sets and the $(r+1)$-sets of an n-set, with vertices adjacent if the corresponding sets differ in exactly 1 element.

It is a well-known conjecture, probably due to Havel, that the graph B_r is Hamiltonian for all r. This is know to be true only for $r \leq 15$. For larger values of r even finding long cycles in B_r seems to be hard. The best previous result (due to Savage and Shields) being that a cycle through 86% of the vertices always exists.

In this talk we describe a construction of cycles in B_r which contain a proportion $1 - o(1)$ of the vertices. One of our tools, giving a Hamilton cycle in the cube with the minimum number of 'changes of direction', may be of independent interest.

Megan Dewar, Carleton University
Title: "Gray Codes of Block Designs"
Abstract: The study of the presence or absence of configurations among consecutive blocks in an ordering of the blocks of a design was initiated by M. Cohen and C. Colbourn in 2003. A (n, l)-configuration is a set system with n elements and l sets in which every element is contained in at least one set. Let $\mathcal{C}$ be a collection of configurations, each having m sets. Let $S = (V, \mathcal{B})$ be a design. A $\mathcal{C}$-ordering of S is a list containing each block of S exactly once, with the property that every m consecutive blocks form a configuration isomorphic to one in $\mathcal{C}$. In this presentation we survey the known results in $\mathcal{C}$-orderings. Many of these results appear under other names or in other fields. We "translate" these results into the language of $\mathcal{C}$-orderings. In the second part the talk, we present a new result: for $v \geq 132$, all $(v, 3, 1)$-designs admit a hut-ordering. Finally, we discuss the relationship of $\mathcal{C}$-orderings to universal cycles. Let B_4 represent the configuration of three triples forming a path and let B_5 be the configuration of triples forming a triangle. We show that a $\{B_4, B_5\}$-cyclic ordering of a triple system is exactly a Ucycle for the triple system.

Monday 6 Dec. 2004

Hal Fredricksen, West Point Naval School.
Title: "The Classical de Bruijn Sequence Problem"
Abstract: de Bruijn sequences and de Bruijn graphs have been studied for considerably longer period than previously thought. There existence was first shown by Flye-Sainte. Marie in 1896 and enumerated by him. Their generation via primitive polynomials was known by Mantel in 1897. With their rediscovery by de Bruijn and Good in 1944 their study began to grow. Algorithms for the generation of these sequences became popular and eventually the first algorithm due to Martin in 1934 was rediscovered. A history of the problem can be found in Fredricksen (SIAM Review, V. 24, # 2, April 1982).

In Figure 39.1 we present a Mind Map of the relationships between the de Bruijn graph and the sequences (Hamilton cycles) in the graph and the decomposition of cycles by the weight of the shift register truth tables that provide for their generation. We give a partial annotation of the Mind Map:

There are 3 areas of study in de Bruijn Graphs/ Sequences:

1. de Bruijn Graphs

2. de Bruijn Sequences

3. The Weight-Cycle Diagram

The mind map in Figure 39.1 shows connections between these topics:

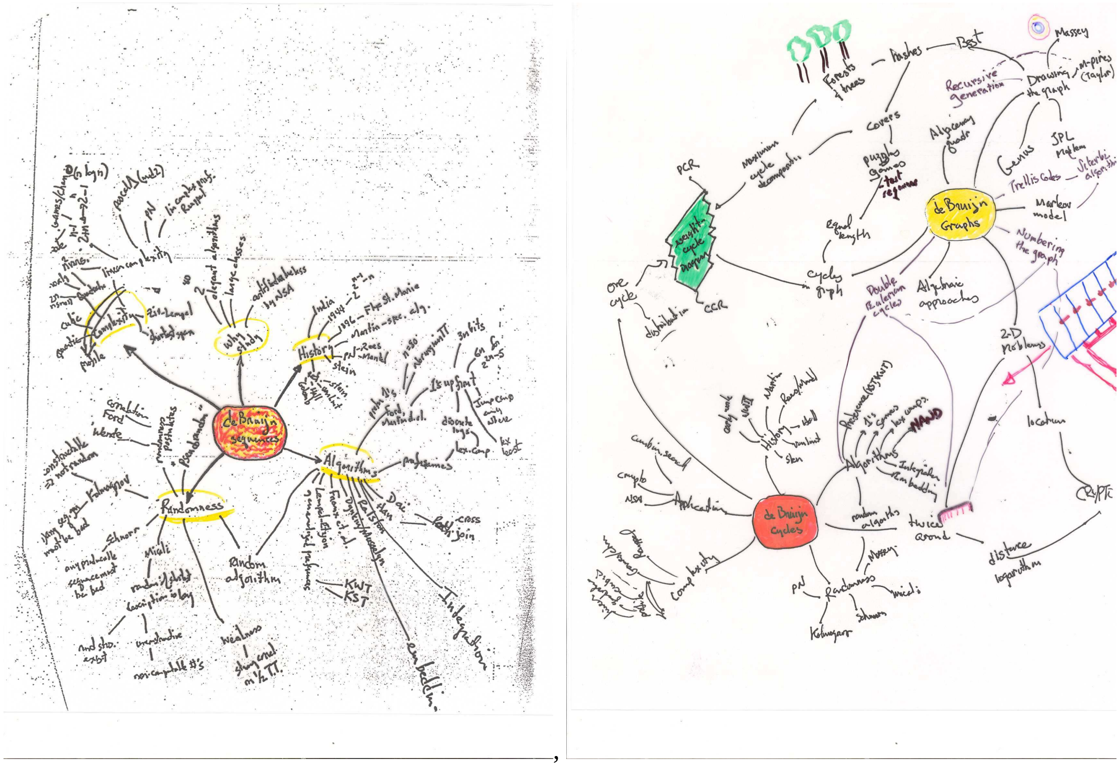


Figure 39.1: A Mind Map of de Bruijn Sequences and their relatives

1. Shift registers and Combs (multiple Eulerian Cycles) connect 1 & 2.

2. Cycles in the de Bruijn graph connect 1 & 3

3. de Bruijn sequences form the left boundary of 3

4. Maximum cycle decompositions of 2 from the right boundary of 3

The graphs (1) are the superposition of adjacency quadruples that connect vertices in the graph via the shift register connections.

There exist many ways to draw the graph and since it rapidly becomes non-planar and complex these methods (recursive techniques, Best diagram, m-pires, etc) become more important for the graph's study.

Graphs are also a representation as a Trellis for Trellis Coding, Viterbi Algorithm, etc. and also provides a nice model of a Markov Process.

Other authors have considered questions of a representation on a higher genus surface (Hales & Butler) and have numbered the graph (Hales & Jewett, Harper).

The graph has many cycle decompositions and decompositions into equal length cycles which provide the impetus for several parlor games and puzzles. Other authors have considered the questions of covers and packings of the graph by maximum independent sets.

The de Bruijn Sequences (2) have a rich and growing history.

A number of elegant algorithms for their generation have been discovered, dissected and improved, primarily the Greedy (Prefer 1s algorithm) and the Lexicographic Compositions (Prefer Sames algorithm).

Some algorithms relate to generation by shift registers (a connection between 1 & 2) and various cycle joining algorithms , e.g. Roth.

Some are elegant combinatorial algorithms – Greedy, prefer 1s, prefer sames, lexicographic compositions, and those of Ralston and Lempel & Etzion as well as generalizations such as the key sequence theorem of Golomb & Welch.

Properties of Randomness make the de Bruijn sequences attractive especially their pseudo-random property and the complexity measures on sequences studied by Kolmogorov/Chaitin, Schnorr, Micali, Maurer, etc.

Complexity of the sequences has been studied by a number of authors including Ziv-Lempel, Reuppel, Games, Chan & Key, and the k-ary complexity profile offers an intriguing possibility for the analysis of these sequences

Persi Diaconis, Stanford University
Ronald Graham, University of California, San Diego
Title: "Some Magic"
Persi Diaconis performed a card trick and discussed the application of universal cycles to card tricks. Both Persi Diaconis and Ronald Graham discussed several open problems which are discussed in Section 39

Informal session: Applications

- Magic tricks
- Erasure correcting codes
- Statistics (permutation tests and randomization procedures)
- Computational efficiency
- analog to digital conversion
- Gray codes and repeated measure designs
- Gray codes with optimized run lengths and light detectors.
- Literary uses of combinatorial orderings

There was also some discussion of various implementations of Gray codes including the various puzzles that incorporate Gray codes, hardware implementations and software implementations.

Brett Stevens, Carleton University
Title: "The Mathematics of Freedom and Constraint"
Abstract: We examine Beckett's play *Quad*. One previous interpretation of *Quad* examines the geometry of movement and suggests a Purgatorial image of simultaneous movement towards and away from God and freedom. We look at the combinatorial aspects of the play and with support from Beckett's other works we suggest a combinatorial analog of this simultaneity of freedom and constraint. Along the way we define a *Beckett-Gray code* which has utility in computer science and computational efficiency.

Eduardo Moreno, University de Chile
Title: "de Bruijn Sequences for General Languages"
Abstract: Let be a language composed by all words of a given length N. A de Bruijn sequence of span N is a cyclic string such that all words in the language appears exactly once as factors of this sequence. This talk shows how to generalize the concept of de Bruijn sequence for a language composed by a subset of words of length N, particularly the languages of all words of length N without factors in a list of "forbidden factors".

We characterize for which languages of words of length N there exists a de Bruijn sequence, and we also study some symbolic dynamical properties of these languages, particularly of the languages defined by forbidden factors. For these kinds of languages, we present two algorithms: one to produce a de Bruijn sequence, using the Lyndon words of the language, and another to construct the lexicographically minimal de Bruijn sequence. These results use the notion of de Bruijn graph and reduce the problem to construct an Eulerian cycle in this graph.

Anant Godbole,East Tennessee State University
Title: "Birthday problem with Dependence"

Abstract: Suppose that the n-tuple X is created by randomly selecting each bit to be 0 or 1 with equal probability. What is the chance that the n corresponding consecutive k tuples are distinct? We exhibit the fact that there is a sharp threshold at $k = 2\log_2 n$. Several open problems will be discussed. This is joint work with Tom Intardonato and Garth Fine.

Tuesday 7 Dec. 2004

Glenn Hurlbert, Arizona State University
Title: "UCycles and Recent Results for $(n-k)$-Subsets of an n-Set"
Abstract: We present a historical survey of Universal Cycles and some new results. We discuss the smallest repetition factor required so that Ucycles of the k-subsets of and n-set exist for $k = 5, 6$ and for k close to n. We present results that show that the length of packing and covering universal words for the k-subsets of and n-set are asymptotically optimal. We finish by discussing de Bruijn Tori and other manifolds.

Brendan McKay, Australian National University
Title: "untitled"
Abstract: Imagine a marine biologist is testing various treatments on plants. He hangs specimens around the edge of a circular tank. There is a distance two neighbor effect. How can the plants be arranged so that this neighbor effect can be controlled for. Three variants of the problem ask for

- A circular list on n elements of length n^2 such that each ordered pair occurs exactly once at distance 1 and distance 2.
- A circular list on n elements of length $n(n-1)$ such that each ordered pair of distinct elements occurs once at distances one and two.
- A circular ordering on n elements of length $n(n-1)/2$ so each unordered pair of distinct elements occurs once at distances one and two.

The first of these problems has some relationship to quasigroups. The second can be solved cyclically with starters for some classes of $n \bmod 8$. The third problem requires that n be odd and also have been solved cyclically for many n.

Karel Casteels, University of Waterloo
Title: "U Cycles of n-1 Partitions of n-Set"
Abstract: In 1992 Chung, Diaconis and Graham generalized de Bruijn cycles to other combinatorial families with *universal cycles*. Universal cycles have been investigated for permutations, partitions, k-partitions and k-subsets. In 1990 Hurlbert proved that there existed at least one Ucycle of $(n-1)$-partitions of an n-set when n is odd and conjectured that when n is even, they do not exist. Herein we prove Hurlbert's conjecture, and establish algebraic necessary and sufficient conditions for their existence. We enumerate all such Ucycles for $n \leq 13$ and give a lower bound on the total number for all n. Additionally we give ranking and unranking formulae. Finally we discuss the structures of the various solutions.

Brad Jackson, San Jose State University
Title: "A Recursive Construction for U Cycles of 2-Subspaces"
Joint work with: Joe Buhler, Centre for Communications Research, La Jolla/Dept. of Math, Reed College, Portland, OR and Ray Mayer, Dept. of Math, Reed College, Portland, OR.
Abstract: Let $F = GF(p)$ be a finite field and suppose that F^n is an n-dimensional vector space over F. We denote by $G(k,n)$, the set of k-dimensional subspaces of F^n. We see that

$$|G(k,n)| = m(k,n) = \frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^np^k-1)}{(p^k-1)(p^k-p)(p^k-p^2)\cdots(p^k-p^{k-1})}.$$

A universal cycle of the k-subspaces is a circular arrangement of $m(k,n)$ elements of F^n, $C = v_1v_2v_3\ldots v_{m(k,n)}$, such that every k-subspace of $G(k,n)$ has exactly one basis that is a block of k consecutive elements of C.

We know that $F^{n*} = F^n \setminus \{0\}$ is a cyclic group. Denote by x_n, the generator of $F^{n*} = \langle x_n \rangle = \{(x_n)^j | 1 \le j \le p^n - 1\}$. In particular, take a generator x_3 of F^{3*}. Then $x_3, x_3^2, x_3^3, , x_3^{m(1,n)}$ is a simultaneous universal cycle of $G(1, n)$ and $G(2, n)$, thus $G(2, n) = \{\langle x_3^j, x_3^{j+1} \rangle | 1 \le j \le m(2, n)\}$ and $G(1, n) = \{\langle x_3^j \rangle | 1 \le j \le m(1, n)\}$. Let $n \ge 3$ be any integer and assume that $C = v_1 v_2 v_3 \dots v_{m(2,n)}$ is a universal cycle of $G(2, n)$ that satisfies $G(1, n) = \{\langle v_j \rangle\}$. We give a recursive construction, which proves the following theorem that universal cycles of the 2-subspaces of a finite vector space always exist.

Theorem: For all $n \ge 3$, there exists a universal cycle $v_1 v_2 v_3 \dots v_{m(2,n)}$ of $G(2, n)$ such that $G(1, n) = \{\langle v_j \rangle\}$ (with repetition allowed).

Similar constructions may exist for $k > 2$, as long as one can construct a simultaneous universal cycle of $(k-1)$-subspaces and k-subspaces of F^{2k-1} to start with. For some small finite fields simultaneous universal cycles of 2-subspaces and 3-subspaces of F^5 have been constructed in an ad hoc manner.

Wednesday 8 Dec. 2004

Carala Savage, North Carolina State University
Title: "Enumeration of Sequences Constrained by the Ratio of Consecutive Parts"
Abstract: This talk will focus on a recurrence that arose in counting nonnegative integer sequences $(a_1, \dots, a_n)$ satisfying the constraints:

$$\frac{a_1}{s_1} \ge \frac{a_2}{s_2} \ge \cdots \ge \frac{a_{n-1}}{s_{n-1}} \ge \frac{a_n}{s_n} \ge 0,$$

for a given constraint sequence $[s_1, \dots, s_n]$ of positive integers. For which sequences $[s_1, \dots, s_n]$ does this recurrence have a nice solution? Examples related to lecture hall partitions, Cayley compositions, and binary partitions will be presented.
Joint work with Sylvie Corteel and Sunyoung Lee.

Mark Weston University of Victoria
Title: "Half-Simple Symmetric Venn Diagrams"
Joint work with Charles E. Killian, Frank Ruskey and Carla D. Savage.
Abstract: A *Venn diagram* for n sets is a collection of n simple closed curves in the plane, with the property that all 2^n possible intersections of curve interiors and exteriors are present exactly once. A *simple* Venn diagram, like the familiar three-circle diagram, has the property that at most two curves intersect at any given point. For n prime, a diagram can be rotationally symmetric: such diagrams have many nice properties as well as being aesthetically pleasing. Until recently, symmetric diagrams were known up to $n = 11$, and simple symmetric diagrams only up to $n = 7$.

A recent paper of Griggs, Killian, and Savage [Elec. J. Combinatorics, 11(1), Research Paper 2, 2004] shows how to construct symmetric Venn diagrams for any prime number of curves. The resulting diagrams contain $\binom{n}{\lfloor n/2 \rfloor}$ intersection points, with exactly n points of intersection through which all curves pass, whereas a simple Venn diagram contains $2^n - 2$ intersection points.

Our work involved modifying their construction to give symmetric Venn diagrams with asymptotically at least 2^{n-1} intersection points, which we thus call "half-simple." The technique is to systematically add extra intersection points by adding extra faces in the dual graph of the Venn diagram. Evidence suggests that the number of intersection points is actually significantly more than half of 2^n: for example, the diagram constructed for 11 curves has 1221 intersection points. Previous research in several papers by Peter Hamburger had found 11-curve Venn diagrams with up to 1001 intersection points.

The question remains as to whether there are *simple* symmetric Venn diagrams for n prime and greater than 7; even the the $n = 11$ case remains open. It would also be interesting to prove an upper bound on the number of symmetric Venn diagrams.

Kevin O'Bryant University of California, San Diego
Title: "The Density of the Outputs of Linear-Shift Registers"
Joint work with Joshua N. Cooper and Dennis Eichhorn
Abstract: For constants $d, a_1, \dots, a_d$, define the 0-1 sequence (with $n > 0$)

$$f_0 = 1, f_{-n} = 0, f_n = a_1 f_{n-1} + a_2 f_{n-2} + \cdots + a_d f_{n-d} \pmod 2.$$

This is a linear shift register, and it is known that for some choices of a_i the output sequence (f_n) is a reduced de Bruijn cycle. In this case, the sequence is 1 and is 0 with nearly equal density. How lopsided can the densities be when the output is not a de Bruijn cycle? We give probabilistic heuristics that the density of '1' in the output is usually near $1/2$. We also the consider nonlinear shift registers with the a_i defined by certain number-theoretically interesting sequences, such as $a_i = 1$ if i is a perfect square, and $a_i = 1$ if the binary expansion of i contains an even number of '1's.

Joshua Cooper, New York University
Title: "Cycles for Other Shapes of Sliding"
Abstract: For a set of integers $\mathcal{I}$, we define a q-ary $\mathcal{I}$ -cycle to be an assignment of the symbols 1 through q to the integers modulo q^n so that every word appears on some translate of $\mathcal{I}$. This definition generalizes that of de Bruijn cycles, and opens up a multitude of questions. We address the existence of such cycles, discuss "reduced" cycles (ones in which the all-zeroes string need not appear), and provide general bounds on the shortest sequence which contains all words on some translate of $\mathcal{I}$. We also prove a variant on recent results concerning decompositions of complete graphs into cycles and employ it to resolve the case of $|\mathcal{I}| = 2$ completely.

Thursday 9 Dec. 2004

Brett Stevens, Carleton University
Title: "Literary Orderings" **Abstract:** We survey the uses that orderings, combinatorial and otherwise, have in Literature. We discuss the Oulipo, Borges, Cortazar, Beckett, and others.

Open problems presented during the workshop

Anant Godbole
Problem: Consider a graph $G = G(k, d, s)$ with the vertex set consisting of all k-letter "words" over an "alphabet" of size d. Furthermore, there will be an edge between vertices $v \neq w$ iff the last $k - s$ letters of v are the same as the first $k - s$ letters of w *or* the first $k - s$ letters of v are the same as the last $k - s$ letters of w. In this note, we show that G is Hamiltonian for all non-trivial values of the parameters. This fact could have been proved on using the standard method, or an extension of the ingenious greedy algorithm proof due to Fredricksen and Maiorana. We offer, however, a proof based on induction on the alphabet size. It is our hope that this method may be of use to exhibit the existence of de Bruijn cycles in other contexts. OPEN PROBLEM: Exploit this technique to its fullest.

Carla Savage
Problem: This is Knuth's Open Problem # 106 in 7.2.1.2 of Volume 4 of *The Art of Computer Programing*.

A *weak order is a relation* $\preceq$ *that is transitive and complete. We can write* $x \equiv y$ *if* $x \preceq y$ *and* $y \preceq x$*;* $x \prec y$ *if* $x \preceq y$ *and* $y \not\preceq x$*. There are 13 weak orders on three elements* $\{1, 2, 3\}$*. A weak order can be represented as a sequence* $a_1 \ldots a_n$ *where* $a_j = k$ *if* j *is preceded by* $k \prec$ *signs. For example, the thirteen weak orders on* $\{1, 2, 3\}$ *are 000, 001, 011, 012, 010, 021, 101, 102, 100, 201, 110, 120, 210, in this form. For all such sequences of length* n *is there a Gray-like code for them?*

Mark Weston
Problem: *"Trotter Gray Code" Is there an* n *bit binary Gray code with the following extra condition: For any* $x, y \in 2^{[n]}$ *and* $x \subset y$ *(where sequences are thought of as the incidence vectors of sets) then* x *preceded* y *in the Gray code, with at most one exceptions with immediately follows* y *in the code. For example:*

$$0000, 0001, 0011, 0010, 0110, 1100, 1000, 1010, 1011, 1001, 1101, 0101, 0111, 1111, 1110$$

The motivation for this come from the fact that such a Gray code gives the chromatic number of the "double shift graph".

Joe Buhler
Problem: *Let* $F = GF(p)$ *be a finite field and suppose that* F^n *is an* n*-dimensional vector space over* F*.*

We denote by $G(k, n)$, the set of k-dimensional subspaces of F^n. The general problem is to find Universal cycles for $G(k, n)$ that are simultaneously Universal cycles for $G(n-k, n)$. As a step in the direction of this, for $k < n/2$, find a map

$$\alpha : G(k, n) \leftarrow G(n-k, n)$$

such that $X \subset \alpha(X)$.

Ronald Graham
Problems:

- *Find a minimal change ordering of all the solutions to Archimedes's Stomachion puzzle.*
- *Show that*

  $$k | \binom{n-1}{k-1}$$

  is a sufficient condition for the existence of the universal cycles of the k-subsets of an n-set provided that $n \geq n_0(k)$.
- *de Bruijn Tori.*
- *Universal tori of the k-subsets of an n-set.*
- *de Bruijn Tori with different window shapes.*

Persi Diaconis
Problems: *Solve*

1. *Enumerate all sequences*
2. *Construct all sequences*
3. *Rank and Unranking problem*
4. *"cutting down problem": can a sequence for a particular object be restricted to smaller order objects and still have a minimal change property?*

For the following problems:

- *Universal cycles of permutations*
- *Universal cycles of set partitions*
- *Universal cycles of permutations with ties.*

Conclusion

The workshop was an excellent and timely opportunity for diverse researchers to meet in a field which is lively and growing. Thus it fulfilled its mandate well. The setting is beautiful, the facilities were excellent. The participants were friendly and excited! We end with a note looking towards the future. The publisher Elsevier and the Journal Discrete Mathematics *have agreed to publish a special issue devoted to the topic of this workshop. The Editors of the issue are Brett Stevens, Brad Jackson and Glenn Hurlbert. We look forward to the exciting new results that will appear there and afterwords.*

List of Participants

Buhler, Joe *(Reed College)*
Casteels, Karel *(University of Waterloo)*
Cooper, Joshua *(New York University)*
Dewar, Megan *(Carleton University)*
Diaconis, Persi *(Stanford University)*
Fredricksen, Harold *(West Point Naval School)*
Godbole, Anant *(East Tennessee State University)*
Graham, Fan Chung *(University of California, San Diego)*
Graham, Ronald *(University of California, San Diego)*
Hurlbert, Glenn *(Arizona State University)*
Jackson, Brad *(San Jose State University)*
Johnson, Robert *(Queen Mary University of London)*
McKay, Brendan *(Australian National University)*
Moreno, Eduardo *(University de Chile)*
O'Bryant, Kevin *(University of California, San Diego)*
Roby, Tom *(California State University)*
Ruskey, Frank *(University of Victoria)*
Savage, Carla *(North Carolina State University)*
Stevens, Brett *(Carleton University)*
West, Julian *(Malaspina University-College)*
Weston, Mark *(University of Victoria)*
Williams, Aaron *(University of Victoria)*

Chapter 40

Numeracy and Beyond (04w5044)

December 4–9, 2004

Organizer(s): *Klaus Hoechsmann (Pacific Institute for the Mathematical Sciences), Tony Gardiner (University of Birmingham), Bernard Madison (University of Arkansas), Yoram Sagher (Florida Atlantic University), Günter Törner (University of Duisburg)*

This was the conclusion of a two part workshop called Numeracy and Beyond begun at PIMS, Vancouver, July 8–11, 2003, and intended not as a gathering of experts offering advice, but one of people with insight and experience in mathematics and its promulgation, soberly examining the question of what level of numeracy might be required of average citizens in the future, and how it would relate to the needs of engineers or scientists.

The first priority was to identify key principles, which are simple, widely acceptable, and fundamental, which could guide teaching and learning, and be largely independent of particular contexts. After detailed presentations and discussions, touching on the various subjects involved, such as the decimal system, fractions, statistics, measurement, graphics, geometry, etc., the following four points had emerged.

1. *Cultivation of numeracy, though built on K–12 education, should continue through the college curriculum in close cooperation with other disciplines.*

2. *Elementary mathematics, being the foundation of numeracy and impinging on many other fields, should be taught with great care and learnt thoroughly.*

3. *Curricula should be sufficiently lean to allow deeper treatment of core topics; their goals should be stated in concise documents with minimal adumbration.*

4. *Teachers should train to be "at home" in basic mathematics, if necessary at the expense of exploring educational theory not closely related to teaching.*

Their implementation was not seen as unproblematic. Given that numbers are indifferent if not repulsive to many (especially Western) cultures, the cooperation mentioned in Item (1) would require considerable effort. Furthermore, Item (3) would go against the almost universal tendency of administrations to issue jargon-laden directives. Straightforward goals such as numerical competence are seldom addressed in today's curricula, which strive instead for "higher" and more nebulous ones. The race toward calculus inhibits the practice of calculation. Overlaps between intended, implemented, and assessed curricula are progressively shrinking (as can be observed even at universities), perhaps mirroring those between the communities of mathematicians, teachers, and policy makers.

The retreat into educational theory, alluded to in Item (4), is as attractive to prospective teachers haunted by "math anxiety" as it harmful to their future performance. However, it is quite legitimately encouraged by Faculties of Education, who naturally see it in a positive light. The resolution of this conflict was seen as a key issue by most participants. Two avenues were described by people with experience in them, and *they both involve the cooperation of mathematicians with a sense of what is relevant in schools: ongoing professional development of teachers and joint seminars with teacher trainers. It was also remarked that elementary mathematics could perhaps be presented in such a way that its common sense roots are clearly visible, so that teachers will learn to master it even without external training.*

The greatest variety of interpretations was attached to the apparently innocuous word "thoroughly" in Item (2). There is a growing recognition (cf." American Educator" Spring 2004) that for knowledge to become long-lasting, sustained practice "beyond the point of mastery" is necessary. Accordingly, many participants felt that teaching should aim at grade-appropriate mastery, including conceptual grasp as well as numerical fluency. While no one disagreed with the proposition that almost all young children have great learning potential, there was some reluctance to have curricular goals defined by international comparisons, such as the performance of Singapore children at Grade 8 (say, in TIMSS 1985). Underdeveloped (e.g., North American) school systems would need time to catch up.

Older students would present even greater challenges, for starters their lack of interest. Here the most common recommendations involved "real life" or "hands on" scenarios. Not all of them were geared toward demonstrating improved numerical know-how, but one presentation showed what was possible, in terms of producing actual, albeit modest, results by applying elementary arithmetic on spread-sheets to simple (simulated) "real life" tasks in "hopeless" Grade 12 classes. This was a useful reality check for the initial visions of wide-spread numeracy.

Like sports, mathematics could serve to equalize chances rather than accentuating differences. In particular, it can open the door to traditionally underrepresented groups into high-level technical professions. If it can ever be made school proof, it will probably be—as in sports—through a return to its rigorous but inherently attractive sources. These are mental—affected, but not altered, by technology.

List of Participants

Askey, Richard *(University of Wisconsin)*
Delord, Michel *(Societe Mathematique de France)*
Ekeland, Ivar *(Pacific Institute for the Mathematical Sciences)*
Friesen, Sharon *(University of Calgary)*
Gardiner, Tony *(University of Birmingham)*
Gaudry, Garth *(University of New South Wales)*
Goldman, Madge *(Gabriella and Paul Rosenbaum Foundation)*
Goris, Tom *(Freudenthal Institute)*
Hoechsmann, Klaus *(Pacific Institute for the Mathematical Sciences)*
Hoogland, Kees *(APS - National Center for School Improvement)*
Hughes Hallett, Deborah *(University of Arizona)*
Katich, Ilija *(Matematichka Gimnazija Sarajevo)*
Keshet, Joshua *(MathCounts Canada)*
Krueger, Katja *(University of Frankfurt)*
Lott, Johnny *(University of Montana)*
Madison, Bernard *(University of Arkansas)*
Mighton, John *(Fields Institute)*
Nicol, Cynthia *(University of British Columbia)*
Packer, Arnold *(Johns Hopkins University)*
Sagher, Yoram *(Florida Atlantic University)*
Scheaffer, Richard *(University of Florida)*
Sirotich, Natasha *(Collingwood School)*
Toerner, Guenter *(University of Duisburg-Essen)*
Wu, Hung-Hsi *(University of California)*

Chapter 41

Resolution of Singularities, Factorization of Birational Mappings, and Toroidal Geometry (04w5510)

December 11–16, 2004

Organizer(s): *Dan Abramovic (Boston University), Edward Bierstone (University of Toronto), Steven Dale Cutkosky (University of Missouri), Kenji Matsuki (Purdue University), Pierre Milman (University of Toronto), Jarosław Włodarczyk (Purdue University)*

The Workshop concentrated on research and goals in three main subjects – resolution of singularities, factorization of birational mappings, *and* toroidalization of morphisms – *as well as on interactions among them. The morning sessions were devoted mainly to mini-series of three lectures each on the topics above, given by Ed Bierstone, Kalle Karu and Dale Cutkosky (respectively). The first day began with an Overview, by Dan Abramovich. There were ten additional invited lectures by participants.*

This report includes an overview, the main themes, and a discussion of perspectives and challenges that emerged during the course of the Workshop.

Overview

Toric geometry is the study of the rich and beautiful geometry of toric varieties, a fairly limited class of rational varieties. Yet this subject interacts in surprising ways with the birational geometry of arbitrary *varieties. A major focus of the workshop was this point of friction between the subjects.*

Hironaka's Theorem

Let us start with a raw statement of Hironaka's theorem on resolution of singularities, as stated in Grothendieck's address at the 1970 International Congress [G]:

Theorem 41.0.1 *Let X be an algebraic variety over a field of characteristic 0, and $U \subset X$ a nonsingular dense open subset. There exists a nonsingular variety X' and a proper morphism $X' \to X$ such that $f^{-1}(U) \to U$ is an isomorphism and $X' \smallsetminus f^{-1}(U)$ is a normal crossings divisor.*

As Grothendieck reported, it was soon recognized that Hironaka's theorem is "not merely a platonic result, giving belated justification for an outdated point of view of algebraic geometry. On the contrary, it is a powerful tool, the most powerful tool we have for studying algebraic and analytic varieties."

Grothendieck goes on to list an impressive array of results we have come to take for granted, all relying on Hironaka's theorem. Clearly,

- *Hironaka's theorem is useful.*

One other point which might get lost in Grothendieck's remark is that:

- *Hironaka's theorem is a marvellously beautiful result.*

Hironaka's theorem is a bit stronger than stated above: the resolution of singularities $X' \to X$ is obtained using a sequence of blowings-up of nonsingular subvarieties lying over the closed subset $X \setminus U$. Grothendieck seemed to think that this is useless, with the exception of preserving projectivity. In hindsight, we know that this is absolutely not true; for many reasons,

- *the use of blowings up is essential.*

This was one theme running through the workshop. It will be discussed below when we relate resolution of singularities to principalization, *and it is the central theme of the discussion of* factorization of birational maps *and of* toroidalization.

Classification of Results

Let us describe some ways we can divide results of these kinds into classes:

1. *We can classify by how well we preserve structure:*
   (a) *We say that a desingularization $X' \to X$ is* dirty *if it pays no regard to the integrity of the nonsingular locus $U \subset X$.*
   (b) *It is* clean *if U remains untouched.*
   (c) *It is* canonical *if it commutes with formally smooth maps $U \to X$; namely, it is a* functor *for schemes with such maps.*
2. *We can classify by what we do to X:*
   (a) *In general, a desingularization $X' \to X$ is by* modification.
   (b) *But it is better to desingularize by* blowings-up *with nonsingular centers.*
3. *We can look at the local picture:*
   (a) valuative *desingularization, generally known as "local uniformization", versus*
   (b) global *desingularization.*
4. *Finally, we have* non-embedded *versus* embedded *desingularization.*

In characteristic 0, the work of Hironaka (1960's) and Bierstone-Milman and Villamayor (last 20 years) provides canonical, embedded global resolution of singularities by blowings-up. *In the 1990's there appeared some proofs of* dirty non-embedded global resolution of singularities by modification. *What could possibly justify this? The point is that there was unfinished work to do, and new approaches were desirable. As we shall see, some of the approaches and results rely heavily on* toroidal geometry.

Unfinished Work (1970)

Looking at Grothendieck's address, we see that things were not quite neat and clean:

1. *Nonzero charateristic: Resolution of singularities in positive and mixed characteristics was, and remains, a major open problem. De Jong's result* – dirty non-embedded global weak desingularization by alteration *is good for some applications. The method of Bogomolov-Pantev reduces "dirty resolution" to understanding:*

(a) resolution of a wild separable cover of a smooth variety, branched over a normal crossings divisor;

(b) resolution of a purely inseparable cover of a smooth variety.

Several lectures in the workshop addressed exciting results and new approaches to the problem, as we report below. Specifically for our theme, Teissier's method reveals the deep importance of toric varieties in the picture of general singularities.

2. *Simplification: While Hironaka's theorem on resolution of singularities [H] is beautiful and powerful, its original proof was so difficult that few have read it, and its original statement left a number of issues unanswered, even in characteristic 0. The current state of the art is very different – resolution of singularities in characteristic 0 is now very well understood, as reported in Bierstone's miniseries. This is again discussed below.*

3. *Generalizations and applications: An ulterior motive for a thorough understanding of resolution of singularities is given by the Toroidalization Conjecture. (See below.) This conjecture is motivated in part by Hironaka's conjecture on* strong factorization of birational maps.

Weak Factorization

Consider a birational map $f : X \dashrightarrow X'$ of smooth complete varieties over an algebraically closed field of characteristic zero, which is an isomorphism on an open set $U \subset X$.

Theorem 41.0.2 ([AKMW, W3]) *The birational map f can be factored as*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{n-1} \dashrightarrow X_n = X', \tag{41.1}$$

where each X_i is a smooth complete variety, and each $X_i \dashrightarrow X_{i+1}$ a blowing-up or blowing-down at a smooth centre that is disjoint from the open set U.

Comparing with our discussion of Hironaka's theorem, we can say that:

- *The theorem is quite useful.*
- *It is quite nice, but we are not fully satisfied – see below.*
- *It is not too difficult, given canonical resolution, which is essential.*

This theorem is one point where toric geometry has so far been absolutely essential, and quite surprisingly so [W2].

Strong Factorization

In order to get a marvelously beautiful result, we want a stronger result:

Conjecture 41.0.3 (Hironaka's strong factorization problem) *Theorem 1.2 holds with (1.1) a succession of blowings-up followed by a succession of blowings-down.*

This is known only in dimension 2. Now:

- *This ought to be quite useful.*
- *It is marvelously beautiful.*
- *We hope that at the end it should not be too difficult.*

Even the toric case is not known in dimensions ≥ 3, though there is an algorithm tested to work in dimension 3 on huge examples. If the toric case works, the non-toric conjecture becomes a special case of the toroidalization conjecture. *(See Sections 3–5 below.)*

Toroidalization

The conjectural strong factorization *is a sort of resolution of singularities of a proper birational map* $X \to Y$*. A natural generalization is the* Toroidalization Conjecture *[AK, AKMW], whose origins go back to Akbulut and King [AkK].*

First, we have:

Theorem 41.0.4 (Dirty toroidalization [AK]) *given a dominant projective morphism of complex projective varieties* $X \to Y$*, there is a diagram*

$$\begin{array}{ccc} X_1 & \to & X \\ \downarrow & & \downarrow \\ Y_1 & \to & Y \end{array}$$

where X_1, Y_1 *are nonsingular varieties with* toroidal structures*,* $X_1 \to X$ *and* $Y_1 \to Y$ *are birational modifications, and the map* $X_1 \to Y_1$ *is a* toroidal morphism.

Here, toric geometry is less a tool and more a goal: to replace an arbitrary map by a toroidal one. This result can be viewed as a desingularization of $X \to Y$ *in the logarithmic sense. (See Section 5 below.)*

The result is useful, and again it is nice but unsatisfactory – we want a clean result using blowings-up:

Conjecture 41.0.5 *In the preceding theorem, we should have the same diagram, where* $X_1 \to X$ *and* $Y_1 \to Y$ *are compositions of clean blowings-up with nonsingular centers.*

The strongest results towards the Toroidalization Conjecture are due to Cutkosky [C1] - [C7]. His lecture series was devoted to his work on toroidalization of a birational map of threefolds. (See Section 4.) This is a beautiful and powerful result, yet its proof is so far extremely difficult. The workshop gave us a chance to peek into this proof and glean some ideas. Hopefully, this will help in simplifying the arguments and generalizing to arbitrary dimension.

Resolution of Singularities

After Hironaka's monumental work on resolution of singularities in characteristic zero in the 1960's, the challenge of finding more straightforward, algorithmic approaches to canonical desingularization was successfully met only in the 1990's, by Bierstone-Milman and Villamayor. The problem is still open in positive and mixed characteristics.

Characteristic Zero

Classically, there are two main theorems: principalization *of an ideal* $\mathcal{I}$*, and* embedded desingularization *of an algebraic or analytic variety* X*. Principalization and embedded desingularization involve two types of transform of the ideal or variety by blowings-up –* weak *and* strict transform*, respectively. In the case that the ideal* $\mathcal{I} = \mathcal{I}_X$ *of* X *is principal (the* hypersurface case*), the two transforms and the two versions of desingularization coincide. There are two local invariants, the* order *of an ideal, and the* Hilbert-Samuel function*, which behave in a nice way with respect to weak and strict transform, respectively.*

The relationship between the local invariants and the corresponding transform is the basis for reducing the two versions of resolution of singularities to an auxiliary problem of canonical desingularization *of a certain collection of local data (generalizing the hypersurface case), originating in Hironaka's idea of an* idealistic exponent*, and called a* presentation *[BM1],* basic object *[EV], or* marked ideal *[W4]. In all cases, canonical desingularization is proved by induction on dimension (passage to a smooth hypersurface of* maximal contact*). But there are important differences in the proofs, reflected in the meaning of "canonical", and in the theorems that can be proved as consequences of canonical desingularization of a "marked ideal".*

The Bierstone-Milman idea of canonicity involves a notion of equivalence of marked ideals by sequences of test transformations, related to the behaviour of the transforms of logarithmic derivatives of an ideal. They prove that desingularization is functorial with respect to the corresponding equivalence relation (a weaker notion than equivalence under smooth mappings, so the equivalence classes are larger). Canonical embedded *desingularization is a consequence of presentation of the Hilbert-Samuel function by a marked ideal. The weaker notion of equivalence allows a comparison of the various algorithms. For example, it shows that the Bierstone-Milman and Włodarczyk algorithms for desingularization of a marked ideal are the same, despite the differences in construction.*

Nonzero Characteristic

In nonzero characteristic, the possibility of reduction to a maximal contact hypersurface fails from the start. For example, consider the hypersurface defined by $f(x) = x_4^2 + x_1^3x_2 + x_2^3x_3 + x_1x_3^7$*, in characteristic* 2*. Then the locus of points of order* 2 *is a curve* $x_1 = t^{15}, x_2 = t^{19}, x_3 = t^7, x_4 = t^{32}$ *that does not lie in any smooth hypersurface [N].*

The most prominent advance in resolution of singularities in positive characteristic presented at the conference was a proof of desingularization of three-dimensional algebraic varieties over an algebraically closed field by Vincent Cossart and Olivier Piltant. The result was previously known for characteristics ≥ 7*, due to Abhyankar (1956), and for three-dimensional hypersurfaces of the special form*

$$y^p - f(x_1, x_2, x_3) = 0, \tag{41.2}$$

due to Cossart (1989). Cossart explained how to prove the local uniformization theorem *(a local version of resolution of singularities, in which the singularities are resolved at the center of a given valuation) for another special class of three-dimensional hypersurfaces, namely the* Artin-Schreier hypersurfaces*, defined by an equation of the form*

$$y^p - g(x_1, x_2, x_3)y - f(x_1, x_2, x_3). \tag{41.3}$$

Roughly, the proof of Cossart–Piltant consists of reducing the case of Artin-Schreier hypersurfaces to that of purely separable ones of the form (2.1). Piltant explained how to reduce the problem of global resolution of singularities to that of local uniformization of varieties of types (2.1) and (2.2), using tricks from birational geometry due to Abhyankar and Zariski. The work of Cossart and Piltant represents major progress in the area of resolution of singularities in positive characteristic.

Franz-Viktor Kuhlmann talked about his results on local uniformization using techniques that come from a branch of logic called model theory (notably the Ax-Kochen principle*) and from Galois theory (inertia and decomposition groups). His two main results are:*

1. *A proof of local uniformization in arbitrary characteristic for Abhyankar valuations; i.e., valuations* ν *centred in a local noetherian domain* (R, m, k)*, which satisfy the equality* $\mathrm{rat.rk}\,\nu + \mathrm{tr.deg}_k\,\nu = \dim R$*.*

2. *A model-theoretic proof of a local analogue of De Jong's celebrated result, which gives local uniformization on a variety with function field* K *after a finite extension of* K*.*

M. Spivakovsky and B. Teissier discussed their ongoing work on the local uniformization theorem in all dimensions and characteristics. Teissier's approach consists in generalizing the method of resolution of plane curve singularities by embedding the singularity in a higher-dimensional space, and then applying a single well-chosen toroidal modification. Spivakovsky's approach uses a generalization of the notion of Puiseux expansion which allows one to relate properties of coordinates before and after blowing up.

Factorization of Birational Maps

A proof of the weak factorization theorem *(Theorem 1.2) was sketched in the mini-series by Kalle Karu. This theorem extends a result of Zariski, which states that any birational map between two smooth complete surfaces can be factored into a succession of blowings-ups at points followed by a succession of blowings-downs at points.*

The proof of Theorem 1.2 relies on the weak factorization theorem for toric varieties. Karu's presentation had three parts:

Polyhedral Cobordisms and the Weak Factorization Theorem

The weak toric conjecture in arbitrary dimensions was proved by Włodarczyk [W1] and later independently by Morelli [Mo] (with a correction in [AMR]). Morelli introduced a notion of combinatorial cobordism*:*

Definition 41.0.6 *Let* $\pi : N^{\mathbb{Q}+} := N^{\mathbb{Q}} \oplus \mathbb{Q} \to N^{\mathbb{Q}}$ *be the natural projection (where* N *denotes a lattice and* $N^{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$*), and* $v = (\{0\} \times 1) \in N^{\mathbb{Q}}$*. A fan* Σ *in* $N^{\mathbb{Q}+}$ *is called a* polyhedral cobordism *or simply a* cobordism *if*

1. *For any cone* $\sigma \in \Sigma$ *the image* $\pi(\sigma)$ *is strictly convex (contains no line).*
2. *The sets of cones*

$$\Sigma_+ := \{\sigma \in \Sigma : \text{ there exists } \epsilon > 0 \text{ such that } \sigma + \epsilon \cdot v \notin |\Sigma|\},$$
$$\Sigma_- := \{\sigma \in \Sigma : \text{ there exists } \epsilon > 0 \text{ such that } \sigma - \epsilon \cdot v \notin |\Sigma|\}$$

are subfans of Σ*, and* $\pi(\Sigma_-) := \{\pi(\tau) : \tau \in \Sigma_-\}$, $\pi(\Sigma_+) := \{\pi(\tau) : \tau \in \Sigma_+\}$ *are fans in* $N^{\mathbb{Q}}$.

Polyhedral cobordisms can be decomposed into elementary collapses. Every elementary collapse defines a simple transformation of fans – either a star-subdivision, a star-assembly or a flip. Note that a flip is a composition of a star-subdivision and a star-assembly. This allows one to connect the fans $\pi(\Sigma_-)$ *and* $\pi(\Sigma_+)$ *by a sequence of star-subdivisons and star-assemblies. The problem is that neither the induced intermediate fans nor the star-subdivisions and star-assemblies are regular even if the fans* $\pi(\Sigma_-)$ *and* $\pi(\Sigma_+)$ *were regular. A solution to the problem is formulated in the following lemma of Morelli.*

Lemma 41.0.7 *Let* Σ *be a simplicial cobordism in* $N^{\mathbb{Q}+}$*. Then there exists a simplicial cobordism* Δ *obtained from* Σ *by a sequence of star-subdivisions, such that* Δ *is* π-nonsingular. *(This means that, for any cone* $\sigma \in \Delta$ *such that* $\dim(\sigma) = \dim(\pi(\sigma))$*, the cone* $\pi(\sigma)$ *is regular.)*

The process of π*-desingularization should be understood as resolution of singularities of the intermediate varieties defined by the factorization. A simplified algorithm for* π*-desingularization was presented in a lecture by Włodarczyk. The process is similar to resolving toric singularities. We apply star-subdivisions to reduce the determinants of the projected cones. Despite the simple underlying idea, the algorithm is much more subtle than usual toric desingularization. The process of* π*-desingularization is based on a classification of cones and their projections. One can distinguish six different configurations of cones and their projections. In the algorithm, these configurations are eliminated one-by-one in a suitable order.*

Birational Cobordisms and the Weak Factorization Theorem

The key notion in the proof of weak factorization is birational cobordism.

Definition 41.0.8 *([W2]) Let* X_1 *and* X_2 *be birationally equivalent normal varieties over a ground field* K*. A* birational cobordism *or simply a* cobordism $B = B(X_1, X_2)$ *between them means a normal variety* B *with an algebraic action of the multiplicative group* K^**, such that the sets*

$$B_- := \{x \in B : \lim_{\mathbf{t}\to 0} \mathbf{t}x \text{ does not exist}\},$$
$$B_+ := \{x \in B : \lim_{\mathbf{t}\to \infty} \mathbf{t}x \text{ does not exist}\}$$

are nonempty and open, there exist geometric quotients B_-/K^*, B_+/K^* *such that* $B_+/K^* \simeq X_1$, $B_-/K^* \simeq X_2$*, and the birational map* $X_1 \dashrightarrow X_2$ *is given by the above isomorphisms and the open embeddings of* $B_+ \cap B_-/K^*$ *into* B_+/K^* *and* B_-/K^**, respectively.*

If B *is a toric variety constructed from a polyhedral cobordism* Σ *in* $N^{\mathbb{Q}+}$*, then* B *defines a birational cobordism with respect to the action of the* 1*-parameter subgroup defined by the vector* $(0,1) \in \mathbb{N} \oplus \mathbb{Z} = \mathbb{N}^+$. *The open subsets* B_-, B_+ *correspond to the subfans* Σ_+, Σ_- *(respectively). The varieties* B_-/K^*, B_+/K^* *correspond to the fans* $\pi(\Sigma_+)$, $\pi(\Sigma_-)$.

The notion of birational cobordism is analogous to cobordism in Morse theory. In the latter, the action of the 1*-parameter group of diffeomorphisms* $G \simeq (\mathbb{R}, +) \simeq (\mathbb{R}^*, \cdot)$*, defined by the gradient field of a Morse function, can be interpreted as above. The bottom and top boundaries determined by a Morse function are isomorphic to the spaces of all orbits with no limit at* $-\infty$ *and* $+\infty$*, respectively (or* 0 *and* $+\infty$*, in multiplicative notation). The critical points of the Morse function are the fixed points of the action. "Passing through" these points induces simple birational transformations analogous to spherical transformations in differential geometry. In the algebraic setting, we replace the Morse function by the moment map for the action of* $S^1 \subset \mathbb{C}^*$*. This defines an order on the fixed-point set components and the corresponding elementary cobordisms. It is convenient to analyze these cobordisms in the language of Geometric Invariant Theory. The elementary cobordisms are merely open subsets of the semistable points defined for different linearizations of the* K^**-action. Varying linearizations gives rise to a sequence of locally toric maps between GIT-quotients. The last problem to overcome is to eliminate singularites of the intermediate varieties (GIT-quotients) and to replace the locally toric morphisms between them by blowings-up and blowings-down.*

There are two approaches to this problem. Kalle discussed the approach of [AKMW] based on the notion of torification *by Abramovich and de Jong. A* torific ideal *is, roughly speaking, a* K^**-equivariant sheaf of ideals whose blowing-up defines the structure of a toroidal embedding compatible with the* K^* *action. The operation of torification (blowing-up of the torific ideal) converts locally toric to toroidal maps, where weak factorization holds. An additional difficulty is that torific ideals can be constructed only locally for elementary cobordisms. Hence torific ideals define local factorizations into blowings-up and blowings-down, which have to be fit together.*

A second approach was discussed by Włodarczyk. Let S *be the stratification determined by the isotropy groups. Let* $s \in S$ *be any stratum with isotropy group* Γ_s*. Then the* Γ_s*-semi-invariant parameters in a neighborhood of a point* $x \in s$ *provide a chart which is locally analytically isomorphic to the tangent space with the induced linear action. Locally, we can identify* B *with an affine space* $\mathbb{A}^n$*, which is a toric variety corresponding to a regular cone* σ_s *in the vector space* $N^{\mathbb{Q}}$*, with a toric action of* Γ_s *on* X_{σ_s}*. The quotient morphism* $X_{\sigma_s} \to X_{\sigma_{s'}}$ *defines a projection* $\pi_s : \sigma_s \to \sigma_s^{\Gamma}$*.*

In particular, if s *consists of fixed points, then the action of* $\Gamma_s = K^*$ *determines a* 1*-parameter subgroup, hence an integral vector* $v \in N^{\mathbb{Q}}$*. The sets* $\mathbb{A}^n_-$ *and* $\mathbb{A}^n_+$ *correspond to the sets* σ_+ *and* σ_-*.The quotient spaces* $\mathbb{A}^n_-/K^*$*,* $\mathbb{A}^n_+/K^*$ *correspond to two subdivisions* $\pi(\sigma_+)$ *and* $\pi(\sigma_-)$ *of the cone* $\pi(\sigma)$*, that are "cobordant" in the sense of Morelli. The problem lies in the fact that the fans* $\pi(\sigma_+)$*,* $\pi(\sigma_-)$ *are singular (i.e., they are not spanned by a part of an integral basis). Consequently, the corresponding birational transformation is a composition of weighted blowings-up and blowings-down between singular varieties. One needs to desingularize the quotient spaces corresponding to these projections.*

If a stratum s *is in the closure of* s' *then* $\sigma_{s'}$ *can be mapped isomorphically onto a face of* σ_s*. Glueing these faces together defines a complex* Σ*. The projections* π *are compatible with face inclusions (up to isomorphisms). One can show that certain decompositions of* Σ *define birational modifications. In particular, if we apply* π*-desingularization to the complex* Σ*, then the corresponding subdivison defines a birational modification* $\overline{B}$ *of* B*.* $\overline{B}$ *is a cobordism all of whose open affine fixed-point free subsets have smooth geometric quotients. The existence of such a cobordism easily implies the weak factorization theorem. (The intermediate varieties are smooth and the blowings-up, blowings-down and flips induced by the elementary cobordisms are regular (smooth).) Since each flip is a composition of a blowing-up and a blowing-down at a smooth centre, we arrive at a factorization of* $X \dashrightarrow X'$ *into blowings-up and blowings-down at smooth centres.*

Strong Factorization Theorem

A local version of the strong factorization problem was solved for 3-dimensional toric varieties by Christensen [Ch]. Cutkosky showed in his monomialization theorem *that a birational map can be locally transformed by blowings-up with smooth centres into a monomial mapping [C2]. In particular, he proved the local strong factorization conjecture in dimension 3 [C2] via Christensen's theorem. Recently, Karu extended Christensen's algorithm to arbitrary dimension. Combining his result with Cutkosky's monomialization theorem gives a proof of the local strong factorization theorem.*

Another approach to stong factorization for toric and arbitrary varieties uses the weak factorization theorem. In fact, one needs to prove strong factorization for varieties which differ by a sequence of blowings-down followed by a sequence of blowings-up. There is a simple algorithm for toric varieties that conjecturally *determines strong factorization. It is not known whether the algorithm terminates even in dimension 3. The algorithm can be reduced to commutativity rules for elementary matrices.*

Toroidalization of Morphisms

Throughout this section, k denotes an algebraically closed field of characteristic zero (unless explicitely stated otherwise). A variety *means an open subset of an irreducible proper k-variety. A toroidal structure on a smooth variety X is a SNC (simple normal crossings) divisor D_X.*

A nonsingular subvariety V of X is a possible centre for D_X if $V \subset D_X$ and V makes SNC with D_X. The blowing-up $\Phi : X_1 \to X$ of a possible centre is called a possible blowing-up. $D_{X_1} = \Phi^{-1}(D_X)$ is then a toroidal structure on X_1.

Recall that $f : X \to Y$ is toroidal (with respect to D_Y and D_X) if $f : (X, D_X) \to (Y, D_Y)$ is locally formally isomorphic to a morphism of toric varieties ([KKMS, AK]).

Toroidalization Conjectures and Results

The following conjectures are more precise versions of Conjecture 1.5 above. The "toroidalization conjecture" of [AKMW] is:

Conjecture 41.0.9 *Suppose that $f : X \to Y$ is a dominant morphism of nonsingular varieties. Then there is a commutative diagram,*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \Phi \downarrow & & \downarrow \Psi \\ X & \xrightarrow{f} & Y \end{array}$$

where Φ, Ψ are products of blowings-up with smooth centres, and there exist SNC divisors D_{Y_1} on Y_1 and D_{X_1} on X_1 such that f_1 is toroidal with respect to D_{Y_1} and D_{X_1}.

There is a stronger version (also stated in [AKMW]) that we will call the "strong toroidalization conjecture":

Conjecture 41.0.10 *Suppose that $f : X \to Y$ is a dominant morphism of smooth varieties. Further suppose that there is a SNC divisor D_Y on Y such that $D_X = f^{-1}(D_Y)$ is a SNC divisor on X, containing the singular locus of the map f. Then there is a commutative diagram,*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \Phi \downarrow & & \downarrow \Psi \\ X & \xrightarrow{f} & Y \end{array}$$

where Φ, Ψ are products of possible blowings-up for the preimages of D_X, D_Y (respectively), and f_1 is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

The characteristic zero assumption on our base field k is necessary in these conjectures. The conjectures fail in positive characteristic even for morphisms of curves (where all blowings-up are trivial). A simple example is

$$t = x^p + x^{p+1}$$

over a field of characteristic p. We have $t = x^p(1 + x)$, but $(1 + x)^{1/p} \notin k[[x]]$.

The (strong) toroidalization conjecture is known to be true in the following cases:

1. $\dim Y = 1$, $\dim X$ *arbitrary. (This case follows from embedded resolution of hypersurface singularities [28, BM1, BEV, W4].)*
2. $\dim X = \dim Y = 2$ *[AkK, CP1, AKMW, Ma].*
3. *Local monomialization (locally along a possibly non-Noetherian valuation) [C1, C2, C5].*

From (3), we get the following:

Theorem 41.0.11 ([C1, C2, C5]) *Suppose that $f : X \to Y$ is a dominant morphism of proper varieties. Then there is a commutative diagram,*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \Phi\downarrow & & \downarrow\Psi \\ X & \xrightarrow{f} & Y \end{array}$$

such that f_1 is toroidal, and Φ, Ψ are locally products of blowings-up with smooth centres. The morphisms Φ, Ψ and f_1 satisfy the existence part of the valuative criterion for properness, but, in general, uniqueness fails (so that these maps are in general not separated).

From (3), we also obtain a proof of "local strong factorization" (conjectured by Abhyankar). Case (3) reduces the proof of the conjecture to the case of a toroidal mapping and a "toroidal" valuation. This was proved in dimension 3, by Christensen [Ch], and extended to arbitrary dimension by Karu [K]. A proof in the style of Christensen's original proof (using determinants and elementary linear algebra) is given in [CS].

Local monomialization along a valuation could possibly be true in positive characteristic. It is certainly true for morphisms of curves, and for morphisms of n-folds to curves in dimensions where resolution of singularities is true. Good progress on this problem has been made for morphisms of surfaces [CP2].

(4) $\dim X = 3, \dim Y = 2$ *[C3] (strong toroidalization).*

(5) $\dim X = \dim Y = 3$, f *birational [C6] (toroidalization), [C7] (strong toroidalization).*

Using (5), we can reduce the "strong factorization" conjecture for birational morphisms of proper 3-folds to the case of toroidal morphisms, so we see that "strong factorization" of birational morphisms of proper 3-folds will follow from the Oda conjecture on "strong factorization" of torodial varieties [O] (cf. §3.3).

We further obtain a new proof of "weak factorization" of birational morphisms of proper 3-folds. Case (5) reduces "weak factorization" to the case of toroidal morphisms, which is solved in [D] (dimension 3), [Mo, W1, AMR] (arbitrary dimension). "Weak factorization" has been proved in all dimensions (using geometric invariant theory) in [W2, AKMW, W3].

Open Problems on Toroidalization

1. *Prove (strong) toroidalization for an arbitrary dominant morphisms of 3-folds.*

   By [C7], we can assume that f is prepared. Much of the proof of toroidalization of birational morphisms of 3-folds [C6, C7] works in the case that f is not birational.

2. *Suppose that $f : X \to Y$ is a dominant morphism from an n-fold to a surface Y. Prove that there is a commutative diagram,*

$$\begin{array}{ccc} X_1 & f_1 & \\ \Phi_1\downarrow & \searrow & \\ X & \xrightarrow{f} & Y \end{array}$$

   such that Φ_1 is a product of possible blowings-up and f_1 is (strongly) prepared. [CK] now implies that f can be toroidalized.

3. *Prove the toroidalization conjecture in all dimensions.*

Related Themes and Challenges

We conclude with a discussion of several interesting directions that emerged in the mini-series and invited lectures, as well as in informal conversations during the Workshop.

Toric Methods in the Study of Singularities

It is an attractive idea to study singularities by analyzing the analytic germs of curves passing through them; i.e., to study the space of "arcs". Nash raised the following question: Is the "Nash map" from the set of maximal irreducible families of arcs passing through the singularities to the set of essential divisors on a resolution bijective? S. Ishii discussed her affirmative answer to this question (joint work with Kollár) in the case of toric varieties, as well as a counter-example in dimension ≥ 4, in general.

Toric geometry provides a dictionary which translates problems in algebraic geometry into problems in the combinatorics of convex bodies in a Euclidean space with a lattice structure. The latter often yields simple answers (e.g, resolution of singularities for a toric variety; cf. [BM3]). So it is particularly interesting in positive characteristic, where the known methods fail, to consider a degeneration of a given variety to a toric variety in the central fibre, and to try to deform (lift) the resolution of singularities of the central fibre to that of the nearby fibres (viewpoint of Teissier).

C. Casagrande's talk presented an elegant example of the power of this dictionary in another context. She has established a conjectured inequality,

$$b_2(X) \cdot (i_X - 1) \leq \dim X,$$

in the case of toric Fano manifolds X over $\mathbb{C}$, where i_X is the minimal degree of a rational curve in X with respect to $-K_X$. The main technique is to study the combinatorial properties of reflexive polytopes, suggested by Batyrev in connection with "mirror symmetry".

Combinatorial Problems Related to Semi-stable Reduction, π-desingularization, and Strong Factorization

Sometimes a problem in algebraic geometry remains of substantial difficulty even when translated into purely combinatorial language. For example, the strong factorization conjecture for toric birational maps in dimension three: We have a simple description of an algorithm for transformation of a weak into a strong factorization (as described by Karu), but we do not know whether it terminates after finitely many steps. Because of Cutkosky's solution of the toroidalization conjecture in dimension 3, this is the only obstruction to the strong factorization conjecture for (general) birational maps in dimension 3.

Abramovich and Karu achieve only "weak" semi-stable reduction in arbitrary dimension. But they settle a combinatorial problem to transform weak into "genuine" semi-stable reduction, only in relative dimension ≤ 3. The problem seems reminicent of π-desingularization (talk of Włodarczyk, simplifying the original algorithm of Morelli and Abramovich-Matsuki-Rashid).

Logarithmic Category

As mentioned above, toroidalization can be interpreted as a problem of resolution of singularities of a morphism, in the logarithmic category. In other words, a morphism $f : (X, D_X) \to (Y, D_Y)$ is toroidal if and only if it has no logarithmic ramification. Cutkosky suggested that some of his techniques for toroidalization might become simpler in the language of the logarithmic category.

One of the differences between the proofs of resolution of singularities by Bierstone-Milman and Włodarczyk (as discussed in Bierstone's lectures), is that the latter uses ordinary differentials, while the former uses logarithmic differentials (in constructing "coefficient ideals").

The logarithmic category may turn out to be a unifying force for the main directions of the workshop.

Computational Aspects, Computer Implementation

M. Rojas discussed the complexity of an algorithm to decide whether a given toric algebraic system has a "torsion" point in its set of solutions. He suggested that a complete understanding of the complexity of this very special problem might lead to a solution of the celebrated $P = NP$ problem.

A. Fruehbis-Krueger presented her computer implementation of Villamayor's algorithm for resolution of singularities (joint work with Pfister). She suggested that her program could be modified to implement the stronger resolution algorithm of Bierstone-Milman.

Concluding Remark

Best wishes for future success to all participants! It would be a great pleasure to meet again at BIRS to discuss developments that have their origins in the ideas communicated in this Workshop.

List of Participants

Abramovich, Dan *(Boston University)*
Ban , Chunsheng *(Ohio State University)*
Berchtold, Florian *(Universitat Konstanz)*
Bierstone, Edward *(University of Toronto)*
Casagrande, Cinzia *(Universita di Pisa)*
Cossart, Vincent *(Universite Versailles Saint-Quentin)*
Cutkosky, Steven Dale *(University of Missouri)*
Eckl, Thomas *(University of Cologne)*
El-Hitti, Samar *(University of Missouri)*
Fruehbis-Krueger, Anne *(Universitat Kaiserslautern)*
Ghezzi, Laura *(University of Missouri)*
Hanumanthu, Krishna *(University of Missouri)*
Ishii, Shihoko *(Tokyo Institute of Technology)*
Jahnke, Priska *(Universitet Bayreuth)*
Karu, Kalle *(University of British Columbia)*
Kawamata, Yujiro *(The University of Tokyo)*
Kawanoue, Hiraku *(Research Institute for Mathematical Scienses)*
Kuhlmann, Franz-Viktor *(University of Saskatchewan)*
Luo, Tie *(National Science Foundation)*
Matsuki, Kenji *(Purdue University)*
McEwan , Lee *(Ohio State University)*
Milman, Pierre *(University of Toronto)*
Piltant, Olivier *(Centre National de la Recherche Scientifique & University Versailles)*
Rojas, J. Maurice *(Texas A&M University)*
Spivakovsky, Mark *(Universite Paul Sabatier)*
Teissier, Bernard *(Centre National de la Recherche Scientifique)*
Wlodarczyk, Jaroslaw *(Purdue University)*

Bibliography

[AK] *Abramovich D., Karu K., Weak semistable reduction in characteristic 0, Invent. Math. 139 (2000), 241 – 273.*

[AKMW] *Abramovich, D., Karu, K., Matsuki, K., and Włodarczyk, J., Torification and factorization of birational maps, J. Amer. Math. Soc. 15 (2002), 531 – 572.*

[AMR] *Abramovich, D., Matsuki, K., Rashid, S., A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension, Tohoku Math. J. 51 (1999), 489 – 537.*

[AkK] *Akbulut, S. and King, H., Topology of algebraic sets, MSRI Publ. 25, Springer-Verlag, Berlin, Heidelberg, New York.*

[BM1] *Bierstone, E. and Milman, P.M., Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128 (1997), 207 – 302.*

[BM2] *Bierstone, E. and Milman, P.M., Desingularization algorithms I. Role of exceptional divisors, Moscow Math. J. 3 (2003), 751–805.*

[BM3] *Bierstone, E. and Milman, P.M., Desingularization of toric and binomial varieties, preprint, math.AG/0411340.*

[BEV] *Bravo, A., Encinas, S., Villamayor, O.,A simplified proof of desingularization and applications, Rev. Mat. Iberamericana (to appear).*

[Ch] *Christensen, C., Strong domination/weak factorization of three dimensional regular local rings, J. Indian Math. Soc., 45 (1981), 21 – 47.*

[C1] *Cutkosky, S.D., Local factorization of birational maps, Adv. Math. 132 (1997), 167 – 315.*

[C2] *Cutkosky, S.D., Local monomialization and factorization of morphisms, Astérisque 260, 1999.*

[C3] *Cutkosky, S.D., Monomialization of Morphisms from 3-folds to surfaces, Lecture Notes in Math. 1786, Springer-Verlag, Berlin, Heidelberg, New York, 2002.*

[C5] *Cutkosky, S.D., Local monomialization of trancendental extensions, Ann. Inst.Fourier Grenoble (to appear).*

[C6] *Cutkosky, S.D., Toroidalization of birational morphisms of projective 3-folds, preprint, math.AG/0407258.*

[C7] *Cutkosky, S.D., Strong toroidalization of birational morphisms of projective 3-folds, preprint.*

[CP1] *Cutkosky, S.D. and Piltant, O., Monomial resolutions of morphisms of algebraic surfaces, Comm. Alg. 28 (2000), 5935 – 5960.*

[CP2] *Cutkosky, S.D. and Piltant, O., Ramification of valuations, Adv. Math. 183 (2004), 1 – 79.*

[CK] *Cutkosky, S.D. and Kascheyeva, O., Monomialization of strongly prepared morphisms from nonsingular n-folds to surfaces, J. Alg. 275 (2004), 275 – 320.*

[CS] *Cutkosky, S.D. and Srinivasan, H., Factorizations of matrices and birational maps, preprint.*

[D] *Danilov, V., Birational geometry of toric 3-folds, Math. USSR Izv. 21 (83), 269 – 280.*

[EV] *Encinas, S. and Villamayor, O., Good points and constructive resolution of singularities, Acta. Math. 181 (1998), 109–158.*

[G] *Grothendieck, A., Travaux de Heisouké Hironaka sur la résolution des singularités, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 7–9, Gauthier-Villars, Paris, 1971.*

[H] *Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Ann. of Math. (2), 79 (1964), 109 – 326.*

[K] *Karu, K., Local strong factorization of toric birational maps, J. Alg. Geom 14 (2005), 165 – 175.*

[KKMS] *Kempf, G., Knudsen, F., Mumford, D. and Saint-Donat, B., Toroidal embeddings I, Lecture Notes in Math. 339, Springer-Verlag, Berlin, Heidelberg, New York, 1973.*

[Ma] *Matsuki, K., A note on toroidalization: the problem of resolution of singularities of morphisms in the logarithmic category, Proc. Barrett Memorial Lectures 2002, University of Tennessee at Knoxville (to appear).*

[Mo] *Morelli, R., The birational geometry of toric varieties, J. Alg. Geom. 5 (1996), 751 – 782.*

[N] *Narasimhan, R., Hyperplanarity of the equimultiple locus, Proc. Amer. Math. Soc. 87 (1983), 403–406.*

[O] *Oda, T., Torus embeddings and applications, Tata Inst., Bombay, 1978.*

[W1] *Włodarczyk, J., Decomposition of birational toric maps in blowups and blowdowns. A proof of the Weak Oda Conjecture, Trans. Amer. Math. Soc. 349 (1997), 373-411.*

[W2] *Włodarczyk, J., Birational Cobordisms and factorization of birational maps, J. Alg. Geom. 9 (2000), 425 – 449.*

[W3] *Włodarczyk, J., Toroidal varieties and the weak factorization theorem, Invent. Math. 154 (2003), 223 – 331.*

[W4] *Włodarczyk, J., Simple Hironaka resolution in characteristic zero, preprint, math.AG/0401401.*

Two-day Workshop Reports

Chapter 42

Human Infant Speech Perception and Language Acquisition: Rules vs. Statistics (04w2552)

March 18–20, 2004

Organizer(s): *Janet Werker (University of British Columbia), Gary Marcus (New York University), Jacques Mehler (University of Trieste), Helen Neville (University of Oregon), Nuria Sebastian-Galles (Universitat de Barcelona)*

During the first year of life, infants become exquisitely tuned to the properties of the native language. This includes sensitivity to the consonant and vowel sounds of the native language, the acceptable sequences of these sounds, the rhythmical properties of the native language, and the multiple cues which signal boundaries between words, phrases, and sentences. Perceptual sensitivity to these properties is essential for word learning, acquisition of grammar, and language acquisition more generally. The predominant theoretical approach guiding this work has, until recently, stemmed from the joint contribution of Chomsky—arguing for a rule based, symbol manipulation, foundation for language acquisition rooted in targeted evolution, and Lenneberg—arguing that language is supported by special areas in the brain. In the past several years, however, a number of studies have documented the role of statistical learning in accounting for the changes in perceptual sensitivity seen in infancy. In this workshop, we explored and debated the extent to which statistical learning vs. rule learning might help account for both perceptual tuning, and subsequent mapping of tuned perceptual categories to the task of language acquisition.

The first evening of the workshop, Wednesday evening, Richard Aslin, a cognitive scientist from the University of Rochester, provided a broad overview of work in statistical learning in infancy. He began with an explication of the original work showing that infants can use transitional probabilities (TPs) to segment words from a stream of speech. In this work, infants were presented with continuous strings of syllables in which the only distinguishing information for positing a "word" was the TP between syllables. Among the strings were three syllable items that had TPs between syllables of 1.0, and three syllable items that had TPs of .33. Following a 3–4 minute familiarization period, infants of 7 months had used these TPs to pull out "words", choosing to listen longer (as indicated by looking time) to "part words" (syllable strings with TPs of only .33) over "words" (strings with TPs of 1.0). Aslin presented a number of updates.

Two kinds of studies generated the most discussion in the debates which began the next morning. 1) those studies comparing the learning of transitional probabilities for linguistic stimuli to learning the transitional probabilities for non-linguistic stimuli such as tones and for visual images. These studies are of interest because they challenge the notion that language learning relies on specialized learning mechanisms, and raise the alternative possibility that generally available learning mechanisms are used in the service of language acquisition. 2) those studies examining the conditions under which infants can learn non-adjacent dependencies. Whereas adjacent TPs are important in word segmentation (and also in learning some aspects of *morphology), non-adjacent TPs are needed if this learning mechanism can contribute to our understanding of how other aspects of morphology as well as syntax are acquired. A number of studies indicate that the learning of non-adjacent dependencies is much more difficult than is learning of adjacent dependencies. Success requires one of a number of manipulations; the insertion of a brief (even undetectable) silent interval between the syllables, high variability of the non-criterial intervening syllables, and/or another linguistic cue such as syllable stress. Vigorous debate centred on the question of why these types of manipulations facilitate learning. Is it because they simplify the pattern detection process that results from the use of statistical regularities? Or is it because these manipulations turn the task from one of statistical learning into one of rule learning? This debate centres on the question of whether language has a specialized rule-based foundation which enables symbol manipulation.*

Two other statistical learning mechanisms were discussed. One was how learning of distributional regularities could account for the building of language appropriate phonetic categories. It was shown recently (by members of our group) that infants of 6-8 months can use distributional regularities to change their phonetic categories. If presented all members of an 8 step continuum, but with more instances of stimuli 2 and 7 in one group (the bimodal group) and more of 4 and 5 in the other group (the monomodal group), infants in the bimodal group divide the continuum into two perceptual categories whereas infants in the monomodal group collapse it into one. New data on the types of information this distributional learning might generalize to, and the ages at which infants might have this mechanism available, were discussed. The other statistical mechanism discussed was associative probabilities. New data were presented showing that learners track the probabilities of word-object associations almost perfectly. Using fMRI data, possible neural systems underlying this learning were discussed, and will be compared the neural mechanisms used in learning other probabilistic associations.

Computational Modelling was another focus of the workshop. A number of researchers in the world are using a variety of computational models to account for and predict the regularities seen in language acquisition. The unique new approach from Gary Marcus in our group is to use computational techniques to formally model the newest findings in developmental neurobiology that might help explain how a brain can become organized for complex cognitive tasks such as language acquisition. The goal behind this work is again to address the fundamental theoretical controversies outlined at the beginning of this report that underly research in language acquisition.

The theoretically guided presentations and debates described above, were informed throughout the meeting with new data in a number of areas which then spawned their own advances in theory. One content area focused on phonetic and phonological perception in bilingual infants and adults, and the link between perceptual changes in infancy and subsequent word learning. This led to the presentation of a new theoretical framework - “shallow” (perceptual) vs. “deep” (functional linguistic use) learning.

A second content area focused on new data showing the different languages young infants can discriminate based on rhythmical properties. Included in the presentation were new steps in the quantification of languages based on rhythmical properties (% vowel and Delta consonant in the words of the language). The novel theoretical contribution here is whether these rhythmical properties are correlated with, and predictive of, the underlying word order of the language. If so, this would be another way in which learning the acoustic properties of the language could help bootstrap its acquisition.

The research presented relied on both behavioural and neuroimaging techniques. One portion of the workshop therefore involved presentation of new team-based advances in the use of these techniques. The two most novel advances discussed were 1) the improvement of optical imaging (near infrared spectroscopy) for imaging the infant brain while listening the language, and 2) the development of a new ERP (electrophysiological, recording of electrical activity over the brain) signature of word segmentation.

The opportunity to bring so many researchers together from so many different fields was essential to the success of this workshop. Moreover, with the help from BIRS, we ensured broad graduate student and postdoctoral fellow involvement as well. The background infrastructure support provided by BIRS was so appreciated, and helped make the meeting a resounding success.

List of Participants

Aslin, Richard *(Brain and Cognitive Sciences, University of Rochester)*
Bird, Sonya *(Department of Linguistics, University of British Columbia)*
Bonatti, Luca *(SISSA/ISAS)*
Byers-Heinlein, Krista *(Department of Psychology, McGill University)*
Dietrich, Christiane *(Department of Psychology, University of British Columbia)*
Fais, Laurel *(Department of Psychology, University of British Columbia)*
Fels, Sidney *(Department of Electrical and Computing Engineering, University of British Columbia)*
Fennell, Chris *(Department of Psychology, University of British Columbia)*
Kemp, Nenagh *(Department of Psychology, University of British Columbia)*
Marcus, Gary *(New York University)*
Maye, Jessica *(Communication Sciences and Disorders, Northwestern University)*
Mehler, Jacques *(University of Trieste)*
Narayan, Chandan R. *(Department of Linguistics, University of Michigan)*
Pakulak, Eric *(Department of Psychology, University of Oregon)*
Pulleyblank, Douglas *(Department of Linguistics, University of British Columbia)*
Sabourin, Laura *(Department of Psychology, University of British Columbia)*
Sanders, Lisa *(Department of Psychology, University of Oregon)*
Sebastian-Galles, Nuria *(Parc Cientific- Hospital Sant Joan de Deu)*
Storjohann, Rasmus *(School of Computing Science, Simon Fraser University)*
Toro, Juan Manuel *(Parc Cientific- Hospital Sant Joan de Deu)*
Vouloumanos, Athena *(Department of Psychology, University of British Columbia)*
Weikum, Whitney *(Department of Psychology, University of British Columbia)*
Werker, Janet *(University of British Columbia)*
Xu, Fei *(Department of Psychology, University of British Columbia)*
Yamada, Yoshiko *(University of Oregon)*
Yeung, H. Henny *(University of Maryland, Dept. of Linguistics)*
Yoshida, Katherine *(Department of Psychology, University of British Columbia)*

Chapter 43

2-day Retreat on Mathematical Ecology and Evolution (04w2540)

March 18–20, 2004

Organizer(s): *Michael Doebeli (University of British Columbia), Thomas Hillen (University of Alberta), Mark Kot (University of Washington), Mark Lewis (University of Alberta), Ed McCauley (University of Calgary)*

Objectives

The aim of the 2-day BIRS retreat was to bring together faculty, post doctoral fellows (PDF) and graduate students from several groups that are involved in the PIMS Collaborative Research Group in Mathematical Ecology and Evolution. The six organizers nominated one PDF and four students from their corresponding research groups to participate in this retreat. The format of the workshop was chosen primarily to initiate discussion, promote exchange of ideas, and encourage collaborations. Each student and some of the PDFs were asked to bring a new and open research problem and present it to a working group of about 8 participants to discuss and work on each problem for about 2 hours. The faculty members of each discussion group guided the discussions so as to encourage students to participate and express their ideas. Although complete solutions of the problems were not expected, progress on the problems was made, while introducing the students to new mathematical approaches to problem solving and science.

The format of the workshop was based on the very successful Woods Hole Oceanographic Institute (WHOI) Nantucket Annual Retreat in Mathematical Ecology, run by WHOI scientists Hal Caswell and Mike Neubert.

Scientific Topics

The topics of this 2-day retreat were exclusively chosen by the students through their contributed research problems. These included questions about predator-prey interactions, harvesting problems, competition, invasion problems, pattern formation, infectious diseases, crop control, fish behaviour, and climate change. During discussion many different forms of models were considered: stochastic models, random walks, ordinary differential equations, partial differential equations and integro-differential equations. Most of the problems were concerned with spatial distribution of species and hence a partial differential approach seemed natural.

In many cases the use on integro-differential equations (IDE) was discussed. This confirmed the need for a more theoretical understanding of IDEs. The theory of IDEs is still not as nearly developed as the theories for reaction-diffusion equations, for example. Almost all of the participating groups of the PIMS CRG in

Mathematical Ecology and Evolution have published research papers about IDE's, are currently working with IDE's, or have used them for modelling. In fact, the theoretical understanding of IDEs seems to be one of the common grounds of this CRG.

Through the discussion of different problems and different modelling tools the students were exposed to a large variety of methods and solution strategies. For all of the discussion groups, several possible model ideas were generated. In many cases a stochastic model was contrasted with a continuous deterministic model and/or with a discrete time model. This showed the students that, in general, there is not a single correct model that must be used, but there maybe many models of very different types that could be used to describe a biological phenomenon. This insight made the students aware to look beyond their own area of expertise and to be open to new methods and ideas. Each participant contributed his knowledge and experiences to the workshop and benefited greatly from each other.

Outreach

The format of the workshop created an intense interactive atmosphere and supported the exchange of ideas between the involved PIMS University groups on all levels (faculty, PDFs, graduate students). The 2-day retreat strengthened the collaborations between Ed McCauley at (U. Calgary) and M. Lewis (U. Alberta) on the modelling of water ecosystems, and many visits took place during the past year. Another collaboration developed between M. Doebeli (UBC), T. Hillen (U. Alberta) and F. Lutscher (U. Calgary and U. Alberta) on the evolution of biodiversity.

List of Participants

Bailey, Susan *(University of Calgary)*
Blachford, Alistair *(University of British Columbia)*
Cobbold, Christina *(University of Glasgow)*
Crumrine, Priscilla *(University of Calgary)*
Davis, Brad *(University of British Columbia)*
Dawson, Andria *(University of Alberta)*
Doebeli, Michael *(University of British Columbia)*
Dore, Rebecca *(University of Washington)*
Eftimie, Raluca *(University of Alberta)*
Flanagan, Kyla *(University of Calgary)*
Hillen, Thomas *(University of Alberta)*
Jerde, Chris *(University of Alberta)*
Kot, Mark *(University of Washington)*
Krkosek, Martin *(University of Alberta)*
Lee, Jung Min *(University of Alberta)*
Lewis, Mark *(University of Alberta)*
Liu, Wenxiang *(University of Alberta)*
Lutscher, Frithjof *(University of Alberta)*
McCauley, Ed *(University of Calgary)*
Medlock, Jan *(University of Washington)*
Merchant, Sandra *(University of British Columbia)*
Nelson, Bill *(University of Calgary)*
Noonburg, Erik *(University of Alberta)*
Pineda, Mario *(University of British Columbia)*
Reluga, Tim *(University of Washington)*
Renclawowicz, Joanna *(University of Alberta)*
Richards, Shane *(University of Calgary)*
Simpson, Karilynn *(University of Calgary)*
Toth, Damon *(University of Washington)*
Tyerman, Jabus *(University of British Columbia)*

Walton, David Brian *(University of Washington)*
Wang, Qian *(University of Alberta)*
Wonham, Marjoree *(University of Alberta)*
de-Camino-Beck, Tomas *(University of Alberta)*

Chapter 44

PIMS PDF Meeting (04w2542)

April 15–17, 2004

Organizer(s): *Manfred Trummer (Simon Fraser University and PIMS)*

This workshop hosted the Annual PIMS Meeting for its postdoctoral fellows. Fourteen PIMS postdoctoral fellows from a huge variety of mathematical sciences research areas and the PIMS Director, Ivar Ekeland, and Deputy Director, Manfred Trummer, met and all talked about their research areas (a list of abstracts is enclosed).

The purpose of the meeting is to allow participants to see what sort of research problems their peers are working on, and to allow participants to make connections if they have common research interests.

The talks were mostly of an introductory nature, but many generated extremely lively discussions. Participants enjoyed the informal discussions, many of them science and research related, many other concerned with all aspects of academic life, including job search strategies, writing of grant applications, and setting up of successful research programmes.

The meeting proved extremely useful to the PIMS postdocs, and BIRS proved to be an ideal setting for this conference.

List of Participants

Boulton, Lyonell *(PIMS, University of Calgary)*
Brecher, Dominic *(PIMS, University of British Columbia)*
Ekeland, Ivar *(PIMS, University of British Columbia)*
Hook, Andrea *(PIMS)*
Jung, Jae-Hun *(PIMS, University of British Columbia)*
Kang, Kyungkeun *(PIMS, University of British Columbia)*
Luke, Russell *(PIMS, Simon Fraser University)*
Lutscher, Frithjof *(PIMS, University of Alberta)*
Manuch, Jan *(PIMS, Simon Fraser University)*
Rangipour, Bahram *(PIMS, University of Victoria)*
Rolland-Lagan, Anne-Gaelle *(PIMS, University of Calgary)*
Shapiro, Jacob *(PIMS, University of British Columbia)*
Tran, Chuong *(PIMS, University of Alberta)*
Trummer, Manfred *(Simon Fraser University and PIMS)*
Vardarajan, Suneeta *(PIMS, University of Alberta)*
Xu, Jian-Jun *(PIMS, Simon Fraser University)*
Zhang, Jianying *(PIMS, University of British Columbia)*

Chapter 45

Mathfair Workshop (04w2600)

April 22–24

Organizer(s): *Ted Lewis (University of Alberta), Andy Liu (University of Alberta), Tom Holloway (University of Alberta)*

From the Introduction of The Math Fair Booklet by Ted Lewis

Everybody knows what a science fair is. Students find projects to work on, they prepare posters and demonstrations, the public is invited to come and see what they have done, and a panel of judges awards prizes for projects that are deemed to be the best.

A math fair is similar, but two important differences set our concept apart. Although mathematics is extremely diverse, our math fairs concentrate on just one aspect of the subject, namely problem solving, and our fairs are officially non-competitive, so there are no awards or prizes. We have chosen to focus on problem solving for several reasons. It is one activity that is common to most of mathematics, it is frequently an explicit part of the mathematics curriculum and it encourages skills in students that can be applied in all areas of their lives.

The problems in this booklet are ones that young students can solve and truly understand with a reasonable amount of work. They will not need a broad educational background, but the problems are not simple and most will have to think before solving them. The same is true about the people who visit the math fair even though they may be adults or students from higher grades. When the paricipants present their problems, they will discover that the visitors need help to work through the solutions, and the presenters will gain the satisfaction and confidence that comes from helping more talented or older persons.

The interaction between the participants and the viewers at a problem-based math fair can have a profound effect on the poise, confidence, communication skills and patience of the participants. The reason for our second difference, that the math fair be officially non-competitive, is so that all students are encouraged to participate and benefit. If some students feel they have little chance of winning they may decline to join in or not put in a full effort.

Even if a math fair is officially non-competitive, informal competition does occur. The participants quickly recognize who among them are good problem solvers, who can explain things well, whose presentations have the best artwork, and which displays attract the most visitors. But this sort of competition is friendly and constructive, and frequently leads to co-operative efforts among the participants. The focus on problem solving and the lack of formal awards are the key parts to our concept of a math fair for children, but otherwise there are many opportunities to creatively adapt the concept to a particular situation. We hope you will find this booklet useful in organizing your own math fair and are looking forward to hearing from you about your experiences.

Math Fairs in Elementary Schools in the Edmonton Area

The Math Fairs in elementary schools in the Edmonton area are gaining in popularity. Initiated upon requests by schools, and supported mainly by PIMS and the Edmonton Public School Board, the Math Fairs were held in previous years at Our Lady of Victories, Parkallen Elementary, and Terrace Heights Schools. The Edmonton Math Fairs are unique in that all students in the school participate. This event is about problem solving, not winning and losing. The schools themselves play a major role in the planning and thus the format can vary from school to school. In some Math Fairs, Education students from the University of Alberta were available to help, primarily by providing a model for a Math Fair that students can emulate in planning their own event. The extensive involvement of students in planning, staging and participating in the Math Fair may be one of the secrets of its success.

Prior to the Math Fair, students choose or are given problems to work on. They work in small groups to solve the problem and subsequently create a tabletop display. On the day of the Math Fair, spectators are invited to tackle the problem, with hints and guidance provided by students in charge. The displays are not poster sessions. Rather, the students are actively involved in the presentations.

The BIRS Workshop

At this workshop the organizers trained the participants so they could run a math fair. Each participant was provided with a copy of the Math Fair Booklet along with a good collection of new puzzles.

List of Participants

Borges-Couture, Paula *(Good Shepherd School)*
Drent, Jason *(Sundre High School)*
Dumanski, Micheal *(St. Gerard School)*
Friesen, Sharon *(GENA)*
Gibbs, Sandy *(Newton School)*
Gibson, Brad *(Lorelei School)*
Hamly, Roxanne *(Newton School)*
Hines, Amanda *(Sundre High School)*
Hohn, Tiina *(Grant MacEwan College)*
Holloway, Tom *(University of Alberta)*
Korf, Lisa *(Seattle)*
Kowalchuk, Auriana *(ESPB)*
Kozak, Carla *(EPSB)*
Lagu, Indy *(Mt. Royal College)*
Levesque, Josée *(Kenilworth Jr. High School)*
Lewis, Ted *(University of Alberta)*
Liu, Andy *(University of Alberta)*
Lore, Pat *(Dovercourt Elementary School)*
Lovallo, Patti *(Killarney Jr. High School)*
Mackie, Jodi *(Bisset Elementary School)*
Mandin-Kelly, Yvette *(Dovercourt Elementary School)*
Mayhew, Dennis *(Lansdowne Elementary School)*
Mertens, Chris *(Sundre High School)*
Mitchell, Shirley *(PIMS)*
Muchena, Sherrill *(Malcolm Tweddle Elementary School)*
Ostopowich , Brad *(St. Mary School)*
Poulin, Tracy *(Lorelei School)*
Prefontaine, Suzanne *(Holyrood Elementary School)*
Price, Tara *(Kenilworth Jr. High School)*
Reichert, Valarie *(Leschi Elementary School, Seattle)*

Slen, Gail *(John Ware Elementary School)*
Steinhauer, Todd *(T. D. Baker Junior High Schoo)*
Sun, Wen-Hsien *(Chiu Chang Publisher)*
Thompson, Tanya *(Pretty River Academy)*
Vaderlind, Paul *(Stockholm University)*
Warfield, Ginger *(University of Washington)*
Whillans, Sharon *(Malcom Tweddle School)*
Whiting, Dolores *(EPSB)*

Chapter 46

Directions in Combinatorial Matrix Theory (04w2525)

May 6–8, 2004

Organizer(s): *Shaun Fallat (University of Regina), Hadi Kharaghani (University of Lethbridge), Steve Kirkland (University of Regina), Bryan Shader (University of Wyoming), Michael Tsatsomeros (Washington State University), Pauline van den Driessche (University of Victoria)*

The Directions in Combinatorial Matrix Theory workshop was held at BIRS May 7–8, 2004, and attracted 29 researchers (10 from Canada, 15 from the U.S. and 4 from abroad) and 7 post-doctoral or graduate students. Talks discussed current developments and open problems in the following emerging themes in Combinatorial Matrix Theory: Spectral properties of families of matrices associated with graphs; Matrix theory and graph theory in the service of Euclidean geometry; Algebraic tools for combinatorial problems; and Spectral properties of classes of matrices. Titles and abstracts of the talks presented can be found at

http://www.pims.math.ca/birs/workshops/2004/04w2525/abstracts.pdf.

Below each of these themes is briefly discussed.

Spectral properties of families of matrices associated with graphs

Numerou connections between the spectrum of the adjacency matrix of a graph G and the graphical and combinatorial properties of the graph have been long, and fruitfully explored. An emerging, promising trend is to study the spectra of an entire class of matrices associated with G. More specifically, let $S(G)$ denote the set of all symmetric matrices whose graph is G (i.e. all symmetric n by n matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if there is an edge in G joining vertex i and j.) Fundamental questions in this area are:

(a) *What combinatorial and geometric properties of a graph G can be ascertained from the invariants of $S(G)$?*

(b) *What constraints are placed upon the spectrum of matrices in $S(G)$ by the graphical properties of G?*

Perhaps the first results along these lines are the early papers of Parter [3] and Fiedler [1] that establish some striking results about the spectrum of acyclic matrices, that is, matrices in $S(G)$, where G is a tree. More recently, the Colin de Verdière invariant [4] (which is the maximum of the second smallest eigenvalue of matrices in a special subset of $S(G)$) has been shown to have deep connections with the embedability, and hence the geometry, of G.

Several conference talks presented new results about the minimum rank, or equivalently the maximum multiplicity of an eigenvalue, of a matrix in $S(G)$, and the inverse eigenvalue problem for $S(G)$ (that is, *determine necessary and sufficient conditions for the ordered sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ to be the ordered eigenvalue list of some matrix in $S(G)$).*

Matrix theory and graph theory in the service of Euclidean geometry

The Gram-matrix $M = [\langle v_i, v_j \rangle]$ of a collection $v_1, v_2, \ldots, v_n$ of vectors in a Euclidean space stores much information about the geometric arrangement of the vectors, and often matrix theoretic properties of M produce startling corresponding geometric properties of the vectors.

One presentation at the workshop, illustrated the promise of this research theme, by deriving necessary and sufficient conditions for the existence of a simplex whose edges make prescribed (perpendicular, acute, or obtuse) angles. Future directions include extending this characterization to other families of polytopes.

Symmetric factorizations $A = BB^T$ (B (entry wise) nonnegative) of a nonnegative matrix A correspond geometrically to a system of nonnegative vectors with prescribed inner products. Such factorizations arise in image processing, physics, and statistical applications. Two fundamental questions are:

(a) If A has a symmetric nonnegative factorization $A = BB^T$ what is the smallest possible number k of columns in such a B? What is the largest k can be for a fixed dimension of A?

(b) If A is an integer matrix, how can one determine whether or not A has a factorization $A = BB^T$ where B is a $(0, 1)$-matrix?

Algebraic tools for tackling combinatorial problems

The Yin of Combinatorial Matrix Theory (CMT) is the use of combinatorial ideas and theorems to more closely analyze problems in matrix theory. The Yang of CMT is to use algebraic concepts and results to tackle combinatorial problems. Perhaps the prime example of the Yang in CMT is the Witsenhausen-Graham-Pollak theorem [2], that asserts that every biclique partition of the complete graph K_n has at least $n - 1$ bicliques, and whose only known proofs are all based on linear algebra.

The Yang of CMT was represented at the conference through talks on a number of subjects. These include: results on the behaviour of the inertia of a matrix when perturbed, and their use in the study of minimum biclique partitions of various families of graphs; a survey of the wide range of matroid theoretic, and graph colouring problems arising in Combinatorial Scientific Computing; a survey of the major questions and a proposed uniform theory for the various generalizations (e.g. Hadamard matrices, weighing matrices, symmetric designs, Type II matrices, etc.) of orthogonal matrices; and a talk using number theory to obtain new results on the long-standing problem of classifying the graphs with integral spectrum.

Spectral properties of classes of matrices

Several promising Yin directions in CMT were discussed.

New families of spectrally arbitrary n by n sign patterns (that is, a sign pattern with the property that every conjugate closed multi-set of n complex numbers is the spectrum of some matrix with the given sign pattern) were presented, and a field theoretic argument was presented to show that if A is an irreducible, spectrally arbitrary sign pattern, then A has at least $2n - 1$ nonzero entries. This leaves the intriguing open problem:

Is there an n by n, irreducible, spectrally arbitrary sign pattern with $2n - 1$ nonzero entries?

A talk concerning the possible Jordan Canonical Forms of a nonnegative matrix with prescribed eigenvalues of largest modulus illustrates that while the Perron-Frobenius theory for nonnegative matrices began over 100 years ago, there are still many interesting open questions about nonnegative matrices to be resolved.

Several results and problems concerning the relationship between the sign patterns of commuting matrices were discussed. For example, an interesting open problem is to determine necessary and sufficient conditions for a pair of sign patterns to allow a pair of commuting matrices.

The Workshop's Open Problem sessions were highly successful, and a list of problems presented will be posted at

http://www.pims.math.ca/birs/workshops/2004/04w2525/openprobs.pdf .

New collaborative efforts resulting from the workshop are already noticeable, especially among the post-docs and graduate students. Results presented at the conference will be disseminated through a special 2005 issue of the Electronic Journal of Linear Algebra.

In summary, the future directions for research in Combinatorial Matrix Theory are abundant, promising, and central to mathematics.

List of Participants

Akelbek, Mahmoud *(University of Regina)*
Anstee, Richard *(University of British Columbia)*
Barioli, Fransesco *(University of Regina)*
Barrett, Wayne *(Brigham Young University)*
Behbahani, Majid *(University of Lethbridge)*
Berman, Abraham *(Technion - Israel Institute of Technology)*
Britz, Thomas *(University of Victoria)*
Brualdi, Richard *(University of Wisconsin, Madison)*
Craigen, Robert *(University of Manitoba)*
Doob, Michael *(University of Manitoba)*
Fallat, Shaun *(University of Regina)*
Fiedler, Miroslav *(Academy of Sciences of the Czech Republic)*
Gibson, Peter *(University of Alabama in Huntsville)*
Grone, Robert *(San Diego State University)*
Hershkowitz, Daniel *(Technion - Israel Institute of Technology)*
Hogben, Leslie *(Iowa State University)*
Ionin, Yury *(Central Michigan University)*
Johnson, Charles *(College of William and Mary)*
Kharaghani, Hadi *(University of Lethbridge)*
Kim, In-Jae *(University of Wyoming)*
Kirkland, Steve *(University of Regina)*
Lancaster, Peter *(University of Calgary)*
Le, Hien *(Washington State University)*
Li, Chi-Kwong *(College of William and Mary)*
Li, Zhongshan *(Georgia State University)*
Liu, XiaoPing *(University of Regina)*
Loewy, Raphael *(Technion - Israel Institute of Technology)*
McDonald, Judith *(Washington State University)*
Olesky, Dale *(University of Victoria)*
Pothen, Alex *(Old Dominion University)*
Schneider, Hans *(University of Wisconsin-Madison)*
Shader, Bryan *(University of Wyoming)*
So, Wasin *(San Jose State University)*
Stuart, Jeff *(Pacific Lutheran University)*
Tsatsomeros, Michael *(Washington State University)*
Vander Meulen, Kevin *(Redeemer University College)*
Varaneckas, Rokas *(Washington State University)*

Bibliography

[1] M. Fiedler, Eigenvectors of acyclic matrices. Czechoslovak Math. J. 25 (1975), 607–618.

[2] R. L. Graham and H. O. Pollak, On embedding graphs in squashed cubes. Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs), pp. 99–110. Lecture Notes in Math., Vol. 303, Springer, Berlin, 1972.

[3] S. Parter, On the eigenvalues and eigenvectors of a class of matrices. J. Soc. Indust. Appl. Math. 8 (1960) 376–388.

[4] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, in: Graph Structure Theory (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, 137–147.

Chapter 47

Decentralized Discrete Event Systems: Structure, Communication, and Control (04w2040)

May 14–15, 2004

Organizer(s): *Peter Caines (McGill University), Stéphane Lafortune (University of Michigan), Laurie Ricker (Mount Allison University), Iakov Romanovski (Queen's University), Karen Rudie (Queen's University), John Thistle (University of Waterloo)*

The workshop themes of structure, communication and control in decentralized discrete event systems were addressed through examinations of modular and hierarchical architectures, language-based theories of distributed control synthesis, and logics for specification, synthesis and verification of control and communication structures. Potential applications were presented in the fields of distributed control of air and watercraft, distributed robotic self-assembly and power system fault detection. Throughout the two-day meeting the sessions were notable for the informal and lively discussions that accompanied essentially all of the talks. From both the presentations and the discussions several key issues emerged, and these are summarized below.

Modularity and Structure:

New unpublished results about the implications of modular structure on the computational complexity of certain verification and control problems were presented. It became apparent that computational issues associated with modular systems are highly sensitive to the inherent symmetry in the system components. In particular, a new undecidability result for certain classes of symmetric modular systems was presented. This result generated considerable discussion among the attendees. On the other hand, it was demonstrated that for another class of symmetric modular systems, significant computational savings could be achieved in solving verification and control problems by exploiting symmetry and building quotient transition structures, reminiscent of partial-order methods in field of formal verification in computer science. It became apparent that these issues are worthy of future investigations, especially regarding the boundary between decidability and undecidability as well as the robustness of quotient structures to the symmetry assumptions.

New work was presented on fundamental properties (controllability, observability, etc.) underpinning control synthesis for multi-agent systems; this included the topic of controller synthesis for vector discrete event systems under partial observations. A promising approach in this direction is the use of hierarchical control architectures presented during the workshop. This method can be effectively used to reduce the computational complexity of various architecturally complex systems.

Logics for Synthesis of Decentralized/Distributed Controllers:

Exciting new results on the use of special logics that allow the solution of controller synthesis problems as certain types of verification problems were presented. These results are very elegant from a theoretical standpoint. They also appear to offer new possibilities for synthesizing control strategies for problems with partial observation. There was a feeling among attendees that this is a promising new avenue, although the computational properties of the approach remain largely unexplored.

Related work on solving control problems for continuous-variable systems subject to temporal logic specifications was presented. In this case, it was shown how the (discrete) specification drives the abstraction of the continuous-variable system to a discrete event system that is sufficiently detailed for controller synthesis purposes.

The use of a special logic for addressing decentralization of information in distributed systems was also presented. This method uses a modal logic for ascribing knowledge to agents and relates the field of discrete-event systems control to recent work in theoretical computer science on formal reasoning about knowledge. One of the goals is to use the knowledge theory approach to aid in the synthesis of communication strategies between controllers in a distributed control setting.

Application Areas:

A grammar-based approach to problems in distributed robotic self-assembly was presented, and various discrete-event control problems associated with Uninhabited Air Vehicles (UAVs) and distributed ship control were discussed. A Petri-net based method of fault detection in electric power distribution system protection networks was proposed. In general, it was felt that in further theoretical developments of existing theories special attention should be paid to applications. For example, more experience with applications could lead to a better understanding of the modelling of incomplete or imperfect system descriptions. It was suggested that a set of benchmark applications be assembled, and that additional energy be devoted to the development of software tools.

Discussion Topics

At the end of the workshop, a free-form session was held to discuss the common themes, open research problems, and debated ideas that arose during the workshop. In addition to the topics and areas in the aforementioned sections of this report, some of the other issues that arose include the following: the tradeoff between expressiveness of a model and its computational capabilities; uncertainty, robustness, unmodelled dynamics and the need to design systems which can tolerate imperfections; the desire for a simple characterization of the fundamental system-theoretic properties of decentralized discrete-event systems; and the relationship between theory and application and to what degree applications should drive future theory.

Future Workshop:

The workshop underlined the vitality, the variety and the pertinence of research on the three broad issues of structure, communication and control. It also highlighted other themes – such as the increasing rapprochement between discrete event control and computer science, which recurred throughout. The attendees unanimously support holding a similar workshop in 2006 to discuss progress made on the key issues summarized above and to identify new strategic research directions in decentralized control of discrete event systems.

List of Participants

Lafortune, Stephane *(University of Michigan, Michigan, USA)*
Ricker, Laurie *(Mount Allison University, Sackville, NB, Canada)*
Rudie, Karen *(Queen's University, Kingston, ON, Canada)*

Boel, Rene *(University of Ghent)*
Caines, Peter *(McGill University)*
Feng, Lei
Giua, Alessandro *(Università di Cagliari)*
Gohari, Peyman *(Concordia University)*
Holloway, Larry
Hubbard, Paul
Klavins, Eric *(University of Washington, Seattle)*
Millan, Jim
O'Young, Siu
Pappas, George
Pinchinat, Sophie *(IRISA)*
Raisch, Joerg *(Universität Magdeburg)*
Rohloff, Kurt
Romanovski, Iakov
Srinivasan, Rama
Thistle, John *(University of Waterloo)*

Chapter 48

Adaptive Wavelet and Multiscale Methods for Partial Differential Equations (04w2055)

June 3–5, 2004

Organizer(s): *Tony Ware (Univerity of Calgary), Manfred Trummer (Simon Fraser University), Bin Han (University of Alberta), Michael Lamouroux (University of Calgary), Elena Braverman (University of Calgary)*

The development of adaptive wavelet methods for partial differential equations has matured significantly in recent years to the point where it is attracting an increasing amount of attention from engineers and from mathematicians. The purpose of this workshop is to bring together mathematicians, engineers, geophysicists and others and provide an opportunity for the participants to get up to date with recent developments in the theory and practice of adaptive wavelet methods, and together to explore potential applications of these new techniques.

List of Participants

Alam, Jarul *(McMaster University)*
Braverman, Elena *(University of Calgary)*
Chen, Wen *(University of Alberta)*
Ditzian, Zeev *(University of Alberta)*
Gibson, Peter *(University of Calgary)*
Han, Bin *(University of Alberta)*
Israeli, Moshe *(Technion - Israel Institute of Technology)*
Kwon, Soon-Geol *(University of Alberta)*
Lamouroux, Michael *(University of Calgary)*
Li, Hua *(University of Calgary)*
Liao, Junlin Robert *(University of Alberta)*
Liu, Songtao *(University of Alberta)*
Oswald, Peter *(Massachusetts Institute of Technology)*
Park, Sang Soo *(University of Alberta)*
Pratt, Aaron *(University of Calgary)*
Vasudevan, Kris *(University of Calgary)*
Wang, Mengzhe *(University of Alberta)*
Ware, Tony *(University of Calgary)*

Zizler, Peter *(Mount Royal College)*

Chapter 49

The Design and Analysis of Computer Experiments for Complex Systems (04w2056)

July 15–17, 2004

Organizer(s): *Derek Bingham (Simon Fraser University), Randy Sitter (Simon Fraser University)*

The design and analysis of computer experiments has become increasingly important to scientists and engineers. Researchers world-wide have recognized this as an important emerging area of research. We propose to hold a 2-day workshop at BIRS aimed at addressing the aforementioned topics. The workshop will be supported, in part, by the National Programme on Complex Data Strucutres (NPCDS). The NPCDS Programme Director (Dr. Jamie Stafford) and the PIMS Director (Dr. Ivar Ekeland) have indicated that NPCDS will be alotted time at BIRS and that this workshop may be suitable.

The workshop has support from the NPCDS and has a secondary goal of identifying participants who will help form an international network of researchers in this area. NPCDS has seed money to help kick-start such a network. This endeavor will also be co-sponsored by the Statistical and Applied Mathematical Sciences Institute (SAMSI) in the USA. SAMSI is an important partner in the formation of an international network and has already indicated interest in activities such as a semester devoted to this topic, with events held at SAMSI and at a math institute in Canada. We have also found interest at Los Alamos National Labs. Industry paricipation will be an important component of the success of the workshop and at making an impact in real scietific investigation. We feel that the BIRS programme will be crucial at meeting the scientific goals of the programme, but also the long-run leadership of Canadian researchers in this emerging area.

We have aimed to be fairly creative with the format of the workshop. The format aims to facilitate interaction among the participants. Like many others, the workshop will have a combination of talks, poster sessions and roundtable discussions. However, the presentations must aim to address the three theme topics of the workshop. Furthermore, the topics presented must meet a secondary criterion in that they must either present new methodology or present an application in one of the theme areas where new methodology is required. In addition, for each topic, a reading list will be created which each of the participants will be expected to be intimately familiar. These unique features are aimed at stimulating research on specific problems.

In the long-run, we hope that the format of the workshop will foster collaboration between the participants. Further, we hope that by partnering with SAMSI and Los Alamos National Labs we will form the framework for future projects that will result in an international network of leading researchers in the design and analysis of computer experiments for complex systems.

List of Participants

Banks, Tom *(North Carolina State University)*
Bayarri, Susie *(University of Valencia)*
Berger, Jim *(Duke University)*
Bingham, Derek *(Simon Fraser University)*
Booker, Andrew *(Boeing)*
Brewster, John *(University of Manitoba)*
Cafeo, John *(NISS/GM)*
Chipman, Hugh *(University of Waterloo)*
Gonzalo, Garcia-Donato *(Statistical and Applied Mathematical Sciences Institute)*
Hengartner, Nick *(Los Alamos National Laboratory)*
Higdon, Dave *(Los Alamos National Laboratory)*
Karuri, Stella *(University of Waterloo)*
Lemieux, Christianne *(University of Calgary)*
Lin, Chunfang *(Simon Fraser University)*
Linkletter, Crystal *(Simon Fraser University)*
Loeppky, Jason *(Simon Fraser University)*
Lu, Wilson *(Simon Fraser University)*
Mcleod, Robert *(University of Manitoba)*
Mease, David *(University of Pennsylvania)*
Meckesheimer, Martin *(Boeing)*
Michailidis, George *(University of Michigan)*
Nakhleh, Charlie *(Los Alamos National Labs)*
Notz, Bill *(Ohio State University)*
Palomo, Jesus *(Statistical and Applied Mathematical Sciences Institute)*
Paulo, Rui *(NISS/SAMSI)*
Pepin, Jason *(Los Alamos National Labs)*
Ranjam, Pritam *(Simon Fraser University)*
Reese, Shane *(Brigham Young University)*
Rillett, Larry *(University of Nebraska)*
Sacks, Jerry *(NISS)*
Santner, Tom *(Ohio State University)*
Sitter, Randy *(Simon Fraser University)*
Spiegelman, Cliff *(Texas A&M)*
Stafford, Jamie *(University of Toronto)*
Steinberg, David *(Tel Aviv University)*
Sudijianto, Agus *(Bank of America)*
Welch, Will *(University of British Columbia)*
Xu, Jiaying *(University of Guelph)*
Ye, Kenny *(SUNY, Stony Brook)*

Chapter 50

Combinatorial and Algorithmic Aspects of Networking and the Internet (04w2059)

August 5–7, 2004

Organizer(s): *Angèle Hamel (Wilfrid Laurier University), Alejandro Lopez–Ortiz (University of Waterloo), Ian Munro (University of Waterloo), Rajeev Motwani (Stanford University), Andrei Broder (IBM T.J. Watson), Srinivasan Keshav (University of Waterloo)*

The workshop, Combinatorial and Algorithmic Aspects of Networking and the Internet (CAAN04), was dedicated to exploring the combinatorics and algorithmics of networking. This interdisciplinary field is a rapidly expanding one, primarily due to the influence of the Internet. The Internet is a global network of 700 million users. An additional 300,000 users are added each day. The Internet itself is in constant flux, with connections and content being added and deleted continuously. How does one study, predict, or even model such an entity? This is the challenge addressed by research in large–scale networks. The unique nature of these networks calls for a variety of techniques from a variety of disciplines. The primary goal of this workshop is to bring together this expertise and provide a snapshot of the cutting edge research in this field.

The workshop was a great success on a number of fronts. First, it brought together a diverse cross-section of researchers in an already scattered and distinctive community. Among the participants were mathematicians, computer scientists in theory and algorithms, computer scientists in networks, physicists, and engineers, as well as researchers from Europe and North America, participants from industry and academia, students and established researchers.

The papers presented were of high quality. The decision was taken to put out a call for papers and select speakers by peer review. The refereeing process led to twelve papers, and up–to–date research was presented. We further took the decision to bracket these cutting–edge talks with two invited survey talks—an opening talk by Ashish Goel and a closing talk by Andrei Broder—that set the area in context and presented an overview of the field. In one of the refereed talks the presenter proposed a solution to a major outstanding problem in the field and there is now ongoing work to further evaluate the correctness of the solution. The Springer–Verlag series, Lecture Notes in Computer Science *(LNCS) has expressed interest in publishing a volume dedicated to the workshop and consisting of the presented papers along with a number of invited survey papers. We anticipate that this volume would become a standard reference or graduate text in this emerging field.*

New collaborations are another possible outcome of the workshop. There was clear interest among the participants for further discussions and collaborations, and, although the format, which included large periods for discussion, was useful, the interaction of the participants was somewhat limited by the short duration of the two–day format.

This workshop may also spawn an annual series of similar workshops. There was also clear interest in future workshops on this topic and the organizers had a number of inquiries from participants about the *possibility of CAAN05. One proposal under discussion is to mount it as a satellite workshop of the Workshop on Algorithms and Data Structures (WADS05) to be held in August 2005 in Waterloo, Ontario.*

The workshop was also greatly enhanced by the wonderful facilities at BIRS, in particular the accommodation, the meeting and collaboration rooms, the easy access to computers, and the proximity to the town of Banff itself.

List of Participants

Barrat, Alain *(Universite de Paris-Sud)*
Bauer, Claus *(Dolby Laboratories)*
Bonato, Anthony *(Wilfrid Laurier University)*
Broder, Andrei *(IBM)*
Bronnimann, Herve *(Polytechnic University)*
Datta , Suprakash *(York University)*
Demaine, Erik *(Massachusetts Institute of Technology)*
Demaine, Martin *(Massachusetts Institute of Technology)*
Dorri, Reza *(University of Waterloo)*
Goel, Ashish *(Stanford University)*
Gr/'egoire, Jean-Charles *(Institut National de la Recherche Scientifique)*
Hall, Alexander *(ETH Zurich)*
Hamel, Angele *(Wilfrid Laurier University)*
Johnson, Matthew *(London School of Economics)*
Karakostas, George *(McMaster University)*
Latapy, Matthieu *(LIAFA - Universite Paris 7)*
Levy, Eythan *(Free University of Brussels)*
Lopez-Ortiz, Alejandro *(University of Waterloo)*
Magnien, Clemence *(Ecole Polytechnique)*
Marbach, Peter *(University of Toronto)*
Munro, Ian *(University of Waterloo)*
Pelsmajer, Michael *(Illinois Institute of Technology)*
Tóth, Csaba *(University of California, Santa Barbara)*
Wilfong, Gordon *(Bell Labs)*

Chapter 51

Linear Operators: Theory, Applications and Computations (04w2063)

August 12–14, 2004

Organizer(s): *Paul Binding (University of Calgary), Peter Lancaster (University of Calgary)*

The theory of matrices and linear operators is going through a highly productive phase driven largely by a great variety of applications. These include magneto-hydrodynamics, vibrations of continua, systems theory, signal processing, for example. They frequently concern the spectral properties of operators on Krein or Pontryagin spaces, and also require modern techniques in perturbation theory and differential equations. The workshop will provide an opportunity for informal discussion and presentation of current research projects in these areas. Participants will include H. Langer (Vienna), A. Markus (Beer-Sheva) and L. Rodman (Williamsburg).

List of Participants

Allegretto, Walter *(University of Alberta)*
Binding, Paul *(University of Calgary)*
Boulton, Lyonell *(University of Calgary)*
Browne, Patrick *(University of Saskatchewan)*
Choi, Man-Duen *(University of Toronto)*
Churchill, Richard *(Hunter College and Graduate Center, CUNY)*
Farenick, Douglas *(University of Regina)*
Gibson, Peter *(York University)*
Lancaster, Peter *(University of Calgary)*
Langer, Heinz *(Vienna University of Technology)*
Markus, Alexander *(Ben Gurion University)*
Rodman, Leiba *(College of William and Mary)*
Sourour, Ahmed Ramzi *(University of Victoria)*
Tretter, Christianne *(University of Bremen)*
Watkins, David *(Washington State University)*
Xu, Hongguo *(University of Kansas)*
Zhou, Fei *(University of Regina)*

Chapter 52

Alberta Topology Seminar (04w2064)

August 19–21, 2004

Organizer(s): *Kristine Bauer (University of Calgary), George Peschke (University of Alberta), Peter Zvengrowski (University of Calgary)*

This 2-day workshop should be considered within the wider context of the Alberta Topology Seminar (ATS). The ATS is an Alberta based regional effort to strengthen the research environment and to foster research activities of mathematicians working in Topology as well as in its interfaces with Algebra, Analysis, Geometry, and Theoretical Physics. Participants come primarily from the Alberta universities, but also from neighbouring provinces and states. The driving force behind this broad endeavour is the basic fact that research thrives in a climate of rich and multidirectional interaction, generating critical mass for discovery amongst the participants in as many ways as possible. The challenge then is to achieve amongst the participants a level of breadth of expertise and familiarity with each other's works to allow such interaction to take place. Towards this end we have had 12 ATS-meetings during the last year with talks comprising a mix of background building educational presentations, as well as conference style presentations about recent discoveries.

Within this context the 2-day workshop we had at BIRS was by far the most important, extensive, and beneficial ATS meeting held to date. The twenty participants formed a healthy mix of established and junior researchers, as well as graduate students representing the Universities of Alberta, Calgary, Lethbridge, and Oregon. We had twelve talks (7 on the first day, 5 on the second) spanning areas that include low dimensional and transformation group topology, representation theory, algebraic geometry, noncommutative geometry, etc. For details, see

http://www.ualberta.ca/dept/math/gauss/AAGT/Topology/BIRS2004.htm

Perhaps contrasting a bit those workshops whose participants come together because they share a common focus, the ultimate significance of this ATS will be measured by the extent to which its very diverse group of participants learn to share and amplify each other's expertise. A number of unexpected exchanges took place which appear very promising in this regard. While the germination of ideas between specialists in neighbouring, but different, fields requires time to nurture, we are quite optimistic that this 2-day workshop will turn out to play a key role towards obtaining tangible results within the near to mid-range future.

The ATS meeting at BIRS also provided an opportunity for the organizers to plan future events for the coming year; for evolving details please consult the above web address.

Finally we wish to express our gratitude to BIRS for hosting this meeting in such a relaxed setting amidst the spectacular scenery of the Canadian Rockies. There was also great praise among the participants for the BIRS facilities, in particular for the excellent computer access available in each room.

List of Participants

Bauer, Kristine *(University of Calgary)*
Budney, Ryan *(University of Oregon)*
Chen, Xi *(University of Alberta)*
Cunningham, Clifton *(University of Calgary)*
Dover, Lynn *(University of Alberta)*
Gannon, Terry *(University of Alberta)*
Krause, Eva Maria *(University of Alberta)*
Krebes, Dave *(University of Calgary)*
Morris, Dave *(University of Lethbridge)*
Nikolaev, Igor *(University of Calgary)*
Oliveros, Deborah *(University of Calgary)*
Peschke, George *(University of Alberta)*
Pianzola, Arturo *(University of Alberta)*
Ruan, Haibo *(University of Alberta)*
Sabbagh, Bouchra *(University of Calgary)*
Tripathi, Satya Prakash *(University of Calgary)*
Zvengrowski, Peter *(University of Calgary)*
von Bergmann, Jens *(Michigan State University)*

Chapter 53

Theoretical Physics Institute (TPI) Symposium 2004 (04w2544)

September 2–4, 2004

Organizer(s): *Frank Marsiglio (University of Alberta), Helmy Sherif (University of Alberta)*

The Theoretical Physics Institute (TPI) at the University of Alberta consists of members from three different departments (Physics, Mathematical and Statistical Sciences, and Chemistry). This is the 2nd time we have held a Symposium at BIRS, with the idea that this was to bring together members of the Institute, their students and postdoctoral fellows, as well as colleagues from other universities in the West, for two days; the intent was to promote the exchange of ideas and collaboration.

This year we had more participants from Eastern Canada, along with keynote speakers from the U.S. and Australia. Our format differed somewhat from last year's in that we had only plenary talks, along with poster sessions in the afternoons.

The first day's talks centred around statistical mechanics, while the 2nd day had a more diverse character. Nonetheless, even then a common theme seemed to be problems that contain many length (or energy) scales, and the means one requires to take proper account of these disparate scales.

A number of participants are actively engaged in many-body problems. We heard about some state-of-the-art calculations on lattices, by straightforward diagonalization. While this techniques continues to provide insights and benchmark checks, it is clear that it can never be used to achieve the thermodynamic limit.

Other speakers (Kadanoff, Plischke, Bowman, Czarnecki) described more complicated problems in that very different scales are involved. The latter actually tied problems in particle physics to those in aerodynamics, connected by a common philosophy/technique of solution.

One speaker (Brown) reviewed some work being done in chemistry at the University of Alberta; it requires the solution of the time-dependent Schrodinger equation for molecules. This work was appreciated by a number of physicists tackling similar problems (there was even a poster involving the time-dependence of a magnetic moment in the presence of a spin current).

Finally a couple of talks focused on issues in entanglement in quantum mechanics. The one by Sanders (Calgary) in particular focused on applications to cryptography and quantum computers. This is clearly an area of burgeoning interest.

Quite a variety of topics was covered in this workshop. We continue to find this annual symposium to be very stimulating, especially in that it points out the many areas in the sciences that share common ground. Such interdisciplinary dialogue is very welcome.

List of Participants

Austen, Dave *(University of Alberta)*
Ballentine, Leslie *(Simon Fraser University)*
Betts, Donald *(Dalhousie University)*
Betts, Patricia *(St. Francis Xavier University)*
Bowman, John *(University of Alberta)*
Brouwer, Wytze *(University of Alberta)*
Brown, Alex *(University of Alberta)*
Buczek, Pawel *(University of Alberta)*
Cheng, Taiwang *(University of Alberta)*
Czarnecki, Andrzej *(University of Alberta)*
Ditzian, Ruth *(University of Chicago)*
Elliott, Chuck *(Kakari Systems)*
Gortel, Zbigniew *(University of Alberta)*
Gribouk, Taras *(University of Alberta)*
Hunter, Doug *(St. Francis Xavier University)*
Isaac, Isaac *(University of Alberta)*
Israel, Werner *(University of Victoria)*
Kadanoff, Leo *(University of Chicago)*
Khanna, Faqir *(University of Alberta)*
Loly, Peter *(University of Manitoba)*
Malbouisson, Jorge *(University of Alberta)*
Marsiglio, Frank *(University of Alberta)*
Oitmaa, Jaan *(University of New South Wales)*
Oliynyk, Todd *(University of Alberta)*
Plischke, Michael *(Simon Fraser University)*
Romanov, Dmitri *(University of Alberta)*
Sanders, Barry *(University of Calgary)*
Santos, Esdras *(University of Alberta)*
Sherif, Helmy *(University of Alberta)*
Strungaru, Nicolae *(University of Alberta)*
Taylor, Keith *(Dalhousie University)*
Vardarajan, Suneeta *(University of Alberta)*
Woolgar, Eric *(University of Alberta)*
Yao, Yushu *(University of Alberta)*

Chapter 54

PIMS Executive Retreat (04w2545)

September 23–25, 2004

Organizer(s): *Manfred Trummer (Pacific Institute for the Mathematical Sciences)*

The PIMS Executive Committee consists of the PIMS Director, Deputy Director, and the Site Directors of the member universities. Invited participants from the mathematics community met with the PIMS Executive Committee to discuss the future directions of PIMS.

List of Participants

Adem, Alejandro *(University of British Columbia)*
Alvarado, Shelley *(Pacific Institute for the Mathematical Sciences)*
Boorman, Michael *(University of Calgary)*
Bose, Chris *(University of Victoria)*
Chen, Gemai *(University of Calgary)*
Cliff, Gerald *(University of Alberta)*
Ekeland, Ivar *(University of British Columbia)*
Ghoussoub, Nassif *(University of British Columbia)*
Goebel, Randy *(University of Alberta)*
Hillen, Thomas *(University of Alberta)*
LeVeque, Randy *(University of Washington)*
Leeming, David *(University of Victoria)*
Lind, Doug *(University of Washington)*
Loyce, Adams *(University of Washington)*
Macki, Jack *(University of Alberta)*
Margrave, Gary *(University of Calgary)*
Moody, Robert *(University of Alberta)*
Perkins, Ed *(University of British Columbia)*
Ripley, Robert *(Martec Ltd.)*
Russell, Bob *(Simon Fraser University)*
Trummer, Manfred *(Pacific Institute for the Mathematical Sciences)*
Williams, Hugh *(University of Calgary)*

Chapter 55

Pacific Northwest Numerical Analysis Seminar (04w2053)

September 30–October 2, 2004

Organizer(s): *Chen Greif (University of British Columbia), Dominik Schoetzau (University of British Columbia), Manfred Trummer (Simon Fraser University)*

The meeting will start at 8pm on Thursday September 30th, and end at noon on Saturday October 2nd. The Banff Centre is available for accommodations for the two nights of Thursday-Friday. Check-in time at the Banff Centre is 4pm and check-out time is noon. Objectives

This meeting is the 18th Annual Pacific Northwest Numerical Analysis Seminar (PNWNAS). It is sponsored by the Pacific Institute for the Mathematical Sciences (PIMS) as an event of the period of concentration in scientific computing, and is hosted by BIRS.

The PNWNAS meeting has been held every year since 1987, and is aimed at bringing together people from the Pacific Northwest who are interested in numerical analysis and scientific computing. Details on previous meetings can be found at http://www.amath.washington.edu/∼pnwnas/

List of Participants

Ascher, Uri *(University of British Columbia)*
Bank, Randy *(University of California at San Diego)*
Braverman, Elena *(University of Calgary)*
Bridson, Robert *(University of British Columbia)*
Calhoun, Donna *(University of Washington)*
Chen, Wan *(University of British Columbia)*
Cormier, Sarah *(University of British Columbia)*
Dennis, John *(Rice University)*
Fagnan, Kirsten *(University of Washington)*
Ferng, William *(The Boeing Company)*
Friedlander, Michael *(University of British Columbia)*
Genz, Alan *(Washington State University)*
Greif, Chen *(University of British Columbia)*
Haws, John *(The Boeing Company)*
Jung, Jae-Hun *(PIMS, University of British Columbia)*
Kropinski, Mary-Catherine *(Simon Fraser University)*
Lamoureux, Michael *(University of Calgary)*
Lehoucq, Rich *(Sandia National Laboratories)*
Li, Dan *(University of British Columbia)*

Minev, Peter *(University of Alberta)*
Mitchell, Ian *(University of British Columbia)*
Oberman, Adam *(Simon Fraser University)*
Overton, Michael *(New York University)*
Quaife, Bryan *(University of Calgary)*
Restrepo, Juan *(University of Arizona)*
Ruuth, Steve *(Simon Fraser University)*
Schoetzau, Dominik *(University of British Columbia)*
Tang, Wei-Pai *(The Boeing Company)*
Taylor, Andrew *(University of Calgary)*
Trummer, Manfred *(Simon Fraser University)*
Varah, Jim *(University of British Columbia)*
Wang, Dong *(Simon Fraser University)*
Ware, Antony *(University of Calgary)*
Watkins, David *(Washington State University)*
Wigton, Laurence *(The Boeing Company)*
Yedlin, Matt *(University of British Columbia)*
Zhang, Jianying *(University of British Columbia)*

Chapter 56

Data Mining MITACS Industry Session (04w2065)

October 14–16, 2004

Organizer(s): *Jim Brookes (MITACS)*

The goals of this workshop include:

- *Network individuals from industry and academia who are interested in both data mining research and the application of advanced techniques in data mining*
- *Share experiences from industrial participants on key issues in the application of data mining and from academia on current research results*
- *Establish future research priorities for data mining*
- *Create new opportunities for research collaborations between industry and academia*

List of Participants

Ben-David, Shai *(University of Waterloo)*
Bengio, Yoshua *(University of Montreal)*
Boire, Richard *(Boire Filler Group)*
Ciampi, Antonio *(McGill University)*
Dzieciolowski, Krzysztof *(Bell Canada)*
Ester, Martin *(Simon Fraser University)*
Huang, Jiayuan *(University of Waterloo)*
Kourti, Theodora *(McMaster University)*
Ling, Daymond *(CIBC)*
Lucas, Jeff *(MITACS)*
Makos, Rick *(Teradata Canada)*
McDonald, Hugh *(Intrawest Corporation)*
Pei, Jian *(University of Buffalo, New York)*
Sander, Joerg *(University of Alberta)*
Sayyad-Shirabad, Jelber *(University of Ottawa)*
Schuurmans, Dale *(University of Alberta)*
Storey, Andrew *(Scotiabank)*
Taciuk, Terry
Tompa, Frank *(University of Waterloo)*

Trummer, Manfred *(Simon Fraser University)*
Wang, Ke *(Simon Fraser University)*
Zamar, Reuben *(University of British Columbia)*

Chapter 57

Canadian Mathematical Leadership Retreat (04w2067)

October 28–30, 2005

Organizer(s): *Nassif Ghoussoub (University of British Columbia)*

List of Participants

Ali, Twareque *(Institut des Sciences Mathematiques)*
Brunner, Hermann *(Memorial University of Newfoundland)*
Campbell, Eddy *(Memorial University of Newfoundland)*
Davidson, Kenneth R. *(University of Waterloo)*
Ekeland, Ivar *(University of British Columbia)*
Ghoussoub, Nassif *(University of British Columbia)*
Gupta, Arvind *(MITACS)*
Jackson, Ken *(Canadian Applied and Industrial Mathematics Society)*
Kane, Richard *(University of Western Ontario)*
Keyfitz, Barbara Lee *(Fields Institute)*
Lalonde, Francois *(University of Montreal)*
Langford, Bill *(Canadian Applied and Industrial Mathematics Society)*
Reid, Nancy *(University of Toronto)*
Rousseau, Christiane *(University of Montreal)*
Thompson, Mary E. *(University of Waterloo)*

Chapter 58

MITACS Theme Meeting: Communication Networks and Security (04w2069)

November 4–6, 2004

Organizer(s): *Evangelos Kranakis (Carleton University)*

List of Participants

Dyer, Danny *(University of Regina)*
Bae, Jung *(Dalhousie University)*
Barbeau, Michel *(Carleton University)*
Clark, Nancy *(Acadia University)*
Coutelen, Thomas *(Université de Montréal)*
Elbiaze, Halima *(Université de Montréal)*
Hahn, Gena *(Université de Montréal)*
Hall, Jeyanthi *(Carleton University)*
Hirt, Andreas *(University of Calgary)*
Hoyer, Peter *(University of Calgary)*
Jaumard, Brigitte *(Universit e de Montréal)*
Kalyaniwalla, Nauzer *(Dalhousie University)*
Kerherve , Brigitte *(Université de Montréal)*
Kranakis, Evangelos *(Carleton University)*
Leung, Henry *(University of Calgary)*
Liu, Hongyu *(Dalhousie University)*
Majumdar, Deyasini *(University of Calgary)*
Metnani, Ammar *(Université de Montréal)*
Nie , Xiaojun *(Carleton University)*
Nowakowski, Richard *(Dalhousie University)*
Scott, Andrew *(University of Calgary)*
Shanmugam, Surendran K. *(University of Calgary)*
Somayaji, *(Carleton University)*
Venkatasubramanian, Vijayaraghavan *(University of Calgary)*
Walgate, Jonathan *(University of Calgary)*
Williams, Hugh *(University of Calgary)*

Wooding, Kjell *(University of Calgary)*
Yang, Boting *(University of Regina)*
Zhao, Yiqiang *(Carleton University)*
Zou, Shaoying *(Carleton University)*

Chapter 59

MITACS Project Meeting: Modelling Trading and Risk in the Market (04w2070)

November 11–13, 2004

Organizer(s): *Tony Ware (University of Calgary)*

The workshop aims to bring academic researchers in mathematical and computational finance and together with risk managers and quantitaive analysts from industry to share new ideas, practical and theoretical questions of the moment, current research, and to foster closer collaboration.

List of Participants

Boland, Jeff *(RBC Insurance)*
Buffington, John *(Epcor)*
Calistrate, Dan *(Direct Energy)*
Chan, Leung Lung *(University of Calgary)*
Chen, Lai *(Enmax)*
Cottrell, Tom *(University of Calgary)*
Davison, Matt *(University of Western Ontario)*
Dmitrasinovic-Vidovic, Gordana *(University of Calgary)*
Elliott, Robert *(University of Calgary)*
Grasselli, Matheus *(McMaster University)*
Hurd, Tom *(McMaster University)*
Kuznetsov, Alexey *(McMaster University)*
Miao, Hong *(University of Calgary)*
Molina, Alberto *(University of British Columbia)*
Powojowski, Miro *(TD Commodity and Energy Trading)*
Quan, Guanghui *(University of Calgary)*
Renee, Garth *(TransAlta)*
Royal, Andrew *(University of Calgary)*
Sadeghi, Ali *(Enmax)*
Seco, Luis *(University of Toronto)*
Simchi, Reza *(Nexen Inc.)*
Swishchuk, Anatoliy *(University of Calgary)*
Walsh, John *(University of British Columbia)*

Ware, Tony *(University of Calgary)*
Weir, Graham
Wong, Hilda *(Direct Energy)*
Yang, Jingping *(McMaster University)*

Chapter 60

Number Theorists Weekend (04w2505)

November 18–20, 2004

Organizer(s): *Mark Bauer (University of Calgary), Michael Bennett (University of British Columbia), Ronald Ferguson (Simon Fraser University)*

This short workshop focused on developments in Number Theory at the interface of Computational Methods with Analytic Number Theory and Diophantine Approximation. These represent two particular strengths of the Canadian mathematical community in general, and of the PIMS region in particular. The workshop was designed as a bridge between the Computational workshop preceding it and that in Analytic Number Theory and Diophantine Approximation which followed. An emphasis was placed on expository talks with graduate student and postdoctoral fellow involvement highlighted.

Computational tools for Diophantine problems include implementations of the Lenstra-Lenstra-Lovascz lattice basis reduction algorithm (typically DeWeger's integer-arithmetic version), Wildanger's algorithm for unit equations in number fields, various techniques for computing information about the unit groups of number fields (typically fundamental units, with or without assuming the Generalized Riemann hypothesis), related algorithms for solving Thue equations, algorithms related to algebraic curves, genus calculations, ranks of elliptic curves and Jacobians (critical for applications of Chabauty-type techniques for bounding rational points on higher genus curves). Modular forms computations à la William Stein (i.e. modular symbols) for computations of Hecke eigenvalues of Galois conjugacy classes of, e.g. weight 2 *cuspidal newforms of level* N *also are critical in modern Diophantine analysis, as are computations of zeros of Dirichlet* L*-functions (see e.g. Rubinstein) and related zero-free regions (Khadiri). These latter results are involved in producing explicit Chebyshev-type bounds for primes in short intervals in arithmetic progressions, which figure in estimating nonarchimedean contributions in the hypergeometric method of Thue and Siegel.*

Multiplicative Number Theory also utilizes a variety of computational tools, both in order to make asymptotic estimates explicit, and also to inspire or provide evidence for conjectures. Some of the major tools of analytic number theory involve the theory of meromorphic functions (which was in large part commenced by the study of the Riemann zeta function in connection with the distribution of prime numbers), the evaluation and estimation of exponential sums, sieve methods, and many techniques from the fields of harmonic analysis, probability, and random matrix theory. In many of these areas, computations inform and suggest directions for future research.

The primary problems to which these computational tools are applied include, on the analytic side, the distribution of prime numbers and of the prime factors of integers, special values of zeta functions (including multiple zeta values) and L-functions, and uniform distribution of arithmetic sequences; and on the Diophantine side, determining the transcendentality of natural constants and of values of modular functions, irrationality measures for these values and for algebraic numbers, and applications to rational points on algebraic varieties and solutions of Diophantine equations (see e.g. [2], [4], [5], [6]).

The workshop kicked off with an expository exploration of open problems by Richard Guy (Calgary) in prime number theory, arithmetic function theory and the arithmetic of elliptic curves. Subsequent talks focused on pseudoprimes (with applications to primality testing), and inequalities for arithmetic functions, *before the direction of the workshop changed on the second day to more algebraic computation. Along these lines, the workshop featured talks on Hilbert modular form computations, with related work on ternary Diophantine equations, via modularity of Q-curves.*

List of Participants

Bauer, Mark *(University of Calgary)*
Bennett, Michael *(University of British Columbia)*
Bernstein, Daniel *(University of Illinois, Chicago)*
Chou, James *(University of Calgary)*
Cohen, Paula *(Texas A&M University)*
Corvaja, Pietro *(Università di Udine)*
David, Chantal *(Concordia University)*
Dembele, Lassina *(University of Calgary)*
Ellenberg, Jordan *(Princeton University)*
Ferguson, Ron *(Simon Fraser University)*
Filaseta, Michael *(University of South Carolina)*
Foster, Chris *(University of Calgary)*
Guy, Richard *(University of Calgary)*
Helfgott, Harald *(Yale University)*
Lemieux, Stephane *(University of Calgary)*
Lobnig, Tanja *(University of Calgary)*
Martin, Greg *(University of British Columbia)*
Mulholland, Jamie *(University of British Columbia)*
Patterson, Roger *(University of Calgary)*
Pinch, Richard *(HMG, Cheltenham)*
Roettger, Eric *(University of Calgary)*
Rozenhart, Pieter *(University of Calgary)*
Scheidler, Renate *(University of Calgary)*
Silvester, Alan *(University of Calgary)*
Sullivan, Nick *(University of Calgary)*
Tang, Adrian *(University of Calgary)*
Venkatesh, Akshay *(Massachusetts Institute of Technology)*
Wooding, Kjell *(University of Calgary)*

Bibliography

[1] *H. Cohen, A Course in Computational Number Theory, (Corrected Third Printing), Graduate Texts in Mathematics 138, Springer 1996.*

[2] *J. B. Conrey, L-functions and random matrices. In* Mathematics unlimited—2001 and beyond, *331–352, Springer, Berlin, 2001.*

[3] *R. Crandall and C. Pomerance, Primes, a computational perspective, Springer 2000.*

[4] *R. Heath-Brown, Rational points and analytic number theory. In* Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), *37–42, Progr. Math., 226, Birkhäuser Boston, Boston, MA, 2004.*

[5] *P. Swinnerton-Dyer, Diophantine equations: progress and problems. In* Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), *3–35, Progr. Math., 226, Birkhäuser Boston, Boston, MA, 2004.*

[6] *M. Waldschmidt, Open Diophantine problems.* Mosc. Math. J. ***4** (2004), no. 1, 245–305, 312.*

Chapter 61

MITACS Environment and Natural Resources Theme Meeting (04w2066)

December 2–4, 2004

Organizer(s): *John Stockie (Simon Fraser University, MITACS)*

List of Participants

Ahn, Hojun *(University of Waterloo)*
Barsky, Sandra *(University of British Columbia)*
Bourlioux, Anne *(University of Montreal)*
Bouzidi, Youcef *(Geo-X Systems Ltd.)*
Bridge, Lloyd *(University of British Columbia)*
Carnes, Brian *(University of Victoria)*
Chang, Paul *(University of British Columbia)*
Chen, Wan *(University of British Columbia)*
Fishman, Lou *(University of Calgary/MDF International)*
Geiger, Hugh *(University of Calgary)*
Gibson, Peter *(York University)*
Josey, Tyson *(Martec Ltd.)*
Lamoureux, Michael *(University of Calgary)*
Lien, Fue-Sang *(University of Waterloo)*
Margrave, Gary *(University of Calgary)*
Montgomery, Patrick *(University of Northern British Columbia)*
Muir, Paul *(Saint Mary's University)*
Nyquist, Brock *(Okanagan University College)*
Patera, Jiri *(Universite de Montreal)*
Petzold, Linda *(University of California)*
Promislow, Keith *(Simon Fraser University/Michigan State University)*
Shah, Akeel *(Simon Fraser University)*
Spiteri, Ray *(University of Saskatchewan)*
Stockie, John *(Simon Fraser University, MITACS)*
Tuy, Rosette *(University of Saskatchewan)*
Tyson, Rebecca *(Okanagan University College)*
Wang, Rong *(University of Saskatchewan)*
Watmough, James *(University of New Brunswick)*
Wetton, Brian *(University of British Columbia)*

Zhao, Xiao-Qiang *(Memorial University of Newfoundland)*

Focused Research Group Reports

Chapter 62

Robust Analysis of Large Data Sets (04frg501)

June 5–19, 2004

Organizer(s): *Ruben Zamar (University of British Columbia), Stefan Van Aelst (University of Ghent, Belgium)*

Robust statistics has been developing for several decades now, and it was time for some reflection on the topic. The informal atmosphere, lengthy discussions and extended talks in our workshop permitted, amongst others things to focus on the following important questions.

- **What are the fields of application where robust statistics can make a difference?** *It became clear that analysis of huge, messy data sets with many variables (the typical data-mining setting) is crucial. Applications in finance, genetics, chemometrics, etc. have been presented at the meeting.*

- **Which kind of methods do we still need to develop?** *From the discussions and talks, it became clear that analysis of datasets that contain several types of data (numerical, categorical, ordinal, etc.), robust statistical inference beyond point estimation, and cleaning of data matrices deserve more attention.*

- **Do we need to put into questions some of the foundations of robustness?** *In the robustness community, high breakdown point and equivariance properties have always been at a central place, but it is not always clear that these properties are required in typical data mining applications. Moreover it is not obvious how to define important robustness notions such as breakdown point in non-standard settings.*

From the discussions at the workshop it became clear that the interplay between robustness and data mining will be an important direction of future research with many applications. In data mining the focus is on extracting valuable information from large databases. Revolutionary progress in digital data acquisition and storage in recent years has resulted in the creation of huge databases. Supermarket transactions, credit card usage, telephone calls details, internet traffic, corporate statistics, astronomical data, gene expression data, medical and clinical data are all examples of such databases. In fact, the production and accumulation of digital databases is occurring at a faster rate than our ability to comprehend and use them. One possible reaction to this avalanche of digital data would be to dismiss them as electronic junk. However, many people including the organizers and most participants of this workshop believe that these databases contain valuable knowledge which can be mined (found, extracted and used).

In the process of mining the data - as in a true mining operation - we go through the following typical *main steps:*

Step I	***Defining mining goal:***	*defining the particular type of structure (or structures) we wish to find.*
Step II	***Scoring mining results:***	*deciding how to quantify - score - the success of a given structure in realizing the mining goal.*
Step III	***Getting it done:***	*designing and implementing an algorithm to optimize the scoring scheme from Step II.*

Since Statistics and Data Mining are both concerned with the analysis and modelling of data and methods to perform these tasks, there is a big overlap between these two disciplines. There is a clear parallel between data mining Steps I, II and III and Statistical Model Building, Model Fitting and Computing. The main difference lies in the fact that, given the magnitude of the datasets encountered in data mining applications, Step III has to be highly automated and run with non or little human intervention. Data miners have always used statistical tools and statistician are now showing an interest in Data Mining problems. The interactions between the two disciplines will be very beneficial to both of them.

It has been discussed by the participants of our workshop (and generally agreed) that there is an opportunity for the application of robust methods and ideas in Data Mining. It is possible, for example that some useful patterns apply to the majority (but not the totality) of the data. Such patterns may never be found by classical statistical methods that attempt to fit the complete dataset. Robust algorithms, on the other hand, search for suitable subsets of the data and therefore can find these partial patterns. But classical robust procedures are computationally intensive and do not scale well to large datasets normally encountered in data mining applications.

Several examples shown at the workshop clearly illustrated that there is a need for robust data analysis techniques that can handle large data sets. However, many of the robust methods that are available cannot be applied directly to large data sets due to practical and theoretical reasons. Practically, robust methods are computationally so demanding that running them on large data sets is not feasible in a reasonable amount of time. From the theoretical viewpoint many robust methods are not suitable for large, high dimensional data sets because they are based on the concept of outlying objects (rows in the data matrix). Most available robust methods treat the measurements for all variables of one object as the basic processing unit. Each object is classified as "good" or "outlying" and if an object is considered outlying, then all measurements for that object are downweighted together. This approach works well for low dimensional data sets because it leads to equivariance properties that are often considered desirable. In high dimensions, on the other hand, it is not reasonable anymore to consider all measurements of an object as deviating from the majority if some of them are outliers. Indeed, often only 1 or a few of the measurements are contaminated while all other measurements of the object are not. In fact, if every variable has a small probability of producing a contaminated measurement, then the probability of having a completely clean object decreases rapidly as the dimension increases. In high dimensions, we can thus be confronted with data sets that contain only few completely clean objects. This violates the basic assumption underlying most available robust procedures: good points form the majority of the data. For high dimensional data sets it is therefore more natural to consider **outlying cells** *instead of* **outlying objects***. A cell is the measurement for one object and one variable. Robust statistics can make relevant contributions to the field of data mining by developing methods and techniques based on outlying cells that are better suited to handle the problems encountered when analyzing large data sets. Research in this direction should focus on developing methods and algorithms that are maybe less refined from the statistical viewpoint but are extremely fast to compute and scale well with growing sample size and dimension.*

List of Participants

Croux, Christophe *(Katholieke Universiteit Leuven)*
Field, Chris *(Dalhousie University)*
Filzmoser, Peter *(Vienna University of Technology)*
Gather, Ursula *(Universitt Dortmund)*
Genton, Marc *(North Carolina State University)*
He, Xuming *(University of Illinois)*
Hennig, Christian *(Universitat Hamburg, SPST/ZMS)*
Maronna , Ricardo *(Universidad Nacional La Plata)*
Martin, Doug *(University of Washington)*
Ronchetti, Elvezio *(University of Geneva)*
Salibian-Barrera, Matias *(Carleton university)*
Tyler, David *(Rutgers University)*
Van Aelst, Stefan *(Ghent University)*
Wei, Ying Willems, Gert *(University of Antwerp)*
Yohai, Victor *(University of Buenos Aires)*
Zamar, Ruben *(University of British Columbia)*

Chapter 63

String Field Theory Camp (04frg538)

July 10–24, 2004

Organizer(s): *Gordon Semenoff (University of British Columbia), Moshe Rozali (University of British Columbia), Mark Van Raamsdonk (University of British Columbia)*

This was an intensive study group of some issues of current interest in string theory and string field theory. Most of the participants of the Camp gave a free form, informal lecture. Typically the lectures lasted for two hours and they inspired lively discussion. Below is a list of the titles and abstracts of the discussions:

1. *Dominic Brecher and Mark van Raamsdonk, University of British Columbia: Generally Covariant Actions for Multiple D-branes: We develop a formalism that allows us to write actions for multiple D-branes with manifest general covariance. While the matrix coordinates of the D-branes have a complicated transformation law under coordinate transformations, we find that these may be promoted to (redundant) matrix fields on the transverse space with a simple covariant transformation law. Using these fields, we define a covariant distribution function (a matrix generalization of the delta function which describes the location of a single brane). The final actions take the form of an integral over the curved space of a scalar single-trace action built from the covariant matrix fields, tensors involving the metric, and the covariant distribution function. For diagonal matrices, the integral localizes to the positions of the individual branes, giving N copies of the single-brane action.*

2. *Anastasia Volovich, KITP: On the Tree-Level S-Matrix of Yang-Mills Theory: We investigate the procedure for computing tree-level amplitudes in Yang-Mills theory from connected instantons in the B-model on $P^{3|4}$, emphasizing that the problem of calculating Feynman diagrams is recast into the problem of finding solutions to a certain set of algebraic equations. We show that the B-model correctly reproduces all 6-particle amplitudes, including non-MHV amplitudes with three negative and three positive helicity gluons. As a further check, we also show that n-particle amplitudes obtained from the B-model obey a number of properties required of gauge theory, such as parity symmetry (which relates an integral over degree d curves to one over degree n-d-2 curves) and the soft and collinear gluon poles.*

3. *Marcus Spradlin, KITP: A Googly Amplitude from the B-model in Twistor Space: Recently it has been proposed that gluon scattering amplitudes in gauge theory can be computed from the D-instanton expansion of the topological B-model on $P^{3|4}$, although only maximally helicity violating (MHV) amplitudes have so far been obtained from a direct B-model calculation. In this note we compute the simplest non-MHV gluon amplitudes (++— and +-+–) from the B-model as an integral over the moduli space of degree 2 curves in $P^{3|4}$ and find perfect agreement with Yang-Mills theory.*

4. *David Berenstein (University of California at Santa Barbara): Deformations of N=4 SYM and integrable spin chain models: Beginning with the planar limit of N=4 SYM theory, we study planar diagrams for field theory deformations of N=4 which are marginal at the free field theory level. We show that the requirement of integrability of the full one loop dilatation operator in the scalar sector,*

places very strong constraints on the field theory, so that the only soluble models correspond essentially to orbifolds of N=4 SYM. For these, the associated spin chain model gets twisted boundary conditions that depend on the length of the chain, but which are still integrable. We also show that theories with integrable subsectors appear quite generically, and it is possible to engineer integrable subsectors to have some specific symmetry, however these do not generally lead to full integrability. We also try to construct a theory whose spin chain has quantum group symmetry $SO_q(6)$ as a deformation of the SO(6) R-symmetry structure of N=4 SYM. We show that it is not possible to obtain a spin chain with that symmetry from deformations of the scalar potential of N=4 SYM. We also show that the natural context for these questions can be better phrased in terms of multi-matrix quantum mechanics rather than in four dimensional field theories.

5. *David Berenstein: University of California at Santa Barbara: A toy model for the AdS/CFT correspondence: We study the large N gauged quantum mechanics for a single Hermitian matrix in the Harmonic oscillator potential well as a toy model for the AdS/CFT correspondence. We argue that the dual geometry should be a string in two dimensions with a curvature of stringy size. Even though the dual geometry is not weakly curved, one can still gain knowledge of the system from a detailed study of the open-closed string duality. We give a mapping between the basis of states made of traces (closed strings) and the eigenvalues of the matrix (D-brane picture) in terms of Schur polynomials. We connect this model with the study of giant gravitons in $AdS_5 x S^5$. We show that the two giant gravitons that expand along AdS_5 and S^5 can be interpreted in the matrix model as taking an eigenvalue from the Fermi sea and exciting it very much, or as making a hole in the Fermi sea respectively. This is similar to recent studies of the c=1 string. This connection gives new insight on how to perform calculations for giant gravitons.*

6. *Taejin Lee, Asia Pacific Center for Theoretical Physics, Seoul: Fermion Representation of the Rolling Tachyon Boundary Conformal Field Theory: A free fermion representation of the rolling tachyon boundary conformal field theory is constructed. The representation is used to obtain an explicit, compact, exact expression for the boundary state. By explicit computation, we show that this boundary state correctly depicts the time evolution of the unstable D-brane in the scalar sector.*

7. *Yutaka Matsuo, University of Tokyo: Cardy states as idempotents of fusion ring in string field theory: With some assumptions, the algebra between Ishibashi states in string field theory can be reduced to a commutative ring. From this viewpoint, Cardy states can be identified with its idempotents. The algebra can be identified with a fusion ring for the rational conformal field theory and a group ring for the orbifold. This observation supports our previous observation that boundary states satisfy a universal idempotency relation under closed string star product.*

8. *Yoshi Kitazawa, KEK Lab, Tsukuba, Japan: Correlators of Matrix Models on Homogeneous Spaces: We investigate the correlators of $TrA_{mu}A_{nu}$ in matrix models on homogeneous spaces: S^2 and $S^2 x S^2$. Their expectation value is a good order parameter to measure the geometry of the space on which non-commutative gauge theory is realized. They also serve as the Wilson lines which carry the minimum momentum. We develop an efficient procedure to calculate them through 1PI diagrams. We determine the large N scaling behaviour of the correlators. The order parameter shows that fuzzy $S^2 x S^2$ acquires a 4 dimensional fractal structure in contrast to fuzzy S^2. We also find that the two point functions exhibit logarithmic scaling violations.*

9. *Washington Taylor, MIT: abelian and nonabelian vector field effective actions from string field theory: The leading terms in the tree-level effective action for the massless fields of the bosonic open string are calculated by integrating out all massive fields in Witten's cubic string field theory. In both the abelian and nonabelian theories, field redefinitions make it possible to express the effective action in terms of the conventional field strength. The resulting actions reproduce the leading terms in the abelian and nonabelian Born-Infeld theories, and include (covariant) derivative corrections.*

10. *Amanda Peet, University of Toronto: Brane-antibrane systems and the thermal life of neutral black holes: A brane-antibrane model for the entropy of neutral black branes is developed, following on from the work of Danielsson, Guijosa and Kruczenski. The model involves equal numbers of Dp-branes and anti-Dp-branes, and arbitrary angular momenta, and covers the cases p=0,1,2,3,4. The*

thermodynamic entropy is reproduced by the strongly coupled field theory, up to a power of two. The strong-coupling physics of the p=0 case is further developed numerically, using techniques of Kabat, Lifschytz et al., in the context of a toy model containing the tachyon and the bosonic degrees of freedom of the D0-brane and anti-D0-brane quantum mechanics. Preliminary numerical results show that strong-coupling finite-temperature stabilization of the tachyon is possible, in this context.

11. *Shiraz Minwalla, Harvard University: Black hole-black string phase transitions in thermal 1+1-dimensional supersymmetric Yang-Mills theory on a circle: We review and extend earlier work that uses the AdS/CFT correspondence to relate the black hole-black string transition of gravitational theories on a circle to a phase transition in maximally supersymmetric 1+1-dimensional SU(N) gauge theories at large N, again compactified on a circle. We perform gravity calculations to determine a likely phase diagram for the strongly coupled gauge theory. We then directly study the phase structure of the same gauge theory, now at weak 't Hooft coupling. In the interesting temperature regime for the phase transition, we may reduce the 1+1-dimensional theory to a 0+1-dimensional bosonic theory, which we solve using Monte Carlo methods. We find strong evidence that the weakly coupled gauge theory also exhibits a black hole-black string like phase transition in the large N limit. We demonstrate that a simple Landau-Ginzburg like model describes the behaviour near the phase transition remarkably well. The weak coupling transition appears to be close to the cusp between a first order and a second order transition.*

12. *Andre Mikhailov, CALTECH: Supersymmetric null-surfaces: Single trace operators with the large R-charge in supersymmetric Yang-Mills theory correspond to the null-surfaces in* $AdS_5 \times S^5$*. We argue that the moduli space of the null-surfaces is the space of contours in the super-Grassmanian parametrizing the complex* $(2|2)$*-dimensional subspaces of the complex* $(4|4)$*-dimensional space. The odd coordinates on this super-Grassmanian correspond to the fermionic degrees of freedom of the superstring.*

A number of research collaborations were either initiated or pursued during the camp. Mark van Raamsdonk, Anastasia Volovich and Marcus Spradlin initiated a project on studying a comparison of the spectrum of the plane wave matrix model and the dilatation operator in a certain sector of supersymmetric Yang-Mills theory. Gordon Semenoff and Taejin Lee worked on fermionization of the rolling tachyon conformal field theory.

List of Participants

Berenstein, David *(IAS Princeton)*
Brecher, Dominic *(University of British Columbia)*
Kitazawa, Yoshi *(KEK Lab)*
Lee, Taejin *(Kangwon University)*
Matsuo, Yutaka *(University of Tokyo)*
Mikhailov, Andre *(California Institute of Technology)*
Minwalla, Shiraz *(Harvard University)*
Peet, Amanda *(University of Toronto)*
Semenoff, Gordon *(University of British Columbia)*
Spradlin, Marcus *(University of California, Santa Barbara)*
Taylor, Washington *(Massachusetts Institute of Technology)*
Van Raamsdonk, Mark *(University of British Columbia)*
Volovich, Anastasia *(KITP, Santa Barbara)*

Chapter 64

Kinetic Models for Multiscale Problems (04frg049)

August 21–September 4, 2004

Organizer(s): *Reinhard Illner (University of Victoria), Shi Jin (University of Wisconsin, Madison), Peter Markowich (Wolfgang Pauli Institute, Vienna), Lorenzo Pareschi (University of Ferrara, Italy)*

General Comments

Besides the four organizers there were five additional "core" participants hosted at the institute. In alphabetical order: Chi-Kun Lin (presently University of Calgary), Dietmar Oelz (Technical University Vienna), Christian Schmeiser (Technical University Vienna), Giovanni Russo (University of Catania) and Holger Teismann (Acadia University). In addition there were two observers, hosted elsewhere in Banff but fully integrated in the scientific activities: Jean Dolbeault (Université Paris Dauphine) and Horst Lange (University of Cologne). There were thus a total of 11 participants.

Each participant gave one or two seminar style presentations on current research. Abstracts of these presentations may be found on the website of Peter Markowich:

http://homepage/univie.ac.at/peter.markowich/

A brief list of the presented topics, in the order in which they were presented, follows:

- *R. Illner: Fokker-Planck type models for multilane traffic flow*
- *S. Jin: Computation of semi-classical limits and multivalued solutions of PDEs*
- *C.-K. Lin: On coupled nonlinear Schroedinger equations, and From compressible to incompressible fluid equations*
- *J. Dolbeault: Entropy/ entropy production methods for degenerate drift-diffusion equations, and Comments on stability and control of quantum equations*
- *H. Lange: Limitations of controllability for linear and nonlinear Schrödinger equations*
- *H. Teismann: An overview of controllability results for Schrödinger equations. The significance of coherent states.*

- *P. Markowich: Bose-Einstein condensates*
- *L. Pareschi: Analytical and numerical results for the Boltzmann equation for Bosons; Bose-Einstein condensation*
- *G. Russo: Implicit-explicit numerical methods for nonlinear hyperbolic systems*
- *C. Schmeiser: Mathematical models for chemotaxis, and Dimension reduction for the Gross-Pitaevski equation*
- *D. Oelz: Nonlinear Diffusion equations as macroscopic limits of generalized BGK models, and Numerical studies on chemotaxis*

These lectures happened in a very informal setting and without serious time constraints, such that detailed discussions were possible during the talks. Everybody attended all lectures and participated in the discussions. Research sessions typically happened immediately after the lectures. These sessions were exceptionally fruitful and entailed many continued and new collaborations. Several research papers were completed or significantly advanced during the workshop. Details are given in the sequel.

Central Research Topics

Among the many research subjects which were considered during the workshop, the following saw the most efforts and progress.

J. Dolbeault, P. Markowich, D. Oelz and C. Schmeiser continued joint work on the mean free path limit of a Boltzmann-type equation with general equilibrium function and a relaxation time approximation collision operator (a generalized BGK model) This advances the theory of diffusion limits, a theory with fundamental roots in a paper by C. Bardos, R. Santos and R. Sentis (Diffusion approximation and computation of the critical size, Trans. Amer. Math. Soc. ***284*** *(1984), no. 2, 617–649). In this project a Boltzmann-type equation with a non-linear relaxation time approximation collision operator involving a general equilibrium function is considered. Under parabolic scaling a rigorous convergence proof of solutions to a macroscopic limit was obtained. The analysis employs compensated compactness theory.*

Different choices for the local equilibrium lead to different macroscopic equations. Most notably, nonlinear diffusion equations ranging from fast diffusion to porous medium equations are reproduced as macroscopic limits by employing different types of equilibrium functions with decreasing rates of decay in terms of energy. The resulting paper is available as a preprint [1].

Chemotaxis was studied by D. Oelz and C. Schmeiser on the kinetic level with particular emphasis on diffusion limits and microscopic modelling of the motion of bacteria. It is conceivable that finite-time blow-up can be avoided on this level by careful modelling of reorientation processes.

Peter Markowich and Lorenzo Pareschi finished work on the numerical solution of the ergodic approximation of the quantum Boltzmann equation. The main difficulty here was finding an efficient scheme which maintains entropy growth, mass conservation and is at the same time able to reproduce a generalized Bose-Einstein equilibrium. The devised scheme is based on appropriate discretisation of the three dimensional integral in the collision operator and was found to be competitive with a Monte-Carlo method. It can be used for both the homogeneous gas dynamics Boltzmann equation and for the Boltzmann equation for Fermions.

Shi Jin and Peter Markowich further completed a paper (with additional authors Sparber, Zheng and Huang) on the numerical solution of the Dirac-Maxwell system. The scheme is based on a spectral discretisation in position space combined with a time splitting method. Test examples are presented in the semiclassical and non-relativistic regimes, focusing on electon-phonon coupling.

Lorenzo Pareschi and Giovanni Russo continued their analysis of stability and accuracy of IMEX-Runge Kutta schemes. Implicit-explicit (IMEX) Runge-Kutta schemes are a very effective tool for the numerical *solution of hyperbolic systems with stiff relaxation. Under mild assumptions on the system coefficients, they provide numerical schemes that are accurate in the limit of both very large and very small relaxation parameters. However, for intermediate values of the parameter the accuracy of such schemes is not known theoretically. Numerical tests show degradation of the accuracy for values of the relaxation parameter of the order of the time step. During the Focused Research Group workshop, asymptotic analysis was used to obtain an estimate of the uniform order of accuracy for the various IMEX-RK schemes developed by the authors. One of the aims of the stability analysis is to exploit the stabilization effect introduced by the dissipative term, when it is treated by an L-stable space discretization. The larger the stabilization, the less severe the restriction on the time step, resulting in a more efficient scheme.*

Jean Dolbeault and Reinhard Illner continued work on entropy methods for (linear) drift-diffusion equations with time-dependent and locally vanishing drift and diffusion coefficients (thus leading to degeneracies in the diffusion). Problems of this nature emerged in the traffic models which Reinhard Illner presented at the workshop; related problems arise in the analysis of flashing ratchets. Many open mathematical questions emerge naturally in this theory, among them the validity of generalized Hardy-Poincare inequalities, the description of asymptotic behaviour of the system if the roots of the diffusion and drift coefficients experience periodic oscillations, and others. A comprehensive paper about these matters was essentially completed during the workshop [2].

Finally, Reinhard Illner, Horst Lange and Holger Teismann completed work on a comprehensive article on the limitations of exact controllability of linear and nonlinear Schroedinger equations, with the Hartree equation and the Gross-Pitaevski equation as the most relevant examples. In particular, a result on non-controllability (complementing the knowledge on optimal control) for the Hartree equation was obtained. The paper is presently undergoing final revisions.

Concluding Observations

1. *The workshop united a small group of researchers with a strong common mathematical culture and very varied applied interests. Such a variety of domains of applications is common among applied mathematicians and especially among people involved in multiscale modelling.*

2. *The main applications considered at the workshop were:*

   - *New classes of kinetic traffic flow models, leading to original research on nonlinear and nonlocal degenerate drift-diffusion equations (Dolbeault, Illner).*
   - *Bose-Einstein condensation: the condensation mechanism, the dynamics of the condensate, and numerical methods (Markowich, Pareschi, Russo, Schmeiser). The numerical methods presented by Russo (IMEX) essentially apply to all the topics considered in the meeting (they are derived to treat multiscale problems).*
   - *quantum control (Dolbeault, Illner, Lange, Teismann)*
   - *multiscale models in biology, in particular the phenomenon of chemotaxis (Markowich, Oelz, Schmeiser)*
   - *Scaling limits in kinetic theory and fluid dynamics (Dolbeault, Jin, Markowich, Lin, Oelz, Schmeiser).*

3. *The format of a small focused research group was universally found to be ideal: long presentations, enough time for detailed discussions, and time and space for active research (there are numerous articles in preparation as a consequence; new projects are being planned)*

The workshop was a resounding success.

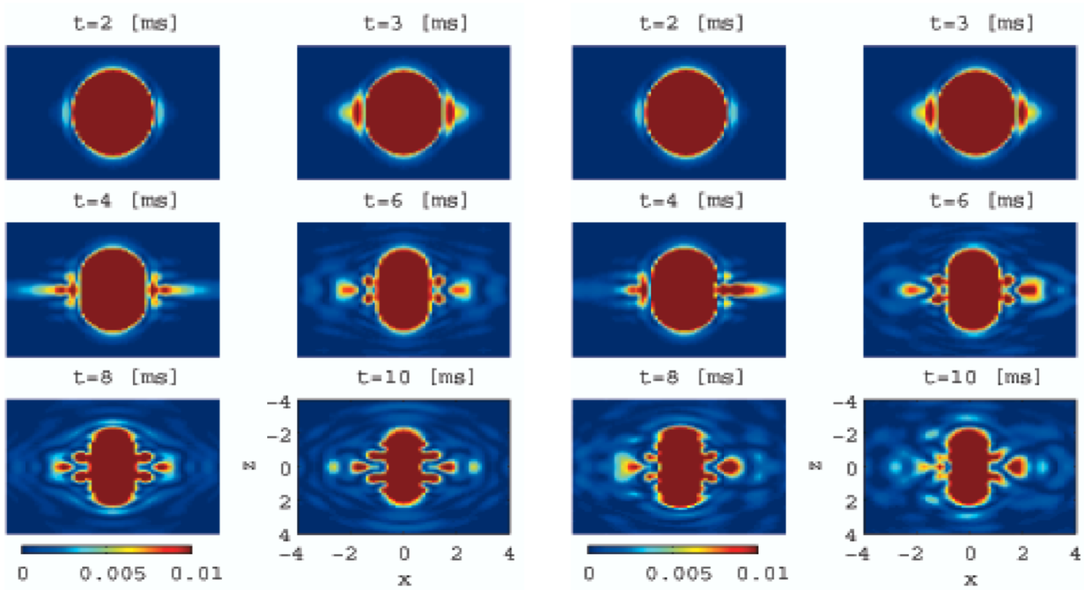


Figure 64.1: Jets in Bose-Einstein condensates

List of Participants

Illner, Reinhard *(University of Victoria)*
Lin, Chi-Kun *(Tainan University)*
Markowich, Peter *(Wolfgang Pauli Institute Vienna)*
Oelz, Dietmar *(University of Vienna)*
Pareschi, Lorenzo *(University of Ferrara, Italy)*
Russo, Giovanni *(Università di Catania)*
Schmeiser, Christian *(Institut fuer Analysis und Scientific Computing)*
Shi, Jin *(University of Wisconsin, Madison)*
Teismann, Holger *(Acadia University)*

Bibliography

[1] J. Dolbeault, P. Markowich, D. Oelz and C. Schmeiser, Mean free path limit of a Boltzmann-type equation with general equilibrium function and relaxation time approximation collision operator. in preparation.

[2] Ph. Bartier, J. Dolbeault, R. Illner and M. Kowalczyk, A qualitative study of linear drift-diffusion equations with time-dependent or vanishing coefficients, in preparation.

[3] R. Illner, H. Lange and H. Teismann, Limitations in quantum control, in preparation.

Research in Teams Reports

Chapter 65

Cohomogeneity One Manifolds With Positive Sectional Curvature (04rit525)

March 13–27, 2004

Organizer(s): *Karsten Grove (University of Maryland), Burkhard Wilking (Universität Münster), Wolfgang Ziller (University of Pennsylvania)*

Background

Since the simplest non-trivial Riemannian manifold is the round sphere of radius r and (constant) curvature $1/r^2$, it is only natural that manifolds with positive curvature have played a central role since the beginning of global Riemannian geometry. In the (complete) non-compact case a theorem of Gromoll and Meyer (1970) asserts that the manifold is diffeomorphic to euclidean space, and in the compact case the classical Bonnet-Myers theorem (1932) imply that the fundamental group is finite. In even dimensions a result of Synge (1934) shows that the manifold is simply connected if it is orientable. The main issue in the subject is therefore to understand (compact) simply connected manifolds with positive curvature. Except for special obstructions for spin manifolds (stemming already from positive scalar curvature) the only known obstruction is the Betti number theorem due to Gromov (1980) which even applies to non-negative curvature: It provides a bound on the total Betti number which depends only on the dimension.

Under natural additional conditions there are celebrated results that identify the manifold with the sphere or with one of the other rank one symmetric spaces, i.e., the complex and quaternionic projective spaces, or the Cayley plane, the so-called CROSSes. It is remarkable that above dimension 24 these are the only known (simply connected) manifolds of positive curvature. Additional examples have appeared in dimensions 6, 7, 12, 13, and 24. These examples include a complete classification of positively curved homogeneous manifolds due to combined work of Berger (1960), Wallach (1970), Aloff-Wallach (1972), and Berard-Bergery (1975) (one in each of the dimensions 6, 12, 13 and 24, and infinitely many in dimension 7). Non-homogeneous examples have been found by Eschenburg (1978) in dimensions 6, 7, and by Bazaikin (1996) in dimension 13 (one in dimension 6 and infinitely many in the other dimensions). All these examples are so-called biquotients, i.e., quotients of a compact Lie group G by a subgroup of $G \times G$ acting on left and right on G.

To advance the theory at this point it seems imperative to find new examples, a task which notoriously is very difficult as indicated above. For simplicity, and since all known examples have a fairly large group of isometries, it seems natural to look for new examples with large symmetry group. Attempts to classify manifolds with positive curvature and large isometry group provide a framework for a systematic search for new examples. One of the natural measurements for the size of the isometry group is its cohomegeneity*, i.e., the dimension of the orbit space. For example, having minimal cohomogeneity 0 means that the isometry group acts transitively on the manifold, i.e., it is homogeneous. In analogy with the case of homogeneous* *manifolds, Wilking recently showed that in any fixed cohomogeneity, sufficiently high dimensional manifolds of positive curvature are CROSSes (up to tangential homotopy equivalence).*

Project Description

The previous section should provide ample justification for a serious investigation of simply connected cohomogeneity one manifolds of positive curvature with a complete classification as the final goal.

It is well known that the orbit space M/G of a simply connected cohomogeneity one G-manifold M, is an interval whose end points correspond to two singular orbits $B_\pm = G/K_\pm$ of codimension at least two, and where each interior point is a principal orbit G/H of codimension one. Moreover, $S^{l_\pm} = K_\pm/H$ are normal spheres to $B_\pm$, and $M = G \times_{K_-} D^{l_-+1} \cup G \times_{K_+} D^{l_++1}$ is the union of tubular neighbourhoods of these orbits. In particular, M is determined by the subgroups $H \subset \{K_\pm\} \subset G$ and vice versa. We point out that in general a cohomogeneity one manifold can support many inequivalent cohomogeneity one actions. In particular there are many (linear) cohomogeneity one actions on spheres and more generally on CROSSes.

A remarkable first step towards the classification of cohomogeneity one manifolds of positive curvature is the recent result of L. Verdiani asserting that in even dimensions *only CROSSes appear with their standard actions. - The same is false in odd dimensions. In fact, one observes that specific infinite subfamilies of the Eschenburg spaces and of the Bazaikin spaces are indeed of cohomogeneity one, as are three non CROSS normal homogeneous spaces, $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3), W^7 = \mathrm{SU}(3)\,\mathrm{SO}(3)/\mathrm{U}(2)$, and $B^{13} = \mathrm{SU}(5)/\mathrm{Sp}(2)S^1$.*

When specialising Wilkings theorem to the case of cohomogeneity one, the dimension beyond which all manifolds are like a CROSS is 72. In recent work by Grove, Verdiani, Wilking and Ziller it was shown that no cohomogeneity one exotic sphere (Kervaire sphere) supports an invariant metric with nonnegative curvature, and neither does any non-linear cohomogeneity one action on a standard sphere. In particular, if a positively curved cohomogeneity one manifold is homotopy equivalent to a CROSS, it is indeed a CROSS with a standard action. It thus remains to classify (simply connected) positively curved cohomogeneity one spaces below dimension 72, that are not homotopy equivalent to a CROSS.

The classification has two natural parts:

- *Find obstructions on the manifold M, i.e., on $H \subset \{K_\pm\} \subset G$, due to positive curvature*
- *Find positively curved metrics on unobstructed manifolds M*

Although we now believe that we have settled the first part, we cannot be sure until the second part has been carried out as well. It is striking that the obstruction we have found in particular imply that the above bound of 72 can be replaced by 13. In fact

***Theorem 1** Let M be a simply connected compact positively curved manifold on which a Lie group G acts isometrically with one dimensional orbit space. Then one has the following possibilities:*

(a) M is equivariantly diffeomorphic to a rank one symmetric space with a linear cohomogeneity one action (all classified by Hsiang-Lawson).

(b) M is equivariantly diffeomorphic to a 7 dimensional positively curved Eschenburg space or 13 dimensional Bazaikin space with their natural isometric cohomogeneity one action.

(c) M is a 7 dimensional manifold on which $S^3 \times S^3$ acts isometrically by cohomogeneity one with finite isotropy group and singular orbits of codimension two.

In parts (a) and (b) we already know the existence of positively curved metrics. In part (c) earlier work of the Grove and Ziller yield the existence of nonnegatively curved metrics. The existence of a positively curved metric is actually further significantly obstructed: If (p_-, q_-) and (p_+, q_+) are the slopes of the circles inside the singular isotropy groups K_-, K_+ as viewed in $S^3 \times S^3$, then either $H = \{\pm 1, \pm i, \pm j, \pm k\}$ or $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. We denote the first family by $M_{(p_-,q_-),(p_+,q_+)}$ and the second one by $N_{(p_-,q_-),(p_+,q_+)}$. The only unobstructed manifolds left (at the moment) are then:

(i) $M_{(1,1),(1+2n,3+2n)}$

(ii) $N_{(1,1),(1+2n,2+2n))}$

(iii) $M_{(1,3),(3,1)}$

(iv) $N_{(3,1),(1,2))}$

Here $M_{(3,1),(1,3)}$ is actually the positively curved Berger space $SO(5)/SO(3)_{max}$, and $M_{(1,1),(1,3)}$ is S^7 with its linear cohomogeneity one action by $SO(4)$ coming from the isotropy representation of the symmetric space $G_2/SO(4)$.

It is also remarkable that the candidates in (i) and (ii) agree precisely with the 3-Sasakian manifolds arising from Hitchin's examples of self dual Einstein orbifolds on S^4. They are therefore $SO(3)$ orbifold principal bundles over S^4 and we observed that the total space happens to be a smooth manifold and not an orbifold.

A further intriguing property of the family $M_{(1,1),(1+2n,3+2n)}$ is that they are 2-connected with $\pi_3 = \mathbb{Z}_r$ and $r = (p_-^2 q_+^2 - p_+^2 q_-^2)/8 = n + 1$ and hence are rational homology spheres. If they have a metric of positive sectional curvature, a theorem of Rong-Petrunin-Tuschmann would imply that the optimal pinching constant of any positively curved metric has to converge to 0 as r increases. This would be the first manifolds with such a phenomenon and would contradict a conjecture due to Fukaya.

Most of our work at BIRS was directed towards the construction of positively curved metrics on the manifolds described in (i) (and hence (ii)) above. We already knew that even this would be a formidable problem since the curvature formulae for invariant metrics are very complicated. Although our understanding and insights deepened significantly as a result of this process (see below), we do not yet know how to construct the desired metrics.

Here is a brief description of the key steps in our search. By invariance, the (most restricted) metrics in (i) on the principal orbits are given by metrics on three orthogonal two dimensional subspaces and hence by 3 symmetric 2×2 block, i.e., by 9 functions on the orbitspace interval. To define a smooth metric on the manifold, specific smoothness conditions at the boundary points where collapse occurs are imposed and completely understood for our candidates. To further simplify our investigations we have made repeated use of a well-known deformation of a G invariant metric first used by Berger for $G = R^1$ and in general by Cheeger. Basically, by shrinking the metric in the direction of the G orbits one generally gets more two planes with positive curvature. In our case we have determined exactly what it takes for a metric to have positive curvature modulo this so-called Berger-Cheeger trick. The full curvature operator for our examples splits into four symmetric 3×3 blocks with complicated expressions in terms of the 9 functions as entries. A sufficient but not quite necessary condition for positive curvature is that each of these blocks are positive definite. Since we already had explicit metrics with nonnegative curvature on our candidates it seemed like a natural attempt to deform these metrics. This turned out to be exceedingly hard, and although we did not prove it, we derived evidence that in fact there are no positively curved metrics on our candidates near the rigid nonnegatively curved metrics constructed earlier. Another intriguing starting point is to use our observation that our candidates coincide with the 3-Sasakian manifolds arising from Hitchin's examples of self dual Einstein orbifolds on S^4. So far, we have been unable to determine if this approach will provide the desired metrics, but the direct procedure does not work. It is striking that among the large class of cohomogeneity one $S^3 \times S^3$ manifolds with nonnegative curvature, our candidates are the only ones that even near the singular orbits admit positive curvature. Our most promising approach so far, and the one we initiated at BIRS towards the end of our stay, is to start with metrics of positive curvature near the singular orbits, and attempt to match them at the boundary of tubular neighbourhoods in a convex fashion. To make this work, it is however clear that much work and new ideas are needed.

List of Participants

Grove, Karsten *(University of Maryland)*
Wilking, Burkhard *(University of Muenster)*
Ziller, Wolfgang *(University of Pennsylvania)*

Chapter 66

Modular Invariants and NIM-reps (04rit048)

March 13–27, 2004

Organizer(s): *Matthias R. Gaberdiel (ETH Zürich, Switzerland), Terry Gannon (University of Alberta)*

During the time of this program we have continued to work on different problems regarding D-branes of WZW models. Let us begin by briefly sketching the context.

One of the best understood string theories are the Wess-Zumino-Witten (WZW) models that describe closed strings propagating on a target space that is a Lie group G. The algebraic structures governing these models are affine Kac-Moody algebras.

As has become clear in recent years, many closed string theories possess D-branes. A D-brane is simply a submanifold of the target space on which the end points of additional open string degrees of freedom can lie. Because of this open string point of view, D-branes can be analyzed and described using conformal field theory techniques (although they are actually 'non-perturbative' objects from the closed string point of view).

D-branes are dynamical objects that can in particular decay in various ways. In order to understand their dynamics it is useful to determine the invariant charges that characterise different configurations of D-branes. It is believed that the corresponding charge groups agree with certain K-theory groups. For the WZW models, the relevant K-groups are specific twisted K-groups that have been worked out in [1, 2].

A lot of work has already been done on D-branes in WZW models. In particular, the D-branes that preserve the full affine symmetry (possibly up to an automorphism) have been constructed. For the case where the group manifold is simply connected their charges have also been determined [3, 4, 5, 6]. (A systematic analysis for the case of non-simply connected Lie groups was recently begun in [7], see also [8, 9].) While the resulting charge groups agree beautifully with the independent K-theory calculations, it has became clear that, apart for some small rank exceptions, the D-branes that preserve the full affine symmetry do not account for all the K-theory charges. [For example, for the case of $G = \mathrm{SU}(n)$ the charge group that is predicted by K-theory consists of 2^{n-2} copies of some finite cyclic group Z_d, while the above D-brane constructions can only account for two of these summands (for $n \geq 3$).] This raises the question of how to construct the D-branes that account for the remaining charges.

This is the problem we attacked during our time at BIRS.[1] [For simplicity, we restricted ourselves to the case of $\mathrm{SU}(n)$, although it should be straightforward to generalise our constructions to arbitrary (simply-connected) groups.] In particular, we managed to find two constructions that seem to generate the remaining D-brane charges. The first construction [10] is based on the suggestion of [4] that the remaining charges should be related by some sort of T-duality to the original untwisted and twisted branes. We have shown that there are precisely 2^{n-2} different constructions that can be obtained in this manner. While these boundary states break in general the affine symmetry, their open string spectra can still be described in terms of twisted

[1]The work was also done in collaboration with a student of one of us.

representations of the affine symmetry algebra. As a consequence their charges can be determined, and we could show that each of the 2^{n-2} *different constructions leads to the same charge group* Z_d*.*

While this construction is quite suggestive, the geometric interpretation of the different D-branes is not obvious. In a second paper [11] we therefore gave a different construction that was more geometrically motivated. It is well known that the group $\mathrm{SU}(n)$ *is rationally homotopy equivalent to a product of odd dimensional spheres*

$$\mathrm{SU}(n) \cong S^{2n-1} \times S^{2n-3} \times \cdots \times S^3 \,. \tag{66.1}$$

Each of these spheres comes from a coset space $S^{2m-1} \cong \mathrm{SU}(m)/\mathrm{SU}(m-1)$*, and thus homotopically we can think of* $\mathrm{SU}(n)$ *as the product of these coset spaces.*

From a conformal field theory point of view we can decompose the space of states in terms of these coset algebras, and we can then construct D-branes that preserve their product. In fact, we managed to find 2^{n-2} *different classes of D-branes that have this property. This multiplicity arises because for each factor of (66.1) (except the* S^3 *factor), there are two possible constructions that seem to describe D-branes whose world-volume does or does not wrap the corresponding sphere. Given the close connection of K-theory to homology [4], this then suggests that these D-branes generate in fact the full K-group.*

Technically, the two constructions [10] and [11] are very similar indeed. The analysis of the charges is more unambiguous for the first construction, while the geometric interpretation is more transparent for the second. Taken together, they therefore give strong support to the assertion that either of them describes a collection of D-branes whose charges generate the full K-group.

Our two weeks at BIRS were again very productive indeed. We found the environment beautifully conducive to research. We also found Andrea Lundquist very helpful. We hope to be able to visit again some time in the future!

List of Participants

Gaberdiel, Matthias *(ETH Zürich, Switzerland)*
Gannon, Terry *(University of Alberta)*

Bibliography

[1] D.S. Freed, M.J. Hopkins and C. Teleman, Twisted K-theory and loop group representations, `math.at/0312155`.

[2] V. Braun, Twisted K-theory of Lie groups, `hep-th/0305178`.

[3] S. Fredenhagen and V. Schomerus, Branes on group manifolds, gluon condensates, and twisted K-theory, JHEP ***0104*** *(2001), 007;* `hep-th/0012164`.

[4] J. Maldacena, G. Moore and N. Seiberg, D-brane instantons and K-theory charges, JHEP ***0111*** *(2001), 062;* `hep-th/0108100`.

[5] P. Bouwknegt, P. Dawson and A. Ridout, D-branes on group manifolds and fusion rings, JHEP ***0212*** *(2002), 065;* `hep-th/0210302`.

[6] M. R. Gaberdiel and T. Gannon, The charges of a twisted brane, JHEP ***0401*** *(2004), 018;* `hep-th/0311242`.

[7] M. R. Gaberdiel and T. Gannon, D-brane charges on non-simply connected groups, to appear in JHEP*;* `hep-th/0403011`.

[8] V. Braun and S. Schäfer-Nameki, Supersymmetric WZW models and twisted K-theory of SO(3), `hep-th/0403287`.

[9] S. Fredenhagen, D-brane charges on SO(3), `hep-th/0404017`.

[10] M. R. Gaberdiel, T. Gannon and D. Roggenkamp, The D-branes of SU(n), `hep-th/0403271`.

[11] M. R. Gaberdiel, T. Gannon and D. Roggenkamp, The coset D-branes of SU(n), `hep-th/ 0404112`.

Chapter 67

Pi in the Sky Meeting (04rit300)

May 13–15, 2004

Organizer(s): *Heather Jenkins (PIMS)*

This was the first meeting of the π in the Sky *Editorial Board. The following points were decided.*

Our Audience

π in the Sky *magazine is primarily aimed at high-school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and topics may also put curriculum topics in a different perspective.*

This will be explicitly written in π in the Sky.

Instructions to Authors

Keep the audience (see above statement) in mind, except for specialized columns/sections.

Normally a paper should not exceed 4 printed pages including pictures. For your information one page is 1200 words without pictures.

Breakdown of a Typical Issue

1. *Editorial by Klaus Hoechsmann*
2. *Thematic article–general, cultural, non-specialized, appeal to broad audience*
3. *Other articles of that kind, including history of math and biographies, applications and uses of math, notably in industry and mathematical careers*
4. *Specialist article(s) for more advanced readers, curriculum topics*
5. *Book reviews (not text books), news items*
6. *Letters to the editor, opinions*
7. *Challenges, strategies*

1–4 make up about 70% of an issue.
5–7 make up about 30% of an issue.

In other words at least 2/3 should be of interest to high school teachers "without their pens".

Editorial Board

Editor in Chief:
Ivar Ekeland (University of British Columbia)

Editorial Board:
John Bowman (University of Alberta)
Dragos Hrimiuc (University of Alberta)
Wieslaw Krawcewicz (University of Alberta)
Alexander Melnikov (University of Alberta)
Volker Runde (University of Alberta)
Michael Lamoureux (University of Calgary)
Klaus Hoechsmann (University of British Columbia)
Florin Diacu (University of Victoria)
David Leeming (University of Victoria)
Len Berggren (Simon Fraser University)
Heather Jenkins (Pacific Institute for the Mathematical Sciences)

Teachers we will ask to join:
Sharon Friesen (Galileo Educational Network, Calgary)
Wendy Swonnell (Lambrick Park Secondary School, Victoria)
John Campbell (Archbishop MacDonald, Edmonton)

A Managing Editor will be appointed.

We will move towards a structure where there are Associate Editors (one per PIMS site).

List of Participants

Jenkins, Heather *(Pacific Institute for the Mathematical Sciences)*
Bowman, John *(University of Alberta)*
Diacu, Florin *(University of Victoria)*
Ekeland, Ivar *(University of British Columbia)*
Hoechsmann, Klaus *(Pacific Institute for the Mathematical Sciences)*
Hrimiuc, Dragos *(University of Alberta)*
Krawcewicz, Wieslaw *(University of Alberta)*
Lamoureux, Michael *(University of Calgary)*
Leeming, David *(University of Victoria)*
Runde, Volker *(University of Alberta)*

Chapter 68

Maximal Functions in Non-commutative Analysis (04rit004)

May 17–June 5, 2004

Organizer(s): *Marius Junge (University of Illinois at Urbana-Champaign), Quanhua Xu (Université Besancon at Franche-Conté)*

In recent years we have seen a very deep connection between the recent theory of operator spaces and quantum probability. This leads to fascinating new ideas and much work to be done. The two organizers are partially involved in this new development.

In our first project we discussed the application of these ideas to noncommutative ergodic theory and maximal functions. One of the main results there is the noncommutative maximal ergodic inequality, which is the noncommutative version of the classical Dunford-Schwarcz maximal ergodic inequality. This solves an old (maybe a main) open problem in noncommutative ergodic theory. This maximal inequality is closely related to as well as inspired by the noncommutative Doob maximal inequality for martingales, recently established by one of the organizers. Another main result of this work is a noncommutative version of Stein's maximal ergodic inequality for symmetric positive Markovian semigroups. As an application, we obtain a maximal inequality for the Poisson semigroup of a free group and a version of the almost everywhere radial convergence for this semigroup. This is the free group analogue of the classical results on the Poisson semigroup on the torus. Some work had been done before the meeting at Banff. However, there were many details that required clarification, discussion and improvement. Let us mention, for example, the notion of almost everywhere convergence. There is a vast literature in the noncommutative setting with at least seven different notions. In our final result on convergence of ergodic averages we could provide a functional analytic formulation which implies all reasonable forms of convergence discussed in the literature before. Finally this project (a paper of approximately 50+ pages) is ready for closure.

The second work achieved during our stay deals with the noncommutative Rosenthal inequalities. This is a continuation of our previous work on the noncommutative Burkholder inequality for martingales. We obtained several noncommutative versions of Rosenthal's inequality on independent mean zero random variables. The applications of this family of inequalities ranges from ℓ_p-norms estimates of the singular values for random matrices, to operator-valued versions in free probability and L_p-estimates in classical Araki-Wood factors obtained from quantum mechanics. Some of these estimates were developed for the understanding of operator space analogue of the work of Rosenthal on the structure theory commutative L_p spaces. For example we applied these results to the study of symmetric subspaces of noncommutative L_p-spaces. Let us mention one of our applications. Let M be a von Neumann algebra and $L_p(M)$ be the associated noncommutative L_p-spaces. Assume $2 < p < \infty$. Let $X \subset L_p(M)$ be a symmetric subspace. Then X is isomorphic either to ℓ_p or to ℓ_2. This is a result in the category of Banach spaces. Its counterpart for operator spaces reads as follows: If X has a completely symmetric basis, then X is completely isomorphic to one of the four spaces: ℓ_p, C_p, R_p and $C_p \cap R_p$, where C_p and R_p are respectively the p-column and p-row space.

The collaboration in the third project on operator space techniques and quantum probability was motivated by the fast current development in the field. Parallel with the work of Pisier and Shlayhtenko, one organizer started to investigate the connection between operator spaces and free probability in type III von Neumann algebras. A similar theory can also be developed for the classical Araki-Woods factors known from quantum mechanics. We developed the material for a book (or memoirs) which should illustrate these new area of research and techniques. This also required the review the material known to either one of the participants. Indeed, due to the extensive exchange in Banff we avoided huge amount of overlap between the two participants. We believe that we have gained a much better and deeper understanding of the very recent works on the operator space Grothendieck inequalities, the embedding of Pisier's operator Hilbertian space OH *into a noncommutative* L_1 *spaces (predual of a von Neumann algebra) by Pisier-Shlyakhtenko and the organizers. We have obtained several applications of these representations. In particular, we proved that the class of completely 1-summing maps on* OH *coincides with the Orlicz-Schatten class* S_Φ*, where* Φ *is the Orlicz function* $\Phi(x) = x^2|\ln x|$ *(*$x > 0$*). This result is in strong contrast with the corresponding Banach space result which asserts that the class of 1-summing maps on* ℓ_2 *is the class of Hilbert-Schmidt operators. Note that the logarithmic factor in* Φ *is at the origin of the same logarithmic factor in the operator space little Grothendieck inequality and the projection constant of* OH_n*.*

List of Participants

Junge, Marius *(University of Illinois, Urbana-Champaign)*
Xu, Quanhua *(Université Besancon at Franche-Conté)*

Bibliography

[1] *M. Junge, Doob's inequality for non-commutative martingales,* J. Reine Angew. Math. ***549*** *(2002), 149–190.*

[2] *M. Junge and Q. Xu,Noncommutative Burkholder/Rosenthal inequalities,* Ann. Probab. ***31*** *(2003), 948–995.*

[3] *G. Pisier and D. Shlyakhtenko, Grothendieck's theorem for operator spaces,* Invent. Math. ***1*** *(2002), 185–217.*

[4] *G. Pisier and Q. Xu,Non-commutative martingale inequalities,* Comm. Math. Phys. ***3****,667–698.*

Chapter 69

Geometrical Analysis in One and Several Complex Variables (04rit524)

May 22–June 5, 2004

Organizer(s): *Joseph A. Cima (University of North Carolina, Chapel Hill), Ian Graham (University of Toronto), Kang-Tae Kim (Pohang University of Science and Technology, Korea), Steven G. Krantz (Washington University, St. Louis)*

Of the topics we initially proposed for study, we spent most of our time considering holomorphic mappings in Banach and Hilbert spaces. Following some early developments dating back to about 1970, there has been a surge of recent activity in infinite-dimensional holomorphy. Of particular interest to us are problems which combine geometrical methods of one and several complex variables with operator theory and functional analysis. For example, the books [6] and [4] are written in this vein, and the papers [11] and [12] show that significant portions of the theory of normal families can be extended to separable Banach or Hilbert spaces. It should also be mentioned that some aspects of univalent function theory (the study of classes of conformal mappings of the unit disc) have been generalized to infinite dimensions (see the recent book [7] and the references there).

The Schwarz lemma is one of the most basic mapping results in one complex variable. Perhaps the most important part of this result is the uniqueness statement: if $f : D \to D$ is a holomorphic mapping from the unit disc D to itself, $f(0) = 0$, and $|f'(0)| = 1$, then f is a rotation. Generalizations to bounded domains in $\mathbb{C}^n$ have been considered by various authors, beginning with H. Cartan around 1930 (see [13], [18]). The result closest in spirit to our work in infinite dimensions is a theorem which states that a holomorphic self-map of a bounded domain in $\mathbb{C}^n$ with a fixed point p at which the eigenvalues of df_p have modulus 1 must be an automorphism of the domain. Using iteration techniques and the Cauchy estimates, one proves that the Jordan canonical form of df_p must be diagonal. Separate arguments are then needed to deal with the case of rational and irrational eigenvalues. Convergence questions for a suitable subsequence of the iterates of the mapping f must then be considered.

During the course of our Research in Teams program we studied versions of the rigidity result in the Schwarz lemma for bounded domains in a separable Hilbert space H. Bounds for the differential of the mapping are known; see [6] or [4]. Also, Harris [9] showed that if the domain in question is the unit ball and df_0 is an invertible isometry then $f = df_0$. We obtained the following theorem:

Theorem. *Let $\Omega \subseteq \mathcal{H}$ be a bounded convex domain. Fix a point $p \in \Omega$. Let $f : \Omega \to \Omega$ be a holomorphic mapping such that*
(a) $f(p) = p$;
(b) the differential df_p is triangularizable;
(c) $\sigma(df_p) \subseteq S^1$.
Then f is a biholomorphic mapping.

Convergence questions in infinite dimensions both for the powers of the elements of the point spectrum *and for subsequences of the iterates of the mapping are quite subtle, and new techniques (whose genesis lies in recent work of Kim and Krantz [12]) must be introduced. We believe that these techniques will have further applications, and plan to explore them.*

We also considered a basic open problem in infinite dimensional holomorphy: to show that the inverse of an injective holomorphic mapping from one bounded domain onto another in a separable Hilbert space H *must be holomorphic. Examples dues to Suffridge [17] and Heath and Suffridge [8] show that some pathological behaviour is possible in non-separable Banach spaces. Some sufficient conditions are known for holomorphy of the inverse mapping [10]; but the general case remains open.*

Members of the group also gave informal talks on topics of mutual interest: a generalization to infinite dimensions of a characterization of the unit ball by its automorphism group [11], the theory of Loewner chains in several variables [7], boundary behaviour of biholomorphic mappings between convex domains [15], and the approximation of vector fields in $\mathbb{C}^n$ *by complete vector fields [2], [5]. We formulated a number of problems for further study in both finite and infinite dimensions, and believe that considering such results simultaneously may lead to a useful cross-fertilization of ideas:*

1. If Ω_1 *and* Ω_2 *are bounded domains in a separable Hilbert space* H *and* $f : \Omega_1 \to \Omega_2$ *is an injective holomorphic mapping of* Ω_1 *onto* Ω_2*, then is* f^{-1} *holomorphic ?*

2. Is there a version of the Hopf lemma for plurisubharmonic functions on bounded domains in infinite dimensions ?

3. If B *is the unit ball in a separable Hilbert space* H*, and if* $f : B \to \Omega$ *is a biholomorphic mapping of* B *onto a bounded convex domain, then does* f^{-1} *extend continuously to* $\overline{\Omega}$ *?*

4. With assumptions as in Problem 3, let U *be a nonisotropic Koranyi ball in* ∂B *and suppose that* $f(U) \subset \partial\Omega$*. With respect to the appropriate Hausdorff measure* μ*, does there exist a constant* $C > 0$ *such that* $\mu(U) \geq C\mu(f(U))$ *?*

5. Suppose that $f : B \subset H \to H$ *is a holomorphic mapping from the unit ball in a separable Hilbert space* H *into* H *with open image and that* X *is an analytic subvariety of* B*. When is* $f(X)$ *a variety ?*

The group was very enthusiastic about the "Research in teams" format and felt that it provided an opportunity to carry out joint work which it would have been very difficult to accomplish without a period of intense concentration with all of us present. The setting, facilities, and staff were wonderful.

List of Participants

Cima, Joseph *(University of North Carolina, Chapel Hill)*
Graham, Ian *(University of Toronto)*
Kim, Kang-Tae *(Pohang Institute of Science and Technology, Korea)*
Krantz, Steven G. *(Washington University, St. Louis)*

Bibliography

[1] *E. Andersen, Volume-preserving automorphisms of* $\mathbb{C}^n$, Complex Variables Theory Appl. ***14****(1990), 223-235.*

[2] *E. Andersen and L. Lempert, On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math. ***110*** *(1992) 371-388.*

[3] *J. B. Conway,* A Course in Operator Theory*, American Mathematical Society, Providence, RI, 2000.*

[4] *S. Dineen,* The Schwarz lemma*, The Clarendon Press, Oxford University Press, Osford, 1989.*

[5] *F. Forstneric and J.-P. Rosay, Approximation of biholomorphic mappings by automorphisms of* $\mathbb{C}^n$, Invent. Math. ***112****(1993), 323-349.*

[6] *T. Franzoni and E. Vesentini,* Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.*

[7] *I. Graham and G. Kohr,* Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker Inc., New York, 2003.*

[8] *L. F. Heath and T. J. Suffridge, Starlike, convex, close-to-convex, spirallike, and* Φ*-like maps in a commutative Banach algebra with identity,* Trans. Amer. Math. Soc. ***250*** *(1979), 195-212.*

[9] *L. Harris, Schwarz's lemma in normed linear spaces,* Proc. Nat. Acad. Sci. U.S.A. ***62*** *(1969), 1014-1017.*

[10] *L. Harris and J.-P. Vigué, A metric condition for equivalence of domains,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. *(8)****67*** *(1979), 402-403 (1980).*

[11] *K.-T. Kim and S. G. Krantz, Characterization of the Hilbert ball by its automorphism group,*Trans. Amer. Math. Soc. ***354*** *(2002), 2797-2818.*

[12] *K.-T. Kim and S. G. Krantz, Normal families of holomorphic functions and mappings on a Banach space,* Expo. Math. ***21****(2003), 193-218.*

[13] *S. G. Krantz,* Function Theory of Several Complex Variables*, American Mathematical Society-Chelsea, Providence, RI, 2001.*

[14] *L. Lempert, The Dolbeault complex in infinite dimensions,* J. Amer. Math. Soc. ***11****(1998), 485-520.*

[15] *P. Mercer, A general Hopf lemma and proper holomorphic mappings between convex domains in* $\mathbb{C}^n$*,* Proc. Amer. Math. Soc.***119****(1993), 573-578.*

[16] *J. Mujica,* Complex Analysis in Banach Spaces*, North-Holland, Amsterdam and New York, 1986.*

[17] *T. J. Suffridge, A holomorphic function having a discontinuous inverse*, Proc. Amer. Math. Soc. ***31*** *(1972), 629-630.*

[18] *H. Wu, Normal families of holomorphic mappings,* Acta Math. ***119*** *(1967), 193-223.*

Chapter 70

Geometry and Deformation Theory of Hyperbolic 3-manifolds (04rit057)

July 3–8, 2004

Organizer(s): *Richard Canary (University of Michigan), Jeffrey Brock (Brown University/University of Texas, Austin), Kenneth Bromberg (University of Utah), Yair Minsky (Yale University)*

Our small group convened to discuss, informally, current and new directions for research in Kleinian groups, in view of the tremendous progress that has occurred in recent years. Major old problems have been solved, and some powerful techniques have been introduced to the field (drilling theorems, model manifolds and curve complexes, and more recently end reductions), which provide new opportunities to obtain a deeper understanding of an intricate subject.

Topology of the Deformation Space

The recent solution of the Ending Lamination Conjecture by Brock-Canary-Minsky [16, 4] gives a complete classification of (finitely-generated) Kleinian groups, but it does not give a topological *description of the deformation space of a group because the invariants involved in the classification do not vary continuously, as discovered and explored by Anderson-Canary [1], Brock [3], and others.*

Bromberg [5] more recently showed that even some remaining optimistic conjectures about this topological structure were false, when he proved that the deformation space of punctured-torus groups is not locally-connected at its boundary.

We discussed at length some approaches to proving similar results for Bers slices*: these are special slices of the representation space where most of the discontinuity phenomena do not occur, and so there was some room for hope that the topology of these slices is tamer than the general case. However it is possible that Bromberg's approach can also prove non-local-connectivity in this case. It may also be that a computer-driven search can produce and verify examples of this.*

On the positive side, we discussed an ongoing project to encode the continuity properties of the ending invariants in a complete way. The machinery in Masur-Minsky [11] and Brock-Canary-Minsky [4] gives some tools for attempting this, but there is so far no single coherent description.

Another promising direction is the question of bumping *and* self-bumping *(of components of the deformation space) in the case of manifolds with compressible boundary. In the incompressible case this phenomenon has been well-studied by Anderson-Canary-McCullough [2], Holt [9], McMullen [13], Bromberg-Holt [6], Ito [10] and others. It is known for example that only finitely many components can meet at one point, but conversely arbitrarily complex finite bumping has been shown to occur. Generalizing this to the compressible case remains a challenge. The discontinuity of topological type of quotient manifold at bumping points can* *be more intricate: handles can switch sides in a compression body, for example. However we still expect that only finitely many components can meet at one point.*

Uniform Models and Quantitative Bounds

An important theme we explored was that of improving our quantitative understanding of hyperbolic 3-manifolds. A number of foundational theorems in the field are proved using "soft" methods, such as compactness of parameter spaces, and hence the bounds obtained are not even in principle computable. Here are some sample questions:

Thurston's bounded image theorem: *A self-map of Teichmüller space $\mathcal{T}(\partial M)$ associated to a 3-manifold M plays a central role in Thurston's geometrization theorem and in subsequent developments [17, 12]. The image of this self-map, in certain cases (M acylindrical and ∂M incompressible) is a bounded set. Nothing is known about the diameter of this image. It is open, in particular, whether a bound exists depending only on the genus of ∂M. We discussed on the one hand the possible construction of examples with arbitrarily large image diameter, and on the other hand we searched for a proof yielding constructive upper bounds. We constructed examples where the complexity of M goes to infinity (with fixed ∂M) while the image diameter remains bounded.*

Uniform models: *The solution of the Ending Lamination Conjecture includes a construction of a* bilipschitz model *for a given hyperbolic manifold, depending only on its topology and its ending invariants. The bilipschitz constants obtained in the proof are non-constructive, but at least in the surface-bundle case they depend only on the topology of the manifold. There is a fairly clear line of argument that we expect will yield similar topologically-dependent bounds in the incompressible-boundary case. However, the case of manifolds with compressible boundary (notably handlebodies) is considerably harder.*

Brock-Souto have made some progress on obtaining uniform models in the compressible case, as has Namazi (a student of Minsky). However in general this area is wide open and promises to involve some delicate topological questions.

Miscellaneous Topics

Infinitely-generated Kleinian groups: *This area remains quite open, except for various constructions of examples. What is a good general theory of such groups? Is there a reasonable setting in which one can establish global rigidity results? McMullen's rigidity theorem controls quasiconformal deformations given an upper bound on injectivity radius. Other infinitely-generated examples, suggested by Bromberg, exhibit* no *deformations of any sort.*

Projective structures and cone manifolds: *Complex projective structures on surfaces are closely related to hyperbolic cone manifold structures on 3-manifolds with boundaries. This relationship figures heavily in Bromberg's work on Bers' density conjecture. Is every complex projective structure induced from some cone manifold? A positive answer would perhaps give new geometric tools for understanding general (not just discrete) representations of surface groups in $PSL(2,\mathbb{C})$, in view of the theorem (Gallo-Kapovich-Marden [8]) that every non-elementary representation of a surface group into $SL(2,\mathbb{C})$ is the monodromy of some complex projective structure.*

Cannon-Thurston maps and local connectivity of limit sets: *The limit set of a finitely-generated Kleinian group is conjectured to be locally connected. This was proved for pseudo-Anosov fibre groups by Cannon-Thurston [7], for bounded-geometry surface groups by Minsky [15], and for punctured-torus groups by McMullen [14] (and there are additional generalizations). The general problem remains open, although it seems that the main tool, which worked in the previous cases, is now available – namely the model manifolds from the solution of the Ending Lamination Conjecture. We discussed some plausible approaches to carrying through a proof; some considerable difficulties remain, but this is a very interesting direction to pursue.*

List of Participants

Brock, Jeffrey *(Brown University/University of Texas, Austin)*
Bromberg, Kenneth *(University of Utah)*
Canary, Richard *(University of Michigan)*
Minsky, Yair *(Yale University)*

Bibliography

[1] *J. Anderson and R. Canary,* Algebraic limits of Kleinian groups which rearrange the pages of a book, *Invent. Math.* ***126*** *(1996), 205–214.*

[2] *J. Anderson, R. Canary, and D. McCullough,* The topology of deformation spaces of Kleinian groups, *Ann. of Math. (2)* ***152*** *(2000), no. 3, 693–741.*

[3] *J. Brock,* Boundaries of Teichmüller spaces and end-invariants for hyperbolic 3-manifolds, *Duke Math. J.* ***106*** *(2001), no. 3, 527–552.*

[4] *J. Brock, R. Canary, and Y. Minsky,* Classification of Kleinian surface groups II: the ending lamination conjecture, *in preparation.*

[5] *K. Bromberg,* The topology of the space of punctured torus groups, *in preparation.*

[6] *K. Bromberg and J. Holt,* Self-bumping of deformation spaces of hyperbolic 3-manifolds, *J. Differential Geom.* ***57*** *(2001), no. 1, 47–65.*

[7] *J. Cannon and W. Thurston,* Group invariant peano curves, *preprint, 1989.*

[8] *D. Gallo, M. Kapovich, and A. Marden,* The monodromy groups of Schwarzian equations on closed Riemann surfaces, *Ann. of Math. (2)* ***151*** *(2000), no. 2, 625–704.*

[9] *John Holt,* Multiple bumping of components of deformation spaces of hyperbolic 3-manifolds, *Amer. J. Math.* ***125*** *(2003), no. 4, 691–736.*

[10] *K. Ito,* Exotic projective structures and quasi-Fuchsian space, *Duke Math. J.* ***105*** *(2000), no. 2, 185–209.*

[11] *H. A. Masur and Y. Minsky,* Geometry of the complex of curves II: Hierarchical structure, *Geom. Funct. Anal.* ***10*** *(2000), 902–974.*

[12] *C. McMullen,* Riemann surfaces and the geometrization of 3-manifolds, *Bull. Amer. Math. Soc.* ***27*** *(1992), 207–216.*

[13] ______, Complex earthquakes and Teichmüller theory, *J. Amer. Math. Soc.* ***11*** *(1998), no. 2, 283–320.*

[14] *Curtis T. McMullen,* Local connectivity, Kleinian groups and geodesics on the blowup of the torus, *Invent. Math.* ***146*** *(2001), no. 1, 35–91.*

[15] *Y. Minsky,* On rigidity, limit sets and end invariants of hyperbolic 3-manifolds, *J. Amer. Math. Soc.* ***7*** *(1994), 539–588.*

[16] ______, Classification of Kleinian surface groups I: models and bounds, *preprint, 2002.*

[17] *W. Thurston,* Hyperbolic structures on 3-manifolds, I: deformation of acylindrical manifolds, *Ann. of Math.* ***124*** *(1986), 203–246.*

Chapter 71

Stability and Computations for Stochastic Delay-Differential Equations (04rit047)

July 24–August 7, 2004

Organizer(s): *Rachel Kuske (University of British Columbia)*

The goal of this research in teams was to bring together a group of researchers working on dynamics of stochastic delay differential equations (SDDE's) from different perspectives: theoretical aspects, nonlinear effects in stochastic dynamics, numerical methods, and mathematical modeling in applications. The goals of the group for the two weeks were to identify new directions in this research area and to begin preliminary work to determine productive directions to pursue. One motivating factor in this process is the increased use of models with memory in applications, resulting in a greater demand to have analysis and computations to understand the behavior of the solutions of these models.

The first direction was the weak convergence of numerical schemes for SDDE's. There have been a number of results for strong convergence, both for Euler-Maruyama methods and multi-step methods, but there are no proofs related to weak convergence. Weak convergence of the methods is important in applications since typically moments or the probability densities are of interest in understanding the behavior of the model. Also, the order of convergence should be better for weak schemes than for strong schemes. Difficulties identified for the proof include the fact that one can not rely on expressions for the generator, as one does for Markovian processes, so that a new approach is necessary. An approach was developed which borrowed from recent results for boundary value problems using Malliavin calculus, incorporated into a convenient formulation for the numerical error, thus making it possible to outline a method of proof which did not rely on the generator. Details in this outline are part of present research.

A second direction involved the development of stochastic models for metal cutting and the phenomenon of chatter. In these models, the position of the cutting tool is dependent on the previous part of the cut, so there is a delay in the forces affecting the tool dynamics. Models of these type are known to have oscillatory instabilities, which in this application is called chatter. The development of the model demonstrated that there is both additive and multiplicative noise, which results in resonance with the natural oscillations, thus amplifying the chatter. Here the goal is to determine how the operating parameter range is changed due to the noise sensitivity in the model.

A third direction identified was complementary empirical studies for the use of numerical methods for SDDE's. While convergence results for numerical methods are typically verified by solving test problems, they may not exhibit the same behavior as the problems observed in practice. For example, analysis for the metal cutting problem suggested several aspects that need to be considered: presence of both multiplication and additive noise, large delays, and oscillatory vs. exponential behavior.

In addition progress was made on some related projects: stochastic dynamics for problems with discon-

tinuous coefficients and models for pricing options on assets with delayed memory. Some of the same ideas are necessary for these problems, since they are infinite dimensional systems with noise sensitivity.

Finally the group outlined possible ways to continue their collaboration. A Trans-Cooperation Program research grant proposal for collaboration between UBC and Humboldt University is being written presently. A number of possibilities were outlined for future meetings. Follow-up work in the directions mentioned above is continuing at individual institutions.

This Research in Teams wishes to thank their hosts at BIRS for a productive two weeks.

List of Participants

Buckwar, Evelyn *(Humboldt University)*
Kuske, Rachel *(University of British Columbia)*
Mohammed, Salah *(South Illinois University, Carbondale)*
Shardlow, Tony *(University of Manchester)*

Chapter 72

Study of Affine Surfaces with Self-maps of Degree > 1 and the Jacobian Problem (04rit553)

July 24–August 14, 2004

Organizer(s): *Rajendra V. Gurjar (Tata Institute of Fundamental Research, India), Masayoshi Miyanishi (Osaka University), Kayo Masuda (Himeji Institute of Technology), Peter Russell (McGill University)*

The four members of our team met at BIRS from July 24 to August 14, 2004. We started with several questions related to our proposed project of study of (proper or étale) self-maps of smooth affine surfaces. It soon became clear that the study of different $\mathbf{A}^1$-fibrations and existence of affine lines on such a surface which are not fibre components of a given fibration is quite important for our purpose. Since our main aim is the study of affine rational surfaces without non-trivial regular invertible functions, the base of an $\mathbf{A}^1$-fibration for such a surface is either the affine line $\mathbf{A}^1$ or the projective line $\mathbf{P}^1$. We realised that the base of the fibration makes a lot of difference to the properties of our surfaces. We defined several classes of smooth affine rational surfaces, depending on how many different $\mathbf{A}^1$-fibrations the surface has and what the base of such a fibration is. The study of interrelations between these classes formed an important part of our work. In the description below, by a surface we mean a smooth complex, affine surface with no non-trivial regular invertible functions.

A surface X is called an ML_0-surface if X has at least two transverse $\mathbf{A}^1$-fibrations with base $\mathbf{A}^1$. It is called an ML_1-surface if it has exactly one such fibration with base $\mathbf{A}^1$ and an ML_2-surface if it has no $\mathbf{A}^1$-fibration over $\mathbf{A}^1$.

The main results which we have proved so far are as follows. The assumption about torsionness of the Picard group of X in some statements below is quite natural in our context since we intend to study surfaces which are as close to the affine plane $\mathbf{C}^2$ as possible, since the famous unsolved Jacobian Problem is always at the back of our mind.

1. (a) *Let X be an ML_0 surface with torsion Picard group. Then any affine line $C \subset X$ is a fiber of an $\mathbf{A}^1$-fibration over X with base $\mathbf{A}^1$.*

   (b) *If X is an ML_1 surface with torsion Picard group then any affine line contained in X is a fiber of the unique $\mathbf{A}^1$-fibration on X.*

   These results generalize the famous Abhyankar-Moh-Suzuki result about uniqueness of embedding of an affine line (upto automorphism of $\mathbf{C}^2$) in $\mathbf{C}^2$. The proof uses full force of the classification theory of non-complete surfaces.

   If the rank of the Picard group of X is > 0 then we found counterexamples to this result. The constructions use special types of surfaces.

2. *Let X be an ML_0 surface with torsion Picard group. If $f : X \to Y$ is either an étale finite or a finite Galois map onto the surface Y then Y is an ML_0 surface. If f is assumed to be only proper then Y is at least ML_1.*

   The question whether the result is true only assuming properness of f is a tantalizing one and is closely connected to our project of proper self-maps.

3. *We found examples of ML_0 and ML_1 surfaces X such that in some cases C^2 is an open subset of X and in some cases it is not. Our understanding of this phenomenon is far from being complete.*

4. *A somewhat unexpected result proved by us says that if X is an ML_0 surface such that the Picard group of X has rank > 0 then X always has an $\mathbf{A}^1$-fibration with base $\mathbf{P}^1$.*

   For the proof we had to study the set of all possible $\mathbf{A}^1$-fibrations on X with base $\mathbf{A}^1$ and an analysis of the ring generated by all the base parameters for these fibrations.

These are some of the main results we have proved during our stay at BIRS. We also raised some new and interesting questions, the answers to which will be quite important for the study of self-maps. Although the members of our team have known each other for a long time, collaborated in pairs, met each other on several occasions for short durations and followed each other's work closely, this is the first time we got together for intense discussions about problems of interests to all of us. The three week period has been mathematically very exhilarating for each of us and has given us much to think about when we go back to our respective places of work and to continue our collaboration by e-mail.

Acknowledgements: *We are very much thankful to BIRS for the generous invitation to spend three wonderful weeks in the beautiful Banff surroundings. Special thanks are due to Ms. Andrea Lundquist, Ms. Jackie Kler and all other staff of BIRS and Banff Center for their warm, friendly help. The buffets in the Cameron Hall were sumptuous and the nearby mountains, lakes and river very inviting for their exploration. All this certainly has had a very positive effect on our research work at Banff.*

List of Participants

Gurjar, Rajendra *(Tata Institute of Fundamental Research, India)*
Masuda, Kayo *(Himeji Institute of Technology)*
Miyanishi, Masayoshi *(Kwansei Gakuin University)*
Russell, Peter *(McGill University)*

Chapter 73

Competing Species, Predator-Prey Models and Measured-valued Diffusions (04rit050)

August 1–14, 2004

Organizer(s): *Richard Durrett (Cornell University), Leonid Mytnik (Technion, Israel), Ed Perkins (University of British Columbia)*

In [5] Evans and Perkins introduced a class of measure-valued branching diffusions which modelled two populations undergoing migration and near critical reproduction which compete for resources. The mathematically challenging part of the model was that competition only occurs when members of the two populations are zero distance apart. This produces a singular interaction involving the collision local time of the two processes. The processes were constructed as solutions of a singular martingale problem in dimensions 3 *or less and it was shown that solutions do not exist in higher dimensions. Uniqueness of solutions was established in one dimension and in some special cases–e.g. when only one population feels the competitive effect. In 1998 Durrett conjectured that in these special cases these processes should arise as the weak limit of a space-time rescaling of a model studied by Durrett and Levin [3] in which two contact processes interact through one having a linear competitive effect on the other. At about the same time Mytnik [7] proved uniqueness of solutions to the original model in the more interesting case of symmetric competition. Earlier Evans and Perkins [6] had shown uniqueness of solutions to an associated historical martingale problem. This is an enriched setting in which one keeps track of the genealogical histories of the interacting populations.*

In 1999, Durrett, Mytnik and Perkins began to work on a general project to show a more general class of competing and cooperating contact processes under rescaling will converge to a more general class of singularly interacting measure-valued diffusions in dimensions 3 *or less. The interactions included both competing species and predator-prey type models. We worked on this pairwise for many years and a lengthy manuscript began circulating between the three of us in the spring of 2003. There were some technical issues to work out still just to get tightness of the approximating systems and to identify the limit points as solutions of the natural martingale problems. A full uniqueness theorem was also missing. A two week meeting at BIRS was proposed as a venue where this project could be completed. It allowed all three of us to work on the problem simultaneously for the first time.*

The second week of the Research in Teams period coincided with the related 5-day workshop on "Stochastic processes in evolutionary and disease genetics" which was of great interest to all of us and to which Rick Durrett was already committed. We met daily during the first week. During the second week Rick Durrett attended the workshop lectures while Leonid Mytnik and myself attended about a third of the lectures. The three of us continued to meet daily. Early in the first week a serious error was found in our earlier work and correcting it took several days. It was very fortunate that we were all together for this period. We extended the known uniqueness results to some additional cases in the predator-prey setting including the general one-

dimensional setting and special cases in the higher dimensional setting. The full uniqueness theorem still remains an open problem although we know how to obtain a general convergence result in the competing species setting by using [6]. A fair amount of work was done on the write-up as well and a completed (as of two days ago) 74-page manuscript is now ready for submission [4].

Several other problems were also discussed including an ongoing project between Mytnik and myself on weak uniqueness for signed solutions of the one-dimensional stochastic PDE for super-Brownian motion.

Having the 5-day workshop overlap with our meeting worked out very well. First it was a superb meeting and a pleasant diversion from our technical problems. Secondly, Ted Cox was attending the meeting and Cox, Durrett and myself had some very fruitful discussions on another rescaling limit theorem for high density Lotka-Volterra models. The issue here was to obtain a limit theorem in the regime where both populations are of the same order of magnitude. In earlier work Cox and myself ([1], [2]) had established a limit theorem in the case where one population is rare and used it to get information on coexistence and survival in the regime where the populations prefer each other to their own type. We were interested in proving an analogous result for the other parameter regime and in particular in showing that the critical survival curve has a discontinuous derivative at the phase transition point where you go from preferring the other type to preferring your own type. Durrett's expertise on related rapidly stirring limits proved to be instrumental in laying out a game plan for getting information on the survival and co-existence questions in this other parameter regime. It involves first showing that in the regime when both populations are large, the appropriate limit is the solution of a particular nonlinear PDE. We believe this leads to a valid approach for establishing the above derivative discontinuity. We feel we only need a bit of time to flesh out the proofs. Perhaps we would be fortunate enough to secure another period at BIRS next year? It was an exciting development in a spectacular setting.

List of Participants

Durrett, Rick *(Cornell University)*
Mytnik, Leonid *(Technion, Israel)*
Perkins, Edwin *(University of British Columbia)*

Bibliography

[1] T. Cox and E. Perkins, Rescaled Lotka-Volterra models converge to super–Brownian motion, to appear Ann. Prob.*, 2003.*

[2] T. Cox and E.Perkins, Survival and coexistence in stochastic spatial Lotka-Volterra models, preprint, 2004.

[3] R. Durrett and S. Levin, Allelopathy in spatially distributed populations, J. Theor. Biol. ***185** (1997), 165–172.*

[4] R. Durrett, L. Mytnik, and E. Perkins, Competing super-Browian motions as limits of interacting particle systems, preprint, 2004.

*[5] S.N. Evans and E.A. Perkins, Measure-valued branching diffusions with singular interactions.*Canadian Journal of Mathematics*, **46(1)** (1994), 120–168.*

[6] S.N. Evans and E.A. Perkins, Collision local times, historical stochastic calculus, and competing super-processes. Electron. J. Probab.*, **3:No. 5** (1998), 120 pp.*

[7] L. Mytnik, Uniqueness for a competing species model. Canadian Journal of Mathematics*, **51(2)** (1999), 372–448.*

Chapter 74

Geometry and Analysis on Cauchy Riemann Manifolds (04rit554)

September 4–18, 2004

Organizer(s): *John Bland (University of Toronto), Tom Duchamp (University of Washington), Peter Garfield (Case Western Reserve University), Robert Hladky (Dartmouth College), Jack Lee (University of Washington)*

One of the famous open questions in several complex variables is the following:

Is every 5 dimensional strongly pseudoconvex CR manifold locally embeddable?

This is a remarkably subtle question relating geometry, complex analysis and partial differential equations. To shed some light on this question, we will briefly recall the definitions and known results.

Let M be a smooth oriented $2n+1$-manifold. An almost CR structure *(for Cauchy-Riemann) on M is a complex subbundle $H_{(1,0)}M$ of the complexified tangent bundle such that $H_{(0,1)}M := \overline{H_{(1,0)}M}$ is everywhere transverse to $H_{(1,0)}M$. The almost CR structure is a* CR structure *if in addition $H_{(0,1)}M$ is integrable as a complex subbundle of the complexified tangent bundle $T_{\mathbb{C}}M$.*

Let η be a real one-form annihilating $H_{(1,0)}M$; it is determined up to multiplication by a nonvanishing function. We choose the sign of η such that the orientation determined by η and the natural orientation for $H_{(1,0)}M$ agrees with the orientation for M. We define the Levi form *associated to η to be the Hermitian form on $H_{(1,0)}M$ determined by*

$$\mathcal{L}(Z,\overline{W}) = -id\eta(Z,\overline{W})$$

for all $Z, W \in H_{(1,0)}M$. If the Levi form is positive definite, then the CR structure is said to be strongly pseudoconvex. *This condition is independent of the choice of η (except for its sign).*

Remark 74.1 *Let U be a smoothly bounded strongly pseudoconvex open set in a complex manifold X. Then the complex structure from X restricts to ∂U as a strongly pseudoconvex CR structure; the Levi form defined here is the same as the usual Levi form defined in complex analysis.*

A CR function *on M is a function $f : M \to \mathbb{C}$ such that $\bar{Z}(f) = 0$ for every $\bar{Z} \in H_{(0,1)}M$. These are the* tangential Cauchy-Riemann equations. *We denote this $\bar{\partial}_b f = 0$.*

One of the most basic questions in the study of CR manifolds is the following:

Question 74.2 *Given a strongly pseudoconvex CR manifold $(M, H_{(1,0)}M)$, is it embeddable? That is, does there exist a smooth embedding $X : M \hookrightarrow \mathbb{C}^N$ such the the components of X are CR functions: $\bar{\partial}_b X = 0$.*

Notice that this is strictly a question in PDEs – the solvability of a linear PDE.

We now recall some well known results concerning the embeddability of strongly pseudoconvex manifolds. The first is a result by Boutet de Monvel.

Theorem 74.3 (Boutet de Monvel [Bouzz]) *Let M be a compact strongly pseudoconvex CR manifold of dimension $(2n+1) \geq 5$. Then M is embeddable.*

The idea of the proof is as follows. Associated to $\bar{\partial}_b$ is a natural subelliptic Laplacian $\Box_b$. Using standard techniques, one solves the $\bar{\partial}_b$ equations with weights in order to specify the differentials at a point, and the value at two points.

The situation is entirely different in three dimensions.

Example 74.4 (Rossi [R]) *There is a real analytic deformation of the standard structure on S^3 which is nonembeddable.*

The basic idea is that the mapping $\Phi : (z,w) \mapsto (z^2 + \epsilon\bar{w}^2/r^4, zw - \epsilon\bar{z}\bar{w}/r^4, w^2 + \epsilon\bar{z}^2/r^4)$ maps $\mathbf{C}^2 \setminus \{0\} \to \mathbf{C}^3$ as a $2:1$-cover of the quadric $XZ - Y^2 = \epsilon$. The induced CR structure on S^3 is strongly pseudoconvex, but for $\epsilon \neq 0$ all CR functions descend to the quotient, hence do not separate points.

In fact, the space of CR structures on S^3 near the standard one is locally a Hilbert space, and the subspace of embeddable structures is a Hilbert subspace of infinite dimension and codimension.

It is natural to ask the same question for local embeddability.

First, it is worth pointing out that for the local question, the answer is always positive in the real analytic case. This follows easily from Cauchy-Kowalewski. This already provides an indication that the question in the smooth case is likely to be either easy or delicate; the latter turns out to be the case.

Here we have the following results.

Theorem 74.5 (Kuranishi [Kur]) *Let M be a strongly pseudoconvex manifold (not necessarily compact) of dimension $2n+1 \geq 9$. Then M is locally embeddable.*

This result was extended to cover the seven dimensional case by Akahori, and simplified by Webster.

Theorem 74.6 (Akahori [Aka], Webster [Web1] [Web2]) *Let M be a strongly pseudoconvex manifold (not necessarily compact) of dimension $(2n+1) \geq 7$. Then M is locally embeddable.*

However, local embeddability fails dramatically in the 3 dimensional case. (See, for example, the example by Nirenberg [Nir]).

This leaves the following famous open question.

Question 74.7 *Does local embeddability hold for 5 - dimensional strongly pseudoconvex CR manifolds?*

This question is remarkably delicate, and resolving it is not simply a matter of obtaining better estimates. Indeed, on the infinitesimal level, there is no obstruction. On the other hand, we have the following example.

Example 74.8 (Nagel-Rosay, [NR]) *On S^5, we consider the one-form*

$$\omega := (\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1)/(|z_1|^2 + |z_2|^2)^2 .$$

This is $\bar{\partial}$-closed (Bochner-Martinelli), but

$$\omega \wedge dz^1 \wedge dz^2$$

is twice the volume form on the 3 spheres $[z_3 = c] \cap S^5$. In particular, it is not in the range of $\bar{\partial}_b$. While it is not smooth when $|z_3| = 1$ (that is, $z_1 = z_2 = 0$), it can be approximated by smooth forms, giving "approximate cohomology"; in particular, it eliminates the possibility of a homotopy formula.

Webster also identifies the special difficulty which arises in dimension 5. One approach to solving the $\bar{\partial}_b$-equations is to use the integral kernels of Henkin. In this case, the $\bar{\partial}_b$ equation is always solvable if there exists a homotopy formula of the form $\alpha = \bar{\partial}_b P\alpha + Q\bar{\partial}_b\alpha$. However, Webster shows that in the 5-dimensional case, the local formula becomes $\alpha = \bar{\partial}_b P\alpha + Q\bar{\partial}_b\alpha + H(\alpha)$; the final term H represents possible obstructions to embeddability. The subtlety of the problem is indicated by the fact that at the infinitesimal level, the obstruction identified by H vanishes for integrable structures.

One of the new ingredients to shed light on the analysis of the situation is the geometry which arises from the compact three dimensional case. This situation can be understood geometrically. If the structure is embeddable, then it is embeddable as the boundary of a convex set in $\mathbf{C}^{n+1}$*. By taking slices of the convex set with complex hyperplanes parallel to the tangent complex hyperplane at a point, we obtain a (singular) foliation of* M *by embeddable CR* 3*-spheres. On the other hand, one can choose a foliation of* M *by* 3*-spheres, and first try to normalize the CR structure on* M *in such a way that the* 3*-spheres are CR* 3*-spheres. Then, roughly speaking,* M *is embeddable if and only if all of the* 3*-spheres in the foliation are embeddable.*

This research problem and the approach outlined was the subject of the RIT.

List of Participants

Bland, John *(University of Toronto)*
Duchamp, Tom *(University of Washington)*
Garfield, Peter *(Case Western Reserve University)*
Hladky, Robert K. *(Dartmouth College)*
Lee, Jack *(University of Washington)*

Bibliography

[Aka] T. Akahori, A new approach to the local embedding of CR structures for $n \geq 4$, *Mem. Amer. Math. Soc.* **366**, *Providence, RI (1987).*

[B] J. Bland, Contact geometry and CR-structures on S^3, *Acta Math.* **172** *(1994), 1–49.*

[Bouzz] Boutet de Monvel, Integration des Equations de Cauchy–Riemann induites formelles, *Seminar Goulaouic–Lions–Schwartz,* **IX** *(1974-75) 1–13.*

[CL] J.-H. Cheng and J.M. Lee, A local slice theorem for 3-dimensional CR structures, *Amer. J. Math.* **117** *(1995), 1249–1298.*

[Kur] M. Kuranishi, Strongly pseudoconvex CR structures over small balls, I, II, III, *Ann. of Math.* **115** *(1982), 451–500;* **116** *(1982), 1–64;* **116** *(1982), 249–330.*

[Le] L. Lempert, On three dimensional Cauchy-Riemann manifolds, *JAMS* **5** *(1992), 1-50.*

[NR] A. Nagel and J.-P. Rosay, Approximate local solutions of $\bar{\partial}_b$, but nonexistence of a homotopy formula for $(0,1)$-forms on hypersurfaces in $\mathbb{C}^3$, *Duke Math. J.* **58** *(1989), 823–827.*

[Nir] L.Nirenberg, On a question of Hans Lewy, *Russian Math Surveys,* **29** *(1974), 251–262.*

[R] H. Rossi, Attaching analytic spaces to an analytic space along a pseudoconvex boundary, *Proceedings of the conference on complex manifolds (Berlin and Heidelberg and New York) (A. Aeppli, E. Calabi,R. Röhrl, ed.), Springer-Verlag, (1965), 242–256.*

[Web1] S. M. Webster, On the local solution of the tangential Cauchy Riemann equations, *Ann. Inst. H. Poincaré* **6** *(1989), 167–182.*

[Web2] S. M. Webster, On the proof of Kuranishi's embedding theorem, *Ann. Inst. H. Poincaré* **6** *(1989), 183-207.*

Chapter 75

Research on Stochastic Models for the Web Graph and Other Scalefree Networks (04rit060)

September 18–25, 2004

Organizer(s): *Anthony Bonato (Wilfrid Laurier University), Jeannette Janssen (Dalhousie University)*

The web graph has nodes representing web pages, and edges representing the links between pages. The web graph is a massive network possessing several billion nodes and edges, with new nodes and edges appearing and disappearing over time. The explosive growth of the web graph itself is mirrored by the recent rapid increase in research on its structural properties, stochastic models, and mining. The web graph is only one example of a massive self-organizing network; other examples are biological networks such as the protein-protein interaction network in a living cell. For a recent survey on the web graph and other massive networks, see [2].

The existing research on models for the web graph deals almost exclusively with finite graphs. However, in the natural sciences, models are often studied by taking the infinite limit. Limiting behaviour can clarify the similarities and differences between models, and show the consequences of the choices made in the model. The first study of limiting behaviour of web graph models was made by the authors in [3]. In that paper, a study was made of deterministic infinite graphs satisfying the locally e.c. adjacency property. As motivation, this adjacency property is satisfied with probability 1 *by limit graphs generated by the copying model of Kumar et al [7]. Subsequent work on limiting behaviour of the preferential attachment web graph models appeared in [6].*

A new copying model for massive networks, written $G(p, \rho, H)$*, was recently introduced by Bonato, Janssen [4], motivated by the copying model of Kumar et al. [7], the generalized copying graphs of Adler, Mitzenmacher [1], and partial duplication model for biological networks in Chung et al. [5]. The three parameters of the model* $G(p, \rho, H)$ *are a probability* $p \in (0, 1)$*, a monotone increasing* random link function $\rho : \mathbb{N} \to \mathbb{N}$*, and a fixed finite initial graph* H*. The new node* v_{t+1} *acquires its neighbours as follows. Choose an existing node* u *from* G_t *uniformly at random. For each neighbour* w *of* u*, independently add an edge from* v_{t+1} *to* w *with probability* p*. In addition, choose* $\rho(t)$*-many nodes from* $V(G_t)$ *uniformly at random, and add edges from* v_{t+1} *to each of these nodes.*

For our week spent at BIRS as part of the Research in Teams program, we investigated the limiting behaviour of the $G(p, \rho, H)$ *copying model. We studied the randomness of the limits for various choices of* ρ*, measured by the* n*-e.c. adjacency properties (the large the value of* n*, the more properties the limit shares with the infinite random graph). Our original approach to this problem involved the use of expectation, and convergence properties of infinite products. During the course of the week, however, it emerged that more advanced techniques from random graph theory were needed. We found that submartingales were the correct* *tool, and were able to prove a key theorem using the Kolmogorov-Doob inequality for submartingales.*

Overall, we found the BIRS environment very conducive to our research methodology, which mainly consists of extended discussions at a whiteboard. On a side note, we had originally planned to work each day from morning until dinner only, but we often found ourselves working into the late evening!

Now that the main theoretical work for our project is complete, we are finalizing a paper for submission to Random Structures and Algorithms*, which is one of the top journals on probabilistic methods in graph theory and combinatorics. Our work has lead to many additional interesting questions surrounding web graph models and their limiting behavior. Our next step is to study the degree distribution of the* $G(p, \rho, H)$*, which we think follows a power law. We also plan on generalizing our approach to other models for the web graph.*

List of Participants

Bonato, Anthony *(Wilfrid Laurier University)*
Janssen, Jeannette *(Dalhousie University)*

Bibliography

[1] A. Adler, M. Mitzenmacher, Towards compressing web graphs, In: Proceedings of the Data Compression Conference, *2001.*

[2] A. Bonato, A survey of models of the web graph, In: Proceedings of the Combinatorial and Algorithmic Aspects of Networking, *2004. In press.*

[3] A. Bonato, J. Janssen, Infinite limits of copying models of the web graph, Internet Mathematics ***1*** *(2004) 193-213.*

[4] A. Bonato, J. Janssen, Limits and power laws of models for the web graph and other networked information spaces, Proceedings of Combinatorial and Algorithmic Aspects of Networking, *2004. In press.*

[5] F. Chung, G. Dewey, D.J. Galas, L. Lu, Duplication models for biological networks, Journal of Computational Biology ***10*** *(2003) 677-688.*

[6] J. Kleinberg, R. Kleinberg, Isomorphism and embedding problems for infinite limits of scale-free graphs In: ACM-SIAM Symposium on Discrete Algorithms, *2005. In press.*

[7] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins, E. Upfal, Stochastic models for the web graph, In: Proceedings of the 41th IEEE Symp. on Foundations of Computer Science, *57-65, 2000.*

Summer School Reports

Chapter 76

MITACS–MSRI–PIMS Special Program on Infectious Diseases, Summer School & Workshop (04ss101 & 04ss100)

June 19–July 2, 2004

Organizer(s): *Fred Brauer (University of British Columbia), Mark Lewis (University of Alberta), Pauline van den Driessche (University of Victoria), James Watmough (University of New Brunswick), Jianhong Wu (York University), Ping Yan (Health Canada)*

The objectives of this special program were to continue the success of the MITACS-PIMS Health Canada Meeting on SARS (held in BIRS, Banff, September 6–7, 2003) in furthering the fruitful interplay among mathematical, statistical and epidemiological sciences and to provide effective training for graduate students and junior researchers on collaborative research and the mathematical modeling and qualitative analysis of infectious diseases.

The program, organized by the MITACS project "Transmission Dynamics and Spatial Spread of Infectious Diseases: Modeling, Prediction and Control", consisted of a Summer School (for graduate students and beginning postdoctoral fellows) June 19–27 followed by a Research Workshop June 28–July 2, 2004.

Admission to the Summer School was competitive since the maximum capacity of the BIRS lecture room was 43. All of the 43 students admitted attended the Summer School: 10 from USA, 32 from Canada and one from China as part of the MITACS exchange program with Chinese Ministry of Higher Education. We were very pleased to have 25 female students for the school. This is a very encouraging sign in our effort to build the interaction of mathematics and epidemiology. Although, several students came from the medical sciences (Health Canada and St. Michael Hospital, for example), most were from graduate programs in mathematical and statistical sciences since the required level of mathematical and statistical background was high. (However, see the comments below about a future course targeted at students and research scientists in the medical community.)

Dr. Yicang Zhou, a MITACS-Chinese Ministry of High Education exchange scholar, attended the school as an observer and took part in the workshop in order to obtain some ideas on organizing such a school at home.

The Summer School was taught by 11 leading and active researchers from Canada and USA. Most instructors were at BIRS during the entire Summer School, enhancing the direct interactions of all students with first-rate scientists in the field.

The Summer School lectures covered a wide range of topics central to the mathematical and statistical modeling of infectious diseases: topics ranged from deterministic and stochastic models to parameter identification, reporting delay adjustments and incubation period estimation; from general theory to specific case studies (of West Nile virus, childhood disease, and of spatial spreading of rabies, for example); from homogeneous mixing to spatial structures (global transportation, spatial dispersal), social networking, age *structures and super-spreading events (of SARS). We also organized a computer tutorial and problem drop-in session.*

The organizers and instructors believe that there is no single textbook or research monograph covering the wide range of topics included in the Summer School. Putting this material together as a book would be a valuable contribution if it could be done relatively quickly. Feedback from students also encourages such an effort.

Group projects were an important part of the Summer School. Students were divided into eight teams working on one of five projects: HIV/AIDS, SARS, Cholera, TB, Malaria. Since students ranged from recent PhD.'s to recent BA's, we made an effort to have each team contain a mixture of experienced and less experienced students and a mixture of students from mathematical, statistical and medical backgrounds. Every afternoon, all instructors made themselves available for advice and help on the projects. The students blended well, and everyone seemed to have benefited significantly from their participation. The group presentations were extremely impressive, and all teams managed very well (within five days) to put together their proposed models, some qualitative analysis, computer simulations, epidemiological background and applications. Two of the teams were later invited to give their presentations in the Workshop and one team (working on modeling Cuba's HIV/AIDS dynamics) had such a wonderful project complete that they were invited on the spot by Dr. Ping Yan to give a formal presentation at a Health Canada meeting. They were also encouraged to submit their work for a journal publication.

Despite the hard work and long hours spent on the projects, participants to the Summer School still found time for group recreational activities including hiking, basketball, volleyball, soccer, and water basketball and excursions to downtown Banff.

Before the Summer School, a list of general references was posted in the Special Program's web page. We also managed to have copies of some standard reference books in the reading room of BIRS during the entire Summer School. These, together with the electronic access to many research papers, provided a temporary library at BIRS.

The Workshop attracted participants from medical schools (UCLA, UC Berkeley/Yale), health research centers (USA CDC, UBC CDC, National Microbiology Laboratory, the National Research Council, Cadham Provincial Laboratory) and Health Canada and mathematical modelers from across Canada and worldwide. This provided a wonderful forum for intensive discussions on how mathematical modeling and analysis are and should be directly related to informed public health policy. Eleven relatively senior students from the Summer School were also invited to participate in the workshop.

During the Workshop, there were invited lectures on various issues including modeling and assessing control strategies and intervention measures, stochastic aspects of disease transmission, sensitivity and uncertainty analysis, vaccination planning and immunity control, drug resistance, social networking, global transportation and the analysis of the National Microbiology Laboratory SARS database. There were also organized discussion sessions on the current status and future directions of mathematics in epidemiology. These discussions focused on how mathematics can make significant contributions to public health policy and how to enhance communication between modelers and epidemiologists. There were also long discussions on how the MITACS team could sustain its current productivity and further its outreach and collaboration with epidemiologists and researchers in public health policy.

The MITACS team members held a joint informal meeting with participants from Canadian health research organizations. It was agreed that we should work closely to make a plan for a similar program targeted at the medical community and graduate students in medical schools.

During the workshop, Fred Brauer and Jianhong Wu were interviewed by Canwest and four newspapers: the Calgary Herald, the Edmonton Journal, the Ottawa Citizen and the Banff Crag & Canyon. Shortly after the Program, Troy Day (Queen's University) was also interviewed on QR77 Radio (Calgary). We believe the media coverage provided some healthy information to the general public why and how mathematical modeling can assist in the prediction and control of infectious diseases.

In summary, we believe

- *The investment by the three funding institutes and BIRS on the Summer School is appreciated by all students and helps Canada build its national capacity for interdisciplinary research in infectious diseases;*
- *The Workshop, with participants from the applied mathematics community, medical schools and health*

research institutes, provided a timely and much needed forum to ensure the current effort in mathematical modeling be directed by the issues important for public health;

- *There has been an increasing demand for the study of infectious diseases to become a predictive science, and there has been a growing need for mathematical modeling and analysis developed on a solid epidemiological and biological foundation. The Special Program represents a welcomed response to the call for closer collaboration between mathematical modelers and epidemiologists for the prediction, prevention and control of infectious diseases. There should be more such special programs.*

List of Summer School Participants (04ss101)

Allen, Linda *(Texas Tech University)*
Arino, Julien *(McMaster University)*
Azer, Nancy *(University of Victoria)*
Bauch, Chris *(University of Guelph)*
Brauer, Fred *(University of British Columbia)*
Castillo-Chavez, Carlos *(Arizona State University)*
Chang, Shan *(University of Victoria)*
Chen, Wei *(Université de Montréal)*
Choo, Karen *(Queen's University)*
Corbett, Brian *(University of Manitoba)*
Davis, Brad *(University of British Columbia)*
Deering, Kathleen
Dueck, Stephen *(University of Manitoba)*
Earn, David *(McMaster University)*
Fan, Guihong *(McMaster University)*
Feng, Runhuan *(Concordia University)*
Heffernan, Jane *(University of Western Ontario)*
Hill, Jessica *(University of British Columbia)*
Hsu, Chia-Yu *(Washington State)*
Jiang, Li *(University of Victoria)*
Klepac, Petra *(Woods Hole Oceanographic Institute)*
Larochelle, Francois *(Carleton University)*
Lewis, Mark *(University of Alberta)*
Lim, Gillian *(University of Toronto)*
Liu, Maoxing *(North China, University of Science and Technology)*
Liu, Rongsong *(York University)*
Ma, Brian *(Simon Fraser University)*
MacKinnon, Ryan *(Carleton University)*
Mapes, Eric *(Washington State)*
Michael, James *(Duke University)*
Mubayi, Anuj *(Arizona State University)*
Nettel-Aguirre, Alberto *(Unviersity of Calgary)*
Ortega, Omayra *(University of Iowa)*
Pereira, Audrey *(University of Guelph)*
Rapatski, Brandy *(University of Maryland)*
Reluga, Timothy *(University of Washington)*
Richards, Stephanie *(University of Manitoba)*
Salmani, Mahin *(University of Victoria)*
Shim, Alicia *(University of British Columbia)*
Teng, Xia *(University of New Brunswick)*
Wagner, Brad *(McMaster University)*
Wang, Lin *(McMaster University)*
Watmough, James *(University of New Brunswick)*

Weiss, Leda Ivic *(York University)*
Williams, Paul *(University of Toronto)*
Wu, Jianhong *(York University)*
Yan, Ping *(Health Canada)*
Zeng, Qingling *(York University)*
Zhang, Fang *(Memorial University of Newfoundland)*
Zhao, Yan *(Vanderbilt University)*
Zhou, Yicang *(York University)*
Zhu, Jiaping *(University of Victoria)*
van den Driessche, Pauline *(University of Victoria)*

List of Workshop Participants (04ss100)

Allen, Linda J. S. *(Texas Tech University)*
Arino, Julien *(McMaster University)*
Bauch, Chris *(University of Guelph)*
Blower, Sally *(UCLA School of Medicine)*
Bowman, Christopher *(National Research Council, Government of Canada)*
Brauer, Fred *(University of British Columbia)*
Choo, Karen *(Queens University)*
Chowell, Gerardo *(Cornell University)*
Cuff, Wilfred *(Health Canada)*
Day, Troy *(Canada Research Chair in Mathematical Biology)*
Dueck, Stephen *(University of Manitoba)*
Earn, David *(McMaster University)*
Galvani, Alison *(University of California, Berkeley)*
Glasser, John *(Centers for Disease Control and Prevention)*
Gumel, Abba *(Institute of Industrial Mathematical Sciences)*
Hsieh, Ying-Hen *(National Chung Hsing University)*
Jolly, Ann *(Health Canada)*
Li, Michael *(University of Alberta)*
Liu, Maoxing *(North China University of Science and Technology)*
Liu, Rongsong *(York University)*
McLeod, Robert *(University of Manitoba)*
Park, Andrew *(York University)*
Pourbohloul, Babak *(UBC Centre for Disease Control)*
Rapatski, Brandy *(University of Maryland)*
Reluga, Timothy *(University of Washington)*
Sahai, Beni *(Cadham Provincial Laboratory)*
Shim, Alicia *(University of British Columbia)*
Trummer, Manfred *(Simon Fraser University)*
Wang, Lin *(McMaster University)*
Watmough, James *(University of New Brunswick)*
Weiss, Leda *(University of Toronto)*
Williams, Paul *(University of Toronto)*
Wu, Jianhong *(York University)*
Yan, Ping *(Health Canada)*
Zeng, Qingling *(York University)*
Zhou, Yicang *(York University)*
Zhu, Huaiping *(York University)*
van den Driessche, Pauline *(University of Victoria)*