Banff International Research Station Proceedings 2003

PIMS

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Foreword

Almost 2000 researchers from over 400 institutions in 30 countries participated in the 68 programmes that ran at the Banff International Research Station in its very first year of operations (2003). These programmes covered almost every aspect of the mathematical sciences and their applications, and this booklet contains summaries of the proceedings from these activities: a dazzling display of the many facets and the intellectual reach of the mathematical sciences of the 21st century.

The ultimate success of BIRS will no doubt play out in the intangible synergies and life-long interactions of the mathematical scientists who had and will have the opportunity to participate in its events. But there are a number of more concrete steps that the BIRS scientific management had decided to pursue from the onset, so as to amplify the impact of the station.

One of them was to require each workshop to provide, after its completion, a 15 page document surveying the state of the art for that subject, for inclusion in a permanent collection of annual BIRS Scientific Reports. These volumes are to give a permanent documentation of what has been done at BIRS, so as to make them available to the world scientific community.

We are very pleased to see that most organizers –95% of them– delivered on their promise. They spent the time and made the effort required to help the station in its task to make these proceedings accessible to the community at large. This book is the fruition of their efforts and we are deeply grateful to them.

The format varied from one presentation to another, but every one of these articles is extremely useful in its own way. An attempt at uniformizing the format will be done for next year's proceedings, and the diversity of this year's contributions will surely help us find the optimal one to adopt for future proceedings.

I hope that you will enjoy reading them as much as I did. They provide a nice glimpse at the state of our art at the beginning of this 3rd millennium.

Nassif Ghoussoub
BIRS Scientific Director

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Five-day Workshop Reports

Chapter 1

Developments in Superstring Theory (03w5024)

March 15–20, 2003

Organizer(s): Jim Bryan (University of British Columbia), Steve Giddings (UC Santa Barbara), Andreas Karch (University of Washington), Moshe Rozali (University of British Columbia), Gordon Semenoff (University of British Columbia), Mark Van Raamsdonk (University of British Columbia)

At present, there are a number of important facets of string theory in which physics is related to deeper mathematics. These include the relationship between Yang-Mills theory and gravity (the AdS/CFT correspondence), work on matrix models and supersymmetric gauge theories, mirror symmetry and classification of string compactifications through techniques such as the derived category, topological string theory, and string compactifications with fluxes, with their relationship to cosmology. The workshop provided an opportunity to survey some developments on these subjects, and to catalyze further work in these directions. The following is a summary of talks given at the workshop.

Hirosi Ooguri gave a talk about the large N duality between topological open string theory and closed string theory. First he reviewed the conjecture by Gopakumar and Vafa about the equivalence between the Chern-Simons gauge theory and the closed string theory near the conifold singularity, and then he presented a field theoretical proof of this conjecture based on his work with Vafa. He discussed how this is related to the correspondence between the matrix models and the N=1 gauge theories in four dimensions. This involves explaining how to use topological string to compute F-terms in low energy effective theories of string compactifications, which was pointed out by Bershadsky, Cecotti, Ooguri, and Vafa. He also discussed meaning of non-planar diagrams of the matrix models as computing effects of the certain deformation of the corresponding gauge theories.

Michael Douglas of Rutgers University gave a review of recent work which shows that the exact superpotential of certain supersymmetric gauge field theories is computed exactly by a certain matrix integral. This is an interesting recent development which has many interesting implications in both mathematical and physical issues surrounding supersymmetric theories and has inspired a large amount of work on matrix models and their relationship with hierarchies of differential equations and integrable models.

Douglas also reviewed recent work with N. Seiberg and E. Witten on anomalies and the chiral ring structure in a supersymmetric U(N) gauge theory. They considered theories with an adjoint chiral superfield and an arbitrary superpotential. He argued that a certain generalization of the Konishi anomaly leads to an equation which is identical to the loop equation of a bosonic matrix model. He argued that this allows one to solve for the expectation values of the chiral operators as functions of a finite number of "integration constants." He showed that this fact can be used to derive the Dijkgraaf-Vafa relation of the effective superpotential to a matrix model. Some of the results are applicable to more general theories. For example, they can determine the classical relations and quantum deformations of the chiral ring of $\mathcal{N} = 1$ super Yang-Mills theory with SU(N) gauge group, showing, as one consequence, that all supersymmetric vacua have a nonzero chiral condensate.

Washington Taylor talked about some recent work on string field theory, in two directions:

a) Performing loop computations in bosonic string field theory, particularly the one-loop open string tadpole calculation, which was work recently published with Ellwood and Shelton. This result shows that the open string tadpole can see the long range gravitational effects of a D-brane usually associated with the closed string tadpole, but confirms the expected picture that the quantum bosonic theory is problematic due to the closed string tachyon; his talk at Banff led to several useful discussions which helped improve the final understanding of the results in the published paper, which appeared a month or so after the meeting.

b) Computing the abelian and nonabelian vector field theories on a D-brane by explicitly integrating out the massive fields in open string field theory. This computation gives a systematic approach to deriving the abelian and nonabelian Born-Infeld actions as well as derivative corrections, and exemplifies the complications of a background-independent theory, which necessitates a complicated field redefinition to get the variables natural for a particular background. Again, several useful discussions at Banff improved their understanding of these results, which were recently published with Coletti and Sigalov.

Paul Aspinwall spoke on mirror symmetry, which forms a relationship between a pair of Calabi-Yau manifolds which appear to be unrelated classically. Originally this relationship was forged by considering closed strings propagating on these target spaces. More recently it has been realized that open strings, and consequently D-branes, give more detailed information of the nature of the mirror relationship. In particular, it demonstrates an association to the derived category conjectured by Kontsevich.

His talk focused on a notion of D-branes stability originally proposed by Douglas, Fiol and Romelsberger. This leads naturally to the idea of a triangulated category which manifests itself in a very different way for the two mirror Calabi-Yau manifolds. Many details of this picture are currently poorly understood.

Ezra Getzler discussed a deformation of the Toda chain, which is called the equivariant Toda lattice, described by a pair of Lax equations, which by the calculations of Okounkov and Pandharipande describes the integrable system associated to the equivariant Gromov-Witten theory of CP^1. This implies (and clarifies) the Toda conjecture of Eguchi, Hori and Yang, which is the non-equivariant limit.

Sheldon Katz's talk on "Geometric Transitions and D-branes" discussed the N=1 field theories arising from wrapping D-branes on the vanishing holomorphic cycles in an extremal transition, joint work with Cachazo and Vafa. A superpotential is generated classically. The vacuum moduli space was identified with the geometric moduli space of appropriate coherent sheaves. Other more recent developments were covered as well.

These results have been combined with large N duality to describe the dynamics of gaugino bilinears, which have since also been described via matrix model techniques.

Anton Kapustin gave a talk on his work on topological A-branes which live on coisotropic submanifolds. He provided string theory arguments which show that A-branes are not necessarily Lagrangian submanifolds in the Calabi-Yau: more general coisotropic branes are also allowed, if the line bundle on the brane is not flat. He showed that a coisotropic A-brane has a natural structure of a foliated manifold with a transverse holomorphic structure. And he argued that the Fukaya category must be enlarged with such objects for the Homological Mirror Symmetry conjecture to be true.

Tuesday's focus was on connections between compactifications and cosmology. There has been a growing consensus that recent cosmological data points to the prevalence of a positive vacuum energy, or cosmological constant. If string theory describes nature, we must find ways to understand this cosmological constant, and physics in spaces with a cosmological constant, within the framework of the theory.

Shamit Kachru reported on recent constructions of string theory vacua with positive cosmological constant. Although aspects of these compactifications are under investigation, they appear to represent first examples demonstrating existence of de Sitter vacua in string theory. The models are constructed from the starting point of orientifold, or more generally F-theory, compactifications with explicit three branes and fluxes present. The fluxes dynamically fix the complex structure moduli of the compactification. In cases where there is a single Kahler modulus, this is generally then fixed by non-perturbative corrections. This typically results in an anti-de Sitter vacuum, but he summarized recent work with Kallosh, Linde, and Trivedi pointing out that adding an anti-D3 brane could lift the vacuum energy to a positive value.

The resulting de Sitter vacuum is, however, only metastable, and will ultimately decay in a decompactification transition. **Steve Giddings** summarized recent work on a very general result of this nature. In particular if 1) there is a positive vacuum energy, as we've apparently now observed in nature and 2) there are extra compact dimensions, then the general result is that the present vacuum state of the Universe is unstable, typically to some form of decompactification transition. However, the lifetime for such a transition is expected to be extremely long as compared to the present age of the Universe.

Eva Silverstein reported on recent work focused in part on the question of understanding de Sitter entropy in the context of D3 branes. In particular, she discussed the entropy of concrete de Sitter flux compactifications and deformations of them containing D-brane domain walls. She summarized the relevant causal and thermodynamic properties of these "D-Sitter" deformations of de Sitter spacetimes, and discussed a string scale correspondence point at which the entropy localized on the D-branes (and measured by probes sent from an observer in the middle of the bubble) scales the same with large flux quantum numbers as the entropy of the original de Sitter space, and at which Bousso's bound is saturated by the D-brane degrees of freedom (up to order one coefficients) for an infinite range of times. From the geometry of a static patch of D-Sitter space and from basic relations in flux compactifications, her work with Fabinger finds support for the possibility of a low energy open string description of the static patch of de Sitter space.

Wednesday included a talk by **Sergei Gukov**, which was based on recent work where he studied three-dimensional Chern-Simons gauge theory with complex gauge group. His main focus was SL(2,C) Chern-Simons theory, which has many interesting connections with three-dimensional quantum gravity and geometry of hyperbolic three-manifolds. In the talk he explained that, in the presence of a single knotted Wilson loop in an infinite-dimensional representation of the gauge group, the classical and quantum properties of such theory are described by an algebraic curve called the A-polynomial of a knot. In particular, quantization of the theory can be formulated in terms of the (Euclidean) effective quantum mechanics on a non-commutative torus, so that the partition function of SL(2,C) Chern-Simons theory in the stationary phase approximation is given by the semi-classical wave function supported on the zero locus of the A-polynomial.

Using this approach, one can obtain some new and rather surprising relations between the A-polynomial, the coloured Jones polynomial, and other invariants of hyperbolic 3-manifolds. These relations generalize the volume conjecture and the Melvin-Morton-Rozansky conjecture, and suggest an intriguing connection between the SL(2,C) partition function and the coloured Jones polynomial.

Eric Sharpe also discussed modelling D-branes with sheaves, and in particular, a powerful computational method that arises when thinking about D-branes in such a mathematical framework. Specifically, he showed how to see directly in BCFT that open string spectra between D-branes corresponding to sheaves are counted by Ext groups. This result has been assumed by various authors for a number of years, and has even been checked in massive theories believed to flow in the IR to the BCFT's in question, but has never previously been understood directly in BCFT. Verifying in BCFT that open string spectra are calculated by Ext groups turns out to involve interesting physical realizations of spectral sequences. He also briefly discussed some permutations of this problem, namely D-branes in flat B fields (where one has a twisted gauge bundle, instead of an honest gauge bundle), and also D-branes in orbifolds, and showed how one sees Ext groups in each of these contexts also.

Thursday began with **Andreas Karch**, who spoke on large N dualities, which have played an important role both in mathematics and physics. In the physical context so far the only regime that is understood is the one of strong coupling on the field theory side (the open string side) which is related to the supergravity limit of the closed string theory. To go beyond this, a fist step is to investigate the limit where the field theory is well understood: the free field theory. Actually one can use the open string side as the definition of the theory and see to what extend it can be rewritten as a closed string theory. He demonstrated that the a free scalar field in light cone frame can be rewritten to become a closed string theory on AdS in the zero radius limit. The surprise is that the closed string worldsheet has to be discretized.

Finally, **Shiraz Minwalla** of Harvard University discussed the some features of the thermodynamic phase diagram of $\mathcal{N}=4$ supersymmetric Yang-Mills theory and its string theory dual, type IIB superstring theory.

The workshop provided a first-class atmosphere for interaction between speakers and for research, and was well received.

Work performed at the conference included Douglas finishing the paper "The statistics of string/M theory vacua," JHEP 0305 (2003) 046, which discusses a new approach to the study of superstring compactification. He had many useful conversations on this with participants at the workshop.

There were a number of other positive comments. For example, H. Ooguri commented "It was a great conference. I learnt a lot by talking to participants, both physicists and mathematicians. The facility at

Banff is very nice and the location is wonderful. I hope there will be more conferences like this. Thank you for organizing it.” W. Taylor commented ”On the whole, I found the Banff workshop to be one of the best physics environments I had attended in some time. Everything was well-organized, the facilities were very good, and the number and quality of people at the workshop was, in my opinion, close to optimal. It was a great workshop, and I hope it happens again!”

In closing the organizers would like to thank the staff at Banff for the excellent facilities and support.

List of Participants

Aspinwall, Paul (Duke University)
Brecher, Dominic (University of British Columbia)
Bryan, Jim (University of British Columbia)
Donagi, Ron (University of Pennsylvania)
Douglas, Michael (Rutgers University)
Furuuchi, Kazuyuki (University of British Columbia)
Getzler, E. (Northwestern University)
Giddings, Steve (University of California, Santa Barbara)
Gukov, Sergei (Harvard University)
Hellerman, Simeon (Stanford University)
Hori, Kentaro (University of Toronto)
Itzhaki, Sunny (Princeton University)
Kachru, Shamit (Stanford University)
Kapustin, Anton (California Institute of Technology)
Karch, Andreas (University of Washington)
Katz, Sheldon (University of Illinois, Urbana-Champagne)
Kutasov, David (University of Chicago)
Liu, Hong (Rutgers University)
Minwalla, Shiraz (Harvard University)
Morrison, Dave (Duke University)
Myers, Robert (Perimeter Institute)
Ooguri, Hirosi (California Institute of Technology)
Page, David (University of Toronto)
Peet, Amanda (University of Toronto)
Randall, Lisa (Harvard University)
Rozali, Moshe (University of British Columbia)
Schleich, Kristin (University of British Columbia)
Schreiber, Ehud (University of British Columbia)
Semenoff, Gordon (University of British Columbia)
Sethi, Savdeep (University of Chicago)
Sharpe, Eric (University of Illinois, Urbana-Champaign)
Silverstein, Eva (Stanford University)
Taylor, Washington (Massachusetts Institute of Technology)
Van Raamsdonk, Mark (University of British Columbia)
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Zaslow, Eric (Northwestern University)

Chapter 2

Scattering and Inverse Scattering (03w5037)

March 23–27, 2003

Organizer(s): Richard Froese (University of British Columbia), Gunther Uhlmann (University of Washington)

In the fields of scattering and inverse scattering theory techniques of microlocal analysis, including the use of Eikonal equations and of complex geometrical optics solutions to Schrödinger and other equations, has led to substantial progress in recent years. The purpose of the workshop was be to bring together people working on different aspects of these fields, to appraise the current status of development and to encourage interaction between mathematicians and scientists and engineers working directly with the applications of scattering and inverse scattering.

Despite close mathematical connections between the fields of scattering and inverse scattering, there has not always been a strong interaction between these fields. Part of the rationale of this workshop was to bring together workers who might not ordinarily interact, but could benefit from sharing ideas.

Basic ideas in direct and inverse scattering

Scattering theory, in the time dependent formulation, is the study of the long time behaviour of solutions of an evolution equation that move out to infinity. The evolution equation might be the time dependent Schrödinger equation, (quantum scattering), the scalar wave equation (acoustical scattering), Maxwell's equations (electromagnetic scattering) or even a non-linear evolution equation. The underlying space might be Euclidean space or a Riemannian manifold. In each problem there is a localized scattering target. Moving in space away from the target to infinity, the equations, or the geometry, become simpler. The idea is that in the distant past and in the far future, the scattered wave will be located in the region where the equation or geometry is simple. It then becomes possible to compare distant past input to the far future output.

Inverse scattering is how we obtain a large part of our information about the world. An everyday example is human vision: from the measurements of scattered light that reaches our retinas, our brains construct a detailed three-dimensional map of the world around us. Dolphins and bats are able to do the same thing from listening to scattered sound waves.

We know about the interior structure of the Earth by solving the inverse problem of determining the sound speed by measuring travel times of seismic waves, the structure of DNA by solving inverse X-ray diffraction problems and the structure of the atom and its constituents from studying the scattering when materials are bombarded with particles.

Medical imaging uses scattering of X-rays, ultrasound waves and electromagnetic waves to make images of the human body which is of invaluable help with medical diagnosis. The oil exploration industry uses the reflection of seismic waves in oil prospecting. Inverse scattering is used in non destructive evaluation of materials to find cracks and corrosions.

In summary the task of direct scattering theory is to determine the relation between the input and output waves, given the details about the scattering target. The task of inverse scattering theory is to determine properties of the target, given sufficiently many input output pairs.

In fact, most often scattering problems are stated in a time independent formulation that results after taking a Fourier transform in the time variable. Although the immediate connection to the original scattering experiments is obscured, it can be easier to state and study scattering problems in the this formulation. Here is the time independent formulation of the obstacle scattering problem, which is relevant to applications in radar and sonar.

The underlying space is $\mathbb{R}^n$ and the obstacle $\Omega \subset \mathbb{R}^n$ is a closed bounded set. We consider solutions to the Helmholz equation

$$\Delta u(x) + k^2 u(x) = 0 \quad \text{for } x \in \mathbb{R}^n \backslash \Omega$$

with either Dirichlet (for a soft obstacle) or Neumann (for a hard obstacle) boundary conditions on $\partial\Omega$. We look for a family of solutions, indexed by ω in the sphere S^{n-1} with the following asymptotic behaviour:

$$u(x;\omega,k) = u(r\xi;\omega,k) \sim e^{ik\langle\omega,x\rangle} + r^{-(n-1)/2} e^{ikr} A(\xi,\omega,k)$$

as $r \to \infty$. Here $r = |x|$ and $\xi = x/|x|$, and $\sim$ denotes equality up to terms of order $r^{-(n+1)/2}$.

For a given obstacle, these equations uniquely determine the function $A(\xi,\omega,k)$, called the *scattering amplitude* or the *far field pattern*..

The *scattering operator* can be defined in the following way. If we consider any solution $u(x) = u(r\xi)$ of the Helmholz equation with asymptotic expansion for large r of the form

$$u(r\xi) \sim r^{(n-1)/2} e^{ikr} a(\xi) + r^{(n-1)/2} e^{-ikr} b(\xi)$$

it turns out that the coefficients $a(\xi)$ and $b(\xi)$ are not independent. Specifying either a or b determines the solution u uniquely and thus the other coefficient. The scattering operator $S(k)$, acting on functions (or distributions) on the sphere describes the relationship between a and b via

$$a = S(k)b$$

To find the relationship between the scattering amplitude and the scattering operator, think of $e^{ik\langle\omega,x\rangle} = e^{ikr\langle\omega,\xi\rangle}$, for fixed ω and k as a family of distributions on the the sphere. The asymptotics of these distributions as $r \to \infty$ are obtained by a stationary phase calculation. As a function of $\xi \in S^{n-1}$ the phase $k\langle\omega,\xi\rangle$ has two critical points, one where $\xi = \omega$ and the other when $\xi = -\omega$. This results in

$$\int_{S^{n-1}} e^{ikr\langle\omega,\xi\rangle} \varphi(\xi) ds(\xi) \sim \left(\frac{2\pi}{r}\right)^{(n-1)/2} \left\{ e^{-i\pi(n-1)/4} e^{ikr} \varphi(\omega) + e^{i\pi(n-1)/4} e^{-ikr} \varphi(-\omega) \right\}$$

or

$$e^{ikr\langle\omega,\xi\rangle} \sim r^{-(n-1)/2} e^{ikr} (2\pi)^{(n-1)/2} e^{-i\pi(n-1)/4} \delta_\omega(\xi) + r^{-(n-1)/2} e^{-ikr} (2\pi)^{(n-1)/2} e^{i\pi(n-1)/4} \delta_{-\omega}(\xi)$$

Comparing asymptotic expansions, we conclude that

$$S(k) = i^{-(n-1)} R + (2\pi)^{-(n-1)/2} e^{-i\pi(n-1)/4} A(k) R$$

where R is the reflection operator sending $\varphi(\xi)$ to $\varphi(-\xi)$ and A is the operator with integral kernel $A(\omega,\xi,k)$.

In this formulation, the direct problem for obstacle scattering is to determine properties of the scattering operator $S(k)$ (or equivalently the scattering amplitude $A(k)$) given the shape of the obstacle Ω and the type of boundary condition imposed. An example of a property of $S(k)$ that has been studied extensively is its meromorphic continuation in k. Poles in the this continuation are called scattering poles or resonances. The goal is to estimate the location of these resonances, depending on the geometry of Ω.

The inverse problem for obstacle scattering is to determine the shape of the obstacle from knowledge of $S(k)$ (or $A(k)$). It turns out that that only partial information about $S(k)$ is required. However, when inverse scattering is used in practical applications, even this partial information is never all available, since only finitely many measurements can be made. Therefore there are many difficult computational issues that arise.

Similar problem direct and inverse scattering problems can be considered for the Helmholtz equation with a variable coefficient index of refraction $n(x)$ which is a positive function equal to one outside a ball. The equation in this case is

$$(\Delta + n(x)k^2)u = 0$$

in Euclidean space.

Recent developments and open problems

Geometric Scattering Theory

In the early 1990's Melrose proposed a far-reaching and novel program to study "geometric scattering theory". The investigations in this area and the main lines of study are influenced both by the classical formulation of mathematical scattering theory and also by many results and methods in the study of the spectral theory of noncompact complete Riemannian manifolds. The point is to develop 'nonrelative' techniques to study questions of a scattering-theoretic nature; even when applied back to classical Euclidean scattering, this perspective has led to some dramatic advances and insights for the direct and inverse problem , e.g. for quantum N-body scattering. and hyperbolic manifolds. The exploration of the various ramifications of this theory, in particular its connection to Hodge theory and index theory, has occupied much of the efforts of Melrose and his coworkers and students over the past decade. The lectures by Mazzeo and Sa Barreto were concerned with this topic.

Inverse scattering in anisotropic media

The medium parameters in anisotropic media depend on direction and position. The inverse scattering problem of determining such media from the scattering amplitude is difficult and closely related to rigidity questions in differential geometry. The index of refraction of the Earth for instance depends on direction; with the fastest the direction of the axis of ration of the Earth. The problem of recovering this anisotropic index of refraction from the singularities of the scattering operator leads to the problem of determining a Riemannian metric on a bounded domain from the lengths of geodesics joining points in the boundary. This corresponds physically to the first arrival times of sound waves. In Riemannian geometry this is known as the boundary rigidity problem. The two dimensional case has recently been solved by Pestov and Uhlmann but the higher dimensional case remains largely open. Stefanov reported progress on this question in his lecture. Hansen considered in his talk an inverse anisotropic scattering problem arising in the mechanics of materials in particular isotropic elastic materials with residual stress. Residual stresses (or initial stresses) in solids and structures generally result from the manufacturing process.

Scattering poles

The scattering operator $S(z)$ is initially defined for real z, but often has a meromorphic continuation to the complex plane. Poles in this continuation are called scattering poles, or resonances. The term resonance comes from physics where they are interpreted as long-lived or resonant states. The lifetime of the resonant state is inversely proportional to its imaginary part. Thus, one is mostly interested in resonances that lie close to the real axis. Often one looks to the corresponding classical mechanical system as a guide to the location of resonances. Classical closed trajectories or stationary orbits should give rise to families of resonances. This was the subject of the talk of Hitrik, who talked about resonances associated to a non-degenerate maximum of a potential in two dimensions.

In the case of a Riemannian manifold with an asymptotically hyperbolic metric, a conformal structure is induced on the boundary at infinity. Zworski and Graham have studied the relation between this conformal structure and the residues of non-spectral scattering poles in the case where the metric is asymptotically Einstein.

There are still many open problems in this area. For example, one can consider the counting function for resonances $N(\lambda)$, defined to be the number of resonances of modulus less than λ. Although upper bounds are known from the work of Melrose, Sjöstrand, Zworksi, Vodev and others, lower bounds are much harder to come by, and exact asymptotics are only known in essentially one dimensional situations.

Inverse Obstacle Problem

This problem is discussed in more detail in the previous section. A recent development in the inverse scattering problem was the introduction by Colton and Kirsch and subsequent versions by Kirsch of the *linear sampling method* which is a numerical reconstruction struction algorithm of the obstacle that relies on the blow up of the associated Green's function at the boundary of the region that one ones to determine.

A very important open theoretical and applied problem is to determine the obstacle from a single incident wave at a fixed frequency measured in every possible direction. This is a formally determined problem since the dimension of the obstacle is the same as the manifold where the scattering amplitude is given. The lectures of Colton and Kirsch dealt with this inverse problem.

Time reversal and scattering in random media

In time reversal experiments waves are emitted from a localized source recorded for an interval of time by an array of receivers-transducers, time-reversed and retransmitted. This process is repeated several times, leading in some cases to refocusing near the original source. This has several applications in nondestructive evaluation and medical techniques such as lithotripsy and hypothermia. This process works because the wave equation is invariant under reversal of time and inhomogeneities and randomness contribute to a better refocusing. The talks by Bal and Ryzhikt with the mathematical analysis of time reversal in random media.

Scattering and inverse scattering in random media is a topic that will attract a lot of attention in the future because the only reasonable mathematical model for very complicated medium is that the medium parameters are random variables. Besides time reversal other applications of scattering in random media include propagation of radar waves through foliage and ultrasound in human tissue.

Complex geometrical optics solutions and inverse scattering at a fixed energy

The inverse scattering problem for the case of the Helmholtz equation with index of refraction one outside a compact set at a fixed energy k has been solved in dimension three or larger. This is in fact a consequence of the fact that the Dirichlet-to-Neumann map associated to this equation in any open and bounded domain uniquely determines the index of refraction. The solution of this latter problem uses complex geometrical optics solutions for the Schrödinger equation. which were developed by Calderón and Sylvester and Uhlmann. Challenges for the future include the two dimensional case and the fixed energy problem for many body scattering, that is the case of several particles interacting. Tamasan reported on progress in the two dimensional case for the two body problem.

Inverse Backscattering Problem

For the inverse backscattering problem for the case of the Helmholtz equation with variable index of refraction the medium is probed with waves in a given direction and the reflection is measured. However very little progress has been made on this very important problem very relevant to remote sensing and ultrasound except for the case that the index of refraction is small.

Optical Tomography

In this relatively new medical imaging modality one attempts to determine the absorption and scattering coefficient of an inhomogeneous medium by probing it with diffuse light. The problem is modelled as an inverse scattering problem for the stationary linear Boltzmann equation or the radiative transport equation.

The information is encoded in the albedo operator, which is the analog of the Dirichlet-to-Neumann map. Although theoretical results are very encouraging there are no numerical reconstruction algorithms which deal with the severe ill-posedness of this problem.

Seismic Imaging

Seismic imaging creates images of the Earth's upper crust using seismic waves generated by artificial sources and recorded into extensive arrays of sensors (geophones or hydrophones). The heterogeneity and anisotropy of the Earth's upper crust require advanced mathematics to generate wave–equation solutions suitable for seismic imaging. In this context, microlocal analysis and Fourier integral operators have been shown to be promising because they offer the potential to manipulate a wavefield directly on its phase space. The talks of Gibson, Lamoureux and Margrave described some of this progress.

However very little is known for the case of anisotropic elastic medium which is a more accurate model of the Earth's subsurface and this is one of the challenges for the future.

Scattering in Complex Materials

When a body is exposed to an incident electromagnetic field, the resulting "scattered" field depends on the electromagnetic properties of the body. By sending controlled incident fields, and measuring the scattered fields (remote sensing), one may obtain information about the makeup of the body with respect to these properties. Alternatively, one may control, say, the electric fields imposed on an accessible boundary and record the magnetic fields on the boundary, again obtaining information about the material makeup of the body.

The material properties one may seek include the body's electric permittivity, ε, magnetic permeability, μ, conductivity, σ, and chirality, β. The relationship between the fields is modelled by Maxwell's system of equations. If we assume that the system is lossless (there is no loss of electromagnetic energy), then in greatest generality these take the form $\text{curl}\, E = -\partial_t B,\ \text{curl}\, H = \partial_t D.$ Here, $E,\ H$ are the electric and magnetic fields, and B, D are the magnetic induction and electric displacement respectively. How B and D depend on E and H, and on the material properties of the body, is described by the constituent relations; these may take a variety of forms depending on the physical assumptions one is willing to make.

Chirality has not been included in the majority of prior analysis. Chirality is an asymmetry in the molecular makeup of the material, and is physically significant and present in many applications. The inclusion of chirality enriches Maxwell's equations and yields the potential to determine practically useful properties of the body.

A chiral molecule (typically a long carbon based corkscrew) is one whose mirror image has reversed orientation. In the absence of life, chiral molecules usually appear in equal amounts in each orientation. Living organisms, on the other hand, almost always produce chiral molecules with a definite orientation.

Chiral molecules scatter light differently. When light is transmitted through chiral material as a phase-coherent wave, left– and right–circularly polarized light are absorbed differently, and propagate at different speeds (thus rotating the plane of polarization of linearly polarized light). These phenomena are collectively termed optical activity consequence of optical activity is that light scattered from chirally structured particles differs characteristically from light scattered from achiral, non-living particles. Understanding chiral materials is of basic importance.

Another fundamental scattering and inverse scattering problem is dispersive materials for which very little is known. These materials are such that the speed of propagation depends on the frequency of the wave. Most materials are dispersive although sometimes one can neglect this dispersion but this is an important effect for broadband signals. A mathematical theory for Helmholtz equations with dispersive index of refraction would have considerable theoretical and practical importance.

The talks

Here is a brief description of the talks presented at this workshop, presented in the order they were given. Instead of giving references to published work, we provide web addresses with the most recent work, where this is available

Rafe Mazzeo reported on his joint work with Andras Vasy on the meromorphic continuation of the resolvent for arbitrary global symmetric spaces of noncompact type. This is related to the meromorphic continuation of the scattering operator $S(k)$. The geometry at infinity for these spaces is complicated, but has an inductive structure that is similar to N-body Hamiltonians. Thus, techniques developed for the study of N-body Hamiltonians can be extended. Mazzeo's recent papers can be found at
`http://math.stanford.edu/~mazzeo/papers/papers.html`.

David Colton considered the inverse scattering problem of determining the shape and surface impedance of a partially coated perfect conductor from a knowledge of the electric far field pattern of the scattered field due to an incident time harmonic plane wave at fixed frequency. Colton's publication list is located at
`http://www.math.udel.edu/~colton/Cbib/ColtonBib.html`

Plamen Stefanov, in his talk about joint work with Gunther Uhlmann, considered the boundary rigidity problem for Riemannian manifolds - is such a manifold uniquely determined, up to an isometry, by the boundary distance function, the function that assigns to two boundary points the distance between them? Selected publications of Stefanov are available at his web page located at
`http://www.math.purdue.edu/~stefanov/`

Sönke Hansen talked about the construction of high-frequency solutions and parametrices for systems of real principal type and indicated some applications. His preprints can be found at
`http://www-math.uni-paderborn.de/~soenke/`

Lenya Ryzhik and **Guillaume Bal** each gave a talk about their work on a mathematical theory of time-reversal experiments. In such experiments a signal emitted by a localized source is recorded by an array of receivers-transducers and re-emitted into the medium reversed in time. The new signal refocuses approximately on the location of the original source despite the small size of the array. Surprisingly, medium heterogeneities improve significantly the quality of refocusing. Ryzhik explained how this phenomenon may be understood in the general framework of refocusing of re-transmitted high frequency acoustic waves recorded at a single time. He discussed in particular the self-averaging properties of the re-transmitted signal that appear due to wave mixing and related them to the self-averaging properties of the phase space energy density of waves in random media. Bal explained why refocusing is improved by random inhomogeneities and related the time-reversal theory to the phase space description of energy propagation in random media, radiative transport and random geometrical optics. He also showed

Ricardo Weder discussed recent results that he obtained on direct and inverse scattering for the forced non-linear Schroedinger equation on the half-line, namely, the construction of the scattering operator and the unique reconstruction of the potential and the non-linearity. The new technical tools that made possible to prove these results are the L^1–L^∞ estimate and the boundedness in the L^p spaces of the wave operators for the Schroedinger equation that is obtained linearizing our problem with force identically zero.

Ira Herbst reported on work with Erik Skibsted about quantum scattering for the Schrödinger equation associated with a symbol $h(x,\xi)$ that is homogeneous of degree zero in x. In classical mechanics, radial points that vary smoothly in energy have corresponding stable and unstable manifolds. Herbst showed that (under certain generic assumptions) as long as there is an unstable manifold, there are no quantum states corresponding to the stable manifold. Selected papers of Herbst are available at
`http://www.math.virginia.edu/Faculty/herbst/`

Robin Graham discussed his joint work with Maciej Zworski on the "non-spectral" poles of the scattering matrix of an asymptotically hyperbolic metric. Such poles always exist and the residues are differential operators. In the case when the metric is asymptotically Einstein, these residues are invariantly associated to the induced conformal structure at infinity and this relationship can be viewed in the framework of the AdS/CFT correspondence of string theory.

Michael Hitrik reported on his work with Johannes Sjöstrand about scattering poles for a semiclassical Schrödinger operator in dimension 2, generated by a unique non-degenerate maximum of the potential. Assuming that the classical frequencies at the critical point satisfy the resonance condition, he obtained a complete asymptotic description of the poles which are at a distance h^δ, $0<\delta<1/2$, away from the critical energy. Hitik's web page is as `http://math.berkeley.edu/~hitrik/`

Alexandru Tamasan talked about the inverse scattering method used to prove (global) uniqueness in the 2D inverse conductivity problem. This method needs a little help when changed to a method of reconstruction. One needs a characterization of the Cauchy data in terms of the measurements. For Nachman's method (which uses a reduction to the Schroedinger equation,) this is given by the Dirichlet-to-Neumann map. In Brown and

Uhlmann's method (which uses a reduction to a first order elliptic hermitian system,) it involves more then just the Dirichlet-to-Neumann map. For the particular case coming from the conductivity equation, such a characterization lead to an algorithm in EIT which allowed $W^{1+\epsilon}$ conductivities. This was joint work with K. Knudsen. Preprints are available at `http://www.math.toronto.edu/tamasan/`

Jenn-Nan Wang discussed recent joint work with Gen Nakamura on the unique continuation property for the two-dimensional inhomogeneous anisotropic elasticity system

Peter Gibson talked about the the use of pseudo-differential operators and the Gabor transform in seismic imaging.

Victor Isakov discussed uniqueness and stability of recovery of two- and three-dimensional domains and of the boundary conditions from the far field data at one or few frequencies. He reviewed old results and gave some new theorems proven by the method of singular solutions. In particular, he considered penetrable obstacles with unknown general transmission conditions Isakov's publication list is at `http://www.math.wichita.edu/∼isakov/`

Antonio Sá Barreto defined radiation fields, showed how to obtain the scattering matrix from them, and used this characterization to study the problem of recovering the metric and the manifold from the scattering matrix at all energies. Sá Barreto's recent papers can be obtained from the web page
`http://www.math.purdue.edu/∼sabarre/papers.html`

Peter Hislop presented some new results on the spectral properties of quantum Hall Hamiltonians associated with unbounded regions in the plane. For one-edge regions that are deformations of the half-plane, he proved a lower-bound on the edge current, and the equality of the edge and bulk conductivity. He also gave a new proof of the existence of bands of absolutely continuous spectrum at energies between the Landau levels. He also studied straight, strip-like regions. This was joint work with J.-M. Combes and E. Soccorsi.

Victor Ivrii considered operators on compact closed manifolds, with coefficients, first derivatives of which are continuous with continuity modulus $O(|\log|x-y||^{-1})$. He derived semiclassical spectral asymptotics with sharp remainder estimate $O(h^{1-d})$. In the process he discussed the question "Where does microlocal analysis start?" and the answer to it "From the logarithmic uncertainty principle". These asymptotics easily yield standard asymptotics with respect to spectral parameter $\lambda \to \infty$. Ivrii's preprints are at `http://www.math.toronto.edu/ivrii/Research/Preprints.html`

Peter Perry, in joint work with Carolyn Gordon, constructed continuous families of metrics on R^n, $n \geq 8$, with the following properties: (1) The metrics are Euclidean outside a compact set, (2) the support of the perturbation may be arbitrarily small, and (3) the continuous families have the same scattering phase. The construction is based on constructions of isospectral compact manifolds by Carolyn Gordon and Dorothee Scheuth. There are examples of pairs of isophasal metrics which have very different isometry groups. Perry's preprints are located at `http://www.ms.uky.edu/∼perry/papers.html`

Oliver Dorn talked about level set method for describing propagating fronts, which has become quite popular in the application of medical or geophysical tomography. The goal in these applications is to reconstruct unknown objects inside a given domain from a finite set of boundary data. Mathematically, these problems define nonlinear inverse problems, where usually iterative solution strategies are required. Starting out from some initial guess for the unknown obstacles, successive corrections to this initial shape are calculated such that the so evolving shapes eventually converge to a shape which satisfies the collected data. Since the hidden objects can have a complicated topological structure which is not known a priori, the shapes usually undergo several topology changes during this evolution before converging to the final solution. Therefore, a powerful and flexible tool for the numerical description of these propagating shapes is essential for the success of the inversion method of choice. In the talk, Dorn presented a recently developed shape reconstruction method which uses a level set representation of the shapes for this purpose. Numerical results were presented for three different practically relevant examples: cross-borehole electromagnetic tomography using a 2D Helmholtz model, surface to borehole 3D electromagnetic induction tomography (EMIT) using a model based on the full 3D system of Maxwell's equations, and diffuse optical tomography (DOT) for medical imaging using a model based on the linear transport equation in 2D. Dorn's publications can be accessed at `http://www.cs.ubc.ca/∼dorn/publications.html`

David Finch considered the problem of inversion of the spherical mean transform on Euclidean space, when the centres are restricted to a hypersurface. He gave some uniqueness results. For the case of functions supported in a ball in an odd dimensional space, he gave closed form inversion formulae analogous to the Radon inversion formula when the spherical means are known for spheres centred on the boundary of the ball. Some applications were presented. This is joint work with Rakesh and Sarah Patch.

A robust approach to seismic imaging can be derived from the Born approximation to wavefield scattering. The resulting imaging algorithm estimates subsurface reflectivity as the ratio of backward extrapolated seismic data to a forward extrapolated model of the seismic source. This places key importance on the ability to extrapolate wavefield through heterogeneous media. In the case where the wavespeed is a function only of the coordinates orthogonal to the direction of extrapolation, an exact solution to the extrapolation problem is available.

Gary Margrave sketched this solution and compare it to two approximations. The first approximation is appears as a Fourier integral operator while the second uses the Gabor transform to approximately factorize this operator. I will discuss this approximate factorization and compare numerical results from all three methods. Selected publications of Margrave can be found at
`http://www.crewes.org/AboutCREWES/Faculty/margrave/margrave.php`

One dimensional scattering theory serves as the basis for an exact wavefield extrapolator in seismic imaging, where the propagation of a seismic wave through the earth's subsurface is modelled by the acoustic wave equation.

Michael Lamoureux showed a comparison between results of exact eigenvalue calculations on a bounded domain, with the results from the scattering theory of a (unbounded) 1D line, as well as with other standard imaging methods. Recent preprints of Lamoureux are available at
`http://www.math.ucalgary.ca/~mikel/papers.html`

Chi-Kun Lin talked about the homogenization of the Dirac type system. It generates memory effects. The memory (or nonlocal) kernel is described by the Volterra integral equation (or Fredholm integral equation depending on the initial or boundary value problem). When the coefficient is independent of time, the memory kernel can be characterized explicitly in terms of Young's measure.

Georgi Vodev discussed some recent results concerning estimates of the local energy decay of solutions to the wave equation on unbounded Riemannian manifolds with non-trapping metrics. These estimates are derived from the properties of the resolvent at high frequency. Applications to a class of asymptotically Euclidean manifolds as well as to perturbations by non-negative long-range potentials were given. Vodev's preprints appear on preprint server of the Nantes mathematics department at
`http://www.math.sciences.univ-nantes.fr/prepub/liste.phtml?annee=2003`

Andreas Kirsch talked about the factorization method, a new approach to solve classes of inverse scattering problems for time harmonic acoustic or elastic waves. In the first part of the talk he explained the idea for the simplest case where one tries to recover an unknown (acoustically soft) obstacle from the far field patterns for plane wave incidence. In the second part he will presented recent extensions to problems with absorption and to mixed boundary conditions and discussed the relationship to the well known MUSIC algorithm in signal processing. Current publications of Kirsch are at `http://www.mathematik.uni-karlsruhe.de/~ma2ki/page.php?id=kirsch&pid=publication`

List of Participants

Ammari, Habib (Ecole Polytechnique)
Bal, Guillaume (Columbia University)
Colton, David (University of Delaware)
Dobranski, Michael (University of Kentucky)
Donaldson, Roger (University of British Columbia)
Dorn, Oliver (Universidad Carlos 3 de Madrid)
Finch, David (Oregon Sate University)
Frigyik, Bela (University of Washington)
Froese, Richard (University of British Columbia)
Gibson, Peter (University of Calgary)
Graham, Robin (University of Washington)
Grossman, Jeff (University of Calgary)
Hansen, Soenke (Universitaet Paderborn)
Herbst, Ira (University of Virginia)

Hislop, Peter (University of Kentucky)
Hitrik, Michael (Mathematical Sciences Research Institute)
Isakov, Victor (Wichita State University)
Ivrii, Victor (University of Toronto)
Kirsch, Andreas (Universität Karlsruhe)
Kusiak, Steven (University of Washington)
Lamoureux, Michael (University of Calgary)
Lin, Chi-Kun (National Cheng Kung University)
Margrave, Gary (University of Calgary)
Mazzeo, Rafe (Stanford University)
McDowall, Stephen (Western Washington University)
Perry, Peter (University of Kentucky)
Ramaseshan, Karthik (University of Washington)
Rundell, William (Texas A & M University)
Ryzhik, Lenya (University of Chicago)
Salo, Mikko (University of Washington)
Skokan, Michal (University of Washington)
Stefanov, Plamen (Purdue University)
Sá Barreto, Antônio (Purdue University)
Tamasan, Alexandru (University of Toronto)
Uhlmann, Gunther (University of Washington)
Vodev, Georgi (Université de Nantes)
Wang, Jenn-Nan (National Taiwan University)
Weder, Ricardo (Universidad Nacional Autonoma de Mexico)
Yedlin, Matt (University of British Columbia)

Chapter 3

Commutative Algebra and Geometry (03w5005)

March 29–April 3, 2003

Organizer(s): Mark Green (University of California, Los Angeles), Jürgen Herzog (Universität Essen), Bernd Sturmfels (University of California, Berkeley)

Workshop Topics

The workshop was intended to focus on the interaction between commutative algebra, algebraic geometry and combinatorics with a special emphasis on Gröbner basis theory, and to give an overview on recent developments and possible future applications. The first lectures by Hibi, Bruns and Conca were of introductory nature and gave surveys on Groebner basis techniques in commutative algebra, on determinantal ideals and lattice polytopes. Most recent results on minors of products of matrices were reported by Miller. New techniques in computing Castelnuovo Mumford regularity with applications to powers of ideals have been presented by Eisenbud, while Cox introduced new computational methods to determine the defining equations of curves and surfaces avoiding elimination theory. Iarrobino and Boij discussed the nature of Hilbert functions of level and Gorenstein algebras. The structure of multigraded and injective resolutions have been the content of the lectures by Charalambous and Huang.

Real algebraic geometry and related subjects have also been one of the main topics of the workshop with lectures by Brenner, Scheiderer, Russell, and Sottile.

The lecture by Zelevinsky on Cluster algebras was one of the highlights of the workshop. There were also quite exciting lectures by Carrell, Khovanskii and Mustata.

Participation

Among the participants there where several postdocs and junior speakers (Blickle, Boijs, Brenner, Maclagan, Sidman, Miller, Yanagawa). One afternoon of the workshop was devoted to give some of the junior participants the possibility to present their research in 50 minutes talks. For all the younger participants the workshop was an excellent opportunity to inform themselves about the newest trends in their field and to discuss with leading researchers.

Among the participants of the workshop we had four women. Two of them, Hara Charalambous and Jessica Sidman, gave a talk. All of them, Charalambous, Dickenstein, Maclagan and Sidman were invited because of their recent scientific contributions in the area of research related to this workshop. Originally two more women were invited but had to cancel their participation by personal reasons.

Outcome

Since the workshop concentrated on a very specific aspects of algebra and algebraic geometry it turned out to be quite efficient it terms of exchange and presentation of new ideas. On the other hand, the invited people represented quite different directions of research, varying from commutative and non-commutative algebra to computational algebra, combinatorics and algebraic geometry, so that different viewpoints on the same subjects could be presented. Also our concept to invite young researchers and first-rate mathematicians turned out to be inspiring for participants.

We decided to have a limited number of talks (at most five a day) to give enough opportunity for scientific discussions. We also decided that the talks should be at least 50 minutes each, so that the speaker not only has the possibility to present his results but also to explain some background and the principle ideas of his approaches. Altogether we had 24 lectures and 37 participants.

The funding was adequate as it covered all local expenses.

List of Participants

Blickle, Manuel (Mathematical Sciences Research Institute)
Boij, Mats (Royal Institute of Technology)
Brenner, Holger (Universitaet Bochum)
Bruns, Winfried (University of Osnabruck)
Carrell, James (University of British Columbia)
Cattani, Eduardo (University of Massachusetts)
Charalambous, Hara (The State University of New York, Albany)
Conca, Aldo (University of Genova)
Cox, David (Amherst College)
Dickenstein, Alicia (University of Buenos Aires)
Eisenbud, David (Mathematical Science Research Institute)
Green, Mark (University of California)
Gubeladze, Joseph (Mathematical Sciences Research Institute)
Herzog, Jurgen (Universitat-Gesamthochschule-Essen)
Hibi, Takayuki (Osaka University)
Hosten, Serkan (San Francisco State University)
Huang, I-Chiau (Academia Sinica)
Hulek, Klaus (Fachbereich Mathematik Universitat Hannover)
Iarrobino, Anthony (Northeastern University)
Khovanskii, Askold (University of Toronto)
Kreuzer, Martin (University of Dortmund)
Littelmann, Peter (Mathematical Sciences Research Institute)
Maclagan, Diane (Stanford University)
Migliore, Juan (University of Notre Dame)
Miller, Ezra (Massachusetts Institute of Technology)
Mustata, Mircea (Clay Mathematics Institute)
Popescu, Sorin (The State University of New York, Stony Brook)
Russell, Peter (McGill University)
Scheiderer, Claus (University of Duisburg)
Schreyer, Frank-Olaf (University of Bayreuth)
Sidman, Jessica (University of California - Berkeley)
Smith, Gregory (Columbia University, Barnard College)
Sottile, Frank (University of Massachusetts)
Stillman, Michael (Cornell University)
Sturmfels, Bernd (University of California - Berkeley)
Yanagawa, Kohji (Osaka University)
Zelevinsky, Andrei (Northeastern University)

Chapter 4

BIRS Workshop on Noncommutative Geometry (03w5060)

April 5–10, 2003

Organizer(s): Alain Connes (Collège de France and Institut des Hautes Études Scientifi ques), Joachim Cuntz (Universitat M ̈unster), George Elliott (University of Toronto), Masoud Khalkhali (University of Western Ontario), Boris Tsygan (Northwestern University)

Noncommutative geometry is a rapidly growing new area of mathematics with links to many disciplines in mathematics and physics. This is a highly interdisciplinary subject which draws its intuitions, ideas and methods from various areas of mathematics and physics and at the same time contributes successfully to the resolution of many of the standard problems and conjectures in these areas. Examples of such interactions and contributions include: operator algebras (K-theory and KK-theory of C^*-algebras and their classification); topology (successful resolution of the Novikov conjecture and the Baum-Connes Conjecture for large classes of groups); global analysis and geometry (various new formulations of index theory problems beyond their classical realms, on singular spaces like the space of leaves of a foliation); algebra (algebraic K-theory computations through topological Hochschild and cyclic homology, the idempotent conjecture for group algebras, the theory of quantum groups and Hopf algebras and their homological invariants); number theory (Connes' new approach to the Riemann hypothesis, relations between Hecke algebras and quantum statistical mechanics via noncommutative geometry, emerging relations with Arakelov theory, hidden quantum symmetries in the space of modular forms recently discovered by Connes and Moscovici); physics, in particular high energy physics (solid state physics of quasi crystals, new formulations of the standard model, relations with string theory, gauge theory, and M(atrix) theory).

It is fair to say that the subject of noncommutative geometry, as we understand it now, started in 1980 with Alain Connes' classic papers on noncommutative differential geometry. His earlier work on the classification of von Neumann algebras, regarded as a kind of noncommutative measure theory, as well as his work on index theory on foliated spaces, showed that there are many situations in mathematics where the spaces one wants to study are badly behaved even as a topological space, let alone being smooth. One of his key observations was that in all these situations one can attach a noncommutative algebra (through a crossed product construction, or variations thereof, called groupoid algebras) that captures most of the relevant information. Examples include: the space of leaves of a foliation, the unitary dual of a noncompact (Lie) group, and the space of Penrose tilings. Thus a pervasive idea in noncommutative geometry is to treat (certain classes) of noncommutative algebras on the same footing as spaces and to try to extend the tools of commutative mathematics (topology, geometry, analysis, commutative algebra) to this new setting.

We should emphasize, however, that this extension has never been straightforward and always involves surprises. An illustration of this is the theory of the flow of weights and the corresponding modular automorphism group in von Neumann algebra theory which has no counterpart in classical measure theory. Similarly, the extension of de Rham homology of manifolds to cyclic cohomology of noncommutative algebras is highly nontrivial.

A new trend in noncommutative geometry is the emergence of a new Hopf algebra as the hidden quantum symmetries of diverse mathematical and physical structures. It is remarkable that this Hopf algebra, namely the Connes-Moscovici Hopf algebra $\mathcal{H}_1$, appears in a natural way as the quantum symmetries of the transverse space of codimension one foliations, as well as the space of modular Hecke algebras in number theory. A closely related Hopf algebra, namely the Connes-Kreimer Hopf algebra, is used to provide a mathematically rigorous treatment of renormalization in quantum field theory.

In the following we will try to summarize the current state of the subject, as reflected in the talks during the workshop, and especially its challenges and also potential for further progress. We will divide this into different subsections.

1. The Baum-Connes conjecture

This conjecture, in its simplest form, is formulated for any locally compact topological group. There are more general Baum-Connes conjectures with coefficients for groups acting on C*-algebras, for groupoid C*-algebras, etc., that for the sake of brevity we don't consider here. In a nutshell the Baum-Connes conjecture predicts that the K-theory of the group C*-algebra of a given topological group is isomorphic, via an explicit map called the Baum-Connes map, to an appropriately defined K-homology of the classifying space of the group. In other words invariants of groups defined through noncommutative geometric tools coincide with invariants defined through classical algebraic topology tools. The Novikov conjecture on the homotopy invariance of higher signatures of non-simply connected manifolds is a consequence of the Baum-Connes conjecture (the relevant group here is the fundamental group of the manifold). Major advances were made in this problem in the past seven years by Higson-Kasparov, Lafforgue, Nest-Echterhoff-Chabert, Yu, Puschnigg and others. The talks by P. Baum, S. Echterhoff, R. Meyer, and G. Yu covered various aspects of the Baum-Connes conjecture.

2. Cyclic cohomology and KK-theory

A major discovery made by Alain Connes in 1981, and independently by Boris Tsygan, was the discovery of cyclic cohomology as the right noncommutative analogue of de Rham homology and a natural target for a Chern character map from K-theory and K-homology. Coupled with K-theory, K-homology and KK-theory, the formalism of cyclic cohomology fully extends many aspects of classical differential topology like Chern-Weil theory to noncommutative spaces. It is an indispensable tool in noncommutative geometry. In recent years Joachim Cuntz and Dan Quillen have formulated an alternative powerful new approach to cyclic homology theories which brings with it many new insights as well as a successful resolution of an old open problem in this area, namely establishing the excision property of periodic cyclic cohomology.

For applications of noncommutative geometry to problems of index theory, e.g. index theory on foliated spaces, it is necessary to extend the formalism of cyclic cohomology to a bivariant cyclic theory for topological algebras and to extend Connes' Chern character to a fully bivariant setting. The most general approach to this problem is due to Joachim Cuntz. In fact the approach of Cuntz made it possible to extend the domain (and definition) of KK-theory to very general categories of topological algebras (rather than just C*-algebras). The fruitfulness of this idea manifests itself in the V. Lafforgue's proof of the Baum-Connes conjecture for groups with property T, where the extension of KK functor to Banach algebras plays an important role.

A new trend in cyclic cohomology theory is the study of the cyclic cohomology of Hopf algebras and quantum groups. Many noncommutative spaces, such as quantum spheres and quantum homogeneous spaces, admit a quantum group of symmetries. A remarkable discovery of Connes and Moscovici in the past few years is the fact that diverse structures, such as the space of leaves of a (codimension one) foliation or the space of modular forms, have a unified quantum symmetry. In their study of transversally elliptic operators on foliated manifolds Connes and Moscovici came up with a new noncommutative and non-cocommutative Hopf algebra denoted by $\mathcal{H}_n$ (the Connes-Moscovici Hopf algebra). $\mathcal{H}_n$ acts on the transverse foliation algebra of codimension n foliations and thus appears as the quantized symmetries of a foliation. They noticed that if one extends the noncommutative Chern-Weil theory of Connes from group and Lie algebra actions to actions of Hopf algebras, then the characteristic classes defined via the local index formula are in the image of this new characteristic map. This extension of Chern-Weil theory involved the introduction of cyclic cohomology for Hopf algebras. The talks by P. Hajac, D. Perrot, B. Rangipour, A. Thom, C. Valqui, and C. Voigt covered various aspects of cyclic cohomology theory and KK-theory.

3. Index theory and noncommutative geometry

The index theorem of Atiyah and Singer and its various generalizations and ramifications are at the core of noncommutative geometry and its applications. A modern abstract index theorem in the noncommutative setting is the local index formula of Connes and Moscovici. A key ingredient of such an abstract index formula is the idea of an spectral triple due to Connes. Broadly speaking, and neglecting the parity, a spectral triple (A, H, D) consists of an algebra A acting by bounded operators on the Hilbert space H and a self-adjoint operator D on H. This data must satisfy certain regularity properties which constitute an abstraction of basic elliptic estimates for elliptic PDE's acting on sections of vector bundles on compact manifolds. The local index formula replaces the old non-local Chern-Connes cocycle by a new Chern character form $Ch(A, H, D)$ of the given spectral triple in the cyclic complex of the algebra A. It is a local formula in the sense that the cochain $Ch(A, H, D)$ depends, in the classical case, only on the germ of the heat kernel of D along the diagonal and in particular is independent of smooth perturbations. This makes the formula extremely attractive for practical calculations. The challenge now is to apply this formula to diverse situations beyond the cases considered so far, namely transversally elliptic operators on foliations (Connes and Moscovici) and the Dirac operator on quantum SU_2 (Connes). The talks by A. Gorokhovsky, J. Phillips, and R. Ponge centred around index theory and noncommutative geometry.

4. Noncommutative geometry and number theory

Current applications and connections of noncommutative geometry to number theory can be divided into four categories. (1) The work of Bost and Connes, where they construct a noncommutative dynamical system (B, σ_t) with partition function the Riemann zeta function $\zeta(\beta)$, where β is the inverse temperature. They show that at the pole $\beta = 1$ there is an spontaneous symmetry breaking. The symmetry group of this system is the group of idéles which is isomorphic to the Galois group $Gal(Q^{ab}/Q)$. This gives a natural interpretation of the zeta function as the partition function of a quantum statistical mechanical system. In particular the class field theory isomorphism appears very naturally in this context. This approach has been extended to the Dedekind zeta function of an arbitrary number field by Cohen, Harari-Leichtnam, and Arledge-Raeburn-Laca. All these results concern abelian extensions of number fields and their generalization to non-abelian extensions is still lacking. (2) The work of Connes on the Riemann hypothesis. It starts by producing a conjectural trace formula which refines the Arthur-Selberg trace formula. The main result of this theory states that this trace formula is valid if and only if the Riemann hypothesis is satisfied by all L-functions with Grössencharakter on the given number field k. (3) The work of Connes and Moscovici on quantum symmetries of the modular Hecke algebras $\mathcal{A}(\Gamma)$ where they show that this algebra admits a natural action of the transverse Hopf algebra $\mathcal{H}_1$. Here Γ is a congruence subgroup of $SL(2, Z)$ and the algebra $\mathcal{A}(\Gamma)$ is the crossed product of the algebra of modular forms of level Γ by the action of the Hecke operators. The action of the generators X, Y and δ_n of $\mathcal{H}_1$ corresponds to the Ramanujan operator, to the weight or number operator, and to the action of certain group cocycles on $GL^+(2, Q)$, respectively. What is very surprising is that the same Hopf algebra $\mathcal{H}_1$ also acts naturally on the (noncommutative) transverse space of codimension one foliations. (4) Relations with arithmetic algebraic geometry and Arakelov theory. This is currently being pursued by Consani, Deninger, Manin, Marcolli and others. The lectures by Alain Connes, Katia Consani and Marcelo Laca during the workshop covered interactions between noncommutative geometry and number theory.

5. Deformation quantization and quantum geometry

The noncommutative algebras that appear in noncommutative geometry usually are obtained either as the result of a process called noncommutative quotient construction or by deformation quantization of some algebra of functions on a classical space. These two constructions are not mutually exclusive. The starting point of deformation quantization is an algebra of functions on a Poisson manifold where the Poisson structure gives the infinitesimal direction of quantization. The existence of deformation quantizations for all Poisson manifolds was finally settled by M. Kontsevich in 1997 after a series of partial results for symplectic manifolds. The algebra of pseudodifferential operators on a manifold is a deformation quantization of the algebra of classical symbols on the cosphere bundle of the manifold. This simple observation is the beginning of an approach to the proof of the index theorem, and its many generalizations by Elliott-Natsume-Nest and Nest-Tsygan, using cyclic cohomology theory. The same can be said about Connes' groupoid approach to index theorems. In a different direction, quantum geometry also consists of the study of noncommutative metric spaces and noncommutative complex structures. The lectures by H. Bursztyn, J. Kaminker, G. Landi, H. Li, I. Nikolaev, A. Polishchuk, I. Putnam, M. Rieffel, and B. Tsygan covered various aspects of deformation quantization and quantum geometry.

Reports of individual speakers

The BIRS Workshop on Noncommutative Geometry took place at a crucial moment in the development of our subject. With many of the leading experts on various aspects of noncommutative geometry attending the conference, the participants and in particular younger researchers got a very good chance to communicate and exchange their ideas. In the following we are attaching abstracts of talks presented during the meeting by individual speakers. These abstracts are written by the speakers themselves.

Paul Baum: Local-global principle for Baum-Connes

Abstract: A group G is a BCC group (or BCC is valid for G) if Baum-Connes with arbitrary coefficients is valid for G. Let F be an algebraic number field (i.e. F is a finite degree extension of the rational numbers Q). Let G be an algebraic group scheme defined over F. $G(F)$ denotes the discrete group of F-rational points of G. For each place v of F, F_v denotes the local field obtained by completing F at v, and $G(F_v)$ is the locally compact group of F_v rational points of G.

THEOREM. If BCC is valid for all of the locally compact groups $G(F_v)$, then BCC is valid for the discrete group $G(F)$.

This can be proved by using the Meyer-Nest point of view that Baum-Connes is a derived functor, or it can be proved by a direct argument due to Baum-Millington-Plymen.

Henrique Bursztyn: Picard groups in deformation quantization

Abstract: The notion of Morita equivalence plays a prominent role in noncommutative geometry; the Picard group of an algebra is its group of self-Morita equivalences. In this talk, I discussed the behaviour of Picard groups when algebras are quantized in the sense of formal deformation quantization. In the case of algebras of functions on Poisson manifolds, the change in the Picard groups can be expressed purely in terms of the geometry of the manifold. I will also report on how these ideas lead to interesting results in Poisson geometry.

Alain Connes: Modular Hecke algebras and their Hopf symmetry

Abstract: This is joint work with Henri Moscovici. We associate to any congruence subgroup of $SL(2, Z)$ a 'modular Hecke algebra' extending both the ring of classical Hecke operators and the algebra of modular forms. These are coordinate algebras for the 'transverse space' of lattices modulo the action of the Hecke correspondences. The underlying symmetry is shown to be encoded by the same Hopf algebra that controls the transverse geometry of codimension 1 foliations. The action of its horizontal generator is given by the Ramanujan operator that corrects the usual differentiation by the logarithmic derivative of the Dedekind eta function, and the action of the vertical generator is given by the Euler operator on modular forms. The other generators of the Hopf symmetry are associated to higher derivatives of the classical Hecke operators. The emerging picture is that of a surprisingly close analogy with the foliation case, with the role of the circle as a complete transversal being assumed by the modular elliptic curve X(6) and with a simple Eichler integral replacing the angular variable. The Schwarzian 1-cocycle gives an inner derivation implemented by the level 1 Eisenstein series of weight 4, and leads to a rational projective structure on X(6). The Hopf cyclic 2-cocycle

representing the transverse fundamental class provides a natural extension of the first Rankin-Cohen bracket to the modular Hecke algebras. Finally, the Hopf cyclic version of the Godbillon-Vey cocycle gives rise to a 1-cocycle on $SL(2,Q)$ with values in Eisenstein series of weight 2, which when coupled with the 'period' cocycle yields a representative of the Euler class, providing an arithmetic formula for the Euler class of $SL(2,Q)$ in terms of generalized Dedekind sums. We then show how to extend the Rankin-Cohen brackets from modular forms to modular Hecke algebras. More generally, our procedure yields such brackets on any associative algebra endowed with an action of the Hopf algebra of transverse geometry in codimension one, such that the derivation corresponding to the Schwarzian derivative is inner. Moreover, we show in full generality that these Rankin-Cohen brackets give rise to associative deformations.

Katia Consani: Archimedean fibers and non-commutative geometry

Abstract: In Arakelov geometry a completion of an arithmetic surface is achieved by enlarging the group of divisors by formal linear combinations of the *closed fibers at infinity*. If one enriches Arakelov's metric structure on a compact Riemann surface of genus at least 2 by choosing a Schottky uniformization, then this extra datum may be combined with the archimedean cohomology theory on the surface to determine the structure of a non-commutative manifold.

Siegfried Echterhoff: Going-Down functors and topological K-theory

Abstract: In this lecture we report on recent joint work with Jerome Chabert and Herve Oyono-Oyono. If G is a locally compact group, a Going-Down functor on G is a family of functors $\mathcal{F}_H : \mathcal{A}(H) \to \mathbf{Ab}$, where H runs through the closed subgroups of G, $\mathcal{A}(H)$ denotes the category of commutative proper H-algebras and $\mathbf{Ab}$ denotes the category of abelian groups, such that $\{\mathcal{F}_H\}_{H<G}$ satisfies certain restriction and inflation axioms. A typical example is the functor $\mathcal{F}_H(C_0(Z)) = KK^G(C_0(Z), A)$ for a fixed G-algebra A. Define $\mathcal{F}(G) := \lim \mathcal{F}_G(C_0(X))$, where X runs through the G-compact subsets of the universal proper G-space $\mathcal{E}(G)$. In the special case $\mathcal{F}_H(C_0(Z)) = KK^G(C_0(Z), A)$, we get $F(G) = K_*^{top}(G; A)$, the topological K-theory of G with coefficient A. We then prove:

Theorem. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two Going-Down functors on G such that there is a natural transformation from F to G respecting the axioms and such that this transformation induces isomorphisms $\mathcal{F}_K(C_0(V)) \cong \mathcal{G}_K(C_0(V))$ for all compact subgroups K of G and for all euclidean linear K-spaces V; then $F(G) = G(G)$.

This result implies, in particular, that any element $x \in KK^G(A, B)$ with $K^G(x) \in KK^K(A, B)$ invertible for all compact $K \subseteq G$ induces an isomorphism $K_*^{top}(G; A) \cong K_*^{top}(G; B)$.

We give several direct applications of the theorem, including a new permanence result of the Baum-Connes conjecture for group extensions and a certain Künneth theorem for topological K-theory and crossed products. As a final application we use our methods to show that the adelic groups $G(\mathbf{A})$ for every linear algebraic group G over a finite extension of $\mathbf{Q}$ satisfies the Baum-Connes conjecture, thus extending an earlier result of Baum, Millington, and Plymen for reductive groups.

Alexander Gorokhovsky: Local index theory over foliation groupoids

Abstract: This is a report on a work joint with J. Lott. We extend methods of our previous work on a superconnection proof of Connes' index theorem for etale groupoids to the case of foliation groupoids. This allows us to give a more canonical proof of the index theorem for foliations.

Piotr Hajac: Hopf cyclic homology with coefficients

Abstract: The purpose of this talk is to outline constructions of cocyclic modules yielding Hopf-cyclic cohomology with coefficients in stable anti-Yetter Drinfeld modules. This gives a common denominator to known cyclic theories, and allows us to extend the Connes-Moscovici formalism, including the transfer map from the Hopf-cyclic to the usual cyclic cohomology. This is joint work with M. Khalkhali, B. Rangipour, and Y. Sommerhaeuser.

Jerry Kaminker: Noncommutative geometry and hyperbolic dynamics

Abstract: Associated to a hyperbolic dynamical system, consisting of a compact metric space with a self-homeomorphism satisfying the axioms of a Smale Space, are two C*-algebras. These algebras satisfy a form of Spanier-Whitehead duality in K-theory. This was worked out several years ago by Ian Putnam and the speaker. Since then some new applications of these ideas have surfaced. In one direction a class of finitely generated groups, each associated to a compact abelian group admitting an expansive automorphism, can be studied. In many cases one can compactify the countable group by adjoining the original compact group as a boundary. The latter is a hyperbolic dynamical system while the countable group is often solvable. Another direction is the general program of extending the connections between dynamics and homology studied by Bowen, Franks, and many others, to the situation where homology is replaced by the K-theory of the C*-algebras associated to the dynamics.

Marcelo Laca: Hecke algebras from Number Theory

Abstract: We first describe the Hecke algebra of a semidirect product group with respect to an almost normal subgroup of the normal part: it is a semigroup crossed product realizable as a corner in the group C*-algebra of another semidirect product. As an application we use this to study the structure of a Hecke C*-algebra naturally associated to an algebraic number field. This structural knowledge, in turn, enables us to characterize the KMS states of the Hecke algebra. When the number field is purely imaginary of class number 1, we are able to establish that the extreme KMS states at low temperature have symmetry group isomorphic to the Galois group of the maximal abelian extension of the field. The talk is based on joint work with Nadia Larsen on Hecke algebras of semidirect products and on joint work with Machiel van Frankenhuysen on phase transitions on Hecke algebras from number theory.

Giovanni Landi: Fredholm modules on deformed spheres

Abstract: The quantum Euclidean spheres, S_q^{N-1}, are (noncommutative) homogeneous spaces of quantum orthogonal groups, $SO_q(N)$. The $*$-algebra $A(S_q^{N-1})$ of polynomial functions on each of these is given by generators and relations which can be expressed in terms of a self-adjoint, unipotent matrix. We explicitly construct complete sets of generators for the K-theory (by nontrivial self-adjoint idempotents and unitaries) and the K-homology (by nontrivial Fredholm modules) of the spheres S_q^{N-1}. We also construct the corresponding Chern characters in cyclic homology and cohomology and compute the pairing of K-theory with K-homology. On odd spheres (i.e., for N even) we exhibit unbounded Fredholm modules by means of a natural unbounded operator D which, while failing to have compact resolvent, has bounded commutators with all elements in the algebra $A(S_q^{N-1})$. This is joint work with Eli Hawkins.

Hanfeng Li: Metric aspect of theta-deformations

Abstract: We show that Connes and Landi's theta-deformations are compact quantum metric spaces under the deformed Dirac operators. We also introduce a new quantum Gromov-Hausdorff distance which is able to distinguish the algebra structure, and show that the theta-deformations are continuous with respect to this distance.

Ralf Meyer: The Baum-Connes conjecture via derived functors

Abstract: This is joint work with Ryszard Nest. We develop a new approach towards the Baum-Connes assembly map that uses the framework of derived categories. One goal of this project is to formulate analogues of the Baum-Connes conjecture for quantum groups. The topological K-theory $K^{\mathrm{top}}(G, A)$ appears as the derived functor of the K-theory of the crossed product, and the assembly map as the natural transformation from a derived functor to the original functor. The main result needed for this is the existence of an

analogue of the Dirac element of the Dirac-dual Dirac method. This is proved using representability theorems in triangulated categories.

Igor Nikolaev: Geodesic laminations and noncommutative geometry

Abstract: Measured geodesic laminations is a remarkable abstraction (due to W. P. Thurston) of many otherwise unrelated phenomena occurring in differential geometry, complex analysis and geometric topology. In this talk we focus on connections of geodesic laminations with the inductive limits of finite-dimensional semi-simple C*-algebras (AF C*-algebras). Our main result is a bijection between the combinatorial presentations of such C*-algebras (the so-called Bratteli diagrams) and measured geodesic laminations on compact surfaces. This link appears helpful indeed as it provides insights to the Teichmuller spaces, Thurston's theory of surface homeomorphisms, Stallings' fibrations to the one side, and noncommutative geometry to the other.

Denis Perrot: Characteristic classes of algebraic KK-theory

Abstract: We propose a definition of algebraic KK-theory based on the category of finitely summable quasihomomorphisms, and construct a Chern character with values in the bivariant cyclic cohomology. One recovers as a particular case the negative Chern character of Hood and Jones. Moreover, the secondary characteristic classes of the corresponding algebraic K-homology are related to the BRS cohomology classes appearing in Quantum Field Theory.

John Phillips: Spectral Flow of Unbounded Self-Adjoint Fredholm Operators

Abstract: We study the gap (= "projection norm" = "graph norm") topology of the space of all (not necessarily bounded) self-adjoint Fredholm operators in a separable Hilbert space. We use the Cayley transform to obtain a uniformly homeomorphic model for our space as a subset of the unitary operators. Using this model, we are able to show the surprising result that this space is path-connected: for example we can connect $+1$ to -1 by a path of self-adjoint Fredholm operators (some of which are necessarily unbounded)! This is in striking contrast to the bounded case where Atiyah and Singer showed that there are three path-components in the space of **bounded** self-adjoint Fredholm operators with $+1$ and -1 being in different contractible components. Moreover, we present a rigorous definition of spectral flow of a path of such operators (actually alternative but mutually equivalent definitions) and prove the homotopy invariance. Thus, as in the bounded case, spectral flow defines a surjective homomorphism from the fundamental group of our space to **Z**. Unfortunately, we have been unable to determine if this homomorphism is one-to-one, or more generally, whether this space is a classifying space for the functor, K_1. As examples, we investigate paths of Dirac type operators on manifolds with boundary. This article has recently been accepted for publication in the *Canadian Journal of Mathematics*.

Alexander Polishchuk: Holomorphic bundles on noncommutative tori

Abstract: In this talk I define the notion of a holomorphic bundle on a noncommutative complex torus and propose a conjecture that relates the category of such bundles to the derived category of coherent sheaves on the associated commutative complex torus.

Raphael Ponge: A new short proof of the local index formula of Atiyah and Singer

Abstract: In this talk it was attempted to give a new short proof of the local index formula of Atiyah and Singer. It combines the Getzler rescaling with Greiner's approach of the heat kernel asymptotics. The latter uses a pseudodifferential representation of the heat kernel and thereby allows a differentiable heat kernel proof of the local index formula. In turn we can easily compute the CM cocycle for Dirac spectral triples, in both the even and odd cases. So we can bypass the use of the asymptotic pseudodifferential calculus of the previous computations of the CM cocycle.

Ian Putnam: Orbit equivalence for Z^2 minimal Cantor systems

Abstract: I will give a short overview describing joint work with Thierry Giordano (Ottawa) and Christian Skau (Trondheim) on the problem of orbit equivalence for Cantor minimal systems. This is a natural extension to the topological category of the program of H. Dye. Specifically, we are now able to enlarge our classification to include certain minimal free actions of Z^2.

Bahram Rangipour: Cyclic cohomology of extended Hopf algebras

Abstract: This is joint work with Masoud Khalkhali. Extended Hopf algebras are a variation of the concept of Hopf algebroids. Hopf algebroids are quantizations of groupoids exactly in the same way that Hopf algebras and quantum groups are quantizations of groups. The definition of an extended Hopf algebra, proposed in our joint paper, is motivated by the need to define a cyclic cohomology theory for them and to extend the formalism of cyclic cohomology of Hopf algebras, defined by Connes and Moscovici, to extended Hopf algebra. The whole theory is motivated by the introduction, by Connes and Moscovici, of the extended Hopf algebra H_{FM} for any smooth manifold M and its cyclic cohomology theory.

Marc Rieffel: Hyperbolic group C^*-algebras as compact quantum metric spaces

Abstract: Let ℓ be a length function on a group G, and let M_ℓ denote the operator of pointwise multiplication by ℓ on $\ell^2(G)$. Following Connes, M_ℓ can be used as a "Dirac" operator for $C_r^*(G)$. It defines a Lipschitz seminorm on $C_r^*(G)$, which defines a metric on the state space of $C_r^*(G)$. We show that if G is a hyperbolic group and if ℓ is a word-length function on G, then the topology from this metric coincides with the weak-$*$ topology (our definition of a "compact quantum metric space"). We show that a convenient framework is that of filtered C^*-algebras which satisfy a suitable " Haagerup-type" condition.

Andreas Thom: Connective E-theory and bivariant homology

We analyze the homotopy theory of sub-homogeneous algebras. A connective version of Connes' and Higson's E-theory is presented. It satisfies universal properties analogous to ordinary E-theory. We give the definition of a bivariant homology theory generalizing cellular homology and analyze the algebra of cohomology operations. It turns out that it serves as a measure of the failure of Bott periodicity in connective E-theory. Finally we give some applications to the homotopy theory of sub-homogeneous algebras and matrix bundles.

Boris Tsygan: Remarks on modules over deformation quantization algebras

Abstract: The theory of modules over algebras of differential operators plays a central role in partial differential equations, representation theory, and geometry. Recently, a wide class of algebras has been studied extensively. These algebras are given by deformation quantization of the algebra of functions on a manifold. They are a far-reaching generalization of the algebra of differential operators. Therefore the structure of modules over them is a natural question. In this talk, we present first results on holonomic modules over a deformation quantization of a symplectic manifold with a metaplectic structure. When the manifold is a cotangent bundle of another manifold, the deformation quantization is closely related to the algebra of differential operators, and our construction yields an asymptotic version of the Hormander's module of distributions whose wave front is the given Lagrangian submanifold. In general, we hope that the category of modules, suitably refined along the lines suggested in the talk, would be related to the objects arising in the theory of Lagrangian intersections, in particular to the Fukaya category of the symplectic manifold.

Guoliang Yu: The coarse Novikov conjecture and convexity

Abstract: In this talk, I explain how the concept of uniform convexity of Banach spaces can be used to study the coarse Novikov conjecture.

Christian Valqui: A categorical approach to excision in bivariant periodic cyclic cohomology

Abstract: This is Joint work with Guillermo Cortinas. We extend the excision theorem of Cuntz and Quillen to algebras and pro-algebras in arbitrary Q-linear categories. For this we define a Waldhausen category structures for various complexes and pro-complexes relevant to cyclic homology and show that this added structure is preserved by them. We establish then a Wodzicki excision theorem for abstract algebras. At last we use the Goodwillie theorem and techniques of the original Cuntz-Quillen proof for periodic cyclic cohomology to get excision in a very general setting.

Christian Voigt: Equivariant cyclic homology

Abstract: We present a general framework in which cyclic homology can be generalized to the equivariant setting. From a conceptual point of view our construction is a (delocalized) noncommutative version of the Cartan model in classical equivariant cohomology. In this talk we focus on actions of discrete groups. We explain in detail the definition of bivariant equivariant periodic cyclic homology HP^G_* in this case. It turns out that HP^G_* is homotopy invariant, stable and satisfies excision in both variables. Moreover analogues of the Green-Julg theorem and its dual hold. We illustrate the general theory by considering group actions on simplicial complexes. In this case equivariant periodic cyclic homology is closely related to a bivariant equivariant cohomology theory which was introduced by Baum and Schneider. As a consequence we see in particular that HP^G_* behaves as expected from the Baum-Connes conjecture.

List of Participants

Baum, Paul (Pennsylvania State University)
Brenken, Berndt (University of Calgary)
Bursztyn, Henrique (University of Toronto)
Connes, Alain (Institut des Hautes Etudes Scientifiques)
Consani, Katia (University of Toronto)
Cuntz, Joachim (Universitaet Muenster)
Dean, Andrew (Lakehead University)
Echterhoff, Siegfried (University Muenster)
Elliott, George (University of Toronto)
Gorokhovsky, Alexander (University of Colorado)
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Chapter 5

Quantum Mechanics on the Large Scale (03w5096)

April 12–17, 2003

Organizer(s): Philip Stamp (University of British Columbia), George Sawatzky (University of British Columbia), Anthony Leggett (University Illinois-Urbana), Timothy Havel (Massachusetts Institute of Technology), Sandu Popescu (University of Bristol), Richard Gill (University Utrecht)

This workshop will bring together people from 2 main areas, viz.,
(i) Physicists working on "macroscopic quantum phenomena", occurring in superconductors, magnets, and other solid-state systems
(ii) Physicists and mathematicians working on "quantum information theory", and quantum computing, and associated problems like decoherence.

In the last year experimental physicists working on superconducting "SQUIDs" have succeeded in preparing superpositions of macroscopic flux states, and work on magnetic systems is not far behind. Apart from the fundamental interest of such macroscopic superpositions, they are of interest to those working in quantum information theory, because it is now believed that only solid state quantum computational or quantum information processing devices will be "scalable" up to sizes where powerful quantum algorithms can be implemented (involving at least 10^4 qubits).

Theoretical and mathematical work in this area concentrates on (i) finding interesting algorithms for quantum computation, and looking at how to implement them using real physical devices (ii) trying to understand the mechanisms of decoherence and how they affect qubit dynamics, and devising schemes for suppressing decoherence (iii) devising error correction codes and understanding how to incorporate them into real devices, and (iv) the mathematical characterisation of the power of quantum algorithms, the degree of entanglement, decoherence measures,etc. Now that solid-state qubits have be made, even materials science problems have become important and of theoretical interest- the field is inevitable becoming rather broad.

The workshop will be bringing together theoretical physicists, mathematicians, and quantum information specialists who wish to focus on this field.

Objectives

There has been great excitement in both the theoretical physics and mathematical communities for some years now, over the possibility of quantum computation and quantum information processing devices. The theoretical interest comes mainly from the promise of enormously enhanced computing power, at least amongst mathematicians and computer scientists. Physicists are excited for an additional reason- the existence of large-scale quantum superpositions, analogous to the famous "Schrodnger's Cat" state, as well as entanglement between many different degrees of freedom in multi-qubit states, is of great fundamental and even philosophical interest, since it challenges basic pre-conceptions about the nature of physical reality.

Until very recently, this discussion was almost entirely theoretical. However macroscopic quantum superpositions of SQUID flux states have now been seen in 2 experiments- the 2 states differing by some 10^8 Bohr magnetons. Large-scale quantum phenomena have also been seen in magnetic systems. This has brought the discussion very much into the real world (as well as silencing many who claimed that such states violated the principles of quantum mechanics and so would never be seen!). It has now become urgent to understand the physical mechanisms of decoherence which have to be tamed, if we are to use superconducting or magnetic qubits to make solid-state quantum computers. The decoherence is typically caused by mobile electrons, as well as by nuclear spins (indeed some of the more esoteric aspects of NMR are becoming important). At the same time the task of harnessing many such qubits together to do computations has become an urgent one. Physicists now need the mathematicians and computer scientists to tell them what sort of processes and algorithms can be used, and how best to design them- and the understanding of decoherence at the multi-qubit entanglement level will require new mathematical ideas. At the same time the mathematicians and computer scientists need to know much more about how to formulate their analyses in a realistic way, since much of the mathematical discussion up to now has centred around idealised and rather unrealistic models. The time is obviously ripe for this meeting of minds- the experimental success has focused everyone's minds in a way that nothing else could have.

The idea of the workshop is to bring the most active people from the various communities involved in this effort, to try and cross some of the bridges between them. These communities are very diverse. Apart from mathematicians involved in quantum information theory and the theory of random processes, and computer scientists working on quantum computational problems, we will bring in theoretical physicists working in solid-state theory (particularly superconductivity and magnetism) in NMR, and in mesoscopic physics. Some of these work on decoherence mechanisms, others on detailed designs for qubits and qubit networks, others on the detailed microscopic theory. Finally we will invite a small number of experimental physicists (maximum 3-4), who will help to focus the meeting on the real world.

The main objectives of the workshop will be (i) to clarify decoherence mechanisms for everyone, and work out strategies for its suppression (ii) to get attendees to devise at least strategies for realistic quantum algorithms, and to work how best these can be realised for real qubits (made from superconductors, magnets, quantum dots, etc.); (iii) to focus attention on the difficult mathematical problem of N-qubit decoherence (ie., for N-qubit entangled states); (iv) to clarify the mathematical techniques used for understanding environmental decoherence, ranging from influence functional techniques to random process theoretical techniques (v) to explore the mathematical limits of quantum computation, and see what this means for real designs, (vi) to look in detail at how error correction codes can be implemented physically, and to look for better ones than those currently in the literature (vii) to better mathematically characterise entanglement and entanglement measures (viii) to understand how theoretical conclusions about decoherence suppression would be reflected in experimental and materials science strategies (ix) to give all the theorists and mathematicians a better appreciation of the experimental picture.

List of Participants

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Chapter 6

Computational Fuel Cell Dynamics-II (03w5039)

April 19–24, 2003

Organizer(s): John Kenna (Ballard Power Systems), Trung Van Nguyen (University of Kansas), Keith Promislow (Simon Fraser University), Brian Wetton (University of British Columbia)

Introduction

The world's major automotive manufactures are engaged in an historic race to develop Proton Exchange Membrane (PEM) fuel cells as clean, high-efficiency alternatives to internal combustion engines for automotive power. PEM fuel cell technology not only holds out the promise of a more environmentally friendly automobile, but also of an extremely versatile power generation system with a broad spectrum of applications.

For such an important application, one rich in interesting phenomena, PEM fuel cells has attracted relatively little interest from modellers, both analytic and computational. Until the last few years, the number of rigorous attempts at modelling fuel cell performance were few (see [10, 6, 8, 12, 4] for some of the "classic" ones). Certainly the field had received little attention from mathematicians. However, the importance of this activity was clear: to develop understanding of fuel cell processes and their interactions; to develop computational fuel cell models to permit faster and more cost efficient design optimization.

The CFCD meeting hosted by PIMS and Ballard Power Systems at Simon Fraser University in June 2001 was an attempt to give focus to this activity. This meeting brought together mathematicians, engineers, and industry representatives in the PEM fuel cell community to exchange expertise and find common ground. It was the goal of the CFCD-II workshop to build upon this effort, uniting researchers from computational and applied mathematics, chemical engineering, mechanical engineering, polymer chemistry, and electrochemical engineering; setting a framework for future research directions, and seeding multi-disciplinary efforts which will lead to the development of a new generation of analytical and computational tools for PEM fuel cell design.

Proton Exchange Fuel Cell Membrane Overview

PEM fuel cells generate electric potential by separating the oxidation of hydrogen into two catalysed steps performed on opposite sides of an electrolyte membrane. The end products are water, water vapour, and heat. A 2D slice of a unit cell is shown in Figure 6.1. The humidified reactant gases, air (the oxidant) and hydrogen (the fuel) flow in channels cut into graphite plates (the 3D structure of the unit cell is shown in Figure 6.2). The gases flow through the gas diffusion electrodes (GDEs) which are often teflonated carbon

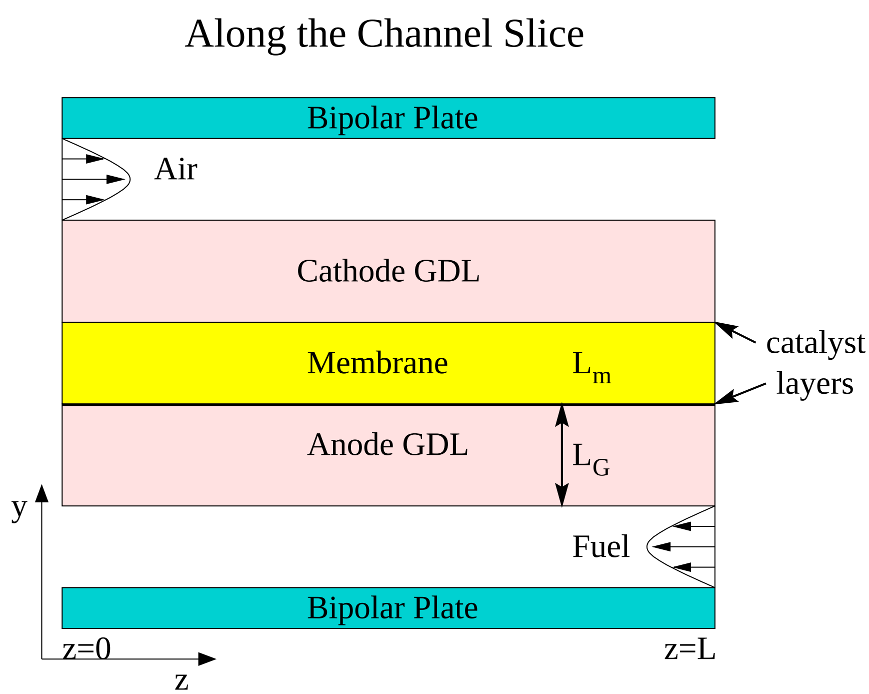

Figure 6.1: Unit Cell Slice.

fibre paper. At the GDE/membrane interface is a Platinum catalyst layer. On the hydrogen (anode) side the catalyst helps the hydrogen more easily disassociate into protons and electrons. The membrane is a good protonic conductor but a very poor electron conductor. The protons can diffuse across the membrane and the electrons go around an external circuit doing useful work. On the other side of the membrane (the cathode) the protons combine with oxygen molecules and the returning electrons to produce water. This reaction is also catalysed by Platinum. It is this reaction which provides the energy to generate the voltage of the circuit.

The unit cell presented here with straight channels is the simplest possible arrangement. In many unit cell designs, the flow fields are arranged in a serpentine way and are the alignment is not the same on anode and cathode. There are additional electrical and thermal coupling effects when unit cells are arranged in series in a fuel cell stack. It should also be mentioned that this type of cell, with pure hydrogen, is only one approach to design.

The electrolyte membrane is a complex polymer comprised of Teflon spines from which typically hydrophilic SO_3^- groups extend. These are arranged in a nanoscale configuration which facilitates the selective diffusivity of the membrane, enabling the fuel cell to perform close to the thermodynamic limit for efficiency. The membrane must be well hydrated to function, however overproduction of liquid water may saturate the surrounding porous electrodes and leads to pronounced drops in power density. The control of the motion and distribution of liquid water in both the nano-structure of the membrane and the surrounding fibrous electrodes is referred to as water management, and is critical to effective cell operation.

Computational Fuel Cell Modelling

A proper modelling of the transport process requires understanding of the interactions of water and ions within the polymer membrane. These issues lead to intriguing mathematical phenomena at the limit of continuum mechanics, including degenerate free-boundary problems requiring novel computational methods. The development of a predictive computational model of water management requires an understanding of the fundamentals of liquid transport in nano-scale pores, in turn demanding development of innovative numerical schemes to adapt to the widely disparate time and length scales present in the system.

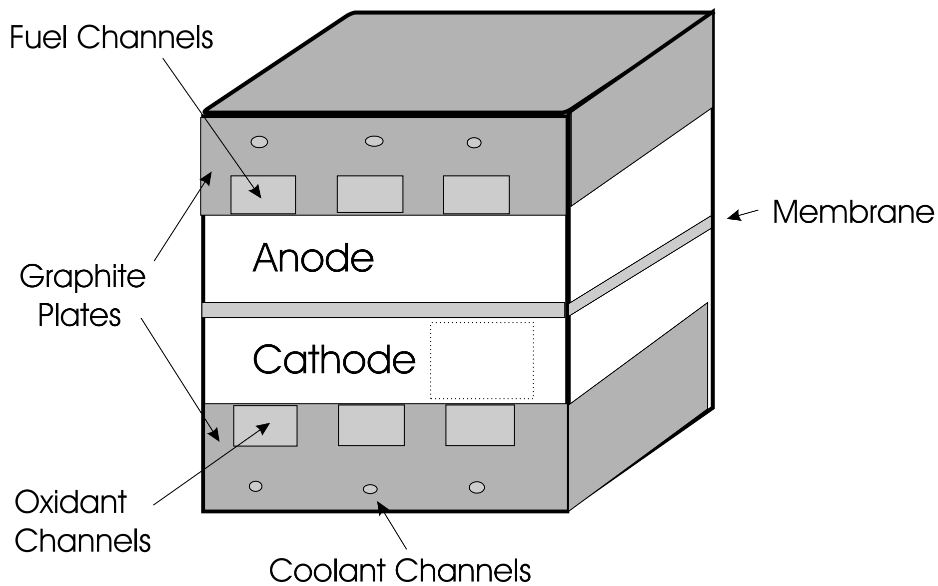

Figure 6.2: The 3D structure of the unit cell.

In the last two years, several large computational fluid dynamics (CFD) code vendors have become interested in developing comprehensive fuel cell computational models. Some examples are the modules developed by CFX [3], StarCD [11] and the more academic FEMLAB [5]. These CFD codes provide convenient 3D meshing and visualization tools and robust solvers for the traditional fluid dynamics elements of fuel cell models. These codes will provide a platform for validated models of elements unique to fuel cells to be integrated into the “big picture”. However, preliminary models suggest that the delicate balance of temperature, condensation and liquid water transport in the GDEs will be difficult to capture accurately in these general packages. Also, larger scale problems such as electrical coupling of cells in stacks and long time transients will have to be handled by specialized codes.

A collaboration between modelling and experimental work is needed in the development of models for fuel cells, as in other similar fields. Experimental work can serve first to guide and validate models and allow parameters to be fit. In turn, models can identify critical parameters that be the subject of experimental measurement or the target for materials engineering.

Some of the more important aspects of fuel cell modelling are listed below.

Mass transport of species in membrane materials: water mobility and proton motion through Nafion and similar PEM products. Some of the questions of interest here have been considered by researchers in biological membranes. Various effects can be considered, ranging from molecular level models, hydraulic pumping, nano-technology and capillary forces.

Condensation and two phase flow models for gas diffusion layers: based on hydrophobicity and capillary forces combined with porosity and permeability factors associated with GDL were presented. This coupling of forces leads to difficulties in predicting water formation within the various regions of the GDL and catalyst areas. These parameters are extremely difficult to measure and correlate to model results.

Capillary, hydrophobicity and rivulet modelling: depicting parameters such as pressure fronts and water droplet formations coupled with various geometry and shapes is an issue. Capillary effects can be a dominating effect when coupled with poorly matched micro-channels and header configuration.

Unit cell modelling: Mathematical solutions with simplified geometry to avoid time consuming CFD solutions. These models can be simplified by using a 2 dimension geometrical model with 1-dimensional transport through the MEA. This ’trick’ allows for quicker solution times assuming minimal channel effects on flow.

Presentation highlights (by alphabetical order)

Jeffery Allen (National Aeronautics and Space Administration) This presentation addressed the need for gravity insensitive fuel cells for space applications. Under these conditions, capillary forces are predominant. However, for terrestrial applications, capillary forces are also important partly due to the small flow channel sizes. Capillary flow in square channels was experimentally investigated and measurements were compared to theoretical analyses. On the basis of the results obtained, further work in several areas were recommended (two phase flow in complex non circular geometries under capillary force control, stability of liquid films, liquid water separation).

Daniel Baker (General Motors) AC impedance was discussed as a potential diagnostic tool (see [2]). to measure in situ local membrane resistance/electrode humidification. It was claimed that separation of the membrane resistance from other resistances (contact, GDL, etc) could be achieved by varying operating conditions). Cable inductance effect at high frequencies was taken into account by adding a small inductance to the model. Current perturbations were used instead of voltage perturbations (more convenient for stack measurements). Measurements were also made with a segmented cell (10 x 10 segments) but only at a single frequency to monitor the membrane resistance. Discrepancies were found between CFD code predictions (StarCD) and measurements which were ascribed to either membrane conductivity values and liquid water presence. The effect of lateral currents between different segments was not assessed.

Jay Benziger (Princeton University) This presentation discussed fuel cell control issues. Most experimental data/models assume either current or voltage control. However, resistance control is more appropriate (a load resistance is changed during transients). This case was studied for a PEMFC CSTR (continuous stirred tank reactor) which is achieved by the use of high stoichiometries and cooling flow rates. A simple model was developed assuming operation in the ohmic regime. Experiments included transients with dry reactants and different initial membrane water contents (obtained with prescribed reactant relative humidities prior to the transient experiments). Interestingly, multiple steady states were found which offer an additional method to probe PEMFC behaviour.

Felix Buchi (Paul Scherrer Institute) A model similar in scope (simple 1+1 D model) to the one presented by Brian Wetton was discussed. However, there were some differences between the models. Buchi used the Stefan Maxwell equations to model gas diffusion in GDLs as opposed to a simple mass transfer coefficient. Also, the electro-osmotic drag coefficient was assumed to be a constant but because of the different membrane behaviour treatment led to the need to discretize the dimension through the membrane which was not necessary in the Wetton's model (analytical equations describe that particular direction).

Ravindra Datta (Worcester Polytechnic Institute) A membrane model was described (water content) but was largely based on a previous paper [7]. The main differences were the inclusion of two modifications: multi-step equilibrium between water molecules and proton, inclusion of a contact angle (claimed to be responsible for the Schroeder's paradox). Also, a membrane conductivity model was also presented which includes the membrane water content model.

Ned Djilali (University of Victoria) A description of a two phase CFD code (steady state, single cell) was discussed. Code convergence was established in 3 steps (no liquid water, liquid water, phase change). The need for a catalyst layer model was highlighted.

Joseph Ferhibach (Worcester Polytechnic Institute) The discussion centred about the use of electrochemical potentials to reduce the number of variables needed for modelling. The molten carbonate fuel cell was previously used to demonstrate the validity of this technique. The PEMFC has not yet been modelled using this approach.

Jurgen Furhmann (WIAS, Berlin) This presentation addressed the mathematical details behind a direct methanol fuel cell model (see next presentation by Klaus Gaertner). This is an academic (not commercial) finite element method (FEM) code that performs sophisticated quasi-Newton iterations using implicit time stepping to achieve fast calculation of the model, which is very stiff due to the liquid water motion in the GDL.

Klaus Gaertner (WIAS, Berlin) A detailed direct methanol fuel cell model was presented (assumptions and physical description of the phenomena). Methanol is fed in the cell as a vapour and as a result some of the conclusions are not directly applicable to PEMFC electrode liquid water accumulation issue at high current densities. However, future work will include an extension to PEMFC. Additionally, the discussion brought up the need to measure the capillary pressure as a function of water content for GDL type materials (a key element to model liquid water in GDLs). This was confirmed by a number of other participants as the only available relationships were derived for soils and related materials which are not directly applicable.

Gerhard Hummer (National Institute of Health) Molecular dynamics was used to understand water movement in pores associated with living cell processes (transfer of water during oxygen consumption along hydrophobic pores). Similar processes occur in kidneys (filtration). Water transport was modelled using carbon nanotubes. Interestingly, proton diffusion in those carbon nanotubes (water transport is one dimensional) occurs 40 times faster than in bulk water. This research could potentially contribute to better designed proton exchange membranes.

Xianguo Li (University of Waterloo) Two models were presented: flow distribution in stacks and cell performance with reformate fuel. The key model parameter was the flow resistance. Model results led to the suggestion of modifying the cell design along the stack to ensure uniform reactant distribution to each cell (impractical from a manufacturing point of view). This work has not yet been published. As for the reformate fuel model, the computed results highlighted at least one deficiency as air bleed values of up to 30% were required to manage trace amounts of CO.

Tim Myers (University of Cape Town) This presentation addressed thin film flows. However, pressure driven flows were not discussed but are considered as future work.

Boaz Nadler (Yale University) This presentation also addressed ion movement in protein channels with the objective to predict their function if the structure is known. The discussion centred around the determination of the proper equations to describe the physical phenomena (extension of the Boltzmann distribution/ Langevin treatment to continuity equations). In effect, the author is trying to include molecular structure effects into a continuum description. This treatment could benefit proton exchange membrane modelling activities.

Stephen Paddison (Los Alamos National Laboratory) This presentation was a summary of previous work aimed at the understanding of membrane properties (electro-osmotic drag, water content and transport, high temperature membranes with low water content requirements) using several techniques which are length scale dependent (ab initio computations, molecular dynamics, statistical thermodynamics) and which can lead to experimental correlations between molecular structures and material properties, and material design specifications. Both Nafion and PEEK (poly-ether-ether-ketones) chemistries were discussed. Nafion side chains do not interact with water (hydrophobic). However, if the side chain is an hydrocarbon, interactions with water are present (hydrophilic). The dissociation of the hydrogen atom from the sulfonate site does not occur with only 1 or 2 water molecules per sulfonate site. Hydrogen dissociation occurs with 3 or more water molecules per sulfonate site. With 6 water molecules per sulfonate site, the proton is separated from the sulfonate site negative charge. Lower water contents at which proton separation occur are better to achieve conductivity at low reactant relative humidities. From this point of view, the separation between proton and sulfonate site is greater for Nafion than PEEK (the strong base behaviour of the PEEK aromatic ring imparts an hydrophilic behaviour which reduces separation). As a consequence, low strength base functionalities should be considered to ensure maximum membrane conductivity. Good agreement was obtained between measured and computed proton diffusion coefficients.

Reginald Paul (University of Calgary) This researcher conducts joint work with Stephen Paddison (modelling proton movement within membrane channels).

Keith Promislow (Simon Fraser University) This presentation set the stage for the conference summarizing fuel cell phenomena and physical models used as representations. There was a brief discussion about the water concentration difference across the membrane which generates a potential difference (about 10 mV). The gap existing between the engineering and modelling world was also highlighted (for example, the use of mathematical approximations which do not necessarily represent physical phenomena). Evidence for the existence of this gap include the difficulty in finding people who are knowledgeable about modelling and experimental aspects of fuel cells. Finally, the need to relate measurement parameters to model variables was highlighted (need to relate macroscopic parameters to microscopic parameters).

Isaak Rubinstein (Ben Gurion University) This presentation addressed a membrane modification for separation processes [9]. The detailed mathematical analysis of ionic transport was new to almost everyone in the audience.

John Stockie (University of New Brunswick) A model for the water content of the electrodes was presented but is not currently validated.

Trung Van Nguyen (University of Kansas) This presentation targeted the use of inter-digitated flow field to characterize the presence of liquid water in GDLs (increased pressure drop). References electrodes were also used (GDEs on the reference electrode side were wider and offset by more than three times the membrane thickness with respect to the GDE on the other side to ensure minimum disturbance [1]). Transient experiments were used and, overpotentials as well as pressure drops were monitored which showed simultaneous increases in value (ascribed to liquid water accumulation). The liquid water accumulation process is slow (approximately 30 minutes).

John Van Zee (University of South Carolina) This presentation summarized efforts to develop a CFD tool to describe PEMFCs. Some model features were questioned or are debatable. An interaction between reactant humidity and cell compression was mentioned. StarCD [11] includes transient functionality coupled with the presence of liquid water (cases with high levels of liquid water were not run). This liquid water is modelled as a thin film. The quantity of liquid water present in cells was not measured. The model is useful to predict cell behaviour but is not currently fully validated. The model showed that liquid water localization is dependent on flow field design. 2° C difference between channel and membrane was also reported.

Brian Wetton (University of British Columbia) This presentation summarized the development of a simple PEMFC model co-developed with Ballard. Henry's law coefficient, presently assuming water as the solvent, was questioned (value should be revised assuming Nafion as the solvent). It was also suggested that model fitting could be carried out in two steps for both low and high current densities.

Jinchao Xu (Pennsylvania State University) This presentation addressed general numerical issues related to adaptive mesh refinement and efficient (multi-grid) solution techniques for a general class of partial differential equation problems. The exciting developments in so-called algebraic multigrid, in which improved solvers can be found automatically from existing discretizations, were outlined. Several impressive examples were shown of the effectiveness of this technique in speeding up the solution of existing discretized problems.

A Personal Note from the Organizers

All the participants thoroughly enjoyed their stay at the BIRS facility. The venue is superb and the scenery, the hospitality and the food were fantastic. The cachet of Banff brought a number of the more important participants in the field that might not otherwise have attended. The beauty (and isolation) of the location kept many there longer than they had originally planned. The scientific discussions continued long after the talks were over, in the meeting rooms, the hiking trails, and by the "donation" fridge. It was a meeting that will shape the future of our fledgling field for years to come. Our thanks to the staff and directorship of BIRS for this opportunity.

List of Participants

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Chapter 7

The Many Aspects of Mahler's Measure (03w5035)

April 26–May 1, 2003

Organizer(s): David Boyd (University of British Columbia), Doug Lind (University of Washington), Fernando Rodriguez Villegas (University of Texas at Austin), Christopher Deninger (Universität Münster)

Introduction

The idea behind the workshop was to bring together experts specializing in many different fields: dynamical systems, K-theory, number theory, topology, analysis, to explore some of the many apparently different ways that Mahler's measure appears in different areas of Mathematics. The hope was to encourage cross-fertilization between these disciplines and increase our understanding of Mahler's measure.

Our plans did not anticipate that on the day that we were to gather for our workshop, nature would decide to dump 60 cm of snow on Calgary, shutting down Calgary International Airport to most incoming flights. As a result, the workshop started a day late with a somewhat diminished attendance. Fortunately, Dale Rolfsen had arrived in Banff a day early and was able to take over until some of the organizers were able to arrive. In spite of the late start, everyone who had planned to speak was able to give a lecture and the outcome exceeded all expectations, as we will see below.

It should be mentioned that many of the participants and many others, including students and junior faculty, took the opportunity to continue their study of Mahler's measure at a PIMS conference in June at Simon Fraser University, organized by Peter Borwein and Stephen Choi. A highlight of this conference was the series of four 90 minute lectures by Jeff Vaaler, each treating a different aspect of Mahler's measure.

Given a polynomial $P(z_1, \ldots, z_n)$ with complex (or for us usually integer) coefficients, the *logarithmic Mahler measure* is defined to be the average of $\log |P|$ over the real n-torus, i.e.

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e(t_1), \ldots, e(t_n))| \, dt_1 \ldots dt_n,$$

where as usual $e(t) := \exp(2\pi i t)$. The quantity actually defined by Mahler was $M(P) = \exp(m(P))$, i.e. the geometric mean of $|P|$ over the n-torus, but it seems that $m(P)$ is really the more fundamental quantity. We will simply refer to $m(P)$ as the Mahler measure of P. Below we will always assume that P has integer coefficients unless otherwise stated. There is no harm in allowing P to be a Laurent polynomial, i.e. a polynomial in $z_1, 1/z_1, \ldots, z_n, 1/z_n$ since these can be converted to ordinary polynomials by multiplication by a monomial in $(z_1, \ldots, z_n)$ and a monomial has logarithmic Mahler measure 0.

Polynomials in one variable

For polynomials in one variable, Jensen's formula shows that if $P(z) = a_0(z-\alpha_1)\dots(z-\alpha_d)$, one has $M(P(z)) = |a_0|\prod_{j=1}^{d}\max(|\alpha_j|,1)$.. If P has integer coefficients, this shows that $M(P)$ is an algebraic integer so $m(P)$ is the logarithm of an algebraic integer. It also shows that $m(P) \geq 0$ and that if $m(P) = 0$ then P is a monic polynomial all of whose roots are roots of unity (briefly, a *cyclotomic* polynomial).

The quantity $m(P)$ for polynomials in one variable occurs naturally in many problems of number theory or dynamical systems, as a growth rate or an entropy. Lehmer [19] encountered $m(P)$ in his study of the integer sequences $\Delta_n = \mathrm{Res}(P(z), z^n - 1) = \prod_{j=1}^{d}(\alpha_j^n - 1)$ for monic P. It is clear that if P does not vanish on the circle then $\lim |\Delta_n|^{1/n} = M(P)$ and this is true even if P does vanish on the circle, but the proof is certainly not obvious. Lehmer was led by this to ask whether there is a constant $c > 0$ such that $m(P) > c$ provided P is not cyclotomic. This question of Lehmer is still unanswered and provided early motivation for the study of $m(P)$. The positive answer is the one expected and is known as ***Lehmer's conjecture*** (although Lehmer did not conjecture this in print). Lehmer provided an example of the polynomial $L(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$ for which $m(L) = 0.162357\dots$ is the smallest value known for non-cyclotomic polynomials. Note that this polynomial is reciprocal (palindromic). Smyth [29] showed in his Ph.D. thesis that if P is non-reciprocal then the smallest value attainable is $m(z^3-z-1) = 0.281199\dots$, almost twice as large.

Lehmer's conjecture is not yet proved and is still a question of much interest. An important and useful result of Edward Dobrowoski [13] gives the estimate $m(P(z)) \geq c(\log\log n/\log n)^3$ for a non-cyclotomic polynomial in one variable with degree n, where $c > 0$ is an explicit constant. This result does not extend to polynomials in many variables, but there is another estimate also due to Dobrowolski [14] that gives an explicit lower bound for $m(P(z))$ depending only on the number of terms in the polynomial. Using the limit theorem described in the next section, this does extend to polynomials in many variables giving exactly the same estimate.

At the workshop, Peter Borwein described work with Kevin Hare and Michael Mossinghoff [2] giving the explicit (and best possible) lower bound of $(1+\sqrt{5})/2$ for the Mahler measure of non-reciprocal polynomials with odd integer coefficients, applying in particular to Littlewood polynomials (those with all coefficients in $\{-1,1\}$. The question of how to extend this to reciprocal polynomials was partially settled during the workshop in a collaboration between Borwein, Mossinghoff and Dobrowolski [3]. They succeeded in proving the result for reciprocal polynomials which have no cyclotomic factor, giving an explicit lower bound which they do not believe to be best possible. The proof uses a clever choice of some auxiliary polynomials and work continues to find an even more clever choice which will improve the estimate.

Jeff Vaaler and his student Shey-Jey Chern [9] have recently proved some remarkable results about the distribution function of the values of the Mahler measure of a polynomial in one variable considered as a function of its coefficients. The methods bear a family relationship to those from the geometry of numbers, but here the fundamental shape is not a convex body. He described these results in one of his lectures at the conference at SFU in June. His student Christopher Sinclair [28] has proved analogous results restricted to the set of reciprocal polynomials of a given degree and spoke on this at the BIRS workshop.

Many Variable Polynomials

Mahler [22] introduced his measure for polynomials in many variables as a device to provide a simple proof of Gelfond's inequality for the product of polynomials in many variables. This uses the obvious $M(PQ) = M(P)M(Q)$ together with the fact that the coefficients of a polynomial can be bounded from above in terms of $M(P)$. This is still an important tool in transcendence theory.

However, it became apparent in the late 1970's that $m(P)$ has has a much more fundamental significance. In studying the *spectrum* (range) of $m(P(z))$ for polynomials in one variable, Boyd noticed that if $P(z_1,\dots,z_n)$ is a polynomial in many variables, then there are sequences of one variable polynomials $P_{\mathbf{a}} = P(z^{a_1},\dots,z^{a_n})$ for which $m(P_{\mathbf{a}}(z))$ converges to $m(P(z_1,\dots,z_n))$ provided $\mathbf{a}\to\infty$ in a suitable way. Some special cases were proved in [4] and the most general case was proved by Lawton [18]. This shows that in order to study the set of values of $m(P(z))$, it is natural to look at the larger set $\mathbb{L}$ of values of $m(P(z_1,\dots,z_n))$ as P varies over all polynomials with integer coefficients in an arbitrary number of variables. It was conjectured in [4] that the set $\mathbb{L}$ is closed. A trivial consequence of this would be a proof of Lehmer's conjecture but without any explicit lower bound.

Lind, Schmidt and Ward [20] used Lawton's limit theorem in their proof that $m(P(z_1,\dots,z_n))$ is the topological entropy of a $\mathbb{Z}^n$ action defined by P. At the workshop, Ward explained how topological entropy is defined, for the benefit of those working in other areas. So the set $\mathbb{L}$ is the set of entropies of these actions and hence occurs in nature. In the final lecture of the workshop, Lind explained what Lehmer's problem should be in the context of compact abelian groups, where the integration over the torus is replaced by integration over the group. He gave an interesting computation showing that the *Lehmer constant* for the group $\mathbb{T}\times(\mathbb{Z}/2\mathbb{Z})$ is strictly less than $m(L(z))$, the presumed minimum for the group $\mathbb{T}$.

Explicit Formulae – 2 variables

At the time Boyd was formulating these ideas it was fortunate that Chris Smyth was visiting on a sabbatical and became intrigued by the question of finding explicit formulae for $m(P)$ for polynomials in more than one variable. During that time, he provided a proof of the remarkable formulae

$$m(1+z_1+z_2)=\frac{3\sqrt{3}}{4\pi}L(2,\chi_{-3}),$$

where $L(s,\chi_{-3})$ is the L-function for the odd quadratic character of conductor 3, and

$$m(1+z_1+z_2+z_3)=\frac{7}{2\pi^2}\zeta(3),$$

where ζ is the Riemann zeta function [30]. Note that in these formulae the measure of a certain 2 variable polynomial is a *di*logarithm and the measure of a certain 3 variable polynomial is a *tri*logarithm, whereas the measures of one variable polynomials are all *uni*logarithms. Since such quantities are widely believed to be algebraically independent, this surely suggests that something deeper is going on.

The recent revival in the interest in $\mathbb{L}$ was due to Christopher Deninger [12], who showed that if $P(z_1,\dots,z_n)$ does not vanish on the n-torus then $m(P)$ is a Deligne period of the motive associated to the variety defined by $P=0$. Thus in this case, $m(P)$ is related to Beilinson's higher regulators. In that same paper, Deninger responded to a challenge from Boyd and produced the following remarkable conjecture:

$$m(1+z_1+1/z_1+z_2+1/z_2)=r\frac{15}{4\pi^2}L(E,2),$$

where E is the elliptic curve of conductor 15 defined by $1+z_1+1/z_1+z_2+1/z_2=0$ and r is an (unspecified) rational number. Calculations verify this formula is correct with $r=1$ to 100 decimal place accuracy. Note that once again we have a dilogarithm of a sort, but here an elliptic dilogarithm.

In a large scale numerical experiment, Boyd [5] computed the measures of polynomials of the form $P_k(z_1,z_2)=k+Q(z_1,z_2)$ where Q is a Laurent polynomial and k is an integer parameter and found families of (conjectural) formulae of this general type relating $m(P)$ to $L(E,2)$, where $P=0$ is of genus 1 or 2 and E is a factor of the Jacobian variety of this curve. The conjectured formulae are true to many decimal place accuracy but for the most part have not yet been rigorously proved. Such formulae fall into the general framework of *periods* introduced by Knotsevich and Zagier [16]. If their meta-conjecture is correct, all these formulae should be provable by *elementary calculus*.

Rodriguez-Villegas [25] carried this study a step further and produced formulae based on the theory of modular forms for $m(P_k)$ for many of the families considered in [5]. He thus obtained expressions for the $m(P_k)$ as Kronecker–Eisenstein series and hence very rapidly converging series for the $m(P_k)$ in terms of a modular parameter. In favourable circumstances, this has allowed him to give rigorous proofs for some of the formulae in [5]. He also made explicit the connection between m and the regulator map from the K-group $K_2(E)$ to $\mathbb{R}$.

Hyperbolic volume

In a different but related direction, again motivated by Smyth's formula for $m(1+z_1+z_2)$, one recognizes that the quantity $3\sqrt{3}L(2,\chi_3)/(4\pi)$ appearing there is, except for the factor π in the denominator, the volume of an equilateral ideal hyperbolic tetrahedron. The set of volumes of hyperbolic 3-manifolds has a well known structure studied by Thurston and Jorgensen. In particular it has a non-zero minimum. An intriguing idea is then that there may be a polynomial P in two variables related to each hyperbolic 3-manifold M in some way so that the relationship $\pi m(P) = \mathrm{vol}(M)$ holds. Perhaps one could go in the other direction from P to M with suitable restrictions on P. For example, what is the manifold related to $1 + z_1 + z_2$?

There are of course many polynomial invariants connected to manifolds. One promising candidate for the relation $\pi m(P) = \mathrm{vol}(M)$ is the so-called A-polynomial $A(x, y)$ defined in [11] for every one-cusped hyperbolic 3-manifold. This does indeed have an intimate connection with volume as explored in [6] and under special circumstances one really does have $\pi m(A) = \mathrm{vol}(M)$, but this is true only under some rather special circumstances. In general all that one can say is that $\pi m(A)$ can be written as the sum of values of the Bloch–Wigner dilogarithm function $\mathcal{D}(z)$ evaluated at certain algebraic numbers (the shapes in certain *pseudo-triangulations* of the manifold) [7]. This result is ultimately a consequence of Schläfli's famous formula for the differential of the volume of a deformed polyhedron.

This connection between $m(A)$ and volume has some bearing on a conjecture of Chinburg that, given an odd primitive quadratic character χ, there is a polynomial $P_\chi(z_1, z_2)$ and a rational number r_χ such that $m(P_\chi) = r_\chi L'(-1, \chi)$, which would generalize Smyth's formula for $P = 1 + z_1 + z_2$. (The relation between $L'(-1, \chi)$ and $L(2, \chi)$ is due to the functional equation for $L(s, \chi)$ and nicely takes care of the factor $d^{3/2}/(4\pi)$ that would otherwise occur. However, it does disguise the fact that the right hand side is a dilogarithm.) At the workshop, Rodriguez-Villegas discussed the proof of a special example of this sort, where one can deduce from K-theory (the theory of the Bloch group) that $m(P(z_1, z_2)) = rL'(-1, \chi)$ for a certain P and χ but with an unspecified rational number that is not computable from a theorem of Borel that is used in the proof. However, in this example one is able to prove that $r = 1/6$ by showing that P is in fact the A-polynomial of a certain hyperbolic manifold constructed by Nathan Dunfield [8].

At the workshop, a number of talks had a bearing on this. Adam Sikora lectured on the A-polynomial and the coloured Jones polynomial J_n of a knot, a *quantum invariant* of a knot. There is a conjecture of Kashaev [15] that the volume of a hyperbolic knot complement can be recovered from the limit of the values of the coloured Jones polynomial evaluated at roots of unity. This has been proved for a few simple knots by direct calculation. Murakami [23] has made the related conjecture that $2\pi m(J_n)/\log n$ converges to the volume of the knot complement as $n \to \infty$. This has not been verified for any knot. It would be very interesting to prove this result at least for those knots for which $\pi m(A)$ is the volume, by relating the coloured Jones polynomial to the A-polynomial.

More Hyperbolic geometry

The Chern-Simons invariant can be considered as a complexification of the Volume. Walter Neumann lectured on the connection between the Chern-Simons invariant and a generalization of the Bloch group which requires a complexification of the Bloch–Wigner dilogarithm [24]. One would like to be able to calculate the Chern–Simons invariant by an integration of some polynomial computable from a triangulation of the manifold similar to that defining Mahler's measure. Since $m(P)$ is real-valued, it is apparent that a complex-valued generalization of $m(P)$ would be required for this purpose. Perhaps the Ronkin function from the new theory of *amoebas* provides a clue [32].

Another polynomial invariant of a 3-manifold is the Alexander polynomial. For example, Lehmer's polynomial turns out to be the Alexander polynomial of the pretzel knot $P(-2, 3, 7)$. At the workshop, Dan Silver and Susan Williams discussed the Alexander polynomial of the complement of a link with n components [26, 27]. This is an n-variable Laurent polynomial. Susan Williams gave a geometric interpretation of the Mahler measure of the Alexander polynomial in terms of the homology of cyclic covers, relating this to the dynamical systems results of Lind, Schmidt and Ward. The fact that Lehmer's polynomial arises in this way suggests that perhaps other small measure polynomials can be constructed in this way, perhaps opening the way to a proof of Lehmer's conjecture.

Remarkably, many of the polynomials identified in [4] as being small limit points of $\mathbb{L}$ turn out to arise as Alexander polynomials of simple links. It was conjectured in [4] that Smyth's numbers $m(1 + z_1 + z_2)$ and $m(1 + z_1 + z_2 + z_3)$ are the minimal elements of the 2nd and 3rd derived sets of $\mathbb{L}$, (the n-th derived set is the set of limit points of the $(n-1)$-st derived set). It would be natural to speculate that $m(1 + z_1 + \cdots + z_n)$

is the smallest element of the n-th derived set for $n \geq 2$, but Silver and Williams cast some doubt on this with their example of the Alexander polynomial of a minimally twisted 4-link which is $1 + z_1 + z_2 + z_3 + z_4 - z_1^{-1} z_2^{-1} z_3 z_4$, which seems to have slightly smaller measure than that of $1 + z_1 + z_2 + z_3 + z_4$. The difficulty in numerically computing the Mahler measure of 4-variable polynomials makes this difficult to verify numerically.

Explicit Formulae – many variable polynomials

Apart from Smyth's formula for $m(1 + z_1 + z_2 + z_3)$, there were for many years no explicit formulae for the Mahler measure of polynomials in more than 2 variables. The situation has changed drastically in recent years. The first formula of this type was again proved by Smyth, [31], another completely different 3 variable polynomial whose Mahler measure is a rational multiple of $\zeta(3)/\pi^2$. He spoke on this and other examples at the workshop and provided the example

$$(z_1 + 1/z_1) \cdots (z_{n-2} + 1/z_{n-2}) + 2^{n-3}(z_{n-1} + z_n),$$

of a polynomial whose Mahler measure can be expressed in terms of an n-logarithm. The workshop provided a forum for further progress along these lines.

In his paper [25] devoted to modular formulae for families of elliptic curves, Rodriguez-Villegas mentioned that the same method would apply to certain modular K3 surfaces, for example that defined by the polynomial $(z_1 + 1/z_1)(z_2 + 1/z_2)(z_3 + 1/z_3) + k$. Marie José Bertin [1] has followed up this suggestion and worked out all the details of two examples different from this. Using results of of Verill, she was able to compute Kronecker–Eisenstein series for the Mahler measure for polynomials

$$P_k = z_1 + 1/z_1 + z_2 + 1/z_2 + z_3 + 1/z_3 - k.$$

For certain values of k the explicit formula can be expressed in terms of the value at $s = 3$ of certain Hecke L-series and modular forms. She spoke on this work at the BIRS workshop and also the SFU conference.

Matilde Lalín described a new approach to evaluating the Mahler measure of some families of polynomials in many variables in terms of polylogarithms [17]. A highlight of this is the formula

$$\pi^4 m((1 + z_1)(1 + z_2)(1 + z_3) + (1 - z_1)(1 - z_2)(z_4 + z_5)) = 93\zeta(5).$$

The connection between the Mahler measure of some 2-variable polynomials and the volume of hyperbolic 3-manifolds leads one to wonder whether this can be extended to higher dimensions. Ruth Kellerhals gave an instructive introduction to hyperbolic volume in higher dimensions explaining how the parity of the dimension plays an important role. The fact that the volume of polyhedra in hyperbolic 5-space can be expressed in terms of polylogarithms of order ≤ 3 suggests a possible connection with $m(P)$ for polynomials in 3 variables. She presented, for example, an orthoscheme with 5-dimensional volume $5\zeta(3)/4608$. Is there a polynomial $P(z_1, z_2, z_3)$ constructible from this orthoscheme for which $\pi^2 m(P(z_1, z_2, z_3)) = 5\zeta(3)/4608$?

Mahler measure and motivic cohomology

In an inspiring lecture, Vincent Maillot went beyond Deninger's [12] framework to provide an explanation of many of the formulae presented by other speakers in terms of the cohomology of the varieties defined by the polynomials. His approach is particularly successful in the case of non-reciprocal polynomials and explains the difference between formulae in which higher L-functions such as $L(E, s)$ appear and formulae in which only only polylogarithms appear. The point is that the Mahler measure only detects the intersection of the variety $P = 0$ with the real torus, and hence the quantities that appear in the right hand side of the formulae should be related to the variety that is the intersection of $P(z_1, \ldots, z_n) = 0$ and $P(1/z_1, \ldots, 1/z_n) = 0$, (an observation that Maillot attributed to Darboux (1875)).

After the workshop, Rodriguez-Villegas pondered what this would mean for the simple polynomials $1 + z_1 + \cdots + z_n$ and came up with the remarkable conjectures that

$$m(1 + z_1 + \cdots + z_4) = L'(f_4, -1),$$

where f_4 is a normalized cusp form of weight 3 and conductor 15, and

$$m(1 + z_1 + \cdots + z_5) = 4L'(f_5, -1),$$

where f_5 is a normalized cusp form of weight 4 and conductor 6. (Note that the functional equation relates these to the values $L(f_4, 4)$ and $L(f_5, 5)$, as one would expect). He verified these results numerically to 28 decimal places. It is a non-trivial problem to compute these Mahler measures numerically. Fortunately, Rodriguez-Villegas, Tornaria and Vaaler had just recently developed a series for $m(1 + z_1 + \cdots + z_n)$ which gave sufficient numerical accuracy for the purpose. (This series was the topic of one of Vaaler's lectures at the SFU conference). These formulae do not seem to extend to $n \geq 6$ because the corresponding space of cusp forms has dimension greater than 1.

In a different direction, Rodriguez-Villegas and Boyd made a search for families of non-reciprocal polynomials in 3 variables for which the intersection of $P(z_1, z_2, z_3) = 0$ and $P(1/z_1, 1/z_2, 1/z_3) = 0$ is an elliptic curve E, in which case it is conceivable that $m(P)$ could be expressible in terms of $L(E, 3)$, that is, as a rational multiple of $L'(E, -1) = (N^2/(8\pi^4))L(E, 3)$, where N is the conductor of E. We found 5 examples where to 40 decimal place accuracy, such formulae seem to be true. For example,

$$m((z_1 + 1)^2 + z_2 + z_3) = L'(E_{24}, -1),$$

where E_{24} denotes an elliptic curve of conductor 24. None of these have yet been rigorously proved.

In one degenerate case, the intersection variety was not an elliptic curve but rather a rational curve, suggesting perhaps a formula in terms of $\zeta(3)$, and leading to the conjecture

$$m((1 + z_1) + (1 - z_1)(z_2 + z_3)) = \frac{28}{5}\frac{\zeta(3)}{\pi^2}.$$

In spite of the intriguing resemblance to some of Lalín's formulae, it appears not to be accessible by her method. Boyd presented this at a lecture at the SFU conference attended by John Condon, a student of Rodriguez-Villegas. Remarkably, a few months later, Condon had found a classical but extremely ingenious (and *long*) proof of this identity which forms the basis for his recent Ph.D. thesis [10].

Conclusions

In the space of 4 days at BIRS, many ideas were exchanged and collaborations formed. The few examples presented above should be evidence of the value of this sort of workshop. Continuing in the spirit of international collaboration, Marie José Bertin and Vincent Maillot were inspired to organize a similar gathering at CIRM in Luminy in May of 2005. It is to be hoped that the research inspired by such workshops will continue to unravel the mysteries of Mahler's marvellous measure.

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Chapter 8

Recent Advances in Algebraic and Enumerative Combinatorics (03w5009)

May 3–8, 2003

Organizer(s): Sara Billey (University of Washington), Ian Goulden (University of Waterloo), Curtis Greene (Haverford College), David Jackson (University of Waterloo), Richard Stanley (Massachusetts Institute of Technology)

A short overview of the subject area

Algebraic and enumerative combinatorics is concerned with objects that have both a combinatorial and an algebraic interpretation. It is a highly active area of the mathematical sciences, with many connections and applications to other areas, including algebraic geometry, representation theory, topology, mathematical physics and statistical mechanics. Enumerative questions arise in the mathematical sciences for a variety of reasons. For example, non-combinatorial structures often have a discrete substructure when they are studied up to topological equivalence. Moreover, some non-combinatorial structures can be discretized, and are therefore susceptible to combinatorial techniques, and the original question is then recovered by the appropriate limit taking (the dimer problem in statistical mechanics is an instance of this).

Algebraic and enumerative combinatorics is a highly active area of the mathematical sciences, and was the subject of a special year at MSRI in 1996/97, organized by Billera, Bjorner, Greene, Stanley and Simion. The one week workshop at BIRS focused on three topics, discussed in detail below, in which substantial progress has been made in the last five years. The topics are interrelated, as is essential for a workshop of this length. The workshop was successful in bringing together a very strong international collection of active researchers in algebraic and enumerative combinatorics, as well as other areas of the mathematical sciences in which substantial enumerative questions with a strong algebraic or analytic foundation have arisen. The connections that have been made between algebraic combinatorics and other areas in the last few years have contributed to rich and significant lines of research. Such research requires a familiarity of several research disciplines, and the workshop facilitated this contact between algebraic and enumerative combinatorics and other fields in an effective way.

Symmetric Functions and Group Representations

Symmetric functions and group representations have played a central role in algebraic combinatorics. This was initiated by the work of Philip Hall [17], which recast much of the classical theory of symmetric functions in terms of linear algebra. Central to this theory is the Schur function, and its generalizations. These include zonal polynomials, Jack symmetric functions, Macdonald symmetric functions, shifted and super Schur functions, and Schubert polynomials. These functions have many equivalent definitions, which bring together different areas of mathematics. For example, on the algebraic side, they are generating functions (via a change of basis) for characters of group representations: for the Schur functions themselves, these are the symmetric group and the complex general linear group; for the super Schur functions, these are Lie superalgebras; for the Schubert polynomials, the group of complex upper triangular matrices. On the combinatorial side, they are generating functions in the monomial basis for two dimensional arrays, called Young tableaux in the case of Schur functions.

The Littlewood-Richardson coefficients are the connection coefficients for expanding the product of two Schur functions into a linear combination of Schur functions. These must be non-negative integers because of their algebraic interpretation as the multiplicity of the irreducible representations of the symmetric group in the tensor product of two irreducible representations. The Littlewood-Richardson rule is a classical combinatorial rule for determining this coefficient, as the number of sequences with certain properties. A new rule for these coefficients, called the honeycomb model, has been used recently by Knutson and Tao [23] to prove the Saturation Conjecture. This result concerns the places where the Littlewood-Richardson coefficient is 0, but relates to a number of other topics, including Horn's Conjecture for the eigenvalues of a sum of Hermitian matrices (shown by Klyachko [19]). A recent paper of Fulton [11] gives a detailed discussion of various situations in which Littlewood-Richardson coefficients play an important role.

Recently, Lapointe and Vinet [25] proved that the coefficients of the Jack symmetric functions are polynomials in the Jack parameter. Knop and Sahi [22] improved this result by showing that the coefficients of the polynomial are nonnegative (conjectured by Macdonald and Stanley).

Goulden, Harer and Jackson [15] have shown, through the use of matrix integrals, that the Jack parameter may be associated with a topological invariant of embeddings of graphs in surfaces and with the Euler characteristic of the moduli space of curves. There is also a connection with finite reflection groups, through the work of Beerends and Opdam [5], and Beerends [3], and with Hilbert schemes, through the work of Nakajima [26]. (The link, from de Concini and Procesi, is that the moduli space can be viewed as a complex vector space with certain hyperplanes removed.) These bring us closer to a combinatorial interpretation of the Jack parameter. Moreover, Knop and Sahi [21] have recently given a tableau interpretation for the Jack function.

Schubert polynomials originally arose in the work of Demazure and Bernstein-Gel'fand-Gel'fand describing the cohomology of flag varieties. The combinatorial theory of in the case of the symmetric group was developed by Lascoux and Schutzenberger and extended by Macdonald, Billey and Haiman, Fomin and Kirillov, Fulton, Pragacz and Ratajski, and others. Many combinatorial problems remain, including finding a generalization of the Littlewood-Richardson rule, for multiplying Schubert polynomials. Also desirable are better formulae for the quantum cohomology, equivariant cohomology and K-theory analogs of Schubert polynomials.

Macdonald polynomials are a two-parameter generalization of Schur functions that also include Hall-Littlewood polynomials, Jack polynomials and zonal polynomials as special cases. Work on these initially focused on Macdonald's 1989 conjectures about the polynomiality and non-negativity of the entries in a certain change of basis matrix, but has expanded into various areas because of the the relationship of these polynomials to, e.g., the representation theory of quantum groups, affine Hecke algebras, and the Calogero-Sutherland model in particle physics [31]. Perhaps the greatest interest in these polynomials has been raised by the manner in which they arise in the study of the diagonal action of the symmetric group on polynomials in two sets of variables. The so-called "n! Conjecture" of Garsia and Haiman has been the most notable result in this area, recently proved by Haiman [16], based on the geometry of the Hilbert scheme of n points in the plane. A number of refinements and extensions of the n! Conjecture (e.g., by F. Bergeron, Garsia, Tesler) are under active study. (The connection with Macdonald polynomials is that the coefficients in the change of basis matrix of Macdonald's 1989 conjectures are thus identified as multiplicities matrix of irreducible representations in a particular representation of the symmetric group, and thus must be non-negative.)

The Littlewood-Richardson coefficients are the connection coefficients for expanding the product of two Schur functions into a linear combination of Schur functions. These must be non-negative integers because of their algebraic interpretation as the multiplicity of the irreducible representations of the symmetric group in the tensor product of two irreducible representations. The Littlewood-Richardson rule is a classical combinatorial rule for determining this coefficient, as the number of sequences with certain properties. A new rule for these coefficients, called the honeycomb model, has been used recently by Knutson and Tao [23] to prove the Saturation Conjecture. This result concerns the places where the Littlewood-Richardson coefficient is 0, but

relates to a number of other topics, including Horn's Conjecture for the eigenvalues of a sum of Hermitian matrices (shown by Klyachko [19]). A recent paper of Fulton [11] gives a detailed discussion of various situations in which Littlewood-Richardson coefficients play an important role. There has been a flourish of activity related to these coefficients in the past year which was a central focus of the workshop. In particular, Fomin, Fulton, Kleber, Knutson, Shimozono, Vakil have made substantial contributions in this area, mostly from the geometric point of view. Vakil's talk on a new geometric realization of the Littlewood-Richardson rule inspired several new directions of generalization. Fomin's talk included two interesting conjectures on these coefficients; the first was easily shown to be false during the meeting but the second one remains open.

Ramified Covers of Surfaces and Hurwitz numbers

There has been a considerable amount of recent work on Hurwitz numbers, enumerating ramified covers of the sphere by surfaces of given genus. The covers are such that there is a unique point with arbitrary ramification type, the remaining branch points having elementary ramification. Work on this problem started with Hurwitz in 1895 and interest in it was reawakened largely by the work of the physicists Crescimanno and Taylor [9] in 1991 (they considered the enumeration of coverings of the sphere as a sort of string theory on the sphere). Central to the combinatorial approach to this question is Hurwitz's original construction [18] in terms of minimal ordered factorizations of permutations into transpositions such that the group generated by the transpositions acts transitively on the sheels. The construction can be obtained by considering the monodromy of the sheets around the branch points. An excellent account of the background and the connections between ramified covers of the sphere and the moduli space of curves has been given very recently by Vakil [33]. Goulden and Jackson [13] have used this approach to obtain a number of explicit expressions for Hurwitz number generating functions for low genera, and have given a polynomiality conjecture as a structure ansatz for Hurwitz numbers. Recently, Ekedahl, Lando, Shapiro and Vainshtein [12] have expressed Hurwitz numbers as Hodge integrals over moduli spaces of curves by a remarkable theorem stating that, for a genus g source and ramification type α over ∞,

$$H_\alpha^g = \frac{r!}{\#\mathrm{Aut}(\alpha)} \prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,m}} \frac{1-\lambda_1+\cdots\pm\lambda_g}{\prod_i(1-\alpha_i\psi_i)}.$$

This was followed with an algebraic proof by Okounkov and Pandharipande [29] of Witten's Conjecture [35] (Kontsevich's Theorem [20]) in Quantum Field Theory. There are combinatorial aspects of this work that are of profound interest in their own right. The construction of Arnol'd Okounkov-Pandharipande ([1, 29]) is highly suggestive of a matrix model through the enumerative theory of graph embeddings.

Bousquet-Mélou and Schaeffer [7] have determined an explicit formula for the number of ramified covers of genus 0 in which ramification is arbitrary, but not specified, over every branch point.

There is considerable interest now in exploring the structure underlying the double Hurwitz numbers. [Mention Okounkov, Ionel, Pandharipande etc?] These numbers count ramified covers of the sphere in which that are two points over which there is arbitrary ramification type. It is believed that the structure underlying these should be even richer, and that the structure associated with the classical case is merely a shadow of this. There is a strongly belief that there should be an ELSV-type theorem that expresses this richer structure, but there is little information about what this theorem should be. For example, polynomiality fails on the double Hurwitz case, and this means that the theorem, if it exists, is significantly different. There were informal discussions about this case, and the prospect for using the Hurwitz encoding and the methods developed within algebraic combinatorics to elicit enumerative evidence for such a theorem. Explicit expression for the double Hurwitz numbers have been given in a number of special cases by Kuleshov and Shipiro [24].

The universal case is, of course, the triple Hurwitz problem, in which there are three points over which arbitrary ramification may be specified. This is because any target surface may be formed as a ramification respecting connected sum of spheres with three punctures (pairs of pants). However, there is evidence that this case, the most general case, does not have as rich a structure as the double Hurwitz case. Progress on coverings of the torus has been made by Dijkgraaf [10].

Random Matrices

Recent breakthroughs in the theory of random matrices have intimate connections with combinatorics. In particular, the definitive work on the largest eigenvalue of a GUE matrix (i.e., a matrix chosen from a certain natural probability distribution on the space of $n \times n$ Hermitian matrices) due to Baik, Deift, and Johansson [2] is connected with (and in fact arose from) increasing subsequences of permutations and the Robinson-Schensted-Knuth algorithm. This work has been further developed by Borodin, Okounkov, Olshanski, Rains, Tracy, Widom, and others, though many interesting questions remain. Moreover, since the pioneering work of Voiculescu [34] it has been known that free probability theory is a fundamental tool for understanding certain aspects of random matrices. Biane [6] has shown deep connections with the "asymptotic representation theory" of the symmetric group, while Nica and Speicher [28, 27] have developed connections between free probability theory, Lagrange inversion, and non-crossing partitions, an object that has been studied extensively in combinatorics. Sniady [30] recently proved a conjecture of Biane [4, Conj. 6.4] concerning Kerov's character generator for values of irreducible characters of the symmetric group S_n on k-cycles, thereby in effect giving the second term in a certain asymptotic approximation to the values of irreducible characters of S_n.

Workshop speakers and talks

- Helene Barcelo: *A Discrete Homotopy Theory and its Application to Hyperplane Arrangements*
- Francois Bergeron: *Diagonal Alternants*
- Nantel Bergeron: *Combinatorial Hopf Algebras*
- Christine Bessenrodt: *Products of Characters of the Symmetric Group and Related Groups*
- Mireille Bousquet-Mélou: *Minimal Transitive Factorizations of Permutations*
- Anders Buch: *Quantum Cohomology of Partial Flag Manifolds*
- Persi Diaconis: *Set Partitions and Character Theory for Upper Triangular Matrices*
- Sergey Fomin and William Fulton: *Eigenvalues, Singular Values, and Schubert Calculus*
- Adriano Garsia: *Cohen Macauliness of the Ring of Quasi-Symmetric Functions*
- Patricia Hersh: *A Hodge Decomposition for the Complex of Injective Words*
- Michael Kleber: *Double-headed LR Coefficients and Type* E_n
- Allen Knutson: *Schubert Polynomials and Quiver Polynomials*
- Christian Krattenthaler: *Symmetric Functions Prove Asymptotic Results for Random Walks in Alcoves of Affine Weyl Groups*
- Ezra Miller: *Combinatorics of Quiver Polynomials*
- Jennifer Morse: *Tableaux Atoms and* k*-Schur Functions*
- Alexandru Nica: *Annular Non-crossing Permutations and Partitions, and Random Matrices*
- Eric Rains: *Vanishing Integrals of Macdonald Polynomials*
- Arun Ram: *The Radical of the Brauer Algebra*
- Victor Reiner: *Conjectures on the Cohomology of the Grassmannian*
- Michael Shapiro and Alek Vainshtein: *Cluster algebras and Poisson geometry*
- Mark Shimozono: *Buch-Fulton Factor Sequence Conjectures*

- Roland Speicher: *Free Probability and Non-Crossing Partitions*
- John Stembridge: *Graded Multiplicities in the Macdonald Kernel and a (q,t)-coinvariant Algebra*
- Craig Tracy: *A Limit Theorem for Shifted Schur Measures*
- Ravi Vakil: *A Geometric Littlewood-Richardson Rule*
- Michelle Wachs: *Posets of Graphs, Partitions and Trees*

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Chapter 9

Statistical Mechanics of Polymer Models (03w5014)

May 10–15, 2003

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Introduction

The standard models used in the statistical mechanics of polymers are combinatorial structures such as self-avoiding walks, lattice polygons and lattice trees. These systems have been studied by combinatorial and probabilistic approaches, by numerical methods including Monte Carlo techniques, and using a variety of techniques from statistical mechanics. There are many challenging open questions, partly motivated by problems from molecular biology, especially for the more physically relevant models in low dimensions. These include questions about entanglement complexity of ring polymers, phase transitions such as polymer adsorption and polymer collapse, and extensions to random copolymers. In this report, we survey this area and highlight aspects that were discussed at the BIRS workshop, May 10-15, 2003, as well as important open questions. There are a number of books [1, 2, 3] and review articles [4, 5] which give useful background information.

Polymer Models

Polymer models have attracted serious attention from mathematicians and physicists since the 1950's. In the simplest case of linear polymer molecules, the polymer consists of a set of identical monomers attached sequentially. The monomers can be numbered $j = 0, 1, 2, \ldots n$ and we say that the polymer has $n + 1$ monomers or a degree of polymerization of $n + 1$. If the polymer is in dilute solution, so that each polymer molecule can be regarded as behaving independently, the polymer will behave differently depending on the quality of the solvent. If the solvent is *good* then the essential features are (i) the flexibility and connectivity of the polymer and (ii) the fact that monomers take up space to the exclusion of other monomers. This space exclusion is called the *excluded volume effect* and it is this phenomenon which makes the problem difficult to model. Random walk models capture the flexibility and connectivity but do not model the excluded volume effect. The model which has been used since the 1950's is a *self-avoiding walk* on a lattice.

Consider the d-dimensional hypercubic lattice $\mathbb{Z}^d$ where the vertices are the integer points in $\mathbb{R}^d$ and the edges connect pairs of vertices which are unit distance apart. An n-edge self-avoiding walk on $\mathbb{Z}^d$ is a set of vertices $j = 0, 1, 2, \ldots n$ such that vertices j and $j+1$ are unit distance apart on the lattice and all the vertices of the walk are distinct vertices in the lattice. Suppose for example that $d = 2$ (ie the square lattice) and, to be definite, let the walk start at the origin. If we write c_n for the number of distinct n-edge self-avoiding walks then $c_1 = 4$, $c_2 = 12$ and $c_3 = 36$. The first interesting question is the number of 4-edge walks, c_4. We have to consider the number of ways for the walk to self-intersect at the fourth step. It is easy to see that $c_4 = 3c_3 - 8 = 100$.

One can construct upper and lower bounds on c_n by counting subsets and supersets. If we attach a coordinate system to the lattice $\mathbb{Z}^d$ in the obvious way then we see that walks which only add steps in positive coordinate directions are self-avoiding so $c_n \geq d^n$. Similarly, if we consider all walks except those with a step which is an immediate reversal of the previous step, then we have $c_n \leq 2d(2d-1)^{n-1}$. An early result about the asymptotic behaviour of c_n is due to Hammersley [6] who showed that the limit

$$\lim_{n\to\infty} n^{-1}\log c_n = \inf_{n>0} n^{-1}\log c_n \equiv \kappa_d \tag{9.1}$$

exists. κ_d is the *connective constant* of the lattice $\mathbb{Z}^d$. The above inequalities imply that $\log(d) \leq \kappa_d \leq \log(2d-1)$ and it is not too difficult to improve these bounds although the precise value of κ_d is not known for any $d \geq 2$. (Using non-rigorous numerical methods we know the value of κ_2 very precisely [7, 8] and we have fairly precise estimates also for $d = 3$ [9].)

A related question is how quickly the limit is approached. For any d, Hammersley and Welsh [10] showed that

$$\kappa_d \leq n^{-1}\log c_n \leq \kappa_d + O(n^{-1/2}) \tag{9.2}$$

so that c_n increases no faster than $e^{\kappa_d n + O(\sqrt{n})}$. Hara and Slade [11, 12] showed that, for $d \geq 5$, $c_n = A_d e^{\kappa_d n}(1 + o(1))$ and it is generally believed that a similar result holds for $d = 4$ but with a log correction and that, for $d = 2, 3$,

$$c_n = A_d n^{\gamma_d - 1} e^{\kappa_d n}(1 + o(1)). \tag{9.3}$$

The critical exponent γ_2 is thought to have the value 43/32 [13, 14] and there is a good numerical estimate of γ_3 [9]. If equation (9.3) were true, then

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = e^{\kappa_d}. \tag{9.4}$$

Equation (9.4) has up to now only been proved for $d \geq 5$ [11, 12]; the best that is known for $d = 2, 3, 4$ is that

$$\lim_{n\to\infty} \frac{c_{n+2}}{c_n} = e^{2\kappa_d}, \tag{9.5}$$

which was proved by Kesten [15] based on a pattern theorem argument. A *pattern* is any fixed self-avoiding walk; a specific pattern P is said to occur in another self-avoiding walk ω if a translate of P is a subwalk of ω. A *proper pattern* or *Kesten pattern* is a pattern which can occur three times (ie as three distinct subwalks) in at least one long self-avoiding walk. Given a proper pattern P, Kesten's pattern theorem says that all but exponentially few sufficiently long self-avoiding walks contain P with a positive density. More precisely, given P there exists an $\epsilon > 0$ such that

$$\limsup_{n\to\infty} n^{-1}\log c_n(\epsilon n, P) < \kappa_d, \tag{9.6}$$

where $c_n(\epsilon n, P)$ is the number of n-step self-avoiding walks in which the pattern P occurs at most ϵn times. Kesten's Pattern Theorem has had many applications beyond the proof of equation (9.5) and some of these are discussed later in this report.

For a polymer in dilute solution in a good solvent, a standard assumption for the self-avoiding walk model is that each n-step walk, starting at the origin, is an equally likely configuration for the polymer. Then, for example, there is interest in the average metric properties of the polymer and their dependence on the excluded volume effect. One metric property is the end-to-end distance of a walk, that is the euclidean distance between the 0'th and the n'th vertex of an n-step walk. Letting s_n denote the root mean square end-to-end distance over all n-step self-avoiding walks, it is believed that

$$s_n = B_d n^{\nu_d}(1 + o(1)). \tag{9.7}$$

For n-step random walks on $\mathbb{Z}^d$, it is known that the root mean square end-to-end distance equals $n^{1/2}$. For $d \geq 5$, it is known that equation (9.7) holds with $\nu_d = 1/2$ but for $d \leq 4$, the best that is known rigorously is that $s_n \leq n$. Numerical and other evidence suggests that $\nu_2 = 3/4$ [13, 14], $\nu_3 \approx 0.588$ [16, 17] and $\nu_4 = 1/2$ with logarithmic corrections. Other measures of the polymer's size (in units of length) such as the root mean square radius of gyration are also expected to have the form given in equation (9.7) where the amplitude B_d may be dependent on the particular measure but the exponent ν_d is not.

Define the generating function

$$C(x) = \sum_n c_n x^n. \tag{9.8}$$

The series converges for $x < x_c = e^{-\kappa_d}$ and diverges for $x \geq x_c$. The functional form given in (9.3) implies that, close to x_c, the generating function $C(x)$ behaves as

$$C(x) \sim \frac{A_d}{(x_c - x)^{\gamma_d}}, \tag{9.9}$$

and this is an example of a *scaling function*. Given a point y in $\mathbb{Z}^d$, let $c_n(0, y)$ be the number of n-step self-avoiding walks starting at the origin and ending at y. The generating function

$$G_x(0, y) = \sum_n c_n(0, y) x^n \tag{9.10}$$

is known as the *two point function* and x_c is also its radius of convergence for any $y \neq 0$. Note that assuming equations (9.3) and (9.7) hold implies [1] that

$$\sum_y |y|^2 G_x(0, y) \sim \text{const.}(x_c - x)^{-2\nu_d - \gamma_d}. \tag{9.11}$$

It was observed numerically [18] that critical exponents such as ν and γ depend on the dimensionality of the problem but not on the particular lattice being studied, although the connective constant is a lattice dependent property. This is the idea of *universality*; certain aspects of the problem (such as critical exponents) do not depend on microscopic details of the model but change if parameters such as dimensionality change. In 1972 de Gennes [19] pointed out that self-avoiding walks are the zero spin space dimension case of a spin model in D spin dimensions and d spatial dimensions. Critical exponents depend on d and D but not on the lattice being considered. Because of the connection to spin models $C(x)$, defined above, is called the *susceptibility* and γ is the *susceptibility critical exponent*. Off-lattice models are thought to have the same exponents as these lattice models provided that the forces are repulsive and short-ranged [17, 20]. Similarly, lattice walks with other forms of short range repulsion, such as neighbour-avoiding walks [21] and spread out walks, have the same critical exponents. The exponent value $\nu_3 \approx 0.588$ has been observed experimentally in light scattering measurements of the radius of gyration of very long polymers in dilute solution in good solvents [22].

Based on the assumption of universality, it is also expected that there is a continuum model which is obtained from an appropriate *scaling limit*, where the step lengths of an n-step walk on the lattice are scaled by a function of n ($(Bn)^{-\nu_d}$, say, for some B) and then $n \to \infty$, and that this model has the same critical exponents as the original lattice model. This assumption combined with nonrigorous renormalization group techniques and conformal field theory arguments led to conjectured values for critical exponents such as γ_d and ν_d [13, 14, 17]. For $d \geq 5$, Hara and Slade [11, 12] proved that such a scaled self-avoiding walk converges in distribution to Brownian motion, i.e. Brownian motion is the scaling limit of self-avoiding walks. For $d = 4$, it is believed, but not yet proved, that the scaling limit is also Brownian motion but with a logarithmic scaling correction. Most recently, for $d = 2$, Lawler *et al* [23] proved that if the scaling limit exists for self-avoiding walks and if it satisfies an appropriate conformal invariance condition, then the scaling limit must be stochastic Loewner evolution with speed $\kappa = 8/3$, $SLE_{8/3}$; SLE_κ is a family of stochastic processes introduced by Schramm [24] that is parametrized by a one-dimensional Brownian motion with speed κ. The critical exponents for $SLE_{8/3}$ can be obtained rigorously and the results of Lawler *et al* [23] predict the same critical exponents as the conformal field theory arguments. The same critical exponents had also been predicted using 2D quantum gravity arguments [25]. For this approach, one considers a polymer model on a random planar graph instead of on a lattice and then Knizhnik *et al*'s [26] KPZ map is used to predict critical exponents [27]. Evidence from numerical studies is consistent with all these critical exponent predictions, and hence the evidence is mounting that the assumptions underlying these predictions for $d \leq 4$ must be valid for self-avoiding walk models.

Ring polymers have a particular interest because of their connection to circular DNA molecules. In three dimensions they can be knotted or linked and we shall return to these phenomena in a later section. In the same way that self-avoiding walks are a model of linear polymers, ring polymers can be modelled by self-avoiding polygons. A *self-avoiding polygon* on the lattice $\mathbb{Z}^d$ is a connected subgraph of the lattice with all vertices of degree 2. Self-avoiding polygons are counted modulo translation. Let p_n be the number of self-avoiding polygons with n edges. On the square lattice ($d = 2$), $p_4 = 1$, $p_6 = 2$ and $p_8 = 7$. We note that for n even

$$\sum_{y:|y|=1} c_{n-1}(0, y) = 2np_n. \tag{9.12}$$

It is known [28] that

$$\lim_{n\to\infty} n^{-1} \log p_n = \sup_{n>0} n^{-1} \log p_n = \kappa_d \tag{9.13}$$

so that the numbers of polygons and self-avoiding walks grow at the same exponential rate. It is believed that $p_n \sim \text{const.}n^{\alpha-3}e^{\kappa_d n}$ where α is the heat capacity exponent in the associated spin model. Self-avoiding walks and polygons are expected to have the same metric exponent ν and hyperscaling connects the exponents α and ν by the relation $2 - \alpha = d\nu$ [1]. Define the generating function

$$P(x) = \sum_n p_n x^n. \tag{9.14}$$

$P(x)$ converges for $x < x_c$ and diverges for $x > x_c$. If hyperscaling holds and the conjectured values for ν are correct then $\alpha < 1$ in all dimensions and $P(x)$ converges at $x = x_c$ [1].

The value of the connective constant κ is not known rigorously for any lattice but there is a non-rigorous argument [13, 14] for the honeycomb lattice that $\kappa = \log\sqrt{2+\sqrt{2}}$. The best estimates of the connective constant for other lattices come from exact enumeration of walks or polygons and, in two dimensions, much longer series are available for the numbers of polygons.

There are corresponding lattice models of branched polymers. The two most widely used are *lattice animals* (ie connected subgraphs of the lattice) and *lattice trees* (lattice animals with no cycles). If a_n and t_n are the numbers of lattice animals and lattice trees with n vertices (counted up to translation) then it is known that the limits

$$\lim_{n\to\infty} n^{-1} \log a_n \equiv \log \lambda \tag{9.15}$$

and

$$\lim_{n\to\infty} n^{-1} \log t_n \equiv \log \lambda_0 \tag{9.16}$$

exist, and that $\lambda_0 < \lambda$. It is believed that $a_n \sim \text{const.}n^{-\theta}\lambda^n$ and $t_n \sim \text{const.}n^{-\theta_0}\lambda_0^n$ and that $\theta = \theta_0$. Parisi and Sourlas [29] gave arguments for the values of θ in two and three dimensions and Brydges and Imbrie [30, 31] have now given a rigorous version of these arguments. For sufficiently large dimensions, Hara and Slade [32] proved $\theta_0 = 5/2$ and that ν, the radius of gyration exponent for trees, is $1/4$. Also, Derbez and Slade [33] proved that the scaling limit for high dimensional trees is integrated super-Brownian excursion (ISE).

Recently an analogue of Kesten's pattern theorem has been proved for lattice animals by Madras [34]. His approach to the proof differs from the approach that applies to the self-avoiding walk case and his approach can also be applied to obtain a pattern theorem for other lattice clusters such as lattice trees. One consequence of his theorem is the following ratio limit:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda. \tag{9.17}$$

We can also consider connected subgraphs of the lattice with fixed homeomorphism type. For given d, suppose that the maximum vertex degree of the graph defining the homeomorphism type τ is less than or equal to $2d$. Let $g_n(\tau)$ be the number of embeddings of τ (lattice subgraphs homeomorphic to τ) in $\mathbb{Z}^d$ with n edges. Then

$$\lim_{n\to\infty} n^{-1} \log g_n(\tau) = \kappa_d \tag{9.18}$$

independent of τ [35]. Given any lattice subgraph homeomorphic to τ, each edge of τ corresponds to a path, called a *branch*, of the lattice subgraph. For the study of polymer networks it is also of interest to consider $\hat{g}_n(\tau)$, the number of n-edge embeddings of τ in $\mathbb{Z}^d$ with the property that each branch of the embedding has $O(n)$ edges. In this case one can also prove that

$$\lim_{n\to\infty} n^{-1} \log \hat{g}_n(\tau) = \kappa_d \tag{9.19}$$

independent of τ [36, 37]. Duplantier [38] has predicted the τ dependence on the critical exponent γ_τ for $\hat{g}_n(\tau)$, assuming that $\hat{g}_n(\tau) = A_d n^{\gamma_\tau - 1} e^{\kappa_d n}(1 + o(1))$. Holmes *et al* [39] have proved Gaussian behaviour in dimension greater than four for a related model, in which the branches of the embedding are composed of mutually avoiding spread-out self-avoiding walks.

Exactly solvable models

From the above discussion it is clear that rigorous results on self-avoiding walks are difficult to obtain and hence there has been some interest in devising simpler models which are exactly solvable and can be used to explore phenomena such as universality and the forms of scaling functions. As an example we shall consider Dyck paths. These are walks on the square lattice where

1. steps are allowed in the $(1, 1)$ and $(1, -1)$ directions,
2. the walk starts at the origin and ends on the line $y = 0$, and
3. no vertex of the walk has negative y-coordinate.

Let the number of Dyck paths with n edges be d_n and define the generating function

$$D(z) = \sum_{n\geq 0} d_n z^n \tag{9.20}$$

where $d_0 = 1$ by convention. By a factorization argument

$$D(z) = 1 + z^2 D(z)^2 \tag{9.21}$$

so that

$$D(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \tag{9.22}$$

from which we see that $d_n \sim$ const. $n^{-3/2}2^n$. For these walks the connective constant is $\log 2$ and the critical exponent corresponding to γ is $-1/2$. In general for such directed models, the value of the connective constant gives a lower bound on the connective constant for self-avoiding walks but the exponent tells us nothing about the value of γ for self-avoiding walks, although it is reassuring to see a form similar to that in equation (9.3). One can extend this type of analysis to other exactly solved sets of walks [3, 40]. If the set is larger (ie grows at an exponentially faster rate) then one obtains a better bound on the connective constant of self-avoiding walks. Another way to extend these models is by adding energy terms to model adsorption or collapse, as discussed in later sections. We note also that some progress has been made recently towards understanding why some lattice models are exactly solvable and others are apparently difficult to solve [41, 42].

Phase Transitions in Polymer Models

For polymers in a poor solvent or interacting with a surface, short range attractive interactions between monomers or between monomers and surface molecules cannot be ignored. For these situations the assumption that each polymer configuration is equally likely is not valid. Instead the standard model is a self-avoiding walk model for which the probability that a particular n-step walk occurs is now assumed to be a function of temperature and one or more appropriate "local" features of the self-avoiding walk such as the number of vertices of the walk in the surface, in the case of a polymer interacting with a surface. These models are called interacting self-avoiding walk models. For such models, one of the main questions of interest is whether or not a phase transition occurs for the model, ie whether or not there exists a temperature, T_c, at which the limiting free energy for the model is non-analytic.

Two interacting self-avoiding walk models have received a great deal of attention, namely, self-avoiding walk models of the adsorption transition and self-avoiding walk models of the collapse transition. There are also versions of these models for self-avoiding polygons, lattice trees, lattice animals, and lattice subgraphs with fixed homeomorphism type. Recently, there has also been much interest in interacting random copolymer models where there is a random distribution in the polymer of more than one type of monomer. There are versions of adsorption and collapse models for random copolymers and in addition there are models of the localization transition. We discuss progress on the study of these interacting polymer models next.

Adsorption at a surface

In this case, the typical situation is to consider a polymer in dilute solution where the polymer is tethered to a surface. It is assumed that there is a short range interaction between the monomers and the surface. For the self-avoiding walk model of this situation, the surface is represented by the hyperplane $z = 0$ in $\mathbb{Z}^d$, where a point in $\mathbb{Z}^d$ is assumed to have coordinates $(x, y, ..., z)$. The main results were obtained by Hammersley *et al* [43] who investigated two self-avoiding walk models of adsorption. For the first model, the walk is tethered to the surface and cannot penetrate it. In this case, $c_n^+(v)$, the number of self-avoiding walks in $z \geq 0$, starting at the origin, and with $v + 1$ vertices in $z = 0$, is considered. Each walk counted in $c_n^+(v)$ is considered to be equally likely and, more generally, the probability of an n-step walk with $v + 1$ vertices (or visits) to $z = 0$ is assumed to have the form

$$\frac{e^{\beta v}}{Q_n^+(\beta)} \tag{9.23}$$

for β real and where the normalization term, $Q_n^+(\beta)$, is given by

$$Q_n^+(\beta) = \sum_v c_n^+(v) e^{\beta v} \tag{9.24}$$

and is known as the *partition function* for the model. The limiting free energy is defined to be

$$\mathcal{F}^+(\beta) = \lim_{n\to\infty} n^{-1} \log Q_n^+(\beta), \tag{9.25}$$

where the limit has been proved to exist for all β and is a convex nondecreasing function of β [43]. Hammersley *et al* proved further that there exists $\beta_c^+ > 0$ for this model such that

$$\begin{aligned} \mathcal{F}^+(\beta) \quad &= \quad \kappa_d \qquad \text{for } \beta \leq \beta_c^+ \\ \text{and } \mathcal{F}^+(\beta) \quad &> \quad \kappa_d \qquad \text{for } \beta > \beta_c^+ \end{aligned} \tag{9.26}$$

and hence $\beta = \beta_c$ is a point of nonanalyticity of $\mathcal{F}^+(\beta)$. Given any β, let $\langle v_n(\beta)\rangle$ be the expected number of visits of an n-edge self-avoiding walk to the surface $z = 0$. Because of the convexity of $\mathcal{F}^+(\beta)$,

$$\lim_{n\to\infty} \frac{\langle v_n(\beta)\rangle}{n} = \frac{d\mathcal{F}^+(\beta)}{d\beta} \tag{9.27}$$

and this derivative exists for almost all β. At each such β value for $\beta > \beta_c^+$, the derivative is known to be positive. It is not known, however, if $\mathcal{F}^+(\beta)$ is continuously differentiable at $\beta = \beta_c^+$. β_c is known as the *adsorption transition critical point*. For $\beta < \beta_c^+$, contacts with the surface are not favoured enough to overcome the loss of entropy that would result from frequent visits to the surface, while for $\beta > \beta_c^+$ this no longer holds and the walk is considered adsorbed to the surface since there is a non-zero fraction of vertices of the walk in the surface.

Hammersley *et al* [43] also considered a related model in which the surface is penetrable. In this case the partition function is given by

$$Q_n(\beta) = \sum_v c_n(v) e^{\beta v}, \tag{9.28}$$

where $c_n(v)$ is the number of n-step self-avoiding walks starting at the origin with $v+1$ vertices in $z=0$. The limiting free energy, given by

$$\mathcal{F}(\beta) = \lim_{n\to\infty} n^{-1} \log Q_n(\beta), \tag{9.29}$$

exists and is a convex nondecreasing function of β [43]. In this case, there exists $\beta_c \geq 0$ such that

$$\begin{aligned} \mathcal{F}(\beta) &= \kappa_d \qquad \text{for } \beta \leq \beta_c \\ \text{and } \mathcal{F}(\beta) &> \kappa_d \qquad \text{for } \beta > \beta_c \end{aligned} \tag{9.30}$$

and it is also known that $\beta_c < \beta_c^+$. For both models, it is believed that at the critical point $\lim_{n\to\infty} \frac{\langle v_n(\beta)\rangle}{n} = 0$, however, this has not been proved. It is also believed, based on numerical evidence, that $\beta_c = 0$.

Exactly solvable, eg partially directed walk and Dyck path, versions of these models have been investigated. As an example we shall consider the adsorption of Dyck paths. Let $d_n(v)$ be the number of n-edge Dyck paths with v visits to the line $y=0$ and define

$$G(x,z) = \sum_{n,v} d_n(v) x^v z^n. \tag{9.31}$$

An equation for G can be obtained by the factorization argument used in section 9, giving

$$G(x,z) = 1 + xz^2 D(z) G(x,z) \tag{9.32}$$

or, equivalently,

$$G = \frac{2}{2 - x + x\sqrt{1-4z^2}}. \tag{9.33}$$

G is singular when the square root is zero (ie when $z = 1/2$) and when the denominator is zero. These two branches meet at $x_c = 2$ which is the location of the adsorption transition in this model. The model shares some of the broad general features of the self-avoiding walk model of polymer adsorption but, of course, the two models disagree in detail. Interestingly, there is evidence that they have the same value (1/2) for the crossover exponent, which describes the shape of the free energy curve in the adsorbed phase, close to the phase transition. More complicated and interesting directed walk models can also be solved exactly using a variety of different and complementary approaches [3, 40].

The Collapse Transition

Experimental evidence [44] shows that for high molecular weight polymers in dilute solution, either as the temperature or solvent quality is reduced the polymers go from typically being spread out in a coil-like configuration to being compact in a ball-like configuration. This transition occurs over a short temperature range, providing evidence for the existence of what's known as the *collapse phase transition*. This transition is thought to be driven by a combination of a short range attraction of monomers with each other and a short range repulsion between monomers and solvent molecules. For the standard self-avoiding walk model of this phenomenon, the energy of a configuration is assumed to be proportional to the number of nearest neighbour contacts of the walk; a *nearest neighbour contact* of a self-avoiding walk is defined to be any edge of the lattice which joins two vertices of the walk but is not an edge of the walk. In this case the partition function of the model is

$$Q_n^{isaw}(\beta) = \sum_{\ell} c_n(\ell) e^{\beta\ell} \tag{9.34}$$

where $c_n(\ell)$ is defined to be the number of n-step self-avoiding walks starting at the origin with ℓ nearest neighbour contacts. The limit which defines the limiting free energy for this model,

$$\lim_{n\to\infty} n^{-1} \log Q_n^{isaw}(\beta) \equiv \mathcal{F}^{isaw}(\beta) \tag{9.35}$$

has been proved to exist for all $\beta \leq 0$ [45], however, the standard approach for proving this limit exists does not work for $\beta > 0$. It is believed that $\mathcal{F}^{isaw}(\beta)$ exists for all β and the numerical evidence suggests that there exists a critical point, known as the *collapse transition critical point* or the θ-point, at $\beta = \beta_c^{isaw}$ where $\mathcal{F}^{isaw}(\beta)$ is nonanalytic and such that

$$s_n = \begin{cases} B_d n^{\nu_d} & \text{for } \beta > \beta_c \\ B_d^c n^{\nu_d^c} & \text{for } \beta = \beta_c \\ C_d n^{1/d} & \text{for } \beta < \beta_c \end{cases}, \tag{9.36}$$

where it is believed that $\nu_d^c = 1/2$ for all $d \geq 3$ and $\nu_2^c = 4/7$. For the corresponding self-avoiding polygon model, it can be proved that the limiting free energy does exist for all β [45], however, the existence of the collapse transition critical point has not been proved for either the walk or polygon model. Numerical evidence [45] suggests that the critical point is the same for the self-avoiding walk and polygon models. For $d > 3$, it is believed that mean field theory should hold for interacting self-avoiding walks but this has not been proved. The $d = 4$ case has been studied by Prellberg and Owczarek [46] using Monte Carlo methods.

In the lattice animal version of this model, it is useful to introduce a two variable partition function

$$Q_n^{ianimal}(\beta_1, \beta_2) = \sum_{\ell,s} a_n(\ell, s) e^{\beta_1 \ell + \beta_2 s} \tag{9.37}$$

where $a_n(\ell, s)$ is the number of n-vertex lattice animals having ℓ nearest neighbour contacts and s solvent contacts, ie edges in the lattice from a vertex in the animal to a vertex not in the animal. In this case, one can prove that the corresponding limiting free energy, $\mathcal{F}^{ianimal}$, exists and is a convex function for all β_1 and β_2 [47]. Furthermore, by exploiting the connection between lattice animals and percolation clusters it is possible to prove that there exists at least one critical point for $\mathcal{F}^{ianimal}$ (corresponding to the bond percolation critical point) where a collapse transition is believed to occur. It is believed that there is a whole curve of collapse transition critical points in the (β_1, β_2)-plane [47, 48].

While the proof of the existence of a collapse phase transition for self-avoiding walks remains elusive, there are some exactly solvable models of collapse. Brak *et al* [49] derived an exact expression for the generating function $G(x, y) = \sum_{n,\ell} e_n(\ell) x^\ell y^n$ where $e_n(\ell)$ is the number of n-step partially directed walks on the square lattice with ℓ nearest-neighbour contacts and showed that this implied that the free energy was non-analytic. There are also other exactly solved models of collapse [50] and also exactly solved models of self-interacting walks interacting with a surface [40]. Dhar [51] has also solved a directed animal model of collapse.

Random Copolymers

Polymers are frequently composed of more than one type of monomer, eg biopolymers such as proteins are composed of different types of amino acids. In some circumstances local interaction properties are dependent on the specific type of monomer involved in the interaction. To investigate the effect this has on phase transitions, there has been much interest in the study of random copolymers.

For the standard self-avoiding walk model, the relevant situation is when the randomness is quenched, that is, we assume that the sequence of monomer types making up a linear polymer is randomly chosen according to a given probability distribution but that once it is chosen the polymer's monomer sequence is fixed. In the simplest model there are two types of monomers, A and B, and the monomer sequence or *colouring* of a linear polymer composed of n monomers is given by a sequence $\chi = \chi_1, \chi_2, ..., \chi_n$, where the χ_i are independent and identically distributed random variables equal to A with probability p and B with probability $1 - p$. Once chosen χ is fixed and then each configuration of the polymer with the same energy is considered to be equally likely. Thus for the case of random copolymer adsorption at an impenetrable surface we suppose that only the A monomers interact with the surface. Given a colouring χ, the partition function is then given by

$$Q_n^+(\beta|\chi) = \sum_{v_A} c_n^+(v_A|\chi) e^{\beta v_A}, \tag{9.38}$$

where $c_n^+(v_A|\chi)$ is the number of n-step self-avoiding walks, starting at the origin, confined to $z \geq 0$, with j'th vertex ($j > 0$) coloured by χ_j, and with v_A A-vertices in $z = 0$. Now in order to investigate the adsorption transition, one is interested in the *limiting quenched average free energy*

$$\bar{\kappa}(\beta) \equiv \lim_{n\to\infty} \langle n^{-1} \log Q_n^+(\beta|\chi) \rangle \tag{9.39}$$

where $\langle\cdot\rangle$ denotes the expected value over all colourings χ. The limit defining $\bar{\kappa}(\beta)$ has been proved to exist and $\bar{\kappa}(\beta)$ is a convex nondecreasing function of β [52]. Furthermore there exists a critical point $\beta = \beta_c^{rca+}(p) \geq \beta_c^+$ corresponding to an adsorption phase transition for this model and $\beta_c^{rca+} > \beta_c^+$ for $0 < p < 1$. It has also been proved [52] that the limiting free energy is *self-averaging* for this model, that is,

$$\lim_{n\to\infty} n^{-1} \log Q_n^+(\beta|\chi) = \bar{\kappa}(\beta) \tag{9.40}$$

with probability one and bounds have been obtained on the extent of this self-averaging [53, 54] for finite values of n. Similar results can be obtained for the case of adsorption at a penetrable surface. It is also known under mild additional conditions that the energy, the average number of surface contacts, self-averages but there remain open questions regarding the self-averaging of other thermodynamic quantities such as the heat capacity [55].

The case of homopolymer adsorption at a penetrable surface can also be generalized to investigate a phenomenon known as *localization at a surface* for random copolymers. In this case we suppose that there is an interface at $z = 0$ separating two different solvents or two liquid phases. It is assumed that A monomers interact with the liquid in $z > 0$, B monomers interact with the liquid in $z < 0$, and all monomers (regardless of type) interact equally with the interface $z = 0$. Experimental evidence for related systems indicates that in the appropriate temperature regime the polymer is localized near the surface and intersects it frequently while for other temperatures the polymer is either found primarily above or below the interface and hence delocalized. Given χ, the relevant three parameter partition function is now given by

$$Q_n^{loc}(\alpha,\beta,\gamma|\chi) = \sum_{v_A,v_B,w} c_n(v_A,v_B,w|\chi)e^{\alpha v_A+\beta v_B+\gamma w}, \tag{9.41}$$

where $c_n(v_A, v_B, w|\chi)$ is the number of n-step self-avoiding walks, starting at the origin, with j'th vertex ($j > 0$) coloured by χ_j, and with v_A A-vertices in $z > 0$, v_B B-vertices in $z < 0$, and w vertices in $z = 0$. The *limiting quenched average free energy*

$$\bar{\kappa}(\alpha,\beta,\gamma) \equiv \lim_{n\to\infty} \langle n^{-1} \log Q_n^{loc}(\alpha,\beta,\gamma|\chi)\rangle \tag{9.42}$$

has been proved to exist and it is a convex nondecreasing function of α, β, γ [56, 57]. Focusing on $p = 1/2$ for convenience, for $\gamma = 0$ there exist two symmetric phase boundaries (one with $\beta \geq \alpha$ and the other with $\beta \leq \alpha$) in the (α, β)-plane that bound the localized phase. The two phase boundaries intersect at the origin in this case and, for example, if one stays in the half-plane $\beta \leq \alpha$ there is a curve of critical points corresponding to a phase transition from the localized phase to the phase in which the polymer is delocalized into $z > 0$. Further rigorous results are known regarding the γ dependence of the phase diagram but many interesting open questions remain. In particular for $\alpha = \beta = 0$, the model is equivalent to homopolymer adsorption at a penetrable surface so that the critical value of γ is β_c which is believed to be zero (but this is not yet proved). For all $\gamma > \beta_c$, the point $(0, 0, \gamma)$ is in the localized phase.

There are no exactly solvable models for localization of a random copolymer at a surface. However, the first models used to study these phenomena were (bilateral) Dyck path models [58, 59]. Qualitatively the results are the same for the Dyck path models and the self-avoiding walk models, but it is possible to prove some stronger results in the case of Dyck path models. In particular, Biskup and den Hollander [59] obtained results about path properties. For the self-avoiding walk model no corresponding path results are available.

As it is difficult to obtain rigorous results for models of homopolymer collapse, the analysis of random copolymer collapse is even more difficult and few results have been obtained [60].

Random Knotting and Entanglement Complexity

Ring polymers in three dimensions behave like simple closed curves in $\mathbb{R}^3$ and can be knotted or linked. Knots and links have been observed in circular DNA molecules and are thought to act as topological obstructions to cellular processes such as replication.

Frisch-Wasserman-Delbruck Conjecture

In the 1960s Frisch and Wasserman, and Delbruck, conjectured that sufficiently long ring polymers would be knotted with high probability. To address this question mathematically we need a model of a ring polymer, and the first model for which a result of this nature was obtained was a polygon on the simple cubic lattice $\mathbb{Z}^3$. Let p_n be the number of n-edge polygons on this lattice and p_n^o be the number of these that are unknotted. It is known [61] that

$$\lim_{n\to\infty} n^{-1} \log p_n^o \equiv \kappa_o < \kappa_3 \tag{9.43}$$

so that unknotted polygons are exponentially rare in the set of polygons, for n large. The key ingredients in the proof of this result are:

(i) There are no "antiknots" - that is, if k is a given knot type and ϕ is the unknot then there does not exist a knot k' such that $k\#k' = \phi$. This means that a knotted circle cannot be unknotted by tying a second knot at a different location in the circle.

(ii) A knotted ball pair is a 3-ball B^3 and a 1-ball B^1, properly embedded in B^3 such that the pair (B^3, B^1) is not homeomorphic to the trivial ball pair (in which B^1 is a diameter of the 3-ball $\{x, y, z | x^2 + y^2 + z^2 \leq 1\}$). Given a knot type k we can construct an embedding (τ) of an arc (ie a 1-ball) in $\mathbb{Z}^3$ such that the union of the dual 3-cells forms a 3-ball and the ball pair is knotted. Moreover, if the vertices of degree 1 of B^1 are connected by an arc entirely outside B^3, the resulting simple closed curve has knot type k.

(iii) Using Kesten's pattern theorem one can proved that the arc τ occurs at least once on all but exponentially few sufficiently long polygons.

These arguments can be extended in various ways. Somewhat weaker results can be obtained for some cases of polygons in $\mathbb{R}^3$ (not lying on a lattice) such as equilateral polygons and Gaussian random polygons [62]. For instance, for Gaussian random polygons, Diao *et al* [62] proved that there exists a constant $\epsilon > 0$ such that Gaussian random polygons with n edges are knotted with probability at least $1 - e^{-n^\epsilon}$ for n sufficiently large. For the lattice case one can prove that prime knots are exponentially rare and, indeed, that the typical knots in long polymers are very complex [35].

Polygons with Fixed Knot Type

Suppose that we fix the knot type of the polygons and write $p_n(k)$ for the number of n-edge polygons in $\mathbb{Z}^3$ with knot type k. Perhaps surprisingly we do not know if the limit

$$\lim_{n\to\infty} n^{-1} \log p_n(k) \tag{9.44}$$

exists (except when $k = \phi$, the unknot). Arguments similar to those outlined in section 9 can be used to establish that

$$\limsup_{n\to\infty} n^{-1} \log p_n(k) < \kappa_3 \tag{9.45}$$

for every k, and a concatenation of polygons with knot types k_1 and k_2 shows that

$$p_{n_1}(k_1) p_{n_2}(k_2) \leq 2 p_{n_1+n_2}(k_1 \# k_2). \tag{9.46}$$

Setting $k_1 = k$, $k_2 = \phi$ then establishes that

$$\liminf_{n\to\infty} n^{-1} \log p_n(k) \geq \kappa_o \tag{9.47}$$

but it is an open question as to whether this is an equality or a strict inequality.

There are many numerical (Monte Carlo) investigations of polygons (both lattice polygons and various off-lattice models in $\mathbb{R}^3$) with fixed knot type [63, 64, 65]. The evidence suggests that, for $k \neq \phi$, the probability that an n-edge polygon has knot type k, increases for small n, goes through a maximum, and decreases for large n. The location of the maximum depends on k and increases as the knot becomes more complex. Very little is known rigorously about this behaviour.

Localization of knots

For a given knot type, is the knot typically localized in a long polygon of this knot type? This question sounds somewhat nebulous (since knotting is a topological property) but it can be made more precise by imagining splitting the n-edge polygon into two arcs of length m and $n-m$, connecting each of these up to form two separate polygons (eg by running parallel rays from the end points of an arc to a point at infinity) and checking the knot types of the resulting (infinite) simple closed curves. If one of them (the one originally having m edges, say) is of knot type k and the other is unknotted we can say that the knot is localized in an arc of length m. If $m = o(n)$ for typical polygons we can regard the knot as typically localized for large n.

This was a contentious question for many years with evidence being presented for both points of view. It is probably fair to say that the numerical evidence now favours localization [66]. Again the evidence is numerical and nothing is known rigorously. A related question is how the mean square radius of gyration depends on knot type. If we write $\langle R_n^2(k)\rangle$ for the mean square radius of gyration for n-edge (lattice) polygons with knot type k, then a reasonable guess is that

$$\langle R_n^2(k)\rangle = A(k)n^{2\nu(k)}(1 + B(k)n^{-\Delta(k)} + ...) \tag{9.48}$$

Numerical evidence suggests that the exponents $\nu(k)$ and $\Delta(k)$ are independent of k and, perhaps, that the amplitude $A(k)$ is independent of k [64]. This would be consistent with knots being localized and all the knot dependence appearing in the correction-to-scaling term, with knot-dependent amplitude $B(k)$.

Topological Restrictions and Excluded Volume

Consider a model of a ring polymer in $\mathbb{R}^3$ where there is no excluded volume. For instance, think of an equilateral ring polymer constrained to form a ring so that its first and last vertices are identical. This is just a random walk in $\mathbb{R}^3$, constrained to return to the origin at its last step. The mean-square radius of gyration $\langle R_n^2\rangle \sim n^{2\nu}$ where n is the number of steps in the walk (or bonds in the ring) and $\nu = 1/2$. Suppose now that the ring is constrained to have a given knot type. Does this topological restriction change the value of ν from its random walk value of $1/2$ to the excluded volume value of about 0.588? This question has been investigated numerically (by Monte Carlo techniques) and there is evidence that the topological restriction changes the universality class to that of polymers with excluded volume (ie $\nu \approx 0.588$) [67].

Connections to DNA experiments

Knots and links can act as topological obstructions to various cellular processes. For instance, in mitosis the two circular DNA molecules cannot separate during the formation of the daughter cells because they are linked, and bacteria cannot express genes on knotted plasmids. Organisms have a collection of enzymes (eg topoisomerases) which effect strand passages in DNA so that DNA molecules can be linked and unlinked, or knotted and unknotted, by these strand passages; the cell abhors topological entanglement in its genetic material, and uses topoisomerases to solve problems arising from such entanglements. There is considerable interest in understanding the mode of action of these enzymes and there has been a fruitful collaboration between topologists and molecular biologists to design and analyse experiments to probe the action of these enzymes [68, 69]. From the topological point of view the circular DNA molecule can be considered as a sum of two string tangles, with a numerator construction to form one or more (possibly knotted or linked) circles. The tangles which typically appear in the biological problems are rational tangles and the classification of these is well-understood. The biological assumption is that the action of the enzyme is confined to the tangle confined in a single 3-ball. By examining the products produced by the action of the enzyme on a given substrate (say an unknotted circular DNA molecule) one can deduce the probable mechanism of enzyme binding and action, in terms of how the broken DNA strands are spatially arranged before religation [70].

Knot Invariants and Knot Energies

A *knot invariant* is any function defined on the set of simple closed curves in $\mathbb{R}^3$ which is a constant for all simple closed curves with a fixed knot type k. If a simple closed curve's knot type is unknown, knot invariants can be useful for either determining the knot type or at least reducing the number of knot type possibilities.

They can also be used as a way to compare two knot types, or, for example, measure the "complexity" of a knot. For example, the *minimum crossing number*, the minimum number of crossings in a regular projection of a knot, is a knot invariant. In the standard knot tables, the knot types are listed according to their minimum crossing number. *Thickness* is a geometrically based knot invariant of recent interest which has been defined and studied in [71, 72]. Recently there has also been much interest in the study of *knot energies*, real-valued and non-negative functions, usually only defined for either the space of smooth knots or the space of polygonal knots. The global minimum of such a function over a given knot type is another example of a knot invariant and any knot which is a global minimizer is referred to as an "ideal" representative of the knot type with respect to the given energy function. Just as with the case of crossing number, these knot energies provide ways to characterize and compare different knot types and sometimes they offer the advantage that they are easy to compute. Some examples of knot energies are möbius energies [73] and rope length [74].

Since a wide variety of knot energy functions have been proposed and studied, it is important to develop criteria for assessing and comparing these functions so that one can use a "good" energy function for a given purpose. Diao *et al* [75] have developed criteria for assessing knot energies defined on equilateral polygonal knots. Equilateral polygonal knots are frequently used in computational studies and the knot energies used for these are often discrete versions of energy functions defined for smooth knots. Ideally a knot invariant derived from a knot energy function would be able to distinguish between all knot types; but this is something that would be difficult to prove. Failing that, one might ask that it would be able to distinguish a knot from the unknot. The minimum energy configuration for a given knot type may not be unique and may even be singular (ie edges may intersect or overlap) and hence one might ask that the minimum configuration at least be non-singular. Also one might prefer the $n \to \infty$ limit (for n the number of edges in the polygon) to be well defined and, if the energy is a discrete version of an energy for smooth knots, that the limit be the smooth knot energy. Not all knot energies in use satisfy all these properties, however, there are some that do [75].

Conclusion

This is a rich field from which challenging combinatorial, probabilistic, topological, and geometrical problems have arisen. Significant advances have been made recently in, for example, understanding scaling limits in high dimensions and in the plane. However, there remain many interesting open problems especially in the study of interacting polymer and copolymer models, in the study of the entanglement complexity of polymers, and regarding finding efficient computational approaches for studying these models.

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Chapter 10

Constraint Programming, Belief Revision, and Combinatorial Optimization (03w5105)

May 24–29, 2003

Organizer(s): Randy Goebel (University of Alberta)

Workshop Summary

This workshop was motivated by the observation that many of the principles underlying the areas of belief revision, constraint programming, and combinatorial optimization are related. While the relationships were more or less obvious to any researcher in any of the three areas, should they make any attempt to browse the other two literatures, there has been very little motivation for interaction.

This BIRS workshop was the first venue to provide the opportunity for these three communities to interact. With this motivation, we were extremely lucky to attract some of the world's best researchers in each of the three areas.

Because of the diversity of the communities, the workshop was organized as a sandwich of two plenary sessions, around a middle "break-out" session, where the three sub area groups worked to summarize the salient principles of their area.

This organization provided the opportunity to acknowledge the overall motivation, and to have R. Goebel present a summary of what he saw as potential points of interaction.

Summaries of Breakout Groups

Belief Revision

The foundations of belief revision rest on the use of mathematical logic as a modelling system for specifying knowledge bases. These knowledge bases are viewed as the knowledge of an artificial agent, and the goal of the discipline of belief revision is to investigate and formalize the properties of how knowledge bases accumulate.

The fundamental challenge arises from the observation that knowledge bases are inherent incomplete, and that as knowledge of the world accumulates, it is often and even typically the case that "pieces" of knowledge, rendered as sentences in a logic, are contradictory.

So the challenge is to make principled decisions about which sentences should be retained, and which should be discarded, in order to maintain a consistent knowledge base.

The day-to-day business of belief revision research is then focused on how to formulate "change" operators which completely specify what do with new information. Underlying each such formulation is the requirement to address the problem of re-establishing consistency. In fact, any algorithm designed to test the consistency of a data or knowledge base is quickly seen to be closely related to the general global search aspect of optimization problems.

Combinatorial Optimization

There are several facets to the discipline of combinatorial optimization, but perhaps the best way to describe the relationship to the other two workshop themes is to focus on the modelling of optimization problems, and the properties of algorithms designed to solve those problems.

The non-expert's view of combinatorial optimization quickly finds that the central machinery of the discipline, which has existed longer than the other two in this workshop, focuses on identifying classes of models for optimization problems, in order to deploy algorithms designed specifically to produce predictable behaviour on those problems. For example, there is considerable discipline expertise in constructing optimization problem descriptions that lie within the scope of specific and well-studied algorithms (e.g., linear models and the Simplex method).

The day-to-day business of combinatorial optimization is largely in addressing two aspects of optimization: 1) the development of modelling principles that help precisely formulate complex problems, usually with the idea that many complex problems can be "factored" into collections of simpler problems, each of which may be solved within predictable algorithmic resources (space, time); and 2) the development of new algorithms (or improvements on existing algorithms) whose formal properties are sufficiently well understood so as to produce predictable performance on a precisely circumscribed problems.

Constraint Programming

The central focus of constraint programming relies on the logical and computational foundations of logic programming. Simply summarized, logic programming replaces the conventional von Neumann sequential programming by casting a programming problem in two parts: 1) a formal specification of the properties of a problem solution, and 2) the application of a general search procedure that can systematically identify the solutions that satisfy the formal specification of solutions.

The relationship to both combinatorial optimization and belief revision is seen at the level of abstraction where a knowledge base or optimization problem model is viewed as the formal specification of a solution (e.g., a set of combinatorial constraints and objective function, or a collection of sentences comprising a knowledge base), and the consistency computation or identification of an optimal solution is a kind of search algorithm.

The day-to-day business of constraint programming researchers is pretty well-aligned with that of combinatorial optimization, at least at the abstract level: first, what are new ways to precisely model an optimization problem, then, what can one do to improve the computational properties of the general constraint search, so as to predictably reduce computational resources needed to find solutions.

Perhaps the difference between combinatorial optimization and constraint programming is revealed at the next level of detail: the specification of models is still carried out in logical formalisms (like belief revision), but there is a lot of effort expended on finding equivalence transformations on problem specifications, in the hope of finding formulations for which existing constraint programming search methods provide improved solutions.

Summary of resulting ideas

There are two aspects of the resulting plenary discussions that are important to report in a summary as brief as this.

Perhaps the first is that, at the highest level of discussion, it is clear that the discipline of belief revision is by far the most abstract of all three areas, connected mostly because of the underlying foundations of logic and logic programming as the computational basis for consistency computations.

The two areas of combinatorial optimization and constraint programming are more clearly related, aligned on problem modelling and precise identification of efficient methods for their solution.

The second, and more important aspect of the plenary session, as well as the fundamental if somewhat vague outcome of the meeting, are the ideas resulting from deeper discussions on both modelling and search.

In the case of modelling, it is clear that the more experienced combinatorial optimization experts have an accumulation of practical knowledge that can be deployed to find modelling "tricks" to reduce apparently intractable problems to solvable problems. In the case where modelling manoeuvres can not provide the required well-behaved problem specifications, the alternative is to work on new and improved algorithms for more complex problems.

Similarly, with constraint programming, there are a growing repertoire of problem transformation methods that help complex problems succumb to standard constraint search algorithms.

Perhaps the potential advantage for constraint programming is that the specification language is a logical language, where the various "tricks" of transformation are potentially explainable in terms of logical semantics and inference, instead of those of combinatorial optimization, which are largely based on mathematical intuition on the underlying computational methods.

In summary, the general ideas of investigating combinatorial optimization factoring methods, and constraint programming model transformations seem like a fruitful avenue of combined research, with very little explored, and much to be done.

Similarly, the wide repertoire of heuristic methods applicable to the search or computation aspect of optimization and constraint programming, also provides a rich area of multidisciplinary investigation. In particular, heuristic methods from Artificial Intelligence have a natural role in constraint programming based optimization. In this regard, the development of predictable robust heuristic optimization algorithms is another avenue of combined research that should prove useful to all three disciplines.

While this summary is necessarily brief, it is clear that there remains much to discuss across the discipline boundaries.

List of Participants

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Chapter 11

Symmetry and Bifurcation in Biology (03w5075)

May 31–June 5, 2003

Organizer(s): Martin Golubitsky (University of Houston), William F. Langford (University of Guelph), Ian Stewart (University of Warwick)

A brief background

Interest in the application of mathematical methods to biology has been growing rapidly, worldwide, in recent years. The great advances in molecular biology have opened up major new areas, but it is becoming recognised that DNA sequences and other molecular information are only one aspect of the understanding of biological systems. Organisms make use of DNA in highly dynamic contexts. For example DNA has a significant role in organising the development of an organism, but that very little information about the form of the organism and its developmental path can be 'read off' from its DNA sequence. Genetics acts in concert with dynamical physical and chemical processes. The more we learn about genes, the more evident becomes the need for a good understanding of dynamic effects in biology— in growth, in development, in the regulation of genetic networks, in ecosystems, and in evolution.

The mathematics of dynamical systems has undergone its own revolution, as the need to consider nonlinear effects has become clear. The theory of dynamical systems is one of the major growth areas of today's mathematical research, and one of its strengths is a strong connection with applied science.

Pattern formation in physical systems is one of the major research frontiers of mathematics and one of the ways that patterns are studied is through symmetry-breaking bifurcations. Standard examples of pattern forming systems include the Taylor-Couette experiment, Benard convection, and combustion [11]. In these systems patterns may be spatial or a combination of spatial and temporal.

Only more recently have patterns been studied in biological systems using symmetry techniques. In physical systems symmetry often appears through homogeneity — the equations of fluid dynamics and reaction-diffusion systems are identical near every point in space. Biological systems are very good at producing (near) identical cells and this is often the source of symmetry in these systems. Examples of pattern forming biological systems appear through Turing bifurcations in the work of Murray, Maini and others (whose mathematical formulation is close to those in physical systems — mainly reaction-diffusion systems) and central pattern generators (that control rhythms in the nervous system). Moreover, in biological systems, finite symmetry groups seem to be as important as continuous ones.

Mathematically, the symmetries of a system can be used to work out a 'catalogue' of typical forms of behaviour and bifurcation theory can be used to decide which of these forms are likely to be seen first. This catalogue is to a great extent model-independent. By this we do not mean that the specific model involved is irrelevant, but that a much can be deduced by knowing only the symmetries of that model. In a sense, all models with a given symmetry explore the same range of pattern-forming behaviours, and that range of behaviours can be studied in its own right without reference to many details of the model. This workshop addressed both the relevant theory and its application to a variety of biological systems.

Here is a short list of biological systems where pattern formation research has been active.

- Pattern Formation: Turing [22], Murray and Maini [16, 17, 15], Gierer and Meinhardt [10].

  Patterns appear in a variety of biological contexts — most notably in the skin of many animals. Attempts to explain these patterns using reaction-diffusion equations and symmetry-breaking bifurcations began with Turing's seminal work.

- Locomotor central pattern generators: Ermentrout and Kopell [13, 14], Buono, Collins, Golubitsky, and Stewart [12, 6].

  Many biologists believe that there is a cluster of neurons (called central pattern generators) somewhere in the nervous system that generate the rhythms seen in animal locomotion. Models for CPGs often involve symmetry and the rhythms themselves are described by spatio-temporal symmetries of periodic states.

- Visual cortex: Ermentrout and Cowan [7]; Bressloff, Cowan, Golubitsky, and Thomas [4, 3].

  Ermentrout and Cowan showed that geometric visual hallucinations could be viewed as pattern formation in an 'activity variable' representing the voltage of neurons in the visual cortex. These pattern formation arguments were based on symmetry arguments just as in Turing bifurcations. However, the story is more complicated as neurons in the primary visual are sensitive to the direction of contours in the visual field. This physiological fact changes the way that symmetry arguments are used in pattern formation arguments leading to new forms of bifurcation and new applications.

- Genetic code: Forger and Hornos [9, 1].

  Can the 'genetic code' (the ways that codons represent amino acids) be a product of symmetry breaking. Hornos and Forger argue yes. It is a speculative but interesting idea based on representation theory (in analogy to the way that elementary particles are viewed as irreducible representations.

- Speciation: Elmhirst, Cohen, and Stewart [18, 19, 20].

  Sympatric speciation (the dividing of co-mingling species into new species) can be viewed as a symmetry-breaking bifurcation. The traditional view that most speciation was due to allopatric speciation (new species forming from two identical but geographically separated species evolving differently) was based in part on the traditional misconception that a symmetric system must have symmetric solutions. Once it is realized that most solutions to a symmetric equation are asymmetric, spontaneous symmetry-breaking can show broadly why sympatric speciation is a reasonable alternative.

- Spiral waves: Barkley [2], Winfree [24], Glass [5], Wulff [8]

  Spiral waves are known to occur in various biological systems including the heart (eg [5]) and to be associated with unwanted dynamics. Thus the description and control of spiral waves could in principle have important applications. Spiral waves are rotating waves (time evolution is the same as spatial rotation) and symmetry is a good way to describe these states [11]. Winfree [24] explored various quasiperiodic meandering spiral wave states. Much about these states were understood phenomenologically using Euclidean symmetry by Barkley and then more rigorously by Wulff *et al*.

- Slime mold: Takamatsu and Tanaka [21], Weijer [23].

  In an unexpected way the growth of slime molds leads to periodic states exhibiting spatiotemporal symmetries.

Objectives of the workshop

There are two distinct ways to encourage interaction between mathematics and biology. 'Horizontal' programs select specific problems in biology (such as protein-folding) and bring many different mathematical methods to bear. Our workshop was the other kind of meeting: a 'vertical' program organized around a package of general methods that apply to many different biological problems. In this case, the package is the exploitation of symmetries in nonlinear dynamical systems, and the strong relation between symmetry and pattern formation.

Over the past 15 years, we and other authors have explored a far more active role for symmetry, in the context of nonlinear dynamical systems. It has become apparent that the symmetries of a system of nonlinear ordinary or partial differential equations can be used, in a systematic and unified way, to analyze, predict, and understand many general mechanisms of pattern-formation.

It is important to understand that 'pattern' here is not restricted to visual patterns such as shape or pigmentation. The structure and function of the visual cortex involves patterns, and can be modelled by a symmetric network of neurons. The formation of new species is a pattern: open group of organisms, a highly symmetric situation, splits into two groups— a less symmetric one. Synchronous firing of neurons, which seems to be an important feature of brain function, is a pattern. Phase relations in biological oscillators are patterns.

In physics, patterns can often be understood by writing down very specific and accurate mathematical models — equations. Few areas of biology are yet equipped with equations of comparable accuracy. It is here the the symmetry approach has major advantages: it is a general method that applies to a variety of models. It can lead to general conclusions even when specific models are unknown, or controversial, or of limited accuracy.

The point here is not the literal symmetry of a biological system, or an organism, or a process. Hardly anything in biology is exactly symmetric. But a huge range of biological systems possess approximate symmetries (for example all organisms in a species are approximately identical), and the best way to model such systems is to exploit the symmetry of an idealised model, and then consider what changes might occur to the conclusions if the symmetry is close, but not exact.

Activities and Developments

Our workshop on *Symmetry and Bifurcation in Biology* brought together mathematicians studying the role of symmetry in pattern formation and more generally equivariant dynamics, with mathematical biologists and neuroscientists studying interesting biological systems, in an attempt to crossfertilize. Participants benefited from the inspirational beauty of the natural setting as much as as from the efficiently run infrastructure of BIRS, that freed them from everyday distractions. Discussions continued out of the lecture room and into the lounge and beyond, to the hiking trails and the excursion to the top of Sulfur Mountain.

In the course of the Workshop several important sub themes emerged, which can be described as follows.

Synchronization and spatio-temporal patterns in coupled oscillator systems

As mentioned above, many biological systems involve networks of symmetrically connected identical "cells", each of which can oscillate. Speakers described many variations on this theme of coupled cells. Peter Ashwin and Mike Field presented new results on coupled cell networks with invariant sets, linked by many heteroclinic connections of which only a small number are selected by the dynamics. Igor Belykh studied models of networks of neurons, oscillating chaotically but capable of synchronization, even when the cells are not identical so that the perfect symmetries are broken. Luciano Buono presented a model of the CPG for locomotion of quadrupeds. The spatio-temporal symmetries of his model correspond to all the observed "gaits" of quadrupeds, and lead to a natural distinction between primary and secondary gaits. Rod Edwards studied gene regulation networks, in the mathematically tractable "hard switching" limit. His theory predicts a rich variety of stable dynamics and allows a natural classification of bifurcations of limit cycles. Marty Golubitsky and Jeroen Lamb presented a potpourri of theory and examples of coupled cell systems and addressed the question of which of the observed dynamics is due to network architecture and which to the specifics of the cells. Yue Xian Li described a model-dependent approach to the modelling of animal gaits leading to a simplest possible network capable of producing the observed animal gaits. Jeff Moehlis presented new results on networks of identically coupled identical oscillators, obtained by reducing the original equations to phase variables that converge to fixed phase differences. The theory is applied to coupled Hodgkin-Huxley neurons. Reiko Tanaka found new hidden symmetric patterns in chains of slime mold oscillators and proposed a new model that can explain all the observed and hidden patterns.

Nonlinear wave patterns and PDE

Among the many nonlinear wave patterns known to occur in biology, those given the most intensive scrutiny in this Workshop were found in the brain and the heart. Paul Bressloff focused on the effects of spatially periodic inhomogeneities in cortical patterns (brain waves) reflecting the underlying crystaline structure of the cortex, and showed how hallucinatory patterns may arise. Leon Glass studied the propagation of waves in the heart and their relationship to abnormally fast cardiac arrhythmias that suddenly start and stop. Carlo Laing developed PDE methods for pattern formation in a spatially extended domain in a plane. Victor LeBlanc presented new results on forced symmetry-breaking for spiral waves in excitable media modelled by reaction-diffusion PDE and Ian Melbourne explained the hypermeander of such spirals, in which the spiral tip undergoes Brownian-like motion. Wayne Nagata showed how the growing tips of plants may be modelled by reaction-diffusion equations, for which bifurcation theory predicts the branching of the growing tips. Peter Thomas and Andrew Torok discussed symmetry-induced coupling of cortical feature maps in the brain and showed how natural symmetry assumptions lead to patterns that are similar to hallucinations observed e.g. by drug users. Lindi Wahl investigated an evolutionary model of the division of labour, in which each individual carries a subset of all genes necessary for survival of the population, and predicts the evolution of the population into generalists, specialist and parasites.

Bursting patterns and neurons

Bursting behaviour of neurons is essential in brain functions such as motor control, information processing and memory formation. Gennady Cymbalyuk explained the interplay between experiments and dynamical systems analysis on the heartbeat motor pattern in leaches. Gerda DeVries presented a bifurcation analysis of the bursting behaviour of pancreatic beta cells, showing that a pair of two cells gives a better representation of a population of cells than a single cell. Brent Dorion studied sensory networks with time delays in the sensory feedback loop and applied this to electric fish. Tomas Gedeon investigated how sensory systems code information about the outside world in spike trains. Frank Hoppensteadt described bifurcation-based modelling of neural networks. Andrey Shilnikov argued that the Lukyanov-Shilnikov bifurcation of a saddle-node periodic orbit explains the bi-stability observed in a neuron model based on a Hodgkin-Huxley formalism.

Iterated function schemes and other variations

Modelling issues in areas such as sympatric speciation suggested some new variations on the existing coupled cell system approach of Cohen, Elmhirst, and Stewart, which employs a network with all-to-all coupling and full symmetric-group symmetry $\mathbf{S}_N$, where N is the number of coarse-grained samples from the population. The first is to form some kind of scaled limit of the ODE model as N tends to ∞. This presumably would be an integro-PDE modelling the time-evolution of the probability distribution of phenotypes, and it would therefore correspond directly and precisely to biological variables. The coupled cell system models would be discretizations of this continuum model. A second variation is to use a random interaction network at each iteration step, chosen (say) so that any given edge occurs with fixed probability p. The resulting model forms a nonlinear iterated function scheme in the sense of Barnsley, with the additional feature that the *set* of functions being iterated is invariant under the action of $\mathbf{S}_N$, even though most of the individual functions are not invariant. The type of model deserves further investigation: it represents the biology more realistically, and it possesses new and interesting mathematical features.

List of Participants

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Chapter 12

Applicable Harmonic Analysis (03w5019)

June 7–12, 2003

Organizer(s): Rong-Qing Jia (University of Alberta), Sherman D. Riemenschneider (West Virginia University), M. Victor Wickerhauser (Washington University)

The stated goal of the conference was to build on the great success of applicable harmonic analysis in the last decade by bringing together first-rate senior experts along with promising young researchers in the theory and application of wavelet analysis, non-linear approximation, computational fluid dynamics and other applications with an emphasis on a combination of theoretical development with practical applications. The mix of topics worked out marvellously. Although talks were originally grouped into sessions according to topic, the resolution of various scheduling conflicts mixed up these groupings quite thoroughly. However, since all talks were plenary and well-attended, the mixing did not prevent anyone from hearing any talk. It also made the workshop quite lively at times, with the participants sometimes gaining unexpected inspiration from superficially unconnected but in fact surprisingly relevant adjacent talks.

Wavelet Frames

Frames were introduced by R. J. Duffin and A. C. Schaeffer in 1952, in the context of non-harmonic Fourier series. But it was the context of multiresolution analysis and the wavelet expansion in applied harmonic analysis that has thrust frames and other means of generating redundant systems into the forefront. Frames are a system of functions that mimic many of the properties of orthogonal systems in the sense that the discrete norm of the coefficient sequence in a frame expansion of a function is equivalent to the norm of the function. However, frames may not be a basis, even when the frame bounds are tight. In other words, frames are often redundant systems. The added redundancy in representation would allow greater flexibility to achieve better approximation or isolation of desired features.

Ingrid Daubechies is famous with her construction of a family of compactly supported orthogonal wavelets with arbitrary smoothness. She is also a pioneering researcher in the area of frames. An important chapter was devoted to frames in her book *Ten Lectures on Wavelets* [3]. She was not able to participate in the workshop. But her co-workers, Bin Han, Amos Ron, and Zuowei Shen all gave remarkable talks on frames at the workshop. Their works represent the forefront of the current research in frames. In addition, Marcin Bownik, Guido Janssen, Qun Mo, and Morten Nielsen discussed their interesting results on the study of frames.

Amos Ron and Zuowei Shen applied shift-invariant systems to both Gabor frames and affine wavelets frames. Consequently, they established a unitary extension principle to construct tight frame systems. At this workshop, Amos Ron gave a talk about his latest joint work with Shen on generalized shift-invariant systems. They provided characterizations of the Bessel property and the frame property in terms of the norms and inverse norms of related matrices. It is expected that their results will have a significant impact on future research in this direction. Guido Janssen in his talk reported the progress he made in his study of Gabor frames. In particular, he emphasized the essential role played by the Ron-Shen criterion for deciding whether a Gabor system is a Gabor frame. Morten Nielsen devoted his talk to nonlinear approximation with tight wavelet frames. He gave characterizations of classical function spaces in terms of the coefficients in the expansion of tight wavelet frames. He also extended the well-known theory of nonlinear approximation of Ron DeVore from orthogonal wavelets to tight wavelet frames.

Zuowei Shen gave an exciting talk on applications of wavelet frames to image processing. He investigated high-resolution image reconstruction, which refers to the reconstruction of high-resolution images from multiple low-resolution, shifted, degraded samples of a true image. In order to achieve the goal of satisfactory reconstruction, he proposed an iterative algorithm based on wavelet analysis. He demonstrated the advantage of redundant systems (wavelet frames) over wavelet bases for the performance of the iterative algorithm. His numerical results showed that the reconstructed images from his wavelet algorithm were better than that obtained from the existent algorithms.

The work of Bin Han was concentrated on wavelet frames. During his visit to Princeton University, Bin Han together with Ingrid Daubechies completed a series of important papers on constructions of wavelet frames. The ideas initiated in those papers were developed further by Bin Han and Qun Mo in their talks. Han proposed several methods to construct dual wavelet frames and tight wavelet frames from refinable functions. He also generalized his methods to refinable vectors of functions and multivariate refinable functions. Starting from two symmetric generators Qun Mo discussed a way to construct tight wavelet frames with the desirable properties such as symmetry and high order of vanishing moments.

Marcin Bownik in his talk presented some recent results on canonical duals of wavelet frames, a topic investigated by Daubechies and Han earlier. He connected the property of shift-invariance with the existence of affine dual frames. Furthermore, he answered affirmatively a conjecture of Daubechies and Han about the period of a dyadic frame wavelet.

The subject of frames is currently a very active research area within the community of applicable harmonic analysis and has captured the attention and interest of many researchers in several related disciplines. The Journal *Applicable and Computational Harmonic Analysis* (ACHA) with C. K. Chui, R. R. Coifman, and I. Daubechies as editors-in-chief is planning to publish a special issue on this area, compiling the ongoing explosive research work and disseminating the important findings to the wide spectrum of ACHA readership. The special issue will be edited by C. Heil. R. Q. Jia, and J. Stoeckler. Some of the results presented in this workshop will be included in the special issue.

Spline Wavelets and Subdivision Schemes

Splines (piecewise polynomials) were introduced by I. J. Schoenberg in 1946 as efficient tools for interpolation and approximation. As the finite element method emerged, splines were soon applied to numerical solutions of differential equations. Being good representations of curves and surfaces, splines were also applied to computer-aided geometric design. The book *A Practical Guide to Splines* by Carl de Boor, [1], provided a guidance for wide applications of splines. In 1990, motivated by rapid development of wavelet analysis, Charles Chui and Jianzhong Wang initiated their study of spline wavelets. Their research and many other important topics were summarized in Chui's book *An Introduction to Wavelets*, [2].

At this workshop, Charles Chui gave a very interesting talk based on his joint work with Qing Tang Jiang on surface subdivision schemes generated by refinable bivariate spline function vectors. They introduced a direct approach for generating local averaging rules for both the $\sqrt{3}$ and 1-to-4 vector subdivision schemes for computer-aided design of smooth surfaces. The innovation of their work was to directly construct refinable bivariate spline function vectors with minimum supports and highest approximation orders and then to compute their refinement masks. The talk of Serge Dubuc was also related to vector refinement equations. In 1987, starting from cardinal splines, Serge Dubuc and Gilles Deslauriers obtained a family of compactly supported interpolatory refinable functions with arbitrary smoothness. It turned out their family of interpolatory refinable functions was closely related to the Daubechies' orthogonal refinable functions and wavelets. At this workshop, Serge Dubuc reported his recent work on Hermite subdivision schemes. His approach was to transform the initial scheme into a uniform stationary vector subdivision scheme. By investigating spectral properties of related matrices, he gave necessary and sufficient conditions for convergence of Hermite subdivision schemes.

Jianzhong Wang introduced some new multi-order spline wavelets in his approach to edge detection in images. Many edge detection methods use multi-scale representation of images in edge detection. His representation uses multi-scales derived from B-splines of different order. He discussed the advantages of the fast multi-order spline wavelet transform, particularly in dealing with edge detection in the presence of noise.

G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust pioneered the development of spline wavelets into multiple wavelets. In particular, by using fractal interpolation functions, they constructed compactly supported orthogonal spline multi-wavelets. At this workshop, Doug Hardin presented a method for generating local orthogonal bases on arbitrary univariate grids via a squeeze map. He also considered wavelets on semi-regular grids in the plane. Another interesting talk on spline wavelets was given by Song-Tau Liu. He reported his joint work with Rong-Qing Jia on wavelet bases of Hermite cubic splines on the interval. On the basis of Hermite cubic splines, they explicitly constructed a pair of spline wavelets which are continuously differentiable and supported on the interval $[-1, 1]$. These wavelets were then adapted to the interval with a very simple construction of boundary wavelets. The wavelet basis was proved to be globally stable and applied to numerical solution of the Sturm-Liouville differential equation with two-point boundary value conditions. Numerical examples were presented to demonstrate the advantage of the wavelet basis.

Concerning spline wavelets and subdivision schemes there are still many open problems that require further investigations. A main issue is how to construct globally stable wavelet bases with desirable properties on arbitrary triangulations. A related problem is subdivision schemes on irregular meshes in high dimensional spaces. These challenging problems would require creative ideas and innovative techniques.

Fractal Geometry and Tiling

Ka-Sing Lau and Yang Wang, two prominent mathematicians in the area of fractal geometry, gave stimulating talks at this workshop. A main issue in fractal geometry is the self-affine tiling. A self-affine pair consists of an expansive matrix with integer entries and a digit set. Such a pair induces a set called the attractor. In fact, the characteristic function of the set induced is the normalized solution of the refinement equation associated with the expansive matrix and the digit set. The question is to characterize the digit set such that the corresponding attractor tiles the underlying Euclidean space by translations. The Cantor set is a well-known self-similar tile. Both Ka-Sing Lau and Yang Wang have done extensive research on this subject. A notable achievement was the use of number theory by J. C. Lagrias and Y. Wang in their study of self-similar tilings. At this workshop, Ka-Sing Lau presented some new results on characterizations of the digit sets with the determinants of the expansive matrices being prime numbers. In particular, he and X. G. He together solved a conjecture of Lagarias and Wang. The talk of Yang Wang connected the theory of tiling and wavelets to spectral measures, which allow orthonormal bases consisting of complex exponentials. His work pointed a new direction in applicable harmonic analysis. Both Ka-Sing Lau and Yang Wang posed some interesting open problems.

Computational and Applied Harmonic Analysis

A key area of applied harmonic analysis is the solution of equations describing fluids. The theoretical school was represented at our workshop by senior researchers Susan Friedlander and Marco Cannone, who discussed their existence and uniqueness results for a dyadic version of Euler's equation and for the mild version of the Navier-Stokes equations, respectively. Cannone's talk at the start of the June 9 sessions was especially helpful to younger researchers, as it began with a clear introduction to Littlewood-Paley decompositions and their use in his powerful existence theory. Friedlander also pitched the first part of her talk to non-experts, describing a modified Euler equation in which derivatives were replaced by sparse matrices acting on Haar wavelets in a similar way. She described initial conditions that evolve to blow up in finite time. Participants thus saw two very different results, existence and blow-up, derived from different simplifications using the same tool from harmonic analysis.

The numerical analysis school of fluid dynamics was more broadly represented. Marie Farge was one of the first researchers to use wavelet decompositions, to enhance numerical simulations of the Navier-Stokes equation, applying the methods in Grossmann and Morlet's seminal 1984 paper. Her talk described a preliminary result on the segmentation of computed vorticity fields using wavelet thresholding. This technique is used in spatially adaptive numerical simulations, as the (large) segments containing no coherent vortices may be approximated more crudely in order to save computing time. Younger researchers Nicholas Kevlahan and Kai Schneider, on the other hand, described wavelet-based feature detection methods for deciding when to refine the grid in order to maintain high accuracy in a Navier-Stokes simulation.

Senior investigator Wolfgang Dahmen then gave a nice survey of his long-running general research program on the multiscale decomposition of linear and nonlinear operators, which has produced results of interest to both the theoreticians and numericists. This time the application was to existence of solutions in the calculus of variations, and their numerical computation. This may have inspired the theoreticians to simplify their model equations into variational form, and the numericists to try some new iterative methods. As if to drive home the point, Yuesheng Xu described how multiscale operator decomposition can be used to solve general second-kind integral equations numerically by iterative refinement from coarse grids.

Thus the talks presented by this very diverse group of theoretical and numerical analysts formed a remarkably coherent collage of the applications of multiscale and wavelet decompositions. Lin Wei, who would have contributed his results on the elasticity equation to this collage, unfortunately could not attend because of visa difficulties.

For the June 11 afternoon session, a number of people were invited to speak on particularly successful applications of harmonic analysis and wavelets, with the idea being to widen everyone's awareness. Image processing is one such application. Peter Oswald gave a survey of wavelet decomposition methods for computer graphics rendering of surfaces in 3-D. Richard Baraniuk described a spatially adaptive algorithm for image compression that made local decisions on whether to encode edges or textures in a small region based on wavelet components. Jacques Liandrat's results hit the same target from a different direction, using new parameters extracted from the "lifting," or predict-and-update implementation of the wavelet transform to make spatially-adaptive choices of local wavelet transform for compact image coding.

Time series analysis is another area where wavelet transforms are a standard tool. Metin Akay described how to detect obstructive sleep apnea from electroencephalograms using filtering in the wavelet domain. Sinan Gunturk took the opposite tack and spoke on best approximation of time series by filtered pulse trains. These two talks were among the most entertaining of the workshop: Akay's because he was working on snoring, and Gunturk because of his amusing analogy with the "fair duel" problem.

Non-linear Approximation

Non-linear approximation has played an ever increasing important role applied harmonic analysis and other areas. It has been particularly important as a means to choose best approximations using a specified number of terms, n-term approximations, from wavelet and frame representations of functions when the functions are believed to belong to some smoothness spaces, such as the Besov spaces. Two of the most prominent researchers in non-linear approximation, Ron DeVore and Vladmir Temlyakov, presented aspects of their work which were complemented by the talks from younger collaborators.

Ron DeVore's talk on adaptive numerical methods for pde's could well have also been place in the last section, showing some of the natural overlap that occurred. The talk centered around the basic Poisson problem

$$-\nabla u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and $\partial\Omega$ is its boundary. Though adaptive methods have been successfully applied to solve such problems, it was only recently that the methods were shown to converge. There had been no analysis on the rate of convergence. DeVore in collaboration with Cohen and Dahmen had recently developed an algorithm with optimal efficiency for adaptive wavelet methods using n-term approximation. For adaptive finite element methods, there was no known algorithm with a proven rate of convergence except in the univariate case. The talk presented work of Ron DeVore, Peter Binev, and Wolfgang Dahmen that provides such a method. The method is not that much different than others based on bulk chasing of a-posteriori estimated except that it introduces a coarsening step. The main ideas of the coarsening step were presented in the talk by Peter Binev.

Best basis selection in greedy algorithms was the basis of the talks of Vladmir Temlyakov and Guergana Petrova. When considering the expansion of functions in terms of basis elements for best n-term approximation there are a variety of settings. A wavelets basis gives a unique expansion, frames allow some redundancy and therefore some additional flexibility, but one can consider a much more general problem of choosing at each step a basis element from a dictionary and its coefficient. Temlyakov takes a *greedy* approach from non-linear approximation in his selection of next basis element, and introduces a new selection of the coefficients. He is about to prove convergence in any uniformly smooth Banach space.

Guergana Petrova in her talk considers the Banach space L_p and considers the problem of approximation of a function class $\mathcal{F}$ in L_p. Here she will first choose a basis and then use n-term approximation with the elements from that basis. An additional degree of non-linearity is added to the problem by allowing the choice of basis to depend on the function class. This requires the determination of what would be a best basis for a particular function class as characterized by intrinsic properties of that function class. David Donoho first studied this problem in Hilbert space (L_2) with the competition taken over all complete orthonormal systems. To extent these ideas beyond L_2 to L_p requires a replacement for the complete orthonormal systems. The proper setting is to use *greedy bases*, which have been characterized by Konyagin and Temlyakov as democratic unconditional bases. In this setting the problem can be successfully handled. The techniques involve metric entropy an encoding and can be applied in a variety of settings.

Sampling Theory and Hyperfunctions

Recently there has been a lot of interest generated by Statistical Learning Theory, especially as a potential tool in bioinformatics. Learning theory studies objects by looking at random samples. The main question that Ding-Xuan Zhou dealt with in his lecture is to determine the number of samples needed to ensure an error bound with a certain level of confidence. These problems often involve a kernel and the study of its associated reproducing kernel Hilbert space. There is a natural connection to approximation theory, harmonic analysis, and wavelets. This interesting talk introduced the audience to the use of multiresolution analysis, wavelets, and sampling theory to the study of problems of learning through support vector machines. This provided some insight into a rapidly developing field in which topics central to the theme of this Workshop could have major implications.

In another direction, Kurt Jetter looked at a generalization of Shannon sampling, the so-called polyphase sampling. Instead of sampling data at the integers using integer translates of fixed function, the polyphase operator uses the translates of a finite set of functions to sample at different phase shifts of the data. The talk set out various approximation properties of polyphase operators and linked this notion to concepts in wavelet theory.

In an entirely different vein, Dohan Kim gave a very interesting lecture on the connections of some fundamental concepts in Fourier analysis, comparing results in hyperfunctions to the Schwartz theory of distributions. The talk was interesting both for its mathematical content and the historical perspective.

List of Participants

Akay, Metin (Dartmouth College)
Baraniuk, Richard (Rice University)
Binev, Peter (University of South Carolina)
Bownik, Marcin (University of Michigan)
Cannone, Marco (Universite de Marne-la-Vallee)
Chui, Charles (University of Missouri-St. Louis)
Dahmen, Wolfgang (RWTH Aachen)
DeVore, Ronald (University of South Carolina)
Dubuc, Serge (University of Montreal)
Farge, Marie (Ecole Normale Superieure)
Friedlander, Susan (University of Illinois at Chicago)
Gunturk, Sinan (Courant Institute of Mathematical Sciences)
Han, Bin (University of Alberta)

Hardin, Douglas (Vanderbilt University)
Janssen, Guido (Philips Research Laboratories)
Jetter, Kurt (Universitaet Hohenheim)
Jia, Rong-Qing (University of Alberta)
Kevlahan, Nicholas (McMaster University)
Kim, Dohan (Seoul National University)
Kim, Jeong eun (Seoul National University)
Lau, Ka-Sing (The Chinese University of Hong Kong)
Lee, Seng Luan (National University of Singapore)
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Temlyakov, Vladimir (University of South Carolina)
Wang, Jianzhong (Sam Houston State University)
Wang, Yang (Georgia Institute of Technology)
Wickerhauser, M. Victor (Washington University)
Xu, Yuesheng (West Virginia University)
Zhou, Ding-Xuan (City University of Hong Kong)

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Chapter 13

Integration on Arc Spaces, Elliptic Genus and Chiral de Rham Complex (03w5065)

June 14–19, 2003

Organizer(s): Mikhail Kapranov (University of Toronto), Anatoly Libgober (University Illinois-Chicago), Francois Loeser (Ecole Normale Superieure)

The main purpose of the Workshop would be to mix people working in the following three closely related recent topics.

Integration on Arc Spaces

Originally introduced by M. Kontsevich, integration on arc spaces ("motivic integration") has become a major tool in algebraic geometry and singularity theory in recent years. Points of the arc space of an algebraic variety just correspond to formal arcs on the original variety (i.e. when the variety is defined by equations, formal power series solutions of these equations). Motivic integration assigns to subsets of the arc space elements of a suitably completed Grothendieck ring of varieties. Classical additive invariants such as the Euler characteristic or the Hodge polynomial factorize through this ring.

Among successful applications of this theory, let us quote:

- work of Batyrev showing the equality of stringy Euler number of orbifolds with the physicists' orbifold Euler number defined by the Dixon-Harvey-Vafa-Witten formula and giving a proof M. Reid's conjecture on the Euler numbers of crepant desingularizations of Gorenstein quotient singularities.
- work of Denef and Loeser on various types of motivic zeta functions and the motivic Milnor fiber.
- work of Mustata on the characterization of locally complete intersection canonical singularities and on the log canonical threshold.
- work of Loeser and Sebag on motivic Serre invariants and birational geometry of degenerations.

Elliptic Genus

Elliptic genera appeared in the mid-80's both in topology and in physics. They are certain holomorphic functions attached to manifolds which interpolate many known genera of manifolds. Their intuitive physical meaning is as the equivariant indices of the Dirac (or Dolbeault) operators on the space of free loops. In the case of Calabi-Yau manifolds these functions are Jacobi forms. Elliptic genera of manifolds attracted much attention due to works of Landweber, Ochanine, Witten, Taubes, Bott, Liu, Krichever, Hohn and Totaro. The latter, in particular, characterized the elliptic genus as the universal genus invariant under the classical flops. Totaro also raised questions about elliptic genera of singular varieties and related the elliptic genus to the Goreski-McPherson problem about Chern numbers defined via small resolutions. L. Borisov and A. Libgober studied the elliptic genera of singular hypersurfaces in toric varieties and showed, using the Chiral de Rham complex, that two Calabi-Yau hypersurfaces in toric varieties which are Batyrev mirror to each other up to sign, which was predicted by physicists viewing the elliptic genus as part of the information which can be read off from the quantum field theory of a manifold. They also introduced elliptic genera for certain class of singular spaces, including orbifolds and varieties with log-terminal singularities, for which the corresponding analogue of Batyrev's result is still a conjecture. These definitions yield a beautiful formula of Dijkgraaf, Moore, E. Verlinde and H. Verlinde giving mysteriously products closely related to Borcherds forms. The conjecture on equality of orbifold and singular elliptic genera will yield a far reaching generalization of Goettsche generating series for the χ_y-characteristic of Hilbert schemes of complex surfaces and probably is related to orbifold cohomology presently developed by Y.Ruan and collaborators. It appears that this conjecture is just a little out of the reach of the methods used by Denef-Loser and Batyrev for proving a similar conjecture for the Hodge numbers and extension of these methods may lead to a insight in the nature of this problem. Elliptic genera and their relations with representations of vertex operator super-algebras are also the subject of important recent work by H. Tamanoi.

More generally, foundations of (generalized) elliptic cohomology are currently being developed in a series of papers by M. Hopkins and coworkers.

Chiral de Rham Complex

In 1998, F. Malikov, V. Schechtman and A. Vaintrob introduced the chiral de Rham complex which is a new kind of cohomology theory for smooth varieties which is a vertex algebra and which has the elliptic genus as its Euler characteristic.

The original construction by Malikov, Schechtman and Vaintrob was quite computational, but recently M. Kapranov and E. Vasserot gave a more conceptual construction using $\mathcal{D}$-modules on the formal loop space. This approach is more in line with the physical intuition about the elliptic genus and its relation to the free loop space. There is some difficulty in constructing an algebro-geometric version of the free space, which is overcome by dealing with formal loops (*i.e* Laurent formal arcs) which are "infinitesimal in the Laurent direction". One the advantages of this new approach is that it allows one to work with a whole category of $\mathcal{D}$-modules on the formal loop space. One can expects an equivalence of the category of such $\mathcal{D}$-modules and a suitable category of vertex modules over the chiral de Rham complex.

In particular, the $\mathcal{D}$-module techniques seem to be a very promising tool for extension of elliptic genera to various classes of singular varieties, establishing relations with crepant resolutions and so on, by drawing on such tools as the decomposition theorem for projective morphisms (Beilinson-Bernstein-Deligne-Saito). Objectives The moment seems very appropriate to bring together people working in the above recent and rapidly developing topics. Many natural interesting open issues remain to be explored and the interplay of techniques and ideas should show extremely useful. Such a workshop will be particularly useful for young researchers and will provide an intensive introduction to an exciting cutting-edge area of modern research.

List of Participants

Batyrev, Viktor (University of Tübingen)
Berglund, Per (University of New Hampshire)
Bittner, Franziska (University of Essen)
Blickle, Manuel (University of Essen)
Budur, Nero (University of Illinois at Chicago)
Chambert-Loir, Antoine (Ecole Polytechnique)
Dong, Chongying (University of California)
Ganter, Nora (Massachusetts Institute of Technology)

Gorbounov, Vassily (University of Kentucky)
Gordon, Julia (University of Michigan)
Guibert, Gil (Université de Nice)
Ishii, Shihoko (Tokyo Institute of Technology)
Kapranov, Mikhail (University of Toronto)
Laszlo, Yves (Universite Paris VI)
Lehn, Manfred (University of Mainz)
Lejeune-Jalabert, Monique (Universite de Versailles)
Libgober, Anatoly (University of Illinois-Chicago)
Loeser, Francois (Ecole Normale Superieure)
Malikov, Fyodor (University of Southern California)
Melle Hernández, Alejandro (Universidad Complutense de Madrid)
Merle, Michel (Universite de Nice)
Mustata, Mircea (Clay Mathematical Institute)
Nieper-Wißkirchen, Marc (University of Cambridge)
Reguera, Ana (University of Valladolid)
Sebag, Julien (Ecole Normale Superieure)
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Vaintrob, Arkady (University of Oregon)
Van Straten, Duco (Johannes Gutenberg-Universität)
Vasserot, Eric (University of Cergy-Pontoise)
Veys, Wim (University of Leuven)
Wang, Weiqiang (University of Virginia)
Yasuda, Takehiko (University of Tokyo)

Chapter 14

Point Processes—Theory and Applications (03w5062)

June 21–26, 2003

Organizer(s): Peter Guttorp (University of Washington), Bruce Smith (Dalhousie University)

The objectives of this workshop were two-fold: firstly, to introduce young researchers to an exciting and broad research area at the interface between statistics and the sciences; and secondly, to review the state of the field since the P.A.W. Lewis point process workshop in 1971.

The workshop was held in honour of David Brillinger, who has spent most of his career working on scientific applications of point processes and time series. His seminal work on stationary interval functions (Brillinger, 1972) has been the basis for much of the theoretical and applied work in the field. Among the forty participants at the workshop, fifteen did their Ph.D. work under David's supervision. It is characteristic of David's students that they have a wide range of scientific interests, and tend to find their methodological problems in trying to solve scientific questions.

There were several sub-themes of the workshop, which illustrate the diversity of point process applications. These included sessions on point process applications in seismology and neurophysiology, and point process models of wildfires. There were sessions focused on theoretical developments, and spatial point processes. Other point process talks included analyses of music, cell motility, traffic, reaction times, and high level crossings. Time series talks included the modelling of chaotic maps, continuous trading under discontinuous dynamics, wavelets, and GARCH residual processes. Other talks included the analysis of software release as a decision process, and independent component analysis.

Overheads of workshop presentations can be found at `bsmith.mathstat.dal.ca/birs.html`.

Feature presentation

David Brillinger, *Regression, mutual information and point processes*

When one studies the strength of association between variables, the object of interest is often the coefficient of determination, which measures the degree of linear dependence. The coefficient of mutual information provides more, including bounds on the squared prediction error. Some mathematical properties of the mutual information were discussed, together with some uses, and issues of estimation. Application to testing for independence was discussed, and a non-parametric estimate of mutual information was set down. The point process case was considered, and an application to the interval analysis of spike trains was studied. Evidence was provided against the assumption of a renewal process. A second example considered records of soccer teams. In which country is the relationship strongest between the number of goals scored, and the playing of games at home? A third example considered the strength of dependence between river flows as a function of the distance between dams, and focused on the point processes of upcrossings of high levels.

In summary, when moving from the coefficient of determination to mutual information, the statement "The hypothesis of independence is rejected" becomes "The estimated strength of dependence is MI."

Applications in earthquake seismology

Jiancang Zhuang, *Stochastic reconstruction of characteristics associated with earthquake clusters*

This presentation described joint work with Yosi Ogata and David Vere-Jones. Among other things, a knowledge of earthquake clustering is necessary for: estimation of background seismicity for zoning, practical risk prediction, or long term earthquake prediction; real time prediction after large earthquakes; and investigation of different physical mechanisms, such as foreshocks and swarms. The Gutenberg-Richter magnitude-frequency distribution, Omori's formula for aftershock frequency decay, and Utsu-Seki's model relating aftershock area and mainshock magnitude were discussed. A space time point process intensity function was set down, with the magnitude distribution assumed independent of other components. A thinning method was described which relates background intensity, intensity of triggered events, triggering probabilities, and magnitudes. A maximum likelihood estimation strategy was discussed, and fitted to Japanese seismicity data for 1926-1995. Estimators for a number of related functions were described, including the magnitude distribution of background events, triggering probabilities, numbers of direct offspring, and temporal and spatial distributions for triggered events. These stochastic reconstruction methods give a visual diagnosis to the ETAS model, and a method for updating the model formulation. Results from reconstruction of clustering features show the same magnitude distributions for background and triggered events, suggest that triggering ability is well modelled by an exponential law, and indicate long range spatial decay of aftershock activity.

Mark Bebbington, *Evaluating point process models for earthquakes'*

Some temporal intensity functions for earthquake models were introduced, including ETAS (epidemic type aftershock sequence) and SRM (stress release) models. The likelihood approach to point process estimation was described and the AIC was proposed as a model selection tool. Residual analysis was based on the Meyer-Papangelou transformation, leading to a unit rate Poisson process. An illustration of the methods was presented using the Wasach fault data, and the SRM model. A linked SRM model was proposed, which divides an area into regions, and allows transfer of stress between regions by earthquakes. The model was fit to data, and influence diagnostics were based on likelihood displacements. The forecasting power of the model was based on information gain per unit time relative to the Poisson process, which converges to the point process entropy and bounds model performance. Some properties of entropy gains for SRM's are discussed, and the methods are used in fitting the Guttenberg-Richter model to seismicity data from north China.

David Vere-Jones, *Prediction of Point Processes*

The talk began by tracing the history of the development of statistical models for point processes, emphasising the motivation provided by earthquake modelling. Ideas associated with linear prediction were presented, including the Bartlett spectrum, and the least squares predictor of the mean rate. Examples are the Hawkes process and a two point cluster process. Non-linear methods were then considered, with the aim of producing probability distributions for quantities such as time to next event, using all available past information. The martingale representation of a point process was set down, in which the compensator determines the process uniquely. The compensator is expressed through its density, the conditional intensity function. Some examples of conditional intensities were provided, and their use in thinning algorithms for point process simulation was discussed. The use of simulations for forecasting was illustrated, and extensions were made to marked point processes. The assessment of probability forecasts was via the entropy score, an the average log probability gain over a sequence of trials, was shown to be the log likelihood ratio. An example was given on information gains for Gamma distributed renewal processes, and Molchan's τ-ν diagram was discussed as another assessment tool.

Rick Schoenberg, *Prototype Point Processes and Applications in Seismology*

Motivation for this work comes from: the description of a "typical" aftershock sequence; the identification of point process outliers; the comparison of foreshock and aftershock patterns; and regional differentiation in spatial point patterns. Metrics were described for assessing differences between point processes on the line, and a prototype was described as a point pattern minimizing sum of squared distances. Examples were given from seismic point processes. Some bounds and special cases were described, and applications to global aftershock point patterns were considered. Local sequences closest to prototypes are identified, together with outliers. In summary, prototypes are found to be useful for summarizing repeated sequences, identifying outliers, finding "typical" point patterns, and comparing subgroups. Some of the aftershock clusters appeared quite atypical, while pre-shock sequences were found to look completely different from aftershock sequences, in having far fewer points, and being much more dispersed.

Yosi Ogata, *Modelling of heterogeneous space-time earthquake occurrence*

Applications in neurophysiology

Three talks based on joint research were given by Rob Kass, Emery Brown, and Satish Iyangar

Rob Kass, *Inhomogeneous Poisson and Markov interval models*

The work was based on many experiments involving records from individual neurons, and multiple, simultaneously recorded neurons. The experimental data was gathered from the primate visual system while the animal was being presented with visual stimuli, and data were collected in replicate trials. The dynamics involve the instantaneous firing rates (intensity functions) of individual neurons. A nonparametric Bayesian adaptive regression spline methodology (BARS) has been developed for intensity estimation, with pooling of data across trials. The Bayesian smoothing leads to large increases in efficiency. The associated spline based smoothing methods were extended from Poisson process to non-Poisson inhomogeneous Markov interval (IMI) models. Bayesian dynamical models were developed, together with methods for multivariate Poisson and IMI models. The BARS model iss computationally intensive, and requires MCMC simulations. Population variation was assessed using functional data analysis. Timing relationships may be studied via inhomogeneous point processes and Poisson regression models. When intensity varies rapidly in one or more time intervals, the BARS method improves fit and inferences. The application of BARS to multiple intensities allows for the assessment of uncertainty. Trial to trial variability was frequently substantial, and may be confounded with non-Poisson structure or other effects. Spline based models incorporate a small number of principal components of trial to trial variability, after removing the effect of firing rate by averaging across trials.

Emery Brown, *State-space models and neural encoding*

Satish Iyengar, *Stochastic integrate-and-fire neurons*

Applications to forest fire prediction

Haiganoush Preisler, *Estimating wildland fire risks*

The problem of interest was the identification of forest fire danger on U.S. federal lands. The national fire danger rating system was described, and types of input data were illustrated. Historical data sources were discussed, together with some recent spatial data on model inputs, including such things as greenness, live moisture, and cover types. Fire risk probabilities were defined, and a space-time point process model was set down, with the conditional intensity of a fire being the object of interest. A discrete approximation to the likelihood was described. An analysis of some data for Oregon was based on generalized linear models, which includes thin plate and periodic spline components.

Roger Peng, *Assessment of a wildfire hazard index in Los Angeles County, California*

The talk began with an introduction to point process modelling, including the interpretation of the conditional intensity function, the general form of the likelihood, other methods of intensity estimation, and the use of the AIC to estimate model dimension. This was followed by an introduction to point process residual analysis, based on the Meyer-Papangelou transformation to a unit rate Poisson process. Forest fire indices were then discussed. These can be thought of as combining information from many variables, and the indices are used to predict fires. Attention was focused on the National Fire Danger Rating System, and the calculation of the Burning Index, which serves as a summary of weather information and fuel properties. Applications were given to a series of wildfire occurrences in Los Angeles county. Fitted models include spatial, seasonal and Burning Index components. Model selection using AIC selected a model using only Burning Index, although a spatial+seasonal performs nearly as well, and residual analysis suggests the need to incorporate spatial and/or seasonal components.

Other point process applications

Rafael Irizarry, *Music and stochastic processes*

Acoustic signals are represented as the time varying amplitude of a pressure wave, and the discretized version of a sound signal provides a times series representation of music. The tatum is the smallest subdivision of the beat, and is mapped to duration in seconds by the performer. An example of discretizing a music score was given using a few bars of Mozart, and it was indicated how the time series representation can be converted to a marked point process. Questions of interest include whether timbre can be quantified, and whether natural sounds can be re-created. The central issue was whether we can find a meaningful parameterization of sound. Is the shape of the periodogram, and in particular, are temporal changes in shape, related to timbre? A dynamic periodogram estimate was constructed, and a locally periodic signal plus noise model was set down. Estimation was via locally harmonic estimation, using an algorithm similar to loess. A number of examples were seen, and heard.

Georg Lindgren, *New concepts and computational tools for extremes and crossings in random fields, with marine applications*

Some problems from fatigue and marine science were discussed, and some remarks were made on the historical analysis of random waves. A Matlab toolbox for the analysis of random waves was described. Several data types were discussed, with the emphasis being on data collected from satellites. Transformations to the Gaussian uni-dimensional were considered, followed by a discussion of extremes and the point processes of level crossings. Point process methods were introduced for the estimation of wave period, amplitude, and steepness. By modelling sea waves as a moving random field, estimates of wave direction, speed and extremes are forthcoming. The ideas were applied to nonlinear wave models, including Stoke's waves.

John Braun, *Analysis of reaction time point processes*

A reaction time experiment involves presenting visual stimuli in the form of light flashes to an observer who pushes a button on observation of a flash. Data consist of the times of the flashes and the response times. By presenting flashes according to a Poisson process, the RT data can be analysed using point process techniques introduced by Brillinger. Nonparametric estimation of second and third order intensities is used in conjunction with a simple parametric model to statistically check for the presence and nature of nonlinear inhibition in the eye-brain-hand system as well as to study the nature of the reaction time delay distribution. Peaks and troughs in the estimated intensity functions were interpreted as evidence of nonlinear thinning. Parametric bootstrap methods were used in estimating confidence bands.

John Rice, *Counting cars*

Some novel data were introduced in the form of a short video of freeway traffic. Among the questions motivated by the data are: how to estimate the velocity field in space and time?; how to estimate the density of cars; how to estimate traffic flow?; how to examine dynamics?; and how to identify accidents? Issues surrounding the initial processing of video were considered, beginning with the basic transformation video to pixel image data more amenable to statistical analysis. Intensity profiles for lanes of traffic can be aggregated over time to give an image of the traffic flow. Visual characteristics of the image correspond, for example, to waves of traffic. A "shift and match" algorithm leads to estimates of the velocity field using first difference approximations. A critique of the method focused on the need for choice of time and space windows, pixelization problems, the smoothness assumption on velocity, and the need for a model. This led to consideration of a dynamical model of motion, and used ideas from the analysis of point process cross spectra. Relationships to generalized additive models and projection pursuit were discussed.

Tore Schweder, *Point process models in line transect studies*

Line transecting is a survey technique aimed at estimating the density of points in an area. An observer travels at fixed speed through an area along a randomly selected line, and records data on sighted points. In the present context, the points are minke whales, which are available for observation only during short surfacings of 2-3 seconds, between dives of variable length, with a mean dive time of about 80 seconds.

A two dimensional detection function was developed by combining a point process model for surfacings in time, and a model for the probability of initially sighting a whale that surfaces at a given relative position. The combined model leads to a likelihood for the clustering of encounters along the track due to spatial clustering of whales. The model incorporates an autoregressive mixing model for dive time series, together with a modulated Poisson process. By allowing for spatial trends in parameters, the Hawkes model provides a promising framework for fitting line transect data.

Theory of Point Processes

Daryl Daley, *Point processes and second order properties*

This presentation began with an historical overview of theoretical developments in point processes, including the theory of product moment densities, stationary interval functions, and spectral representations. The problem of characterizing the class of Bartlett spectra for weakly stationary mean square continuous random measures was introduced. An application to SARS was discussed, and age dependent branching processes and Hawkes processes were introduced as possible models. The history of the SARS epidemic was summarized, and some epidemiological data were presented, leading to some estimates of mortality. This was followed by a discussion of long range dependence, beginning with a discussion of the Hurst index, and concluding with a condition for long range dependence of of a point process.

Ricardas Zitikis, *Nonparametric estimation of Poisson intensity functions*

Spatial point processes

Adrian Baddely, *Residuals and diagnostics for spatial point processes*

The presentation was based on joint work with Rolf Turner, Jesper Moller and Martin Hazelton. Some examples of spatial point and line patterns were used to motivate the work. In general, one wishes to accommodate spatial inhomogeneity, interpoint interactions, and covariate effects. Some estimation methods were discussed, including MM, ML, pseudolikelihood, and MCMC approaches to evaluation. The principal modelling tool was the Papangelous conditional intensity of a point, given the rest of the process. A fitting algorithm was described, together with some examples of fits. The issue of interest was how to assess the fit of a point process to spatial data. What are the residuals for a spatial point process, and how can they be effectively used? For a spatial point process, the residual random measure is a set indexed martingale. A diagnostic procedure of Stoyan and Grabarnik was described, and applied to some test data sets. In general, the weights assigned to individual points are difficult to interpret. Kernel smoothing of the weights leads to a diagnostic plot which can be used to differentiate good and bad models. A lurking variable plot was introduced, and a form of generalized residual was introduced. Residual plots were constructed for data from a Strauss process, and QQ plots constructed. General conclusions were that: residuals for spatial point patterns are defined at all locations, not just the data points; there are useful analogies to residual plots in classical settings; systematic departures (trends) can be identified by plotting kernel smoothed residuals against covariates; interpoint interactions can be identified through QQ plots of smoothed residuals.

Rolf Turner, *Strauss revisited: using cumulants to fit models to point process patterns*

The form of Strauss' intensity function for a spatial point process in a finite window was introduced. The model depends on the number of r close pairs in the pattern. Strauss' estimate of model parameters is based on the cumulant generating function. Some theoretical results for cumulants and moments were set down, and estimating equations were developed. The calculations involve U-statistics. Some analysis was carried out on Strauss' data on California redwoods. A simulation based estimation algorithm was developed, which can be used to estimate moments, cumulants, probability functions, and fit distributions. A number of estimation methods were applied to Strauss' data. All methods give reasonable results, with the exception of an empirical moments method. An unconditional procedure was developed which does not fix the number of points in the region. The method uses an exponential family model for the point process density, and sufficient statistics were identified for some models. The multivariate cumulant generating function was set down, and an associated estimating equation ensued. Simulation based methods were considered in the multivariate setting, but were out performed by the cumulant based method. Relationships with importance sampling were identified.

Jesper Moller, *Statistical inference and simulation for spatial point processes*

Time Series and Related Talks

Knut Aase, *Continuous trading in an exchange economy under discontinuous dynamics*

Beginning in the 1960's, a discussion arose regarding the so-called "mean-variance" analysis in economics. Since that time, dynamic models have been utilized in economics and finance, and continuous time models have gained prominence, for example, the Merton, Black, Scholes model. Uncertainty in such models is typically modelled by a d-dimensional Brownian motion, leading to stochastic differential equations. The modelling framework was extended to include models with discontinuous dynamics, and model fitting strategies were addressed. By introducing jump dynamics into the noise term of the dynamic stochastic DE, it was found that more than second moments are needed for effective estimation. Some discussion of the "riskfree rate puzzle" followed.

Rajendra Bhansali, *Chaotic maps with slowly decaying correlations and intermittency*

It has been recognized that many observed time series display "long-memory" in the sense that they have correlations that decay to zero at a polynomial or slower rate. Many of the stochastic models for characterizing such behaviour are linear, and related to the popular class of non-stationary integrated processes. Another popular approach is through the consideration of long memory ARCH models. This paper presented a new and entirely different approach to modelling long-memory, by considering a deterministic sequence generated by iteratively applying a one dimensional map, leading to chaotic dynamics. This is in contrast to a random system. There has lately been much work in generating chaotic intermittency maps which exhibit correlations functions decaying at rate $u^{-\alpha}$, for $0 < \alpha < 2$, or even more slowly. The objective of this talk was to introduce an algebraice definition of the intermittent family of maps.

Ed Ionides, *Models for Cell Motility*

Animal cells move actively. Questions of interest include: what makes a cell move? how does a cell respond to a stimulus? how should we best model cell motion in the design of bio-artificial tissues? Different scales for studying motility might include locomotion, translocation, or migration. Migration and translocation can be modelled using stochastic DE's, while locomotion is the proper domain for biophysical modelling. Qualitative responses to a stimulus could be one of several types of taxis, for example, kinesis, which is movement according to a concentration gradient. One would like to model directionality as some combination of taxis motions. A popular description of cell motion in the absence of stimulus is the Ornstein-Uhlenbeck process, and with concentration gradients, one can extend to a non-central OU model. A stochastic DE model was set down for topotaxis, in which a cell turns preferentially towards a stimulus. Parameter estimation for the associated partially observed nonlinear diffusion process was based on particle filters. Statistical analysis of some experimental data suggested that the OU model was preferred to the topotaxis model.

Reg Kulperger, *Garch residual processes*

Nonlinear time series are important in financial modelling and other areas. The use of ARCH models is particularly common, and GARCH models offer a different form for the conditional variance. This work studies residual processes and functionals for such models. The EDF process for residuals from ARCH models is easier to work with than for AR models, but this does not easily extend to GARCH models. From a data analytic standpoint, heavy tails of ARCH models make QQ plots difficult to interpret. The EDF process for a stationary ARCH-M model was studied in detail, and it was found that the residuals do not behave as the iid innovations. Applications include the Jarque-Berra test for normality, and omnibus tests based on skewness and kurtosis. An asymptotic $\chi^2(2)$ distribution for GARCH residuals was derived.

Other presentations:

Jostein Lillestol, *Bayesian estimation of NIG-models via Markov chain Monte Carlo methods*
Anthony Thrall, *Software release as a statistical decision*
Alan Izenman, *Independent component analysis*
Andrey Feuerverger, *Wavelets and parametric inference*

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Chapter 15

Joint Dynamics (03w5122)

June 28–July 3, 2003

Organizer(s): Douglas Lind (University of Washington), Daniel Rudolph (University of Maryland), Klaus Schmidt (University of Vienna), Boris Solomyak (University of Washington)

This workshop focused on three themes: recent advances in ergodic theory, rigidity, and tiling systems. Therefore, we divide our report accordingly.

ERGODIC THEORY

Ergodic theory, as it arises in dynamics, concerns itself principally with the existence and structure of invariant probability measures for the dynamics. This study breaks naturally into two parts, the nature and origin of invariant Borel measures for some group or semigroup of continuous actions on some space and the structure and invariants of such an action, up to measurable conjugacy, for some fixed invariant measure. Both these areas were represented in the presentations on the Ergodic Theory day at the workshop.

From the Ergodic Theorem, the study of the origin and nature of invariant measures, at least for amenable group actions, is really the study of the statistical properties of orbits. The shift action on tiling systems provide excellent examples of this. As the space of tilings under the usual local finiteness assumptions, form a compact metric space, one is guaranteed the existence of invariant probability measures. Penrose tilings tilings, for example, are uniquely ergodic, there is only one shift invariant and ergodic probability measure. If we let B_r be the ball of radius r in $\mathbb{R}^2$ and $\sigma_{\vec{v}}$ be the shift action of tilings, then this unique ergodicity is equivalent to saying that for all continuous functions f the ergodic averages

$$A(x, B_r) = \frac{1}{V(B_r)} \int_{\vec{v} \in B_r} f(\sigma_{\vec{v}}(x))\, d\vec{v}$$

converge uniformly as $r \to \infty$. The limit of course is the integral of f. When such an average at a point x converges to the integral with respect to some measure μ for all continuous f, the point is said to be "generic" for μ. In fact most systems that can be described a substitution systems are uniquely ergodic. At the other extreme, there are $\mathbb{R}^n$ tiling systems whose invariant measures model all $\mathbb{R}^n$ dynamical systems of entropy below the topological entropy of the tiling system.

Two of the presentations at the workshop concerned the exotic statistical behaviour of orbits and the corresponding exotic behaviour of invariant measures for natural systems. Anthony Quas, with his coauthors Emmanuel Lesigne and Mate Wierdl, has been investigation the behaviour of orbits of Cartesian powers of the Morse system. This system is a uniquely ergodic substitution and is perhaps the simplest system more general than a compact group rotation as it is a 2-point extension of the dyadic odometer. They have shown that up to the third power all points are generic for a measure, the natural extension of the corresponding measure for the odometer factor. For the 4th power though points arise which are not generic for any measure.

Chris Hoffman, with his coauthors N. Berger and V. Sidoravicius discussed certain generalizations of Markov processes. We remind the reader that a finite state Markov process is given by a transition matrix of conditional probabilities and that usually, by the Perron-Frobenius theorem, these transition probabilities can arise from only one possible invariant measure. What has been shown here is that, generalizing the Markov case, if the dependence of the transition probabilities on the distant past decays rapidly enough, then again one has this uniqueness. On the other hand, if it does not, then a large number of rather exotic measures can arise, all with this common transition probability.

The remaining three presentations concerned the structure of systems with a fixed measure. This other half of ergodic theory then concerns investigations of ergodic theorems, recurrence properties, mixing properties, entropy and the nature of invariant sub-algebras and joinings.

Bryna Kra discussed a tremendously exciting new development in the study of Furstenberg's theory of multiple recurrence, due to her and B. Host. In his classical work from the 70's, leading to the ergodic proof of Szemeredi's theorem, Furstenberg showed that for T an ergodic map, for all $k \in \mathbb{N}$ and sets $A_1, \ldots, A_k$ of positive measure that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \prod_{n=1}^{k-1} \chi_{A_i}(T^{ji}(x)) > 0.$$

This is saying that intersections of the form

$$T^j(A_1) \cap T^{2j}(A_2) \cap \cdots \cap T^{kj}(A_j)$$

have positive measure for many values of j.

What Kra and Host have show is that this $\limsup$ is actually a limit in L^2 and moreover this limit function is measurable with respect to a sub-algebra where the dynamics is that of a rotation on a k-step nil-manifold. This is probably just the opening chapter in a long story to be developed on their techniques.

A significant and fundamental direction of ergodic theory in the past fifteen years has been an attempt to lift the deeply understood theory of actions of a single transformations to actions of larger groups. Joint dynamics would concern this for higher rank Abelian groups and semigroups. The work of Ornstein and Weiss in the late 80's has shown that the natural limit for natural generalizations are the amenable groups. Dan Rudolph described the current state of a broad approach to such an effort through what is called the "orbit transference method". The core observation here is that all ergodic and free measure preserving actions of discrete amenable groups are orbit equivalent. Based on this Rudolph and B. Weiss developed an approach to lifting results for $\mathbb{Z}$ actions to the general discrete amenable action. Results concerning mixing properties are natural candidates for this method. For example using it one can show that K-systems (systems of completely positive entropy) are mixing of all orders and have countable Haar spectrum. The work discussed concerned a class of theorems showing that mixing properties of a base system lift to isometric extensions of that system, if the extension is weakly mixing. Interestingly, for the mixing property itself the method has not yet proven successful.

The talk of Jean-Paul Thouvenot, "A new information theoretical viewpoint in ergodic theory" presented a structure of ideas, that if successful, would finally offer tools to understand one of the deepest open areas of ergodic theory: To what extent is the weak Pinsker property universal? This property says that for every $\varepsilon > 0$ the system is measurably conjugate to the direct product of a Bernoulli system and a system of entropy at most ε. The structure outlined by Thouvenot would indicate the property is not universal.

The talks themselves and the problem session after offered a broad range of open questions from study of explicit systems arising from tilings and trees to the general theory of measure preserving dynamics. It would not be appropriate to go into further detail here. In addition private discussions among the participants were active and vital. Some very hopeful developments arose from these discussions in a number of areas.

RIGIDITY PHENOMENA

Rigidity is a highly successful theme in several mathematical disciplines, which roughly means that a weak assumption about a mathematical objects (e.g. a map between two spaces) implies a much stronger conclusion (e.g. that this map needs to be of a particularly nice and regular type which is easy to classify).

In the context of this workshop, a particular class of this general theme was considered, namely regarding natural actions of commutative semigroups (such as $\mathbb{Z}_+^d$, $\mathbb{Z}^d$ or $\mathbb{R}^d$ with $d > 1$) which are neither cyclic nor a finite extension of a cyclic group or semigroup.

Specifically considered were two types of such higher rank actions:

1. $\mathbb{Z}^d$ or $\mathbb{Z}_+^d$-actions by endomorphism of a compact Abelian group X. The simplest example is when X is a torus. By suitably extending X to a larger compact Abelian group $\tilde{X}$ one can reduce most aspects of the study of $\mathbb{Z}_+^d$-action by endomorphisms of X to a $\mathbb{Z}^d$-action by automorphisms of $\tilde{X}$.

2. Actions of multidimensional $\mathbb{R}$-diagonalizable subgroups of an Algebraic group G with $\mathbb{R}$-rank at least 2 on quotients G/Γ with $\Gamma < G$ discrete.

A basic problem for these actions is to classify the invariant probability measures (if one works in the measurable category) or the closed invariant sets (in the topological category). Results of these type can be used to prove other rigidity statements, for example partial results towards classification of invariant measures have been used to show that isomorphisms between some such systems need to be algebraic, as well as information about joinings. The converse is also true: in Furstenberg's original paper introducing the subjects, joinings were used to classify closed invariant sets.

The prototype example of such action is the action generated by the maps $x \mapsto nx \bmod 1$ and $x \mapsto mx \bmod 1$ with n, m relatively prime (or more generally multiplicatively independent, i.e. $\log n / \log m \notin \mathbb{Q}$) on the one torus $\mathbb{R}/\mathbb{Z}$. For this example, the closed invariant sets have been completely classified by Furstenberg in 1967, but understanding the invariant probability measures is one of the big open problems in the subject. If μ is such an invariant measure, and if one assumes one of the maps, say $x \mapsto nx \bmod 1$, has positive entropy then it is a theorem of Johnson-Rudolph that μ is Lebesgue.

New results discussed. The last year has been an exciting time with much progress, which was discussed at the workshop. A very partial (and somewhat arbitrary) list of these results is the following:

1. progress regarding classification of measure theoretic isomorphisms of such higher dimensional Abelian actions was reported, both in the context of symmetric spaces and for much more general actions on compact Abelian group by commuting automorphisms (even when each automorphism has infinite entropy).

2. There are fundamental difficulties in extending the Johnson-Rudolph theorem to more general situations. Katok and Spatzier have been able to do so some years ago, but in most situations they obtained a substantially weaker result than Rudolph's theorem, with substantially reduced applications.

   recently, there has been substantial progress in this direction. Applications include a proof of arithmetic quantum unique ergodicity, and a partial results towards Littlewood's conjecture regarding Diophantine approximations.

3. new and exciting results regarding mixing properties of actions on zero dimensional compact Abelian groups by commuting automorphisms were presented, with applications to rigidity.

Partial list of problems from problem session. A fruitful problem session was held. Among the problems presented were the following:

1. $\times n, \times m$-action on $\mathbb{R}/\mathbb{Z}$, with n, m multiplicatively independent.

   (a) Furstenberg's original conjecture was that any nonatomic $\times n, \times m$ invariant measure is Lebesgue. Mixing properties of the action seem to play a role. The question was raised whether assuming additional mixing properties, for example mixing of all orders, implies anything about the measure without an entropy assumption.

   (b) Host has an alternative approach to Rudolph's theorem which gives a stronger results, namely that if μ is a $\times m$-ergodic measure on $\mathbb{R}/\mathbb{Z}$ with positive entropy then for μ almost every x, $\left\{n^j x \bmod 1\right\}_j$ is a quick distributed (with respect to the uniform measure) on $\mathbb{R}/\mathbb{Z}$. His proof works only for m, n relatively prime; more recently it has been shown that a similar result works unless n divides a power of m. It is interesting to check whether a Host type theorem holds also for n,m multiplicatively independent but n does divide a power of m (e.g. $m = 30; n = 2$).

(c) An alternative formulation of Furstenberg's conjecture can be given in terms of since joinings: any measurable self joining of the measure preserving $\mathbb{Z}^2$ action generated by $\times n, \times m$ on $\mathbb{R} divided \mathbb{Z}$ equipped with Lebesgue measure is either the trivial joinings or a joinings which is given by a graph of a measure theoretic endomorphism commuting with $\times n, \times m$ (which in this case can only be the map $\times k$ for some integer k).

(d) In the topological category, the following seemingly simple question is surprisingly challenging: what are the possible limit sets of $\left\{n^i m^j \mathbf{x} \bmod \mathbb{Z}^2\right\}_{i,j\to\infty}$ for $\mathbf{x} \in \mathbb{R}^2/\mathbb{Z}^2$? For $n = 10, m = 20$ it is known this limit sets can be L-shaped. It is possible for n, m relatively prime?

2. As mentioned earlier, many problems about actions of semigroups of endomorphisms can be translated two questions about actions of groups of automorphisms. An interesting possible direction of research is trying to study intrinsic properties of the non invertible case of actions by endomorphisms.

3. For $\mathbb{Z}^d$ actions on zero dimensional compact Abelian groups (such as Ledrappier's example), it is well-known that there can be many irregular invariant measures. Some classification or natural conjecture about these measures is required. In particular, for example, it is known such Haar measure is the only measure which is mixing (indeed only measure which is mixing in a certain critical direction). It at the only measure was full support? Is it the only measure with a single element of the action acting ergodically?

TILING DYNAMICAL SYSTEMS, SUBSTITUTIONS, AND RELATED PROBLEMS

Let M be a Riemannian homogeneous space (such as $\mathbb{Z}^d$, $\mathbb{R}^d$, or the hyperbolic plane), and pick a point to be the origin. Let G be a closed subgroup of the group of isometries of M. Let X be a collection of tilings of M, with the topology that two tilings are ϵ-close if they agree on a ball of size $1/\epsilon$ around the origin, up to the action of an ϵ-small element of G. We assume that X is closed under the action of G, and that X is compact. This implies that X has finitely many distinct tiles and finitely many "patches" of any given size, up to the action of G. A tiling dynamical system is the action of G on X, or the action of a closed subgroup $\Gamma \subset G$, when X is Γ-invariant. In fact, it is not necessary to restrict oneself to tilings: this construction easily extends to spaces of Delone sets (i.e. uniformly discrete relatively dense sets), packings, coverings, etc.

Tiling dynamical systems can be viewed as generalized symbolic dynamical systems in which the tiles play the role of the symbols. Both multidimensionality and the connectedness (in most cases) of the acting group play important roles in the theory. In particular, the theory of multidimensional shifts of finite type can be regarded as a part of this more general theory.

Considering the transitive dynamical system generated by a single tiling, it is possible to express classical symmetry theory in dynamical terms. For instance, when the group acting on the tiling space is $\mathbb{R}^d$, freeness of the action is equivalent to the tiling being aperiodic. Most of the examples that have been studied fall into one of three classes: finite type tilings, substitution tilings or quasiperiodic tilings. The case which has been studied the most is $M = G = \Gamma = \mathbb{R}^d$ (such tilings are often called *translationally finite*). However, one of the active areas of current research deals with the case of $M = \mathbb{R}^d$ and G the full Euclidean group, with the action of $\Gamma = \mathbb{R}^d \subset G$. Such tilings are exemplified by the pinwheel tiling of the plane in which tiles occur in infinitely many orientations. Another area of recent activity concerns the case of $M = \mathbb{H}^2$ and $G = PSL(2, \mathbb{R})$.

Of the three classes indicated above, **finite type tilings** are the broadest, and there are consequently fewer general results than for the others. Finite type tiling spaces are defined by "local rules," by analogy with shifts of finite type. The theory splits into a zero entropy theory, exemplified by the "marked" Penrose tilings, and a positive entropy theory, exemplified by the Rudolph tilings. The positive entropy theory has mostly been studied in the context of multidimensional shifts of finite type. However, the universal modelling property exhibited by Rudolph tilings suggests many rich possibilities for positive entropy results. The zero entropy theory has been studied more extensively. Many examples tend to be strictly ergodic, a phenomenon that is strictly multidimensional (in the finite type case). This is perhaps explained by the fact that most of the examples in this class arise from the other two general classes discussed below.

Substitution tilings generalize one dimensional substitution dynamical systems. Examples in this class are strictly ergodic and entropy zero. They can have pure point spectrum, mixed spectrum, or be weakly mixing. In the translationally finite case, they are never strongly mixing (metrically at least). Penrose tilings, with their substitution rule, are a pure point spectrum example in this class. This class of examples has interesting connections with number theory (the theory of beta expansions in particular) and with the theory of Markov partitions. A remarkable fact is that every substitution tiling dynamical system has a finite type almost 1:1 cover. This provides a huge number of examples of finite type tilings.

Here, we take **quasiperiodic tilings** to mean a projection of a "higher-dimensional" periodic structure to $\mathbb{R}^d$. One way to think of these examples is as generalizations of Sturmian dynamical systems. They are all translationally finite. Examples in this class are strictly ergodic and have pure point spectrum. Since pure point spectrum can be interpreted in terms of diffraction, these tilings have been a popular source of models for quasicrystals. The question of when examples in this class are finite type has been answered in a variety of special cases, and seems to depend in a sensitive way on number theoretic properties of the parameters. Some tilings of this class are also known to arise from substitutions. For example, the Penrose tilings belong to this class too!

Next we mention several specific areas of recent activity, with interesting open problems. There is no way that this account can be comprehensive, so we mostly limit ourselves to the directions discussed at the workshop. Obviously, we will have to describe them in very general terms omitting all the technicalities.

Complexity. The complexity $C(n)$ is the function which counts the number of distinct n^d patches in a tiling space. Entropy is zero if and only if complexity is sub-exponential in n^d. Quasiperiodic and substitution tilings all have sub-polynomial complexity. However, while any strictly ergodic finite type tiling has entropy zero, it seems possible *a priori* for it to have complexity higher than any polynomial. Do such examples really exist?

Computation of algebraic topological invariants (e.g., cohomology groups) for tiling spaces. Technically these are homeomorphism invariants, but any homeomorphism of tiling spaces is actually an orbit equivalence, so these groups carry a great deal of dynamical information. Anderson and Putnam (1998) associated a stationary inverse limit space with a translationally finite substitution tiling, which allowed them to compute the cohomology and K-theory of the space of tilings. This construction turned out to be very useful and was extended to much more general settings by Gähler, Bellissard-Benedetti-Gambaudo, Kellendonk, Ormes-Radin-Sadun, and others. In particular, tiling spaces are inverse limits even when there is no substitution involved. Moreover, in the translationally finite case, the tiling spaces are Cantor set fiber bundles (Sadun-Williams 2003), but the general case is not known.

Conjugacies. For symbolic subshifts, the classical Curtis-Lyndon-Hedlund theorem states that a topological conjugacy can be represented by a sliding block code. For tiling dynamical systems there is a natural analog, usually called "mutual local derivability." Mutually locally derivable tilings generate topologically conjugate systems. However, the converse is false, in general, as was shown by Radin-Sadun (1998) and Petersen (1999). Recently, Holton-Radin-Sadun proved that the converse does hold for pinwheel-like spaces. Along the way they proved the invertibility of the substitution in this setting, complementing the results of Solomyak (1998) for translationally finite substitution tilings. In another direction, Clark and Sadun explored when "deformed" tilings give rise to topologically conjugate systems and found conjugacy invariants using cohomology groups mentioned above.

Optimally dense packings. L. Bowen, Radin, and their co-authors used the ergodic theory approach to investigate optimally dense packings in the Euclidean and, most notably, the hyperbolic, spaces. They obtained uniqueness and symmetry results, organizing the solutions to the densest packings problems through invariant measures on the topological space of packings.

Model sets and pure point diffraction. In the theory of aperiodic systems, which is motivated by the physics of quasicrystals, one of the central questions is to classify (describe) systems with pure point spectrum. Physicists are concerned with the *diffraction spectrum*, but as demonstrated by Dworkin (1993) and Lee-Moody-Solomyak (2002), under general assumptions pure point diffraction is equivalent to pure discrete spectrum in the sense of ergodic theory. A general "cut and projection formalism" (extending the quasiperiodic tilings mentioned above) was developed by Moody, Baake, Schlottmann, and their collaborators, building on earlier work of Meyer. Its main object is a "model set" obtained by projecting a (subset of a) lattice in $\mathcal{G} \times \mathbb{R}^d$ to $\mathbb{R}^d$, where $\mathcal{G}$ is a locally compact Abelian group. All such systems have pure discrete spectrum. On the other hand, all the main examples of substitution systems with pure discrete spectrum seem to be representable in the model set framework, often with $\mathcal{G}$ being non-Archimedean! How general this phenomenon is remains an open problem. Baake and Moody recently developed an approach to systems with pure point diffraction via the theory of almost periodic measures, which leads to the cut and project formalism and may help resolve this question.

Spectrum and geometric models for substitution tiling systems. There are still many open problems, even for one-dimensional symbolic substitution systems. In particular, we do not know when the spectrum is (a) pure discrete, (b) pure singular (but there is always a singular component). A class of nontrivial higher-dimensional systems with a Lebesgue component in its spectrum was recently found by Priebe Frank (2003). Geometric models for substitutions with pure discrete spectrum are closely related to model sets discussed in the previous item. Much less studied are geometric models for weakly mixing systems. Recently, Fitzkee-Hockett-Robinson (2000) showed that certain examples may be realized as almost 1:1 factors of pseudo-Anosov diffeomorphisms. Very little is known when the substitution tiling is not translationally finite; in particular, it would be interesting to decide whether the pinwheel tiling dynamical system is strongly mixing (or even topologically mixing).

List of Participants

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Chapter 16

Mathematical Biology: From Molecules to Ecosystems: The Legacy of Lee Segel (03w5080)

July 5–10, 2003

Organizer(s): Leah Keshet (University of British Columbia), Simon Levin (Princeton University), Mark Lewis (University of Alberta)

Mathematical Biology is a rich and vibrant field, spanning numerous mathematical and biological disciplines, and attracting bright and promising young people, particularly in the post-genomic era.

In recent years, there has been a much greater level of awareness of the role that mathematics can play as a research tool in biology, and as a full and equal partner in furthering an understanding of biological systems. Mathematics can help to elucidate the unity of biology, from cells to ecosystem, by showing how phenomena at one level of organization relate to those at others. Many of the classical applied mathematics techniques (differential equations, dynamical systems, asymptotic analysis, and scaling methods) have been harnessed in exploring important problems at all size and time scales in biology: How neuronal signalling conveys information, how cell motility is controlled, how differentiation and development produces the full complexity of cells and tissues in a mature individual, how populations of individuals interact to form social aggregates, competing or interacting parts of the web of ecosystems, what governs the transmission of disease and epidemics, how evolution shapes these interactions and ecosystems.

While the field itself is quite broad, there are many common themes and shared approaches. In this workshop we explored a unifying common link that highlights how seemingly diverse areas of biological application are interwoven in the discipline, a theme that centres on the work and contributions of Lee Segel.

Lee Segel has pioneered research in Mathematical Biology since the early 1970's. He is a respected authority in the applied mathematics/mathematical biology community who has mentored countless colleagues and students. This conference, aimed at honouring his continuing distinguished career brought together people working on related theoretical and mathematical approaches to a variety of biological phenomena. Unlike many conferences that concentrate exclusively at one level of organization, this workshop aimed to reinforce the idea that evolutionary thinking transcends scales. One of the key themes was to show the usefulness of scaling methods in Applied Mathematics, and specifically in some of the biological applications mentioned above. We bridged work done at the level of molecules and cells (e.g. cell motility, cytoskeleton dynamics, development and differentiation) with that at the level of organisms and populations in ecological settings.

Represented at this workshop were scientists across all ages (from graduate student to emeritus professors), with significant representation from both genders, and with a mixture of talks from novice and experts alike. A uniting common thread linking various contributions was the relationship (via collaboration, citation, extension, or simple admiration) to Prof Segel's numerous areas of expertise.

The formal workshop started on Sunday morning with a session on Molecular Biology chaired by George

Oster (UC Berkeley) highlighting the motion of microorganisms, and physical and mathematical models of their shapes and movements. Here the outstanding questions stem from the interaction between mathematical physics and life at low Reynolds number, where inertial forces are negligible [23]. How are the shapes of organisms such as *Spiroplasma* governed by stresses and strains a the micro level? Here the answer comes from a geometric analysis of forces acting on *Spiroplasma* [35]. How are the forces generated cause cells to divide during the cell cycle? [7, 6, 30] How do animals such as leeches propel themselves forward through movement? (early analyses of leech movement predicted movement should be in the backwards direction, for example.) This question was addressed through using a sophisticated numerical simulation approach (generated by the Peskin method of immersed boundary [4, 10]). How do group-living organisms such as myxobacteria, interact with each other through modifying their spatial environment? The motion of a population of bacteria was described and analyzed [11].

Continuing this theme on Sunday evening, was the session on Cellular Slime Moulds (Chaired by Albert Goldbeter, U Brusselles). These moulds self-organize into spatial patterns via movement in response to chemical signalling [40]. Providing a mathematical or computational explanation of the mechanism by which the patterns are formed has been a major challenge for those in mathematical biology. Methods range from the mathematical analysis of chemotaxis models such as the Keller-Segel model for chemotaxis (a set of partial differential equations), to the development of complex computer simulation code which can be used to track movement of individual cells.

The Keller-Segel equations exhibit a rich variety of mathematical behaviours, ranging from blow-up in finite time to the formation of stable spatial aggregations via Turing-like pattern formation. Such spatial aggregations are interpreted biologically as the necessary ingredient for social interactions between the amoebae.

A new generation of numerical methods have been developed to model amoeboid pattern formation. At the workshop several methods were presented and discussed in detail, including three-dimensional models which track the forces acting on each cell (Palsson model) and the Potts cellular automaton model for (presented by Stan Maree). The Potts model provided a stunning simulation of the patterns observed in slime moulds, culminating in the formation of fruiting body in *Dictyostelium discoideum* [22, 21, 19, 20]. However, an intense discussion ensued on the validity of Potts model formalism for simulating the motion of a cell aggregate. A point made by Oster and Odell is that this formalism is not grounded in Newtonian mechanics, and thus problematic in allowing for unrealistic mechanical effects. The discussion and valuable scientific exchange based on these comments continued informally, during meals, discussion groups, and over many day's worth of coffee. The most positive aspect of this exchange is that it revealed what must be done to put such models on a firmer footing, to the benefit of practitioners relying on this convenient, if problematic simulation method.

One of the focal points of the workshop was the opportunity presented to junior scientists to present their work. Talks were followed by ample and lively discussion, suggestions for aspects to consider, and ideas about extensions or simplifications of the models. The collegial group of participants, and the supportive senior researchers kept the atmosphere open, relaxed, and positive. Many new connections and important future contacts were forged during these five days.

A session on Pattern formation took place on Monday morning [17]. Here mechanisms underlying the formation of patterns in biological tissues were analyzed. Typically there is more than one possible explanation on how such patterns arise. One explanation relies on the presence of an existing chemical 'prepattern' which is read off by the developing tissue. Another explanation posits that the patterns are formed dynamically, through the interaction between different components of the tissues. Typical methods for analyzing pattern formation include nonlinear partial differential equations (PDEs) and numerical cellular automata models. In this workshop, both areas were represented, with Philip Maini (Oxford U) speaking on modelling Biological Pattern Formation through analysis of PDEs [5, 32], and Paulien Hogeweg (Utrecht) on development of multicellular 'model' organisms with numerical models [12, 14]. Details of techniques that can be used to understand the nonlinear aspects of pattern formation were discussed by David Wollkind. Here 'weakly nonlinear' analysis near a bifurcation point is possible if one uses singular perturbation and multiple time scales. The result is a Landau-type which describes the evolution of unstable modes on a long time scale [34, 38].

The first Perspectives on Ecology and Evolution provided a change of biological scale, from the cellular to the environmental. Here, one major problem where mathematical theory can play a role is in understanding the processes that govern the introduction and spread of invading (non-native) species. These species range from diseases (such as West Nile virus) to pest species (such as zebra mussel). Here mathematical models have a central role to play. When the processes of invasions can be described using dynamical systems it is possible to mathematical deduce conditions on model parameters that allow the introduced population to grow and spread, versus die out. These conditions relate to steady-state analyses and to the calculation of a mathematically-defined ‘basic reproduction number’ [2, 36]. Marjorie Wonham (postdoctoral research associate, U Alberta) presented an analysis of conditions under which West Nile virus can be locally controlled. Here, the relevant variables relate to the relative densities of mosquitoes and birds (disease vectors) [39].

Once invading species establish they can have a major impact on the ecosystems that they come into contact with. The impact of invaders and the role they play in their new environment was discussed in detail by Fred Adler who showed how the interaction between evolutionary and dynamical processes creates a so-called ecological ‘balance of terror’ [1].

Formal talks were scheduled on mornings and evenings, leaving afternoons open for outdoor activities, self-organized sessions, and informal discussions. Our group hike took place on Monday afternoon, at a leisurely pace along one of many trails by the river. We passed by the Fairmont Banff Springs Hotel, admired the Bow River Falls, and continued behind the gold course into woodsy trails.

Tuesday was a heavy day, with sessions on Immunology chaired by Alan Perelson (Los Alamos), and Ecology II Chaired by Mark Lewis in the morning and evening. In the morning session, Prof Lee Segel spoke about his own recent research on ‘How the immune system can cope with its multiple overlapping and conflicting goals’ (video available on BIRS web site) [29, 31]. Unlike some of the other areas in the workshop, here the major challenge is to deduce what the appropriate or correct equations are. The area of mathematical immunology is new enough to not have established sets of equations. Interactions are extremely complex. If all interactions are included in a model, then it would be so complex as to defy analysis. On the other hand, it is not yet known which interactions are insignificant enough to leave out of a model. Prof Segel introduced the workshop participants to these modelling challenges.

A New Mexico contingent of theoretical immunologists (and computer scientists), Alan Perelson, Stephanie Forrest, and Christy Warrender presented their work [26, 27, 25, 37]. Here problems in preventing intrusion in computer security can be likened to problems the body faces in keeping out foreign cells. Forrest and coworkers have developed a program to produce bio-based solutions to computer security [13, 24, 33]. Here they provide the computer analogs to cellular antibodies and immune cells to combat possible intruders.

On Tuesday afternoon, we also held an informal session for contributed talks (organized and chaired by UBC graduate student, Adriana Dawes). This very productive session gave students and postdocs an opportunity to share ideas and feedback on their research projects.

Evening talks on ecological dynamics fellows Frithjof Lutscher [18] and Christina Cobbold (U Alberta) were well-received, eliciting discussion and suggestions by Fred Adler. Here changes in the populations of organisms, such as the pest species forest tent caterpillar, can be described by discrete-time, continuous space dynamical systems called integro-difference equations. The mathematical analysis of these dynamical systems, using bifurcation theory, operator theory and numerics, is useful in determining the qualitative behaviour of the biological populations. The study of integrodifference equations is a new field, with many outstanding mathematical challenges. Some of these challenges were outlined by Lutscher and Cobbold.

The session on Modelling Diseases on Wednesday morning (Chaired by Fred Adler, Utah) featured a lecture by David Earn on the dynamics of childhood diseases [3, 9]. Here, the puzzling features of data on periodic disease incidence, and the ability (or inability) of models to explain this data formed the main theme of this talk. Earn developed a series of models from the simple to the more complex. The goal of his talk was to illustrate the appropriate level of complexity and analysis needed to explain complicated temporal patterns of disease outbreak in childhood disease. Is it sufficient to have a simple nonlinear dynamical model, or should additional features be added? The approach Earn took was to start with a simple model, and to build up a series of increasingly realistic nested models, with a view to finding the level of complexity that was sufficient to explain the qualitative features in childhood disease oscillations. Each model was analyzed using the methods of dynamical systems theory.

A duet by Princeton trainees, Joshua Plotkin and Jonathan Dushoff, on the evolutionary ecology of influenza viruses followed. This research looks at disease through the eyes of evolutionary biology, plotting the mutation of the flu virus through changes in the folding structure of their protein [8, 15, 28]. Here the challenge is to successfully plot likely future changes (mutations) in the protein folding, so as to be able to predict likely changes in the flu strain. The concepts outlined in their analysis can be applied more widely to potential disease threats such as avian bird flu.

On Wednesday evening, we held a second session on Immunology, with emphasis on autoimmunity, and Type 1 diabetes (chaired by Rob de Boer, Utrecht). Two representatives of a MITACS team on biomedical modelling (Stan Maree, departing PDF from UBC, and Diane Finegood, SFU) presented results of a two-year effort on modelling auto-immune (Type 1) diabetes. An afternoon MITACS working group meeting on type 1 diabetes preceded the evening session, with participation by researchers from four Alberta and British Columbia universities and discussions about how to coordinate future collaborative work on that subject.

An underlying theme in the workshop was the role that complex adaptive systems play in understanding biological phenomena, ranging from the immune system to cell to ecological interactions [16]. These complex adaptive systems pose a new set of challenges: new models are needed to capture the right level of complexity, without becoming intractable; sophisticated mathematical and numerical approaches are needed for the analysis of the models.

The BIRS workshop included some of the very top mathematical biologists, worldwide. The breadth of the areas covered is testament to the ever widening application of the area of mathematics to biology. Throughout the meeting the scientific insight and gentle humour provided by the guest of honour, Lee Segel, was a constant inspiration. As a finale for this last evening together, we staged an impromptu session composed of limericks and ‘famous quotations’ to recap and summarize workshop highlights.

List of Participants

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Chapter 17

Perspectives in Differential Geometry (03w5074)

July 12–17, 2003

Organizer(s): Richard M. Schoen (Stanford University), Gang Tian (Massachusetts Institute of Technology, Princeton University), Jingyi Chen (University of British Columbia)

Calibrated geometry

Professor Robert Bryant from Duke University spoke on "Resolving a coassociative cone"

Abstract: I construct some new examples of coassociative submanifolds in R^7 that give resolutions of some homogeneous cones that were already known. (The point is that there is an irreducible $SO(3)$ action that fixes the coassociative calibration and I have been able to figure out (by a rather involved ODE analysis) how to compute the $SO(3)$-invariant coassociative submanifolds. Two of these are cones and were already known. The rest are asymptotic to these cones in one way or another, but the topology of the examples can be quite different from what you might expect. These are interesting partly because they give resolutions of isolated 'orbifold-type' coassociative singularities. However, the main tool is a very delicate ODE analysis rather than any PDE methods. It turns out that the ODE system is completely integrable, though this is far from obvious at first glance.

Professor Alexei Kovalev from Cambridge University gave a progress report on "A special Lagrangian fibration of smooth compact Calabi-Yau 3-fold".

Abstract. We construct a compact smooth Ricci-flat Kähler 3-fold as a carefully chosen 'generalized connected sum' of two asymptotically cylindrical manifolds taken at their cylindrical ends. A special Lagrangian (SL) fibration is obtained on each of the two non-compact pieces with typical fibre a 3-torus (and some singular fibres). By proving a gluing theorem for these fibrations a SL fibration is obtained on the connected sum. This is, to the author's knowledge, the first example of a SL fibration of a compact non-singular simply-connected Calabi–Yau threefold.

SL fibrations are important for the Strominger-Yau-Zaslow (SYZ) conjecture which proposes to explain the 'mirror symmetry' duality between Calabi-Yau (CY) threefolds, so that the mirror Calabi-Yau is obtainable by replacing each non-singular SL torus fibre by the dual torus. It is therefore a rather suggestive further direction to identify the mirror CY for this example of SL fibration, according to the SYZ. Another direction of this project is to generalize the construction and find further examples of SL-fibered irreducible Calabi–Yau threefolds (and verify SYZ for them too).

The abstract of Professor Yann Rollin's, from M.I.T., talk is: We prove a Bennequin inequality for Legendrian knots K in 3-dimensional contact manifolds Y, under the assumption that Y is fillable by a 4-dimensional manifold M with a non zero Seiberg-Witten invariant. The proof requires an excision result for Seiberg-Witten moduli spaces; then, the Bennequin inequality is just a manifestation of the adjunction inequality for surfaces lying inside M. As a corollary we find a new proof of an Eliashberg's theorem without using the theory of pseudo-holomorphic curves, namely, that symplectically fillable contact structures must be tight. (Joint work with Tom Mrowka)

Mean curvature flows

Professor Gerhard Huisken from Max Planck Institute for Gravitational Physics, Germany, spoke on "Surgery for geometric evolution equations".

Abstract: Mean curvature flow and Ricci flow are quasilinear parabolic evolution equations describing a deformation of embedded hypersurfaces and Riemannian metrics respectively. They were used in the past to evolve certain initial geometries into known shapes just like a heat equation deforms an initial temperature into an equilibrium. During the last few years and months possible singularities of the flows above have become understood in great detail, creating a chance to extend the flows past singularities and to prove uniformisation results for large classes of manifolds. The lecture discusses recent results of Huisken and Sinestrari on mean curvature flow and their relation to the results of Hamilton and Perelmann on Ricci flow as well as the relation to the geometrisation of 3-manifolds. A particular focus is the surgery for 3-surfaces of positive scalar curvature.

Interesting open problems:

a) Mean curvature flow with surgery for two-dimensional surfaces of positive mean curvature.

b) Convergence of surgery solutions to weak solutions of the flow.

Professor Jiayu Li, from Institute of Mathematics at Academia Sinica, China, discussed some recent developments of mean curvature flows with codimension greater than one. In particular, he considered motions of Lagrangian submanifolds in a Calabi-Yau manifold and symplectic surfaces in a Kähler-Einstein surface. Certain natural convexity assumption is made on the initial submanifolds: in the Lagrangian case one considers an initial Lagrangian submanifold which is almost calibrated by the real part of the holomorphic n-form of the Calabi-Yau n-fold. Mean curvature flow preserves this convexity and the Lagrangian property. In the symplectic case, mean curvature flow preserves the symplectic property. Structure of the tangent cones at the first singular time of these flows was studied. He also discussed the long-time existence and convergence in special cases.

Open problems:

a) what is the size of the singular set of the symplectic or the Lagrangian mean curvature flows at the first singular time?

b) how does one run the flows in the above two cases after the first singular time?

Ricci flow

Professor Huai-dong Cao, from Texas A&M University, spoke on some recent work on Kähler Ricci flow. The abstract of his talk is: An important open problem in the study of the Ricci flow on compact Kahler manifolds is whether a solution to the normalized flow is nonsingular. In this talk I will show how to combine the recent local injectivity radius estimate of Perelman with my Li-Yau-Hamilton estimate for the Kaehler-Ricci flow to derive uniform curvature estimate for solutions to the Ricci flow on compact Kähler manifolds with positive bisectional curvature, i.e., such solutions must be nonsingular. This is a joint work with B. L. Chen and X.P. Zhu.

Harmonic mappings and harmonic functions

Professor Chikako Mese of Connecticut College spoke on "Harmonic maps from a 2-complex to a R-tree".

For a finitely generated group acting an a R-tree T, the equivariant harmonic map from the universal cover of a flat admissible 2-complex X to T defines a holomophic quadratic differential on each face of X. This in turn defines a measured foliation on each face of X which piece together to give a foliated 2-complex. This foliation gives rise to a R-tree T'. We say the action on T is geometric if T' is isometric to T. This generalizes the notion of geometric action of a surface group.

Open questions and research directions:

a) What is the proper definition of a holomorphic quadratic differential on a 2-complex?

b) What properties are needed so the a holomorphic quadratic differential defines a measured foliation on a 2-complex?

The title of Professor Peter Topping from Mathematics Institute at University of Warwick is: Harmonic map flow update.

He discussed some recent developments in the study of the two dimensional harmonic map heat flow. These included the discovery and partial understanding of the 'reverse bubbling' phenomenon, and a negative resolution of the problem of whether the flow $u(T)$ at a singular time T need be continuous. We also discussed some new hypotheses on flows which guarantee that $u(T)$ must be continuous.

Professor Lei Ni from University of California at San Diego discussed a new monotonicity formula for the plurisubharmonic functions on complete Kähler manifolds with nonnegative bisectional curvature. As applications we derive the sharp estimates for the dimension of the spaces of holomorphic functions (sections) with polynomial growth, which in particular, partially solve a conjecture of Yau. The finite generation of the ring of polynomial growth holomorphic functions is still open.

Fully nonlinear equations and conformal geometry

Professor Pengfei Guan of McMaster University discussed certain class of locally conformally flat manifolds with positive σ_k scalar curvature, relies a fundamental result of Schoen-Yau on developing maps. For $k \geq 1$, we say (M^n, g) has σ_k-positive scalar curvature if $\sigma_j(S_g) > 0$ for all $j \leq k$, where S_g the Schouten tensor. If (M, g) has positive σ_k scalar curvature for some $k \geq \frac{n}{2}$, then it must be conformally equivalent to a spherical space form. We address the question: *for $k < \frac{n}{2}$, given a locally conformally flat manifold (M, g_0), when there is a k-admissible metric, i.e., $g \in [g_0]$ such that (M, g) has σ_k-positive scalar curvature?*

First, as in the scalar curvature case, there are some topological obstructions. We prove that $H_p(M) =$ for an optimal range of p depending on k. On the other hand, we obtained a sufficient condition for the existence of k-admissible metrics according to certain Yamabe type functionals. Recall the Yamabe constant of $[g]$ can be defined as

$$Y_1([g]) = \inf_{g\in[g]} (vol(g))^{-\frac{n-2}{n}} \int_M \sigma_1(g) dvol(g).$$

We define a sequence of conformal invariants for $2 \leq l \leq n/2$ by letting

$$Y_l = \begin{cases} \quad \inf\limits_{g\in\mathcal{C}_{l-1}} (vol(g))^{-\frac{n-2l}{n}} \int_M \sigma_l(g) dvol(g), & \text{if } \mathcal{C}_{l-1} \neq \emptyset, \\ \quad -\infty, \quad \text{if } \mathcal{C}_{l-1} = \emptyset, \end{cases}$$

where $\mathcal{C}_k = \{\hat{g} \in [g] \,|\, l(S_{\hat{g}})(x) \in \Gamma_k^+, \forall x \in M.\}$, $\quad \Gamma_k^+ = \{\lambda \in \mathbf{R}^n \,|\, \sigma_j(\lambda) > 0, \forall 1 \leq j \leq k\}$. Our main result is: if $Y_k([g]) > 0$, then $\mathcal{C}_k \neq \emptyset$, i.e., there is a k-admissible $g \in \mathcal{C}_k$. A direct consequence is: if $dimM = 2m$ and $\mathcal{C}_{m-1} \neq \emptyset$ and the Euler characteristic of M is positive, then (M, g_0) is conformally equivalent to S^{2m}.

We prove that, like in the case of the Yamabe problem, one can deform the metric to an "extremal" metric with $\sigma_k(S_g) = constant$ which minimizes the corresponding curvature functional

$$\mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dg$$

if there is a k-admissible metric.

Similar to a result of Schoen-Yau on scalar curvature case, we show that if M_1, M_2 are two locally conformally flat manifolds of same dimension with σ_k ($k < \frac{n}{2}$) positive scalar curvature, one may assign a metric on the connected sum $M_1 \# M_2$ such that it is locally conformally flat with positive σ_k scalar curvature. This leads to conjecture that: for $k = [\frac{n-1}{2}]$, if (M, g) is a locally conformally flat manifold with positive σ_k scalar curvature, then M is of the following form: $L_1 \# \cdots \# L_i \# H_1 \# \cdots \# H_j$, where $L_i' s$ and H_j' are quotients of the standard sphere S^n and $S^{n-1} \times S^1$ respectively. When $n = 3, 4$, it is true following the results of Schoen-Yau and Izeki.

Professor Jeff Viaclovsky from M.I.T. spoke on “Fully nonlinear equations in conformal geometry”. The abstract of the talk is: We present a conformal deformation involving a fully nonlinear equation in dimension 4, starting with positive scalar curvature. Assuming a certain conformal invariant is positive, one may deform from positive scalar curvature to a stronger condition involving the Ricci tensor. A special case of this deformation gives a more direct proof of the result of Chang, Gursky and Yang. We also give a new conformally invariant condition for positivity of the Paneitz operator, which allows us to give many new examples of manifolds admitting metrics with constant Q-curvature. Another problem is to find a conforml deformation to make the k-th elementary symmetric function of the eigenvalues of the Schouten tensor equal to a constant. In dimension 3, this is solved if the manifold is not simply connected, and in dimension 4 the problem is completely solved. This is joint work with Matt Gursky.

Professor Jose Escobar from Cornell University talked about uniqueness and non-uniqueness of metrics with prescribed scalar curvature and mean curvature on compact manifolds with boundary. He also discussed a joint work with Garcia on the prescribed zero curvature and mean curvature problem on the n-dimensional Euclidean ball for $n \geq 2$. They considered the limits of solutions of the regularization obtained by decreasing the critical exponent and characterize those subcritical solutions which blow-up at the least possible energy level, determining the points at which they can concentrate and their Morse indices. When $n = 3$ this is the only blow-up which can occur for solutions. When $n \geq 4$ conditions are given to guarantee there is only one simple blow-up point.

General relativity and Einstein metrics

Professor Hugh Bray of Columbia University talked about “Quasi-Local Mass in General Relativity”. Abstract: we discuss quasi-local mass funcitonals in general relativity, with an emphasis on explicit functionals such as the Hawking mass and the Brown-York mass. We note how it is possible to generalize the Hawking mass to compute generalized Hawking masses, each of which is monotone with respect to certain flows which are themselves generalizations of inverse mean curvature flow. We will comment how the constructions of Shi-Tam have allowed for upper bounds for the Bartnik mass of a region, and discuss the conclusion that the Hawking mass of an outer-minimizing region is less than or equal to the Brown-York mass. Finally we will discuss “inverse mean curvature vector” flows of surfaces in a spacetime and how it relates to observations made by Frauendiener, Mars, Malec, Simon, and Hayward.

Professor Rafe Mazzeo from Stanford University demonstrated how to use the gluing technique to produce Poncaré-Einstein metrics.

List of Participants

Arezzo, Claudio (Universita Degli Studi di Parma)
Bray, Hubert (Massachusetts Institute of Technology)
Bryant, Robert (Duke University)
Cao, Huai-Dong (Institute for Pure and Applied Mathematics)
Chen, Jingyi (University of British Columbia)
Chen, Xiuxiong (University of Wisconsin)
Choe, Jaigyoung (Stanford University)
Escobar, Jose (Cornell University)
Fraser, Ailana (University of British Columbia)
Guan, Pengfei (McMaster University)
Gulliver, Robert (University of Minnesota)
Huisken, Gerhard (Max Planck Inst for Gravitational Physics)
Isenberg, Jim (University of Oregon)
Karigiannis, Spiro (McMaster University)
Knopf, Dan (University of Iowa)
Kovalev, Alexei (University of Cambridge)
Li, Jiayu (Acadimic Sinica)
Mazzeo, Rafe (Stanford University)

Mese, Chikako (Connecticut College)
Ni, Lei (University of California, San Diego)
Pacard, Frank (Universite Paris 12)
Paul, Sean (Columbia University)
Qing, Jie (University of California-Santa Cruz)
Qiu, Weiyang (Harvard University)
Rollin, Yann (Massachusetts Institute of Technology)
Schoen, Richard (Stanford University)
Smoczyk, Knut (Max Planck Gesellschaft)
Tian, Gang (Massachusetts Institute of Technology)
Topping, Peter (University of Warwick)
Viaclovsky, Jeff (Massachusetts Institute of Technology)
Wang, Mckenzie (McMaster University)
Wang, Xiaodong (Massachusetts Institute of Technology)
White, Brian (Stanford University)
Yamada, Sumio (University of Alabama, Birmingham)

Chapter 18

Differential Invariants and Invariant Differential Equations (03w5100)

July 19–24, 2003

Organizer(s): Niky Kamran (McGill University), Peter J. Olver (University of Minnesota)

The workshop succeeded in its goal to bring together leading geometers, group theorists, analysts and applied mathematicians to discuss the state of the art in the fields of differential invariants and invariant differential equations. A wide range of stimulating issues and applications was presented in the talks and discussed among the participants. Indeed, it was impressive to see so many participants working together late in the evening or early in the morning in the BIRS lounge. The BIRS facilities provided an ideal setting for this kind of exchange, and it was particularly gratifying to see geometers, analysts and applied mathematicians succeeding in communicating their ideas to each other.

The spectacular recent advances in the analytic study of the mean curvature flow for submanifolds of Euclidean space and of smoothing by the flows associated to some non-linear invariant diffusion equations made the time for this meeting very ripe. Indeed, the connections between integrable soliton equations, Poisson geometry, and geometric motions of curves and surfaces, can be traced back to the remarkable Hasimoto transformation between the equation of motion of a vortex filament and the nonlinear Schrödinger equation. Hierarchies containing both curve shortening flows, vortex filament equations, equations of elastic rods, and thermal grooving have now been established. The past decade has seen a steady increase in the range of theoretical developments, and practical applications in fluid mechanics, elasticity, geometry, computer vision, and soliton theory. Recent work has highlighted the role of differential invariants and invariant evolution equations in understanding and extending this connection to other geometries and other transformation groups. In particular, operators appearing in the invariant Euler-Lagrange complex constructed with the moving frame theory govern the bi-Hamiltonian structure, and hence integrability of these geometrical motions. Moving frames, in their original form as introduced by Elie Cartan, and in their more recent generalization discovered by Fels and Olver provide an extraordinarily powerful approach to these questions. The participants and speakers at the workshop were therefore carefully selected so as to reflect much of the recent activity in these subjects.

A first group of talks was concerned the interplay between group actions, differential geometry, and the profound connections between classical differential geometries—Riemannian, affine and conformal—and integrable (soliton) differential equations. The power of the method of moving frames as a tool for computing differential invariants and invariant differential operators appeared as a recurrent theme in several of these talks.

Tom Ivey lectured on the vortex filament equation for curves in three-dimensional space as a geometric counterpart to the integrable nonlinear Schrödinger equation arising in nonlinear optics. Connections with the Kirchhoff elastic rod and knot theory were presented, with many unresolved issues in the case of higher genus knotted solutions remaining.

Keti Tenenblat presented her recent results on differential systems describing spherical and pseudo-spherical surfaces of constant nonzero Gaussian curvature. Examples include the nonlinear Schrödinger equation and the Landau–Lifschitz equation.

Gloria Marí–Beffa described how to find an invariant moving frame along a curve in a manifold with a conformal structure, and, as a result, a Poisson bracket defined on the space of conformal differential invariants of curves. Certain conformally invariant curve flow induces Hamiltonian and integrable flows on the differential invariants. Jan Sanders further developed the connections between Cartan geometry and integrability in the conformal case, by showing that the structure equations for the flow of the parallel connection of a curve embedded in an n-dimensional conformal manifold leads to integrable bi-Hamiltonian scalar-vector equations.

Stephen Anco discussed applications of frames to nonlinear wave equations and integrable evolution equations. The frame formulation of arclength-preserving curves in Riemannian geometries has been shown to give a geometrical derivation of integrable evolution equations, such as vector mKdV equations, and their associated Hamiltonian operator structures.

Eugene Ferapontov discussed the integrability of (2+1)-dimensional quasilinear systems of hydrodynamic type through decoupling, in infinitely many ways, into a pair of compatible n-component one-dimensional systems in the Riemann invariants. Exact solutions described by these reductions, known as nonlinear interactions of planar simple waves, can be viewed as natural dispersionless analogs of n-gap solutions. As an example of this approach is a complete classification of integrable (2+1)-dimensional systems of conservation laws possessing a convex quadratic entropy.

Robert Bryant presented recent progress in Finsler geometry, a far-reaching generalization of classical Riemannian geometry. Cartan's generalization of Lie's third theorem is applied to study the space of Finsler metrics of constant curvature. The analysis of the geometry of these equations leads to some very interesting Kähler metrics on the complex n-quadric.

Joel Langer showed how Schwarz reflection geometry of analytic curves in the complex plane can be interpreted as an infinite dimensional symmetric space geometry. In the continuous limit, he obtains the geodesic equation, a second order PDE describing conformally invariant evolution of analytic plane curves.

The second group of talks dealt with the more classical topic of symmetries of differential equations, differential invariants and moving frames.

Ian Anderson proposed a general framework for group theoretical origins of superposition principles for nonlinear differential equations. His examples indicate several intriguing open research problems and, in particular, suggest deep relationships between integrable exterior differential systems and group actions on jet spaces.

Michael Eastwood gave an analysis of the higher symmetries of the Yamabe Laplacian, leading to intriguing connection with conformal geometry and the AdS/CFT correspondence.

Boris Doubrov demonstrated how the classical Wilczynski invariants of linear ordinary differential equations can, through linearization, be used to construct differential invariants for classifying nonlinear ordinary differential equations. Applications in the inverse problem of the calculus of variations were presented.

Peter Olver and Juha Pohjanpelto introduced new computational algorithms for infinite-dimensional Lie pseudo-group actions based on the equivariant theory of moving frames and a new, direct construction of invariant Maurer–Cartan forms for infinite-dimensional pseudo-groups. New applications include complete classifications of differential invariants, recurrence relations and syzygies, the invariant bicomplex, resolution of equivalence and symmetry problems, and the calculus of variations.

Harvey Segur posed a question in the study of partial differential equations with multi-symplectic structure, which can have practical value in stability calculations, as in the nonlinear Schrödinger equation in two or more dimensions. He posed the question of why such an extra Hamiltonian structure is sometimes present, and sometimes not. This led to an extensive discussion of the issue with several of the participants.

The third theme was the interaction of symmetry and geometry with numerical and symbolic computational algorithms. This tied in to the rapidly evolving field of geometric numerical integration.

Peter Hydon and Elizabeth Mansfield have developed a new theory of "difference forms" that are designed to play the same role for difference equations, including discretization of continuous systems, that the classical exterior calculus plays for differential equations. They showed how to derive analogues of Stokes' Theorem, de Rham and Çech cohomology, and how to compute nontrivial cohomology groups of rectangular lattices with holes. These constructions have direct interpretations and consequences for various finite difference schemes, including collocation methods. Applications in digital image analysis were indicated.

The key point of Greg Reid's talk on numerical jet geometry was that, in real world applications, one only deals with approximations and so one must rework the calculus of symmetry groups, conservation laws, explicit solutions, etc. taking into account the limited precision of numerical parameters in the system. His goal is to develop a numerical version of algebraic methods such as Gröbner bases, relying recent developments in numeric algebraic geometry and the singular value decomposition. Applications to computer vision and the determination of approximate symmetries of partial differential equations were discussed.

Pavel Winternitz explored the Lie point symmetries of difference schemes involving one dependent and one independent variable. The symmetries act on the equation and on the lattice. If the lattice is suitably adapted, difference schemes have essentially the same symmetries as differential equations obtained as their continuous limits. He showed how symmetries can be used to classify difference schemes, to decrease the order of the scheme, and to obtain exact solutions. Variational symmetries are particularly useful in this context. Consequences for numerical analysis will be discussed, as well as generalizations to multivariable difference schemes.

Thomas Wolf demonstrated his computer packages for determining the existence of higher order symmetries and first integrals for classifying integrable systems. In particular, all quadratic Hamiltonians of Kowalevski type having additional first integral of third or fourth degree were found and quantum analogs of these Hamiltonians given.

Finally, two talks on applications, in quantum mechanics, and in computer vision, rounded out a successful workshop program.

Federico Finkel survey recent results on Sutherland spin models from quantum mechanics. Complete integrability and exact solvability of this model can be established by relating it to a set of differential-difference operators known as Dunkl operators, coming from the theory of orthogonal polynomials.

Kaleem Siddiqi explained the use of flux invariants for shape in image processing. He considers the average outward flux through a Jordan curve of the gradient vector field of the Euclidean distance function to the boundary. In the zero-area limit, this Euclidean-invariant measure serves to distinguish medial points, providing a theoretical justification for its use in a Hamilton–Jacobi skeletonization algorithm. In the case of shrinking circular neighbourhoods, the average outward flux measure also reveals the object angle at skeletal points. Hence, formulae for obtaining the boundary curves, their curvatures, and other geometric quantities of interest, can be written in terms of the average outward flux limit values at skeletal points.

List of Participants

Anco, Stephen (Brock University)
Anderson, Ian (Utah State University)
Boutin, Mireille (Max-Planck-Institut Fur Mathematik)
Bryant, Robert (Duke University)
Doubrov, Boris (International Students at Lewis & Clark)
Eastwood, Michael (University of Adelaide)
Ferapontov, Eugene (Loughborough University)
Finkel, Federico (Universidad Complutense)
Gomez-Ullate, David (Centre de Recherches Mathématiques)
Hydon, Peter (University of Surrey)
Ivey, Thomas (College of Charleston)
Kamran, Niky (McGill University)
Karigiannis, Spiro (McMaster University)
Khovanskii, Askold (University of Toronto)
Kruglikov, Boris (University of Tromsoe)
Langer, Joel (Case Western Reserve University)
Mansfield, Elizabeth (University of Kent at Canterbury)
Mari-Beffa, Gloria (University of Wisconsin)
Olver, Peter (University of Minnesota)
Phillips, Carlos (McGill University)

Pohjanpelto, Juha (Oregon State University)
Reid, Greg (University of Western Ontario)
Robart, Thierry (Howard University)
Sanders, Jan (Vrije Universiteit)
Segur, Harvey (University of Colorado)
Siddiqi, Kaleem (McGill University)
Smaili, Fatima Drissi (McGill University)
Tenenblat, Keti (Universidade de Brasilia)
The, Dennis (McGill University)
Wang, Jing Ping (Brock University)
Weiss, Isaac (University of Maryland)
Winternitz, Pavel (Centre de Recherches Mathematiques)
Wolf, Thomas (Brock University)
Yamaguchi, Keizo (Hokkaido University)

Chapter 19

Analysis and Geometric Measure Theory (03w5046)

July 26–31, 2003

Organizer(s): Ana Granados (University of Washington), Herv´e Pajot (Universit´e de Grenoble I), Tatiana Toro (University of Washington)

Presentation of the workshop

Objectives

The workshop has been dedicated to problems where there is strong interplay between analysis (in particular harmonic analysis and complex analysis) and geometric measure theory (in particular rectifiability and variational methods).

Topics to be covered include

(i) Analytic capacity and rectifiability
The classical Painlevé problem consists in finding a geometric characterization for compact sets of the complex plane which are removable for bounded analytic functions. The methods used to study this problem come from complex analysis (analytic capacity), harmonic analysis (Cauchy singular integral operator) and geometric measure theory (rectifiability). In 1998, G. David solved the Vitushkin conjecture which provides an answer to Painlevé's question for sets with finite 1-dimensional Hausdorff measure. His work relied on the ideas of many mathematicians among others M. Christ, P. Jones, P. Mattila, M. Melnikov and J. Verdera. Recently, X. Tolsa proposed a solution for the Painlevé problem in terms of Menger curvature.

Problems to be discussed during the workshop include:

- Discussion of Tolsa's conditions;

- Bilipschitz invariance of the class of removable sets for bounded analytic functions in the complex plane;

- Relationship between analytic capacity and Favard length;

- Harmonic analysis in nonhomogeneous spaces;

- The higher dimensional case, namely the study of removable sets for Lipschitz harmonic functions in $\mathbf{R}^n$ (the main problem is that although there exist analogs of the Menger curvature for $n-1$-dimensional sets, they are not adapted to the study of the Riesz transforms. Hence, only a few basic things about this problem are known).

(ii) Analysis and rectifiability in singular metric spaces
Partially motivated by questions arising in classical differential geometry, several authors have begun developing theories of analysis and rectifiability in metric spaces. To this effect basic tools of geometric function theory, for example Poincaré inequalities or quasi-conformal mappings, have been introduced and studied in general metric spaces. Counterparts to the classical theorems in Euclidean spaces have been proved in metric spaces with bounded geometry. For instance, in 1999 J. Cheeger proved a version of Rademacher's theorem on the differentiability of Lipschitz functions on metric spaces where Poincaré inequalities hold. The tools from non-smooth analysis play a crucial role in understanding limiting phenomena arising from smooth geometry.

Problems to be discussed during the workshop include:

- Geometric analysis (Poincaré inequalities, Sobolev spaces, ..) and applications to PDE and geometry;

- Basic tools of geometric measure theory (Sets of finite perimeter, area and co-area formulae, ..) in metric spaces;

- The Kakeya problem;

- Definitions of rectifiability in metric spaces (for instance, in Carnot groups).

(iii) Mumford-Shah functional
This functional was introduced in connection with image segmentation. Let Ω be a bounded domain in the plane and let g be a bounded function on Ω. The Mumford-Shah functional is given by

$$J(u,K) = \int\int_{\Omega\sim K} |u-g|^2 + \int\int_{\Omega\sim K} |\nabla u|^2 + H^1(K).$$

The existence of minimizers (u,K) (in a reasonable sense) is known, but the main problem consists in studying the geometric properties of the set of singularities K. The Mumford-Shah conjecture states that K should be the finite union of C^1 arcs. Recent progress have been made by G. David, A. Bonnet, L. Ambrosio, S. Solimini, N. Fusco among others, but the conjecture is still open.
The study of the Mumford-Shah functional in higher dimensions is a vibrant new question which seems to be related to the theory of minimal surfaces.

Problems to be discussed during the workshop include:

- Complete classification of the global minimizers of the 2-dimensional Mumford-Shah functional;

- Study of cracktips (C^1 regularity, calibration,...) for the 2-dimensional Mumford-Shah functional;

- Study of the 3-dimensional Mumford-Shah functional, in particular connexions with minimal surfaces, complete classification of global minimizers.

Geometric Function Theory (written by J. B. Garnett)

1. Sunhi Choi: Lower Density Theorem for Harmonic Measure

Let Ω be a simply connected domain in $\mathbb{C}$ and let $\omega(w, \cdot, \Omega)$ denote the harmonic measure on $\partial\Omega$ for $w \in \Omega$. If f is a conformal mapping from the unit disk $\mathbb{D}$ onto Ω with $f(0) = w$, then the angular limit $f(\zeta)$ exists at almost every $\zeta \in \partial\mathbb{D}$ and the harmonic measure of a set $E \subset \partial\Omega$ is the normalized linear measure of $f^{-1}(E) \subset \partial\mathbb{D}$. We have the following theorem:

Theorem 1: $\omega \ll \Lambda_1$ on the set

$$\{x \in \partial\Omega \,|\, \liminf_{r\to 0} \frac{\omega(B(x,r))}{r} > 0\}.$$

Theorem 1 was conjectured by C. J. Bishop in 1991. It has several corollaries.

Corollary 1: Let F be a subset of $\partial\Omega$ and assume that there exists a constant $M(F)$ such that

$$\sum \mathrm{rad}(B_i) \leq M(F) < \infty$$

for every disjoint collection of balls $\{B_i\}$ with $\mathrm{center}(B_i) \in F$ and $\mathrm{rad}(B_i) < \mathrm{diam}(\partial\Omega)$. Then, $\omega \ll \Lambda_1$ on F.

Conversely, Theorem 1 can be easily derived from Corollary 1.

The next corollary, first proved by Bishop and Jones by much different methods, resolves a conjecture of ∅ksendal.

Corollary 2: Let F be a subset of a rectifiable curve Γ, then for any simply connected domain Ω, $\omega \ll \Lambda_1$ on $F \cap \partial\Omega$.

The last corollary was also conjectured by Bishop.

Corollary 3: At ω-almost every McMillian twist point $x \in \partial\Omega$,

$$\liminf_{r\to 0} \frac{\omega(B(x,r))}{r} = 0.$$

The corollaries follow from the theorem by covering lemmas, Lebesgue density arguments and theorems of Makarov and Pommerenke relating harmonic measure to linear measure. The proof of the theorem uses extremal length and some explicit constructions of Lipschitz domains.

2. John Garnett: Analytic Capacity, Cantor Sets, Menger Curvature and Bilipschitz Maps

The talk was a survey of the theory of analytic capacity, emphasizing the important recent work of Melnikov and Verdera, of Tolsa and of Volberg.

The *analytic capacity* of a compact plane set E is

$$\gamma(E) = \sup\big\{|a_1| : f(z) = \frac{a_1}{z} + \cdots \in H^\infty(\mathbb{C} \sim E), ||f||_\infty \leq 1\big\}.$$

Thus $\gamma(E) = 0$ if and only if there are no non-constant bounded analytic functions on $\mathbf{C} \sim E$. The main question is to give a geometric necessary and sufficient conditions for $\gamma(E) > 0$. In particular if T is a bilipschitz homeomorphism of the plane, is there a constant $C = C(T)$ such that

$$C^{-1}\gamma(E) \leq \gamma(T(E)) \leq C\gamma(E)? \tag{1}$$

It is classical that $\gamma(E) = 0$ if the Hausdorff measure $\Lambda_1(E) = 0$ and $\gamma(E) > 0$ if $\Lambda_\alpha(E) > 0$ for some $\alpha > 0$, i.e. if the Hausdorff dimension of E exceeds 1.

For sets of dimension 1, more recent work of Calderón, Mattila-Melnikov-Verdera, and David, using some ideas of Christ and Jones, show that if $0 < \Lambda_1(E) < \infty$ then the following three conditions are equivalent:

(i) $\gamma(E) > 0$

(ii) there is a rectifiable curve Γ such that $\Lambda_1(E \cap \Gamma) > 0$

(iii) there is $F \subset E$, $\Lambda_1(F) > 0$ and the Cauchy integral

$$Cf(z) = \text{p.v.} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\Lambda_1(\zeta),$$

is bounded $L^2(F, ds) \to L^2(\Gamma, ds)$.

The proof that (iii) implies some local rectifiability hinges on the notion of *Menger curvature*. For three complex numbers named x, y, and z, let $c(x, y, z)$ be the reciprocal of the radius of the circle through x, y and z, and take $c(x, y, z) = 0$ if the points are co-linear. Let μ be a finite positive Borel measure of *linear growth:* $\mu(B(z, R) \leq R, \quad \forall z, \ \forall R$. The Menger curvature of μ is defined to be

$$c^2(\mu) = \int \int \int c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z).$$

The connection between Menger curvature and the theorem rests on a remarkable discovery of Melnikov and Verdera: If μ is a positive measure of linear growth, then

$$\frac{c^2(\mu)}{6} = \int \Big| \int \frac{d\mu(w)}{w - z} \Big|^2 d\mu(z) + O(1).$$

Hence by (iii) $\mu = \chi_F \Lambda_1$ has $c^2(\mu) < \infty$, and an argument using the P. Jones β-numbers shows there is some rectifiable curve Γ such that $\Lambda_1(F \cap \Gamma) > 0$.

The assumption (2) was later removed by G. David, and later by Nazarov, Triel and Volberg by a different method.

The remaining case, E of Hausdorff dimension 1 but $\Lambda_1(E) = \infty$ was recently resolved by X. Tolsa, who proved that $\gamma(E) > 0$ if and only if E supports a positive measure of linear growth and finite Menger curvature and if and only if E supports a positive measure of linear growth and bounded Cauchy potential $\int \frac{d\mu(w)}{z-w}$.

The speaker described his result with Verdera that (1) holds for all bilipschitz images of planar Cantor sets, a new stronger theorem of Tolsa that proved (1) for *all* compact plane sets, and the generalizations by Volberg (without Menger curvature!) to the case of Lipschitz harmonic capacity in $\mathbb{R}^n$. In particular, Volberg has proved that for $E \subset \mathbb{R}^n$ compact, if

$$\Gamma_n(E) = \sup\big\{ | < \Delta f, 1 > | : f \text{ is harmonic off } E, ||\nabla f||_\infty \leq 1 \big\}$$

and

$$\Gamma_n^+(E) = \sup\big\{ \mu(E) : \mu > 0, \int_E \frac{d\mu(y)}{|x - y|^{n-2}} = f(x) \in \mathrm{Lip}_1, \ \ ||\nabla f||_\infty \leq 1 \big\},$$

then $\Gamma_n^+ \leq \Gamma_n \leq C_n \Gamma_n^+$.

3. Marie Jose Gonzalez: Geometry of Curves and Beltrami-Type Operators

A rectifiable plane curve Γ is called a *chord-arc* curve (or Lavrientiev curve) if each sub-arc $\gamma \subset \Gamma$ with endpoints a and b has length

$$\ell(\gamma) \leq C|b - a|.$$

We assume that $\infty \in \Gamma$, and that Γ is chord-arc. Then $\Gamma = \varphi(\mathbb{R})$ where φ is a bilipschitz homeomorphism of the plane to itself and if Φ is a conformal map from the upper halfplane to either component of $\mathbb{C} \sim \Gamma$, then $\log(\Phi') \in BMO$.

This talk gives the latest news on three related problems:

Problem 1: If $\epsilon > 0$ and if f is a bilipschitz homeomorphism of the plane, can f be factored

$$f = f_1 \circ f_2 \circ \cdots \circ f_n$$

where each f_j and f_j^{-1} has Lipschitz constant bounded by $1 + \epsilon$?

Problem 2: If Γ is a chord-arc curve, is there a deformation from Γ to $\mathbb{R}$ through which $\log(\Phi')$ varies continuously in BMO?

Problem 3: Is the subset $\big\{ \log(\Phi') : \Gamma \text{ chord} - \text{arc} \big\} \subset BMO$ connected?

A combination of theorems by Astala and Zinsmeister, MacManus, and Bishop and Jones shows that when Γ is a quasicircle the following are equivalent:

(i) $\log(\Phi') \in BMO$;

(ii) $\Gamma = \rho(\mathbb{R})$ where ρ is a quasiconformal map for which $\frac{|\mu^2|}{y}$ is a Carleson measure in the upper half plane and $\mu = \frac{\rho_{\bar{z}}}{\rho_z}$ is the Beltrami coefficient of ρ.

(iii) Γ contains big pieces of chord-arc curves.

Because of (iii) such curves are called "BJ curves". It follows from (ii) that the set of BJ curves is connected in the $\log \Phi' - BMO$ topology. An important related result is the 1988 theorem of Semmes: The quasicircle Γ is chord-arc if $\frac{|\mu^2|}{y}$ has small Carleson measure constant.

The speaker discussed her two recent theorems with K. Astala.

Theorem 1: Γ is a BJ curve if and only if there exists a quasiconformal map ρ such that $\Gamma = \rho(\mathbb{R})$, ρ has Beltrami coefficient μ and $I - \mu S$ is bounded on $L^2(\frac{dxdy}{y})$, where S is the Beurling transform

$$Sf(z) = \frac{1}{\pi}\int\int \frac{f(w)}{(w-z)^2} dA(w).$$

Theorem 2: Γ is chord arc if and only if there exists such ρ and μ such that $I - \mu S$ is invertible on $L^2(\frac{dxdy}{y})$.

The speaker also discussed further connections between $I - \mu S$ and the Semmes result above.

4. Pekka Koskela: Metric Sobolev Spaces

This talk gives an approach to Sobolev spaces in metric spaces based on point-wise Lipschitz constants. The *point-wise Lipschitz constant* of u is

$$\mathrm{Lip} u(x) = \limsup_{r\to 0} \sup_{d(y,x)<r} \frac{|u(y) - u(x)|}{r}.$$

(X, d, μ) is a *doubling space* if (X, d) is a metric space and μ is a Borel measure on X such that for constant C_d,

$$\mu(B(x, 2r)) \le C_d \mu(B(x, r)).$$

We say X supports a $p-$*Poincaré inequality* if there exist constants C_p and $\lambda \ge 1$ such that

$$⨍_B |u - u_B|\, d\mu \le C_p \,\mathrm{diam}(B) \left(⨍_{\lambda B} (\mathrm{Lip} u(x))^p \, d\mu\right)^{1/p}, \tag{19.1}$$

for all balls B and for each Lipschitz function u.

This should perhaps be called a weak Poincaré inequality, but it turns out that the Poincaré inequality always improves itself to a (p, p)-inequality, perhaps with larger C and λ. Indeed, even a (q, p)-inequality follows with an optimal $q > p$. Also, the constant λ can often be taken to be 1 by enlarging C. This holds if the metric d is a *path metric* (i.e. infimum of lengths of paths joining the points) and *geodesic*: in this case the geometry of balls can be controlled and one can iterate the Poincaré inequality so as to decrease λ. We call such a metric a *length metric* and the corresponding space a *length space*. If we assume that X is *proper* (i.e. all closed balls are compact), then it follows from the Poincaré inequality that we can replace the metric d with a bi-Lipschitz equivalent length metric.

In $\mathbb{R}^n$ every Sobolev function has a gradient almost everywhere. In our general situation a version of this persists, if we use the concept of upper gradient.

Let $u : A \to \overline{R}$, $A \subset X$. Any Borel function $g : A \to [0, \infty]$ such that for each rectifiable path $\gamma : [0, l] \to A$

$$|u(\gamma(l)) - u(\gamma(0))| \le \int_\gamma g ds$$

is called an *upper gradient* of u on A. We now define, for given $1 \le p \le \infty$,

$$N^{1,p}(X) = \{u \in L^p(X) : \ u \text{ has an upper gradient } g \in L^p(X)\},$$

where the L^p-spaces are taken with respect to our measure μ and the concept of an upper gradient is with respect to our metric d. The norm on $N^{1,p}$ is

$$||u||_{1,p} = ||u||_p + \inf_{g_u} ||g_u||_p,$$

where the infimum is taken over all upper gradients of u, and as usual one needs to consider equivalence classes in order to obtain a normed vector space. In the Euclidean setting

$$N^{1,p}(R^n) = W^{1,p}(R^n)$$

when both the metric and the measure are the usual Euclidean ones.

Theorem 1: Suppose that (X, d, μ) is a doubling length space that supports a p-Poincaré inequality. Let B be a ball and $u \in N^{1,p}(B)$. Suppose that $\mu(B(x, r)) \geq C_b(r/\operatorname{diam}(B))^s \mu(B)$ whenever $B(x, r) \subset B$.

1. If $p < s$, then
$$||u - u_B||_{L^{p^*}(B)} \leq C \operatorname{diam}(B)\mu(B)^{1/p^*-1/p}\|g\|_{L^p(B)}, \tag{19.2}$$
where $p^* = ps/(s-p)$.

2. If $p = s$, then
$$\fint_B \exp\left(\frac{C_1\mu(B)^{1/s}|u - u_B|}{\operatorname{diam}(B)\|g\|_{L^s(B)}}\right)^{s/(s-1)} d\mu \leq C_2.$$

3. If $p > s$, then $|u(x) - u_B| \in L^\infty(B)$ and
$$\|u - u_B\|_{L^\infty(B)} \leq C \operatorname{diam}(B)\mu(B)^{-1/p}\|g\|_{L^p(B)}.$$

Here $C_i = C_i(\lambda, s, C_p, C_b, C_d)$.

Theorem 2: Let X be a proper doubling space that supports a p-Poincaré inequality, $p \geq 1$. Then $N^{1,p}(X)$ consists precisely of those functions in $L^p(X)$ that are L^p-limits of sequences of Lipschitz functions for which also the sequence of the point-wise Lipschitz norms converges in $L^p(X)$. Moreover, when $p > 1$, the space $N^{1,p}(X)$ is reflexive.

The approximation result here is essentially due to Shanmugalingam and the reflexivity is due to Cheeger.

It is often convenient to know that the Poincaré inequality can be characterized by a point-wise inequality. We recall that for every $R > 0$ the *restricted maximal operator* is

$$M_R u(x) = \sup_{0<r<R} \fint_{B(x,r)} |u(x)| d\mu,$$

where u is a measurable function. Because the proof of the point-wise inequality is somewhat easier when $p > 1$ and works for pairs of functions, not only pairs of functions and upper gradients, we first only state this case.

Lemma 1: Let (X, d, μ) be a doubling space, u be locally integrable and $g \geq 0$ measurable. If $p > 1$, then the following conditions are quantitatively equivalent:

1. There exist $C > 0$ and $\lambda \geq 1$ such that
$$\fint_B \left|u - u_B\right| d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu\right)^{1/p} \tag{19.3}$$
for every ball B.

2. There exist $C > 0$ and $\tau > 0$ such that
$$|u(x) - u_B| \leq C \operatorname{diam}(B) \left(M_{\tau \operatorname{diam}(B)} g^p(x)\right)^{1/p}$$
for every ball B and a.e. $x \in B$.

3. There exist $C > 0$ and $\sigma > 0$ such that

$$|u(x) - u(y)| \leq Cd(x,y)\left(M_{\sigma d(x,y)}g^p(x) + M_{\sigma d(x,y)}g^p(y)\right)^{1/p}$$

for almost every $x, y \in X$.

Moreover, even when $p = 1$, condition 1 implies condition 2 which yields condition 3.

Lemma 2: Let (X, d, μ) be a proper doubling space. Then the three conditions of Lemma 2. are quantitatively equivalent for functions $u \in N^{1,p}_{loc}(X)$ and their upper gradients.

We say that X is *quasiconvex* if there exists a constant $C \geq 1$ such that each pair $x, y \in X$ can be joined with a rectifiable curve γ such that

$$\operatorname{length}(\gamma) \leq Cd(x,y).$$

Lemma 3: Assume that (X, d, μ) is a doubling space that supports a p-Poincaré inequality and that X is proper. Then X is quasiconvex.

The previous result allows one to replace the metric of a proper space that supports a Poincaré inequality with a bi-Lipschitz equivalent path metric.

Corollary: Suppose that (X, d, μ) supports a p-Poincaré inequality and that X is proper. Define $\hat{d}(x,y) = \inf_\gamma \operatorname{length}(\gamma)$, where the infimum is taken over all curves that join x and y. Then $\hat{d}$ is a geodesic metric and there exists a constant C so that

$$d(x,y)/C \leq \hat{d}(x,y) \leq Cd(x,y)$$

for all $x, y \in X$.

There is yet another way to characterize the Poincaré inequality. Following Semmes we define, for given $\epsilon > 0$ and measurable $u : X \to \overline{R}$,

$$D_\epsilon u(x) = \sup_{y \in B(x,\epsilon)} \frac{|u(x) - u(y)|}{\epsilon},$$

for every $x \in X$. The following result is due to Keith and Rajala.

Theorem 4: Let X be a proper doubling space. Then X supports a p-Poincaré inequality if and only if there are constants C and λ so that

$$\fint_B |u - u_B|\, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} (D_\epsilon u)^p\, d\mu\right)^{1/p}, \tag{19.4}$$

for each ϵ and each ball $B \subset X$ of diameter at least 2ϵ and all u.

We have mentioned that the Poincaré inequality is not destroyed by by bi-Lipschitz changes of the metric. The Poincaré inequality turns out also to persist under convergence of spaces. if we use the notion of (pointed) measured Gromov-Hausdorff convergence.

Theorem 5: Suppose that $(X_i, x_i, \mu_i, d_i)_i$ is a sequence of geodesic, pointed, proper doubling spaces so that each space is doubling with the same constant C_d and so that each of them supports a p-Poincaré inequality with fixed constants C_P, λ. If this sequence converges in the pointed, measured Gromov-Hausdorff sense to a proper space (X, x, d, μ), then (X, d, μ) is a doubling space that supports a p-Poincaré inequality. Moreover, (X, d, μ) is geodesic.

5. Joan Mateu: Signed Riesz Capacities

This talk represents joint work with Laura Prat and Joan Verdera. If $K \subset \mathbb{R}^n$ is a compact set and $0 < \alpha < n$ we define

$$\gamma_\alpha(K) = \sup |T(1)|,$$

where the supremum is over all distributions T supported on K such that for $1 \leq i \leq n$,

$$||T * \frac{x_i}{|x|^{1+\alpha}}||_{L^\infty(\mathbb{R}^n)} \leq 1.$$

For $n = 2$ and $\alpha = 1$, γ_1 is essentially the same as analytic capacity, and for $n \geq 2$ and $\alpha = n - 1$ it is essentially Lipschitz harmonic capacity.

Prat had showed in her thesis that if $0 < \alpha < 1$ then every set K with finite α-dimensional Hausdorff measure has $\gamma_\alpha(K) = 0$. The case $\alpha > 1$ is not so well understood, but here Prat also showed that $\gamma_\alpha(K) = 0$ if K is α Ahlfors-David regular.

The Riesz capacity is

$$C_{s,p}(K) = \inf\{||\varphi||_p^p : \varphi * \frac{1}{|x|^{n-s}} \geq 1 \text{ on} K,\ \varphi \in C_0^\infty\},$$

where $1 < p < \infty$ and $0 < s < \frac{p}{n}$.

Theorem: For every n and $0 < \alpha < 1$ there is a constant C depending only on n and α such that for all compact $K \subset \mathbb{R}^n$,

$$C^{-1} C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K) \leq \gamma_\alpha(K) \leq C C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K).$$

The Prat result about sets of finite α measure follows from the theorem and known estimates for $C_{s,p}$. It is also known that $C_{s,p}$ is subadditive, and hence the theorem implies that

$$\gamma_\alpha(K_1 \cup K_2) \leq C\gamma_\alpha(K_1) + \gamma_\alpha(K_2),$$

with constant C depending only on n and α. Moreover, since $C_{s,p}$ is bilipschitz invariant, the theorem also implies that

$$C^{-1}\gamma_\alpha(K) \leq \gamma_\alpha(T(K)) \leq C\gamma_\alpha(K)$$

for every bilipschitz homeomorphism T of $\mathbb{R}^n$, where the constant C depends only on n, α and the Lipschitz constants of T and T^{-1}.

The proof of the Theorem has two main steps. The first step is to compare γ_α with the corresponding "positive" capacity $\gamma_{\alpha,+}$. Here $\gamma_{\alpha,+}(K) = \sup \mu(K)$, where the supremum is over all positive measures μ supported on K such that for $1 \leq i \leq n$, $||\mu * \frac{x_i}{|x|^{1+\alpha}}||_{L^\infty(\mathbb{R}^n)} \leq 1$, and the first step of the proof is to show

$$C^{-1}\gamma_\alpha(K) \leq \gamma_{\alpha,+}(K) \leq C\gamma_\alpha(K).$$

The proof of this somewhat resembles Tolsa's proof of the corresponding result for analytic capacity, and uses Prat's earlier proof of the positivity of the symmetrization of the Riesz kernel k_α and a localization result for the kernel k_α. This localization result for $\alpha < n$ is non-trivial and constitutes the main technical difficulty of the proof. The second step of the proof is to use Wolff potentials to compare $\gamma_{\alpha,+}(K)$ to $C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K)$.

6. Daniel Meyer: Quasisymmetric Embeddings of Self Similar Surfaces

A *quasiconformal* map $f : X \to Y$ is a homeomorphism of metric spaces (distance written as $|x - a|$) such that for all $x \in X$

$$\limsup_{\varepsilon \to 0} \frac{\max_{|z-x|<\varepsilon}|f(z) - f(x)|}{\min_{|z-x|<\varepsilon}|f(z) - f(x)|} \leq K$$

with K independent of x. If $K = 1$ f is *conformal*. The homeomorphisms $f : X \to Y$ is η-*quasisymmetric* if there is an increasing homeomorphism $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, a,$ and $b \in X$,

$$\frac{|f(x) - f(a)|}{|f(x) - f(b)|} \leq \eta\Big(\frac{|x - a|}{|x - b|}\Big).$$

Every quasisymmetric map is quasiconformal and every quasiconformal self map of $\mathbb{R}^n$ is quasisymmetric.

In dimension 2, the images of the unit circle under a global quasiconformal mappings are characterized by the *Ahlfors three point condition*: if ζ lies on the smaller diameter arc with endpoints z and w, then

$$|z-\zeta| \le C|z-w|.$$

The von Koch snowflake curve is an example. However, for $n \ge 3$ no characterization of the images of S^{n-1} under quasiconformal self maps of $\mathbb{R}^n$.

The speaker gives explicit constructions of quasiconformal maps from S^2 to certain 2-dimensional topological surfaces, analogous to the snowflake curve, known as "snowball" surfaces. These maps are constructed by iterating specific rational maps. The speaker also proves that the maps constructed above have extensions to quasiconformal self maps of $\mathbb{R}^3$, again by explicit construction.

7. Cristian Rios: The L^p Dirichlet problem and nondivergence harmonic measure

For $k = 0, 1$ let $A_k(x) = \{a_k^{i,j}(x)\}$ be a symmetric $n \times n$ complex matrix function on $\mathbb{R}^n$ for which there exists $0 < \lambda < \Lambda < \infty$ such that for all $x, \xi \in \mathbb{R}^n$,

$$\lambda|\xi|^2 \le \xi \cdot A_k(x)\xi \le \Lambda|\xi|^2,$$

and let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Consider the Dirichlet problem

$$\mathcal{L}_k u = \sum_{i,j=1}^{n} a_k^{i,j}(x)\partial_{i,j}u(x) = 0, \quad x \in D;$$

$$u = g, \quad x \in \partial D.$$

Let σ be surface measure on ∂D and let $1 < p < \infty$. Say that $\mathcal{D}_p$ holds for $\mathcal{L} = \mathcal{L}_k$ if the solution u has nontangential maximal function $N(u)$ satisfying

$$||N(u)||_{L^p(\sigma)} \le C_p||g||_{L^p(\sigma)}$$

for all $g \in C(\partial D)$. Define

$$a(x) = \max_{1\le i,j\le n} ||a_1^{i,j} - a_0^{i,j}||_{L^\infty\left(B(x,\frac{\text{dist}(x,\partial D)}{2}\right)}.$$

Theorem: Assume there is $\rho > 0$ such that $A_k \in BMO_\rho$ where BMO_ρ is the Sarason class

$$\inf_c \frac{1}{(\sigma(\partial D \cap B(x,r))} \int_{\partial D\cap B(x,r)} |f(y) - c|d\sigma(y) \le \rho\big(\sigma(\partial D \cap B(x,r)\big)$$

for all $x \in \partial D$, and assume that $\mathcal{L}_0$ satisfies D_p. Then $\mathcal{L}_1$ satisfies D_p if

$$\sup_{Q\in\partial D}\sup_{r>0} \frac{1}{\sigma(\partial D \cap B(Q,r))} \int_{D\cap B(Q,r)} \frac{a^2(x)}{\text{dist}(x,\partial D)}dx < \infty.$$

A similar result for elliptic operators of divergence form was proved by Fefferman, Kenig and Pipher in 1991.

8. Nages Shanmugalingam; The Dirichlet Problem for Domains in Metric Measure Spaces

Let Ω be a domain in the Euclidean space $\mathbf{R}^n$. For $1 \le p < \infty$ the classical $W^{1,p}(\Omega)$ is the collection of all $u \in L^p(\Omega)$ with distributional derivatives $\partial_i u$, $i = 1, ..., n$, in $L^p(\Omega)$, under the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{i=1}^{n} \|\partial_i u\|_{L^p}.$$

Some classical properties of Sobolev functions $u \in W^{1,p}(\mathbf{R}^n)$ include:

1. Poincarè inequality: For each ball B, radius r,

$$\fint_B |u - u_B|\, dx \leq Cr \left(\fint_B |\nabla u|^p\, dx \right)^{1/p}.$$

   where C depends only on n and p.

2. If $p > n$, it is Hölder continuous with exponent $\alpha = 1 - n/p$.

3. If $p < n$, $u \in L^{p^*}$, where $p^* = \frac{np}{n-p}$.

4. *the weak upper gradient inequality:* there exists a zero p-modulus (defined below) curve family Γ so that if $\gamma \notin \Gamma$,

$$|u(x) - u(y)| \leq \int_\gamma |\nabla u|\, ds.$$

5. *The Hajłasz Inequality:*

$$|u(x) - u(y)| \leq C|x - y| \left(M_p|\nabla u|(x) + M_p|\nabla u|(y) \right), a.e.$$

   Let X be a metric space equipped with metric d and measure μ, and $1 \leq p < \infty$. We say a function u is in the *Hajłasz space* $H^{1,p}(X)$ if $u \in L^p(X)$ and there exists $g \geq 0$ $g \in L^p(X)$ such that

$$|u(x) - u(y)| \leq d(x, y) \left(g(x) + g(y) \right).$$

The definition of Hajłasz space has three advantages: when $p > 1$ it yields the same space as classical Sobolev space on $\mathbf{R}^n$; it is defined by a pointwise inequality; and the Poincaré inequality holds:

$$\fint_B |u - u_B| \leq Cr \left(\fint_B g^p \right)^{1/p}$$

whenever B a ball in X of radius r. It has has two disadvantages: for general domains in $\mathbf{R}^n$ it is not the same as the classical Sobolev space, and the Hajłasz gradient g may not have the truncation property.

Now let X have metric d and measure μ. The p-modulus of a path family Γ is

$$\mathrm{Mod}_p\Gamma = \inf_\rho \|\rho\|_{L^p}^p,$$

where the infimum is taken over all non-negative Borel-measurable functions ρ such that for each rectifiable γ in Γ

$$\int_\gamma \rho ds \geq 1.$$

Fuglede proved that modulus is an outer measure on the collection of all curves in X, and Fuglede and Koskela-MacManus proved that a curve family Γ has zero p-modulus if and only if there exists $L^p(X) \ni \rho \geq 0$ such that for each rectifiable $\gamma \in \Gamma$,

$$\int_\gamma \rho\, ds = \infty.$$

Any property holding on all compact curves except for a zero modulus family of curves is said to hold on *p-almost every curve*, or p-a.e.

Definition: A Borel function $\rho \geq 0$ on X is an *upper gradient* if $u : X \to [-\infty, \infty]$ if on all curves γ,

$$|u(x) - u(y)| \leq \int_\gamma \rho. \tag{19.5}$$

If ρ satisfies (1) only p-a.e., it is called a *p-weak upper gradient* of u.

We say (X, μ) has the *truncation property* if whenever u is constant on a closed (or open) set E, and $\rho \in L^p(X)$ is a p-weak upper gradient of u, then

$$\rho_{\text{new}}(x) = \begin{cases} \rho(x) & \text{if } x \notin E \\ 0 & \text{if } x \in E \end{cases}$$

is a p-weak upper gradient of u.

We say a function u is in the *Newtonian space* $N^{1,p}(X)$ if $u \in L^p(X)$ and if u has an upper gradient $\rho \in L^p(X)$, and we define its "norm" by:

$$\|u\|_{N^{1,p}} = \|u\|_{L^p} + \inf_{\rho} \|\rho\|_{L^p},$$

where we identify u and v if $\|u - v\|_{N^{1,p}} = 0$.

Koskela and MacManus proved that every p-weak upper gradient in $L^p(X)$ can be approximated to desired accuracy by upper gradients in $L^p(X)$. We use p-weak upper gradients rather than upper gradients because minimal p-weak upper gradients exist and are unique, and if $||f_n - f||p \to 0$, if f_n has upper gradient g_n and if and $||g_n - g||_p \to 0$, then g is a weak upper gradient of f.

Properties of Newtonian Spaces:

- If u is in $N^{1,p}(X)$, it is absolutely continuous on p-a.e. curve.
- $N^{1,p}(X)$ is Banach.
- Every Cauchy sequence in $N^{1,p}(X)$ has a subsequence that converges uniformly outside arbitrarily small p-capacity sets, where $\text{Cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}}$, the infimum being over all $u \in N^{1,p}(X)$ s.t. $u\big|_E = 1$.
- If X is any domain in $\mathbf{R}^n$ and $1 \le p < \infty$, then $N^{1,p}(X)$ is isometrically the classical Sobolev space.

We say μ is *doubling* if there exists $C > 0$ such that for all $x \in X$ and all $r > 0$

$$\mu(B(x, 2r)) \le C\mu(B(x, r)),$$

and we say X supports (q, p)-*Poincaré inequality* if there is $C > 0$ such that for all balls $B \subset X$ and for all $u :\in N^{1,p}(X)$ and all weak upper gradients ρ of u,

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{1/q} \le C \text{rad}(B) \left(\fint_B \rho^p \, d\mu \right)^{1/p}.$$

Semmes has proved that if μ is doubling and X supports $(1, p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$, and X is a quasiconvex space. Hajłasz–Koskela and Shanmugalingam proved that if μ is Ahlfors regular and if X supports a $(1, q)$-Poincaré inequality for some $q < p$, then $N^{1,p}(X)$ satisfies the Sobolev embedding theorems. Cheeger proved that if μ is doubling and if X supports a $(1, p)$-Poincaré inequality, then $N^{1,p}(X)$ is reflexive and admits a natural derivation.

Ohtsuka showed that for every $L^p(\mathbf{R}^n) \ni \rho \ge 0$ there is a set G_ρ such that $|\mathbf{R}^n \backslash G_\rho| = 0$, and for all $x \ne y \in G_\rho$ there is a rectifiable curve γ connecting x to y with

$$\int_\gamma \rho \, ds < \infty.$$

Suppose $L^p(\mathbf{R}^n) \ni \rho \ge 0$. Then ρ partitions X into equivalence classes via the following equivalence $x \sim y$ if and only if $x = y$ or there exists a rectifiable $\gamma_{x,y}$ such that $\int_{\gamma_{x,y}} \rho \, ds < \infty$. X is said to have the MEC_p property if for each such ρ there exists an equivalence class G_ρ with $\mu(X \backslash G_\rho) = 0$. For example $\mathbf{R}^n$ is MEC_p for all $1 \le p < \infty$, $\mathbf{R}^n$ with the snowflake metric ($d(x, y) = |x - y|^\epsilon$, $0 < \epsilon < 1$ fixed) is not MEC_p for any p, and if X supports local $(1, p)$-Poincaré then X is MEC_p. **The Dirichlet Problem:** Given a domain $V \subset \mathbf{R}^n$ and $f \in W^{1,p}(\mathbf{R}^n)$, we seek $u \in W^{1,p}(\mathbf{R}^n)$ so that:

(i) $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ on V, and
(ii) $f - u \in W_0^{1,p}(\mathbf{R}^n)$.
Condition (i) is equivalent to:

$$\int_V |\nabla(u+h)|^p \geq \int_V |\nabla u|^p,$$

for all $h \in W_0^{1,p}(\mathbf{R}^n)$. Known results on the Dirichlet problem include:

- If $1 < p < \infty$ μ doubling, and if X is proper and supports the $(1,p)$-Poincaré, and $E \subset X$ is open with $\mathrm{Cap}_p(X \backslash E) > 0$, then minimizing u exists for boundary data $f \in N^{1,p}(X)$ and satisfies Harnack.
- If $1 < p < \infty$ $E \subset X$ is a bounded open set, and $f \in N^{1,p}(X)$ is bounded, then minimizing u exists.
- Cheeger (1998): If X is MEC_p, then given the "boundary value" f, the solution u is unique.
- If X is MEC_p, such solutions satisfy the maximum principle: If u, v are solutions on E to two problems involving (possibly) different boundary functions, and $u \geq v$ p-q.e. on $X \backslash E$, then $u \geq v$ p-q.e. on E.

Definitions: $u : E \to (-\infty, \infty]$ is p-*superharmonic* if u is lower semicontinuous, $u \not\equiv \infty$, and if for all $\Omega \subset\subset E$ and all $v \in N^{1,p}(X)$ p-harmonic in Ω: $v \leq u$ in $E \sim \Omega \Longrightarrow v \leq u$ on Ω.

$u : E \to \mathbf{R}$ is a p-*superminimizer* if $u \in N_{loc}^{1,p}(E)$ and for all $\Omega \subset\subset E$, and all $\phi \in N_0^{1,p}(\Omega)^+$,

$$\int_\Omega g_{u+\phi}^p \, d\mu \geq \int_\Omega g_u^p \, d\mu.$$

Given Borel $f : \partial E \to \mathbf{R}$, define the Perron families

$$\begin{aligned} U_f := \{u : E \to (-\infty, \infty] \; : \; & u \; p\text{-superharm. on } E, \\ & u \text{ bdd below on } E, \liminf_{E \ni x \to y \in \partial E} u(x) \geq u(y)\}, \end{aligned}$$

and

$$L_f := -U_{-f},$$

the *Upper Perron solution*

$$\overline{P} f(x) := \inf_{u \in U_f} \tilde{u}(x),$$

and the *Lower Perron solution*

$$\underline{P} f(x) := \sup_{u \in L_f} \widehat{u}(x) = -\overline{P}(-f)(x).$$

Say f is *resolutive* if $\overline{P} f = \underline{P} f$.

Theorem (Björn-Björn-Shanmugalingam): If μ is doubling and X supports the $(1,p)$-Poincaré, then the following are resolutive:

- $f \in N^{1,p}(X)$ ($Pf \in N^{1,p}(X)$).
- continuous functions.
- If E is p-regular, then bounded semicontinuous functions.
- If $K \subset \partial E$ is compact and F zero p-capacity set containing all p-irregular boundary points, then $\chi_{K \cup F}$ ($P\chi_{K \cup F} = \overline{P} \chi_K$).

The Mumford-Shah Problem and Minimal Surfaces (written by G. David, T. De Pauw and B. Hardt)

1. The Mumford-Shah functional in dimension 3

Guy David's lecture focused mainly on open problems connected too the Mumford-Shah functional. This functional was introduced in image processing, and is a reference tool in image segmentation, but the main concern here is the study of its minimizers. It is given by

$$J(u,K) = \int_{\Omega\sim K} |\nabla u|^2 + \int_{\Omega\sim K} |\nabla u - g|^2 + H^{n-1}(K),$$

where Ω is a simple bounded domain in $\mathbf{R}^n$, g is a given bounded function on Ω, and the competitors are pairs (u,K) such that K is closed in Ω, with finite Hausdorff measure $H^{n-1}(K)$ of codimension 1, and u is, say, locally of class C^1 away from K.

A rapid account of recent results of regularity for K was given (C^1 regularity in many places by Ambrosio, Fusco, Pallara, Rigot; blow-up techniques and regularity for the isolated components of K, recent work by Léger and David), but the main point of the lecture was open questions on the functional itself and on its global version on $\mathbf{R}^n$ obtained by blow-up.

The most the well-known problem is the conjecture of Mumford and Shah, which concerns minimizers in dimension 2, and says that if (u,K) is a reduced minimizer for J, then the singular set K is a finite union of curves of class C^1, which may only meet by sets of 3 and with 120 degrees angles. But David mostly wanted to convince the audience that there are other, equally interesting and perhaps easier questions, mainly in dimension 3.

Of course many of the known theorem in dimension 2 become questions in higher dimensions, because Bonnet's monotonicity argument and Léger's magic formula do not seem to have counterparts, but let us name a few.

First, is the function u essentially determined by K when we know that (u,K) is a global minimizer in space?

Also, a perturbation result of Ambrosio, Fusco, and Pallara says that of in a small ball B, K is very flat (i.e., close to a hyperplane) and $\int_{B\sim K} |\nabla u|^2$ is very small, then K is a nice C^1 surface in half the ball. It would be interesting to know whether in this result, planes can be replaced with the other minimal sets in $\mathbf{R}^3$, like the product of a Y and a line.

Finally, we are lacking a precise analogue of the Mumford-Shah conjecture in 3-space: we know precisely a few global minimizers, but we are probably missing a last basic one.

There are connections between this and other lectures of the conference (such as Thierry De Pauw and Robert Hardt's), not only because the techniques mostly belong to Geometric Measure Theory, but also because a good understanding of the minimal sets in 3-space, for instance, will almost surely help with the perturbation results. Conversely, one can hope that Mumford-Shah techniques will be used in other parts of Geometric Measure Theory.

Some references:

L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, Clarendon Press, Oxford 2000.

A. Bonnet, On the regularity of edges in image segmentation, Ann. Inst. H. Poincaré, Analyse non linéaire, Vol 13, 4 (1996), 485-528.

G. David and J.-C. Léger, Monotonicity and separation for the Mumford-Shah functional, Annales de l'I.H.P., Analyse non linéaire 19, 5, 2002, 631-682.

G. David, Singular sets of minimizers for the Mumford-Shah functional, book in preparation, some parts can be found at http://www.math.u-psud.fr/ gdavid/

2. On minimizing Scans

For two integers $1 \leq m \leq n$ the problem of Plateau can be stated as follows. Given an $m-1$ dimensional *boundary* $B \subset \mathbf{R}^n$, we seek an m dimensional *surface* $S \subset \mathbf{R}^n$, *spanning* B, having least *area* among all such surfaces. Solving the problem consists partly in making sense of the italicized words.

H. Federer and W. Fleming introduced the *integral currents* in $\mathbf{R}^n$ (the surfaces), their *mass* (the area) and their boundary. We now briefly review their theory. An m dimensional rectifiable current consists in the following data:

(1) a Borel $\mathcal{H}^m$ rectifiable set $M \subset \mathbf{R}^n$;

(2) a Borel map $\xi : M \to \wedge_m \mathbf{R}^n$ such that for $\mathcal{H}^m$ almost every $x \in M$ a simple m vector $\xi(x)$ associated with the approximate tangent space to M at x, of length $|\xi(x)| = 1$;

(3) a Borel function $\theta : M \to \{1, 2, 3, \ldots\}$.

We moreover assume that this triple (M, ξ, θ) is such that its *mass*

$$\int_M \theta d\mathcal{H}^m < \infty \, .$$

Therefore we can associate with this data an m current T (in the sense of de Rham) in the following way:

$$T : \mathcal{D}^m(\mathbf{R}^n) \to \mathbf{R} : \phi \mapsto \int_M \langle \phi, \xi \rangle \theta d\mathcal{H}^m \, .$$

Now the *boundary* of a current T of degree $m \geq 1$ is the $m-1$ dimensional current ∂T defined by $\langle \partial T, \zeta \rangle := \langle T, d\zeta \rangle$ whenever ζ is a compactly supported differential form of degree $m-1$ with smooth coefficients. An m dimensional integral current T is an m dimensional rectifiable current such that also ∂T is rectifiable.

The Theorem of Federer and Fleming proves the existence of a mass minimizing current T among all those having boundary $\partial T = B$ for some $m-1$ dimensional compactly supported rectifiable current B with $\partial B = 0$. These mass minimizers model some but not all soap films when $n = 3$ and $m = 2$.

Given $0 < q < 1$ we let the q *mass* of a triple (M, ξ, θ) as above be

$$\int_M \theta^q d\mathcal{H}^m$$

Requiring that the q mass be finite does not imply that the mass is finite. Therefore one cannot interpret anymore the triple (M, ξ, θ) as a current, and we simply call it a *scan*. Nevertheless it is still possible to define an appropriate notion of boundary for these objects. We prove that given B as before there exists a q mass minimizing scan (M, ξ, θ) whose boundary is B. In case B is associated with a smooth embedded submanifold of $\mathbf{R}^n$ (without boundary) then the minimizing scan we obtain is in fact a current (that is it has finite mass). In general we prove that its underlying set M enjoys the following regularity: there exists an m dimensional properly embedded $C^{1,\alpha}$ submanifold $W \subset \mathbf{R}^n$ such that the Hausdorff dimension of the symmetric difference $M \triangle W$ is at most $m-1$.

Geometric Measure Theory in Singular Metric Spaces

There are several totally different approaches of the notion of rectifiability in singular metric spaces, in particular Carnot groups (for instance, Heisenberg groups). Three talks were about possible definitions:

- by B. Kirchheim (joint work with L. Ambrosio) in the setting of general metric spaces. For them, a Borel subset S of a metric space E is d-rectifiable if there exists a (countable) sequence of Lipschitz mappings $f_j : A_j \subset \mathbf{R}^d \to E$ such that $H^d(S / \cup_j f_j(\mathbf{R}^d)) = 0$. From this, they develop a rather complete theory of rectifiable sets. As applications, they get a version of the Rademacher theorem (differentiability of Lipschitz functions), area and co-area formulae, ... They also developed a theory of currents supported on rectifiable sets in metric spaces.

- by R. Serapioni (joint work with B. Franchi and F.Serra-Cassano) in the case of Heisenberg groups (and some special Carnot groups). For them, rectifiable sets in the Heinsenberg group are defined modulo a set of zero measure as subsets of the union of C^1-manifolds (with respect to the Carnot-Caratheodory structure of the group). As application, they get a version of the famous theorem of E. De Giorgi about sets of finite perimeter.

- by S. Pauls in the case of Carnot groups. In his definition, he replaces Lipschitz images of subsets of Euclidean spaces by Lipschitz images of some fixed subgroup of the original Carnot group.

In their talks, V. Magnani, P. Mattila J. Tyson discussed classical tools in (euclidean) geometric measure theory (as weak tangent measures, area and co-area formulae, ...) in the setting of Carnot groups.

It should be mentioned that there was a lot of discussions about this subject between the talks. This area of research is quite new and most of the definitions are not totally satisfactory.

Other classical problems of geometric measure theory (in Euclidean spaces) have been discussed by F. Germinet (comparison of dimensions), T. O'Neil (Visible sets), I. Laba (The Kakeya problem and related topics), N. Zobin (Whitney-type extension theorems for functions).

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Chapter 20

Monge-Ampere Type Equations and Applications (03w5067)

August 2–7, 2003

Organizer(s): Alice Chang (Princeton University), Pengfei Guan (McMaster University), Paul Yang (Princeton University)

The workshop on Monge-Ampere type equations at BIRS was held from August 2 to August 7 in the summer of 2003. It is a focused meeting in the rapidly developing areas related to Monge-Ampère equation, and fully nonlinear equations in general. The workshop presents a unique and timely opportunity to stimulate important new mathematical research and exposition on the subjects. There were 20 one-hour talks during the 5-day workshop to present important recent works. The setting of BIRS is ideal for promoting informal interaction that has proved so fruitful in the past in the field. There has been increasing interaction among researchers on the field In recent years. In a more general context, the workshop served as viechle to foster mathematical interaction between researchers in the field in different areas of the world.

The participants consist of mathematicians from Canada, United States, France, Germany, Australia and China. The participants of the workshop includes both some of the most distinguished mathematicians in the world and many of the top young researchers (including 2 graduate students) in the field. That's a good mixture of both senior and junior researchers. There were great interactions between senior and junior mathematicians working in the area. This kind interaction at the research level is crucial to the continuing high level of advancement of the field. The workshop provided a wonderful educational opportunity for many junior researchers in the field. The combination of talks, problem sessions, informal discussions, and the special environment of BIRS provided this opportunity.

Because of the geographic diversity of the researchers in the field, bringing the participants together for the 5-day workshop strongly facilitate the dissemination of the most recent research ideas and results, which otherwise might not be possible. The atmosphere of the workshop and its surroundings may lead to new collaborations during the workshop and especially in the years following the workshop. Neil Trudinger of the Australian National University, one of the prominent mathematicians in the field, told the organizers that this workshop is the best meeting he has ever attended. The combination of the breadth and the cohesiveness of the field of Monge-Ampere type equations certainly has made the 5-day workshop at BIRS have significant impact on the field.

In the following, we summarize the main mathematical activities during the workshop around the main theme of Monge-Ampère and fully nonlinear equations. We divide it into to several sections according to the emphasis on different aspects of the field.

Monge-Ampère equation and classical differential geometry

Alexandrov type inequalities for Cartan-Hadamard manifolds.

In his study of elliptic equations and geometry, the great Russian geometer A.D. Alexandrov introduced the simple but important idea of the lower (upper) contact set and the associated normal mapping (generalized gradient map). For a function $u \in C^0(\Omega)$ (Ω a bounded domain in $\mathbb{R}^n$), the lower contact set

$$\Gamma^- = \{y \in \Omega | u(x) \geq u(y) + p \cdot (x-y) \text{ , for all } x \in \Omega \text{ , for some } p = p(y) \in \mathbb{R}^n\} .$$

Note that u is convex if and only if $\Gamma^- = \Omega$ and if $u \in C^1(\Omega)$, $p = Du(y)$ in the above definition. Further if $u \in C^2(\Omega)$, the Hessian $D^2 u \geq 0$ on Γ^-. Associated to $y \in \Omega$ is $\chi(y)$, the set of slopes of lower supporting planes to the graph of u at y (so $\chi(y) = \emptyset$ if $y \notin \Gamma^-$). It is not difficult to see that for $u \in C^2(\Omega \cap C^0(\overline{\Omega})$,

$$|\chi(\Omega)| = |\chi(\Gamma^-| \leq \int_{\Gamma^-} \det D^2 u(y) \, dy$$

The extension of these ideas to Riemannian manifolds is surprisingly difficult. Joel Spruck discussed this extension for a large class of Cartan-Hadamard manifolds where we replace the linear functions in the definition of the lower contact set by Busemann functions associated to a given point at infinity. The proof uses in an essential way the ergodic theory of the geodesic flow to compute the Jacobian for the generalized gradient map. In a similar way, he derived for a compact hypersurface inequalities for a generalized Gauss map on the outer contact set. These new inequalities are sharp for radial functions and geodesic balls in H^n.

Convex solutions of elliptic PDE in classical differential geometry.

There is a vast literature on the convexity of solutions of quasilinear elliptic equation in $\mathbb{R}^n$, especially the deformation technique of Caffarelli-Friedman and Korevaar-Lewis via the strong minimum principle. In some recent work of P. Guan-X. Ma on classical problems in differential geometry [7], some fundamental questions the convexity problem for fully nonlinear equations have been put into spotlight. It is needed to generalize results for quasilinear equations to fully nonlinear equations. Some of the work has been done for a large general fully nonlinear elliptic equations. Some applications have been given to problems in classical differential geometry and convex bodies, for example Christoffel-Minkowski problem, prescribed Weingarten curvature and prescribed curvature measure in convex body.

The Minkowski Problem for hedgehogs

In differential geometry, the Minkowski problem is that of the existence, uniqueness and regularity of closed convex hypersurfaces of $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed as a function of the unit normal vector. More generally, the Minkowski problem concerns the existence and uniqueness of convex bodies of $\mathbb{R}^{n+1}$ whose area measure of order n is prescribed on the unit sphere $\mathbb{S}^n$. This classical Minkowski problem is equivalent to the question of solutions of a Monge-Ampère PDE of elliptic type on $\mathbb{S}^n$. Hedgehogs of $\mathbb{R}^{n+1}$ are the geometrical realizations of formal differences of convex bodies of $\mathbb{R}^{n+1}$. In the case of differences of convex bodies of class C^2_+, these geometrical realizations are (possibly singular and self-intersecting) envelopes parametrized by their Gauss map. They constitute a vector space $\mathcal{H}^{n+1}$ in which one can study a given convex body by splitting it judiciously (i.e. according to the problem under consideration) into a sum of hedgehogs.

The Minkowski problem has a natural extension to hedgehogs. For non-convex hedgehogs, this extended Minkowski problem is equivalent to the question of solutions of a Monge-Ampère PDE of mixed type on $\mathbb{S}^n$.

Yves Martinez-Maure expounded the present state of our knowledge on the subject and will consider certain particular cases. In particular, he illustrated the interest of hedgehogs by showing how to construct counter-examples to an old conjectured characterization of the $2-$sphere related to this extension of Minkowski's problem.

Regularity of Monge-Ampere equations and applications in affine geometry

In X.J. Wang's talk he reported some recent advances by N.S. Trudinger and himself on the regularity of Monge-Ampere equations and applications to the affine Plateau geometry. The affine Plateau problem concerns the existence of smooth affine maximal hypersurfaces subject to certain boundary restrictions. The affine maximal surface equation is a fourth order nonlinear PDE which can be written as a system of two Monge-Ampere type equations. The existence of solutions was obtained by the upper semi-continuity of the affine surface area functional and a uniform cone property of locally convex hypersurfaces. The a priori estimates of strictly convex solutions was established by using Caffarelli and Gutierrez's interior estimates for Monge-Ampere equations. To prove that a maximizer can be approximated by smooth solutions, a boundary Schauder estimate for the Monge-Ampere equation and the global regularity to a second boundary value problem of the affine maximal surface equation were also established. The strict convexity of solutions remains open, except in the two dimensional case.

John Urbas proved the existence of smooth solutions of a class of Hessian equations on a compact Riemannian manifold without imposing any curvature restrictions on the manifold.

Local Isometric Embedding Problem.

The celebrated Nash's isometric theorem states that any n-dimensional Riemannian manifold can be isometrically embedded in to $\mathbb{R}^N$. In general, N is a very big number compare to n. For local isometric embedding, the classical Janet-Cartan theorem tell us that any analytic n-dimensional Riemannian manifold can be isometrically embedded into $\mathbb{R}^{\frac{n(n+1)}{2}}$, the number $\frac{n(n+1)}{2}$ is optimal. One would like to remove the analyticity assumption. The fundamental question regarding the local isometric problem for smooth surface is: can it be locally isometrically embedded into $\mathbb{R}^3$? There are important works of C.S. Lin, which dealt with the cases: (i) Gauss curvature nonnegative and (ii) Gaus curvature changes sign cleanly. The problem is closely related to the local solvability of Monge-Ampère equation. In a surprise development, Nikolai Nadirashvili and Yu Yuan constructed smooth metrics on 2-manifold with nonpositive Gauss curvature which cannot be C^3 locally isometrically embedded in $\mathbb{R}^3$. Moreover, the Gauss curvature of the metric can be made negative except for one point.

On the other hand, Qing Han reported his joint work with J. X. Hong and C.S. Lin. They obtained a result on the local isometric embedding of surfaces, which seems to be a compliment of the examples of Nikolai Nadirashvili and Yu Yuan. They proved that there is a such embedding if the Gauss curvature of metric is non-positive, and the null set of the Gauss curvature has nice structure. In particular, if the Gaus curvature is non-positive and of finite type, the local isometric embedding problem for surface is solvable. In our view, these results present some important behaviour of degenerate hyperbolic Monge-Ampère equation. We should also relate this local problem to the similar global problem on $\mathbb{S}^n$ posted by Yves Martinez-Maure.

This is a new direction of Monge-Ampère equation: try to deal with hyperbolic type of this equation. Almost nothing is known at the moment. Any progress in this direction will be important to the theory of Monge-Ampère equation and will certainly yield some geometric applications.

Prescribing Curvature problem.

Bo Guan presented results from his joint work with Joel Spruck on the Plateau type problem of finding hypersurfaces of constant curvature with prescribed boundary. More precisely, the problem can be formulated as follows: given a smooth symmetric function f of n ($n \geq 2$) variables and a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of closed smooth embedded $(n-1)$ dimensional submanifolds of $\mathbb{R}^{n+1}$, one asks whether there exist (immersed) hypersurfaces M in $\mathbb{R}^{n+1}$ of constant curvature $f(\kappa[M]) = K$ with boundary $\partial M = \Gamma$ for some constant K, where $\kappa[M] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of M. Important examples include the classical Plateau problem for minimal or constant mean curvature surfaces and the corresponding problem for Gauss curvature. In this and some earlier work, they introduced two different approaches to the problem: the Perron method and the volume minimizing approximation. These methods are based on the solvability of the problem in the non-parametric setting (the Dirichlet problem) and an important uniform local graph representation property of locally convex hypersurfaces. It would be interesting to extend the methods to other classes of hypersurfaces.

The work of Bo Guan and Joel Spruck relies on the existence of a "subsolution". A natural geometric question is: when there is a such "subsolution"? Using methods of the h-principle of Gromov, specifically the holonomic approximation theorems, Mohammad Ghomi proved that any compact hypersurface with boundary immersed in Euclidean space is regularly homotopic to a hypersurface whose principal directions have a prescribed topological type, and whose principal curvatures are prescribed to within an arbitrary small error. Further he described how to construct regular homotopies which control the principal curvatures and directions of hypersurfaces. These results have been obtained in his recent joint work with Marek Kossowski, and generalize theorems of Gluck and Pan on positively curved surfaces in 3-space, which had been proved by explicit constructions. Also they are somewhat reminiscent of the classical continuity method used to obtain some of the recent results on locally convex hypersurfaces with boundary.

Fully nonlinear equations in conformal geometry.

Nonlinear partial differential equations have played a role in the problem of uniformization of conformal structures since the work of Poincare. In dimensions greater than two, the analysis of conformal structure is approached from two perspectives. The first deals with conformally flat structures where the theory of Kleinian groups provide a large family of examples and hyperbolic geometry provided the basic tools for analysis. The second perspective deals with general conformal structures by studying the metrics from the view point of partial differential equations. The well known Yamabe equation, is among a family of conformally covariant partial differential equations which are geometrically natural and analytically interesting. The curvature tensor Rm associated to a Riemannian metric g may be decomposed as

$$Rm = W + \frac{1}{2} A \odot g$$

where $\odot$ indicates the Kulkarni-Nomizu product of bilinear forms, where W is the Weyl tensor, $A = \frac{1}{n-2}(Rc - \frac{R}{2(n-1)} g)$ is the Schouten tensor, and R the scalar curvature. We say A belongs to the k-positive cone if the k-th symmetric function $\sigma_k(A)$ of the eigenvalues of A is positive, and that A may be joined to the identity matrix by a path of matrices A_t along which $\sigma_k(A_t) > 0$. We say that a metric belongs to the k-positive cone, if the Schouten tensor belongs to the k-positive cone at each point. Under a conformal change of metric $g' = v^{-2} g$, the Weyl tensor transforms by scaling, while the Schouten tensor transforms by a complicated expression involving the Hessian as well as the gradient of the conformal factor:

$$A' = A + \frac{1}{v}\{\nabla^2 v - \frac{\Delta v}{n} g\} + \frac{1}{2n(n-1)}\{R + 2(n-1)\frac{\Delta v}{v} - n(n-1)\frac{|\nabla v|^2}{v^2}\}g.$$

- The symmetric functions σ_k of the eigenvalues of the Schouten tensor may be expressed then in terms of a fully nonlinear expression involving up to two derivatives of the conformal factor e^w. When $k = 1$ the equation is known as the Yamabe equation, which is a semi-linear equation with a critical nonlinearity. A great deal is understood about this equation. When $k \geq 2$ the equation is fully nonlinear thus the study of this equation requires techniques coming from two previously disjoint area in nonlinear partial differential equations. There are several motivation to strive for an understanding of the fully nonlinear equations originating from geometry as well as analysis. The most important reason is that the higher degree equation yields stronger control of the Ricci tensor, and hence stronger control of the geometry. The first evidence in this direction is provided by the work of Gursky-Viaclovsky [11] in which they characterize the space forms in 3-D as the critical points of the functional $\int \sigma_2(A_g) dV$ among the metrics of normalized volume. This work points to the possibility to understand the three dimensional geometrization problem via the study of the σ_2 equation. As a strong confirmation of this possibility, the work of Chang-Gursky-Yang ([1, 2, 3]) provided the necessary and sufficient condition in four dimensions for the existence of metrics with Schouten tensor A_g to lie in the 2-positive cone, as well as solvability of the the equation to prescribe $\sigma_2(A_g)$, and as a consequence a sphere theorem in 4-D characterizing the 4-sphere by two conformally invariant conditions in terms of the positivity of two conformally covariant linear operators. The key that ties the topology to the analysis is provided through the Chern-Gauss-Bonnet formula, in which the expression $\sigma_2(A_g)$ plays the leading role.

Following this work, there has been a lot of activity in extending this result in two directions. The first is to provide similar sufficient conditions for existence of metrics with Schouten tensor belonging to the k-positive cone in which they provided an alternative argument to the main result of [1] without appealing to the regularization procedure via higher order equations, and actually yielded further criteria for positivity of the Paneitz operator. Matt Gursky spoke about this new result at the workshop. The second main direction is directed at the solvability of the σ_k equations once the k-positive cone is known to be nonempty. In this direction, there is a lot of work in the past two years. For general conformal structures, Gursky-Viaclovsky [13] proved the necessary a priori estimates in dimensions three and four for nearly all of the prescribed σ_k equations, thus nearly answering almost completely the solvability question, once the k-positive cone is non-empty. At the workshop Viaclovsky spoke about this work, and related questions dealing with the case where $-A$ belongs to the k-positive cone.

On the other hand, for conformally flat structures in all dimensions, there is rather satisfactory progress in obtaining apriori estimates for all the σ_k equations. This is the result of a large collaborative effort by several competing groups of analysts including P. Guan, G. Wang, C.S. Lin, Y.Y. Li, A. Li in which the main technical tools are (i) the local estimate of Guan-Wang [8], (ii), the conformal fully nonlinear flow, (iii) the method of moving planes coupled with some inspired computations deriving largely from previous experience with the Yamabe equation as well as the fully nonlinear equations of Monge-Ampere type. In particular, when the k-positive cone is non-empty, there is a complete existence theory ([9, 15]). Y.Y. Li was invited to present the current state of the problem, but unfortunately was not able to make the workshop.

Among the conformally flat structures, the ones arising from Kleinian groups provide a potential area for application of the newly developed fully nonlinear equations. Traditionally, the study of Kleinian groups had proceeded either through the study of Beltrami equations and thus restricted to the case of 2-D, or by doing the analysis on the hyperbolic structures which the Kleinian quotient bounds. The condition that the Schouten tensor lies in the k-positive cone, i.e. $\sigma_j(A) > 0$ for $j = 1, ..., k$ places strong restriction on the Ricci part of the curvature tensor. In particular, the well known work of Schoen-Yau says that the conformally flat structures with $\sigma_1(A) > 0$ are Kleinian quotients, with limit set whose Hausdorff dimension is bounded by $(n-2)/2$, and hence certain homotopy groups as well as homology groups of the Kleinian quotient vanishes. In recent development, this result has its counterpart for the metrics in the k-positive cone. Guofang Wang presented a joint work with P.F. Guan and C.S. Lin [10, 6] on certain further vanishing of cohomology, and G. Del Mar presented her forthcoming thesis work [16] on further bounds for the Hausdorff dimension of the limit set, resulting in further vanishing of homotopy groups. A consequence of these work is the beginning of a classification theory of Kleinian groups whose Kleinian quotients have metrics belonging to certain k-positive cones.

There is further anticipated development of these conformally invariant equation that relates to the recently developing scattering theory of convex co-compact hyperbolic manifolds, as well as the more general conformally compact manifolds that is playing an increasing role in the holographic principle as predicted by the mathematical physicists. Alice Chang presented the recent joint work with J. Qing and P. Yang [4] on a generalized Gauss-Bonnet formula for conformally compact manifolds in which the principal term involves the renormalized volume, a subtle quantity that is directly related to the scattering pole of the conformally compact manifold. This presents possibilities for further interaction of the conformally invariant nonlinear PDEs with the scattering theory, and the theory of conformal invariants that is under active development by C. Fefferman and his students.

Analysis on subelliptic Monge-Ampère equations

A higher dimensional partial Legendre transform and regularity of degenerate Monge-Ampère equations.

Eric Sawyer presented his joint work with Rios and Whedeen on the C^∞-regularity of degenerate Monge-Ampère equations in higher dimensions. In general, we know (under some reasonable conditions) that solution to degenerate elliptic Monge-Ampère equations is $C^{1,1}$. This regularity is sharp. But, in differential geometry, one needs to known when such a solution is in fact C^∞. In 2-case, there exists a previous result of P. Guan says that if u is $C^{1,1}$, $u_{11} \geq c > 0$, $\det(u_{ij} \geq 0$ is of finite type, then $u \in C^\infty$. Sawyer-Rios-Whedeen generalized that result to higher dimensions under stronger assumption that u is $C^{2,1}$. The important part of their result is the higher dimensional partial Legendre transform. They were able to relate the partial Legendre transform in a way similar to Cauchy-Riemann type equation. They work relies on the important work of Sawyer-Whedeen on regularity of degenerate quasilinear elliptic equations.

Convex functions in the subelliptic setting and applications.

In the past decade, research for fully nonlinear equations in Euclidean spaces has made considerable progress. We refer to the two monographs in this direction by Caffarelli-Cabré and by Gutierrez. The simplest example is the so-called Monge-Ampère equation which for smooth function u is given by

$$det(D^2u) = f.$$

In considering solutions to the above equation, the notion of convex functions plays a crucial role. The notions of generalized weak solutions and viscosity solutions which were introduced by A.D. Aleksandrov rely on the properties of convex functions.

Convex functions in Euclidean space can be characterized as universal viscosity subsolutions of homogeneous fully non-linear degenerate elliptic equations of second order.

Motivated by the role that convex functions play in the theory of fully nonlinear equations, Guozhen Lu and his collaborators formulate several notions of convexity in the subelliptic structure on the Heisenberg group towards the aim of developing an intrinsic theory of subelliptic fully non-linear equations. We will discuss the notion of group convexity, horizontal convexity and viscosity convexity. These definitions can be considered in the general case of Hörmander vector fields. Their proofs strongly use the viscosity theory for subelliptic equations. They study convex functions defined by requiring that their symmetrized horizontal second derivatives are non-negative in the viscosity sense. They call these functions *v-convex*. The main result is to establish their local Lipschitz continuity. They introduced the notion of *horizontal convexity,* which has many interesting properties. The upper-semicontinuous horizontally convex functions are v-convex, and therefore Lipschitz continuous. These two notions of convexity are equivalent. The notion of convexity independently defined by Cabré and Caffarelli is equivalent to horizontal convexity. Another approach of horizontal convexity is also considered by Danielli-Garofalo-Nhieu.

Using the fact that symmetrized second order derivatives of convex functions are signed measures, the recent results of Gutierrez-Montanari of Monge-Ampere operators on H^1, the gradient estimates for convex functions], and the weak differentiability of convex functions by Ambrosio-Magnani, they derived the second order differentiability a.e. of convex functions on H^1. Thus generalizing the well-known Alexandrov's theorem to the subelliptic setting. This proof works in general Carnot groups as long as one can show the second commutators of convex functions are signed measures (thus each mixed second order derivatives are signed measures and then we can apply the theorem of Ambrosio-Magnani to get weak differentiability of second order).

Fully nonlinear subelliptic theory is at the beginning stage now. It is desirable to develop a satisfactory theory of existence, uniqueness and regularity theory for the Monge-Ampere operator recently discovered.

Boundary value problems for complex Monge-Ampere equation and characterization of the ball using Kahler–Einstein metric

In the complex analysis, one of major problem is classifying domains under biholomorphism. Riemann mapping theorem in complex analysis of one variable play the exactly role. However, when dimension is greater than one, the problem become more complicated and very interesting. Strongly related problem is the Fefferman mapping problem: Every biholomorphism between two smoothly bounded pseudoconvex domain in complex Euclidean space must be a diffeomorphism between the closures of the domains. Many progress has been made by many excellent mathematicians started with C. Fefferman in 1974. In general, the problem is still open. Boundary regularity of complex Monge-Ampere equations is strongly related to the Fefferman mapping problem. In part one of Song-Ying Li's talk, he gave a sharp regularity for degenerate complex Monge-Ampere equation on pseudoconvex domain of finite type in n-dimensional complex Euclidean space.

Bergman metric and Kahler-Einstein metric are very important metrics in complex analysis and complex geometry, they have certain boundary behaviour near boundary of a strictly pseudoconvex domain in n-dimensional complex Euclidean space. A general open question was asked by S. T. Yau: To classify the domains have Bergman Kahler-Einstein metric. Only very special case was known. The second part of his talk was to present a theorem to characterize the unit ball in n-dimensional complex Euclidean space by using Kahler-Einstein metric, which is the first step of approaching Yau's question in a long term project.

Monge-Ampère equation and mass transportation

Geometric properties of the set of probability densities of prescribed second moments.

The set of probability densities can be endowed with the so-called Wasserstein metric. The geodesic between two probability densities can be explicitly written using the convex function that appear in the Monge-Ampere equation involving these densities. Motivated by applications in kinetic theory, Wilfrid Gangbo and Eric Carlen analyze the induced geometry of the set of densities satisfying the constraint on the variance and means, and they determine all of the geodesics on it. It turns out, for example, that the entropy is uniformly strictly convex on the constrained manifold, though not uniformly convex without the constraint.

Semigeostrophic fluid flow in an elliptical ocean basin.
Robert McCann's talk describes joint work with Adam Oberman concerning exact solutions to the semi-geostrophic model for atmospheric and oceanic flows. This rotating fluid model involves an active scalar transport problem, much like the vorticity formulation of Euler's equation, but with Monge-Ampere instead of Poisson's equation relating stream function to advected scalar. Affine invariance of the determinant provides exact solutions in which both pressure and stream function remain quadratic but evolve nonlinearly. The finite dimensional dynamics are expressed canonically in Hamiltonian form.

List of Participants

Bland, John (University of Toronto)
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Chapter 21

Localization Behaviour in Reaction-Diffusion Equations (03w5078)

August 10–16, 2003

Organizer(s): Thomas Hillen (University of Alberta), Michael J. Ward (University of British Columbia), Juncheng Wei (Chinese University of Hong Kong)

The purpose of this workshop was to discuss advances in the mathematical study of localized structures in reaction-diffusion systems arising in applications. In recent years there have been many important developments in the mathematical treatment of localization phenomena in reaction-diffusion problems, especially those connected with the equilibrium or static theory associated with certain traditional classes of elliptic PDE's. However, many open problems remain, most notably those in the realm of nonlinear dynamics, bifurcation behaviour, and the numerical computation of localized structures in reaction-diffusion systems. In addition, there is a strong need for researchers in this area to be exposed to new classes of PDE's involving localization phenomena that arise from a more sophisticated modelling of biological and physical problems.

To illustrate the importance and relevance of this research area, over the past four years there have been many conferences devoted to the analysis of localization phenomena in reaction-diffusion systems. A sample of these include a one-week conference in Crete, Greece in June 1999 on nonlinear dynamics for PDE's related to materials science (organized by N. Alikakos of U. Tennessee), a one-week conference on reaction-diffusion systems at CUHK in Dec. 1999 (organized by J. Wei of CUHK and M. Mimura of U. Hiroshima), a two-week conference in pattern formation at the Lorentz Institute in Leiden, Holland in March 2001 (organized by D. Hillhorst of U. Paris-Sud and H. Matano of U. Tokyo), the PIMS conference on point-condensation phenomena in Vancouver, Canada in July 2001 (organized by C. Gui of U. Connecticut and N. Ghoussoub of UBC), and a six-month programme (Jan. 2001-Jun. 2001) in reaction-diffusion equations at the Newton Institute in Cambridge, England (organized by N. Dancer of U. Sydney and H. Brezis of Paris 6).

In this workshop we focused on three main sub-areas of localization behaviour in reaction-diffusion systems: (1) localized patterns in chemotaxis; (2) localized patterns in activator inhibitor systems including the Gierer-Meinhardt and Gray-Scott models; (3) localized patterns in emerging areas, including di-block co-polymers, chemical reactions on surfaces, combustion etc.

In this workshop we solicited four survey talks: *Horstmann* on chemotaxis; *Nishiura* on the Gray-Scott model; *Ren* on the di-block co-polymer problem, and *Wei* on the Gierer-Meinhardt model. In addition, each of these people also gave a more specialized lecture. There were 13 other lectures given by other participants. We now summarize results and open problems that were brought forth to each of the three themes of the workshop.

Chemotaxis and Oriented Movement

One major focus of the meeting was to investigate pattern formation in reaction-advection-diffusion systems that occur in population dynamics, and in chemotaxis.

An essential characteristic of living organisms is the ability to sense signals in the environment and adapt their movement accordingly. This allows for the location of food, the avoidance of predators, or the search for mates. When the response involves the detection of a chemical, it is termed *chemotaxis*, *chemokinesis*, or generally *chemosensitive movement*. The term *chemotaxis* is used broadly in the mathematical literature to describe general chemosensitive movement responses, and it is in this context that we use the term here. Models for chemotaxis have been successfully applied to bacteria, slime molds, skin pigmentation patterns, leukocytes and many other examples.

The following participants gave presentations about population dynamics *Chris Cosner (Miami)*, *Thomas Hillen (Edmonton)*, *Dirk Horstmann (Köln, Germany)*, *Hans Othmer (Minneapolis)*, and *Angela Stevens (Leipzig, Germany)*.

Diffusion Based Models for Chemotaxis

Patlak (1953) and Keller and Segel (1970) were the first to derive a mathematical model for chemotaxis. The Keller-Segel model in it's general form consists of four coupled reaction diffusion equations. It can be reduced to "only two" essential parameters, the population density $u(t,x)$ and the concentration of a chemical signal $v(t,x)$. The Keller-Segel model reads

$$\begin{aligned} u_t &= \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v) \\ v_t &= k_c\Delta v - k_3(v)v + uf(v). \end{aligned} \tag{21.1}$$

This system has been studied on unbounded and on bounded domains with various boundary conditions (Dirichlet, Neumann, mixed). In his survey talk, *Dirk Horstmann* gave an excellent review of the existing literature and the analytical results on the Keller-Segel model (21.1). Most results deal with finite time blow-up solutions. In the case of constant k_1, k_2, k_3, and f it is known that the qualitative behaviour strongly depends on the space dimension. In 1-D the system has globally existing solutions. The 2-D case in ambiguous and thresholds have been found. If the total initial mass exceeds its threshold, then the solution blows up in finite time. When the initial mass is below this threshold, then solutions exist globally in time.

The blow-up solutions of (21.1) show the existence of a very strong instability and a large aggregational force. In certain situations, however, it is desirable to obtain stable aggregation patterns, which do not blow up in finite time. *Thomas Hillen* gave a continuation of *Horstmann's* review in the sense that he discussed various mechanisms which prevent blow up. The mechanisms were (i) saturation effects, (ii) volume filling effects, (iii) quorum sensing effects, and (iv) finite sampling radius. Ad (i): Saturation effects in $k_2(u,v)$ occur very naturally if cell surface receptor kinetics are taken into account. Chemotaxis models with saturation effects have been studied analytically and have been used in many applications. Ad (ii): The volume filling effect was introduced by Hillen and Painter (2002). It is assumed that particles have a finite volume and that cells can not move into regions which are already filled by other cells. In a simple form this leads to a term $k_2(u,v) = \chi u(1-u)$. It was shown analytically that this form of k_2 leads to globally existing solutions in all space dimensions. Ad (iii): Quorum sensing occurs if the cells release an extra chemical which is repulsive to other cells. The resulting equation has two competing drift terms, chemotactic attraction and quorum sensing repulsion. It is an open mathematical challenge to find general conditions such that solutions blow up, or exist globally. Ad (iv): Also the inclusion of a finite sampling radius leads to global existence, at least in 2-D, as was shown by Hillen and Schmeiser.

Transport Models for Chemotaxis

The models mentioned so far are based on the assumption that the particles carry out an uncorrelated random walk, which is modelled by diffusion. However, some species movement can be characterized better by a transport equation. The bacterium *Eschirichia coli* moves via rotation of flagella and, when rotating anticlockwise, these flagella bundle together resulting in a period of smooth swimming - a "run". Clockwise rotation, however, results in the flagella spraying outwards resulting in a random reorientation, or a "tumble". Normal swimming is characterized by periods of smooth runs punctuated by tumbling. In the presence of a chemoattractant, *E. coli* bias their behaviour by tumbling less frequently in an increasing attractant gradient, resulting in the general movement toward high concentrations. Detection of the attractant is made by the binding of attractant molecules to cell surface receptors, which subsequently initiates a cell internal pathway which transduces the signal to the movement machinery.

Both of these aspects, the microscopic level of signal transduction, and the macroscopic level of population movement have been modelled separately. *Hans Othmer*, in his talk, attempts to merge these levels of modelling to find one model for the whole process of bacterial chemotaxis which also includes the internal dynamics. The analysis is based on the velocity jump process for describing the motion of individuals, wherein each individual carries an internal state that evolves according to a system of ordinary differential equations forced by a time- and/or space-dependent external signal. He derives a macroscopic system of hyperbolic differential equations from this velocity jump process using moment closure techniques. He also reduces this macroscopic system to a single second order hyperbolic equation which, in a suitable limit, reduces to a classical chemotaxis equation in which the chemotactic sensitivity is now a known function of parameters of the internal dynamics.

Angela Stevens also studies transport equations for chemotaxis. She discusses in detail how the variety of evaluations of the chemical stimulus influences the motion of the respective species on the macroscopic level. It is known that in a formal limit the transport equations lead to chemotaxis models of Keller-Segel type. Chalub, Markowich, Perthame and Schmeiser (2003) rigorously proved that in three dimensions, these kind of kinetic models lead to the classical Keller-Segel model as its drift-diffusion limit, when the equation for the chemical signal is of elliptic type and local and non-local effects are taken into account.

In the talk of *Stevens* the rigorous derivation of Keller-Segel type systems and their variants is given in case of parabolic or elliptic signal equation (also in 2 dimensions). Under suitable structure conditions existence of global solutions for the kinetic model can be shown.

Forward-Backward Diffusion

Chris Cosner presented reaction diffusion population models which lead to aggregations. This includes models with taxis-terms (chemotaxis or prey-taxis). Classical models for population dynamics with spatial dispersal typically assume that dispersal occurs by passive diffusion, perhaps with advection. Replacing passive diffusion with nonlinear diffusion of the type proposed to model aggregation can lead to changes in the possible dynamics supported by the models. In logistic models it can create an Allee effect, where small populations go extinct but populations above some threshold persist. Stronger versions of nonlinear diffusion can lead to ill-posed problems. Cosner described some nonlinear diffusion models in population dynamics, presented analytical results, and discussed some topics for further research on such phenomena. The analytic results are based on bifurcation theory and classical methods in partial differential equations.

In a second talk, *Dirk Horstmann* studied a very specific reduced Keller-Segel model, where forward-backward diffusion arises. The equation takes the form

$$p_t = \Delta(T(p)p), \qquad \text{with} \qquad T(p) = \frac{1}{K + p^2}\,.$$

This is an ill-posed forward-backward problem. However this equation can be approximated by a systems of a PDE and an ODE that takes time delays into account. Horstmann presented joint results with H.G. Othmer and K.J. Painter where he compared for one specific model the asymptotic behaviour of the solution of the non-local in time or time delay problem with the asymptotic behaviour of the solution of the corresponding single equation problem, which is a forward-backward problem in this specific case. Horstmann discussed the variational structure of the problem above and he drew connections to the Perrona-Malik formalism in image processing.

Conclusion

It appears that the pattern forming mechanisms are related to those for the Gray-Scott model or for Cahn-Hillard type problems. Whereas in many physical contexts there is usually an energy functional which guides the dynamics, in many biological application there is no such energy functional. Nevertheless, it is an open challenge for the future to describe the analogy between Cahn-Hillard patterns and chemotaxis patterns. Moreover, some chemotaxis patterns appear to be metastable. In the context of Gierer-Meinhardt systems a theory of non local eigenvalue problems and metastability has been developed recently. We expect that this theory can be adapted to the biological applications as well.

An Update on the Study of the Gierer-Meinhardt and Gray-Scott Systems

Through the use of a linearized analysis, Alan Turing in 1952 showed how stable spatially complex patterns can develop from small perturbations of spatially homogeneous initial data for a coupled system of reaction-diffusion equations. He then proposed that this type of localization could be responsible for the process of morphogenesis. Since that time, there have been many reaction-diffusion models proposed for pattern formation, including the following well-known activator-inhibitor system of Gierer and Meinhardt 1972

$$\text{(GM)} \qquad \frac{\partial a}{\partial t} = \epsilon^2 \Delta a - a + \frac{a^p}{h^q}, \qquad \tau \frac{\partial h}{\partial t} = D\Delta h - h + \frac{a^r}{h^s}, \qquad \frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0 \ \text{ on } \partial\Omega,$$

where

$$\epsilon << 1\,, \quad \frac{\epsilon^2}{D} << 1 \quad \tau > 0\,, \quad \frac{qr}{(p-1)(s+1)} > 1 \quad p > 1\,, \quad q > 0\,, \quad r > 1\,, \quad s \geq 0\,. \tag{21.1}$$

Extensions of the Gierer-Meinhardt model have been used to model localization phenomena in developmental biology and pattern formation on sea shells. Previous studies have been concentrated on using various types of weakly nonlinear theories for the onset of instabilities of spatially homogeneous steady-state solutions. However, such a theory no longer works in the study of highly inhomogeneous solutions, notably, spike-type solutions.

A related system, introduced by Gray and Scott (1988), models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a reservoir of one of the two chemicals in the reaction. In nondimensional variables the resulting system, known as the Gray-Scott model, can be written as

$$\begin{aligned} v_t &= \epsilon^2 \Delta v - v + Auv^2\,, \qquad x \in \Omega\,, \quad t > 0\,,\\ \tau u_t &= D\Delta u + (1-u) - uv^2 \qquad x \in \Omega\,, \quad t > 0\,,\\ \frac{\partial v}{\partial \nu} &= \frac{\partial u}{\partial \nu} = 0\,, \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

Here $A > 0$, D, $\tau > 1$, and $\epsilon \ll 1$, are constants. For various ranges of these parameters, the Gray-Scott model is known to posses a rich solution structure including the existence of stable standing pulses, the propagation of travelling waves, pulse-replication behaviour, and spatio-temporal chaos (cf. Pearson (1993), Reynolds et al. (1997), Nishiura and Ueyama (1999,2001), Doelman, Gardner, and Kaper (1998)).

Following on the many studies on spikey phenomena for singularly perturbed scalar quasilinear elliptic PDE's originating in the papers of W. M. Ni and collaborators, there has been considerable focus on the mathematical study of spike-type solutions for activator-inhibitor systems. This meeting has been an excellent platform for mathematicians from different areas including, dynamical systems, nonlinear PDEs, variational methods, and numerical analysis, to share their insights and results on pattern formation in the GM and Gray-Scott models, and related systems. Many open questions on mathematical problems related to spikey patterns were presented in the survey talk of *Wei*.

There were six lectures related to spikey patterns in the GM and Gray-Scott models, and related systems. *Dancer's* talk concerns steady-states of shadow systems, *Kolokolnikov's* lecture dealt pulse-splitting behaviour in the Gray-Scott model, *Kaper's* talk was focused on semi-strong interactions in the Gray-Scott model, *Nishiura* introduced the notion of scattors in dissipative systems and he gave a survey talk of pulse-type behaviour in the Gray-Scott model, *Ward's* discussed Hopf bifurcations of spike solutions and the dynamics of spikes, while *Wei* gave a survey lecture highlighting open problems and accomplishments to date.

He also gave a lecture on spotty patterns in the two-dimensional Gray-Scott model and an analysis of the instability of ring solutions in the Gray-Scott model and their evolution to spotty patterns.

For the GM model, we begin with the Shadow System obtained by letting $D \to \infty$;

$$\frac{\partial a}{\partial t} = \epsilon^2 \Delta a - a + \frac{a^p}{\xi^q}, \qquad \tau \frac{\partial \xi}{\partial t} = -\xi + \xi^{-s} \frac{1}{|\Omega|} \int_\Omega a^r, \qquad \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial\Omega .$$

By the scaling $a = \xi^{\frac{q}{p-1}} u$, the corresponding steady-state problem becomes

$$(I) \;\; \epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \qquad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Since the fundamental works of Ni-Takagi (91-93), many authors have studied problem (I) in search of multiple interior or boundary spike solutions. They include:Alikakos, Bates, Cao, Dancer, del Pino, Felmer, Fusco, Ghoussoub, Grossi, Gui, Kowlaczyk, Y.Y. Li, A. Pistoia, C.-S. Lin, Ni, Takagi, Noussair, XB Pan, Shi, Ward, Wei, Winter, Yan... Gui-Wei's result asserts that for any fixed integers K, l, there is a solution to (I) with K boundary spikes and l interior spikes.

Dancer reported his recent study on (I), where he proved that if the Morse index is finite and dimension N is $N = 2, 3$, then all solutions must have multiple spikes. His result also extends to the study of a bistable-type nonlinearity. There are still many questions remained towards understanding (I):

Question 1: uniqueness and Morse index of multiple spike solutions.
Question 2: characterize and construct solutions that concentrate on a higher-dimensional subset of Ω.
Question 3: do concentrated solutions exist for the supercritical case $p > \frac{N+2}{N-2}$?
Question 4: characterize the global bifurcation branches of solutions as a function of ϵ.

The stability of multiple spikes solutions for the shadow system has been studied by Ni-Takagi-Yanagida in the one-dimensional case and by Wei (1999) in the 2D case. Wei's result only applies to the following exponent case

$$(*) \text{ either } r = 2, \; 1 < p \leq 1 + \frac{4}{N} \quad \text{or} \quad r = p + 1, 1 < p < \frac{N+2}{N-2} .$$

Question 5: develop a general method for studying the spectrum of the following nonlocal eigenvalue problem (NLEP) in the right half plane $\mathrm{Re}(\lambda) \geq 0$:

$$\Delta\phi - \phi + p w^{p-1} \phi - \gamma(p-1) \frac{\int_{R^N} w^{r-1} \phi}{\int_{R^N} w^r} w^p = \lambda \phi .$$

Here w is the ground-state solution.

The most interesting case is $r = p$ which arises in the study of Selkov's model, the Keller-Segel model in chemotaxis, an Epidemic SIS model, and in the study of hot-spot behaviour in microwave heating models.

Next we discuss the near-shadow system case $D \sim e^{\frac{d}{\epsilon}}$. Kolokolnikov and Ward (preprint:2003) have done some bifurcation analysis of one-spike solutions for dumbbell shape domains. Still the following question remains to be resolved:

Question 6: under what condition on D, can one construct multiple boundary or multiple interior spike solutions to the GM model without any additional condition on Ω? Where are the locations of spike equilibria in an arbitrarily shaped two-dimensional domain as a function of D.

For the strong-coupling case $D = O(1)$, there have been many results in one and two space dimensions. Takagi (1986) constructed multiple symmetric spike solutions in 1D, M. Ward and Wei (2001) constructed multiple asymmetric spikes in 1D generated by exactly two types. Wei and Winter 99-2002 constructed multiple symmetric and asymmetric spots in 2D. Doelman-Kaper-Ploeg, Chen-Del Pino-Kowalczyk (2000) constructed multiple pulse ground states in 1D. Del Pino-Kowalczyk-Wei (2000) constructed multi-bump ground state solutions on a regular $K-$polygon or concentric polygons or honey-comb. Wei and Winter (2002) constructed multiple symmetric and asymmetric clusters in 1D. Ni-Wei (2003) constructed ring-like solutions in higher dimensions.

Question 7: why are there only two types of patterns? Can one construct a reaction-diffusion systems with more than two types of spikes in a given spike pattern? The phenomena of only two types of spikes is believed to be related to properties of certain Green's functions.

Question 8: Suppose $N \geq 3$, are there ground state solutions with a single bump? This problem can be reduced to the study of the following simple-looking elliptic system

$$\Delta A - A + \frac{A^2}{H} = 0\,, \qquad \text{in } R^N$$
$$\Delta H + A^2 = 0 \qquad \text{in } R^N,$$
$$A = A(|y|), \quad H = H(|y|), \quad A, H \to 0 \quad \text{as} \quad |y| \to +\infty\,.$$

There have been many recent works on the stability of spike patterns in the strong coupling case. In 1D, Iron-Ward-Wei (2001) (matched asymptotics), Wei-Winter (2002) (rigorous) showed that there exists a sequence of decreasing numbers:$D_1 > D_2 > D_3 > ... > D_K > ...$ such that for $D < D_K$, $K-$symmetric spikes are stable when $\tau = 0$ and for $D > D_K$, K-symmetric spikes are unstable for any $\tau \geq 0$. Similar results are also established in 2D by Wei-Winter (2002). In that case,

$$D_K \sim \frac{|\Omega| \log \frac{\sqrt{|\Omega|}}{\epsilon}}{2\pi K} \tag{21.2}$$

Question 9: What about stability of multiple asymmetric K-spikes in 1D? Numerically, they are believed to be all unstable.

Wei mentioned an interesting comparison between D_K in 2D with Ozawa's asymptotic expansion of small eigenvalues for domains with small holes.

Question 10 : What is the $O(1)$ term in the asymptotic expansion of $D_K(\epsilon)$?

For the case $\tau > 0$, the stability of spike patterns for the 1-D GM model has been studied in Ward-Wei (2003) using a combination of rigorous and asymptotic analysis. This was the subject of *Ward's* lecture. The stability diagram in the $\tau - D$ parameter plane has been now largely understood. For a fixed $D < D_K$, it was shown that there is a Hopf bifurcation as τ increases past some critical value τ_h. This oscillatory instability has the effect of synchronizing the amplitude and phase of the amplitude oscillations of the spikes.

The nonlocal eigenvalue problem that is central to the analysis in Ward and Wei (2003) has the form

$$\Delta\phi - \phi + pw^{p-1}\phi - \chi(\tau\lambda)(p-1)\frac{\int_{R^1} w^{r-1}\phi}{\int_{R^1} w^r} w^p = \lambda_0 \phi, \qquad \phi \in H^2(R^N)\,.$$

There are many open questions regarding eigenvalue problems of this form in one space dimension

In *Ward's* talk, it is shown that for multiple spikes in 1D, one has

$$\chi = \chi(z; j) \equiv qr \left(s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)} \left[\tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)} \right] \right)^{-1} ,$$

where

$$z \equiv \tau\lambda\,, \qquad \theta_\lambda \equiv \theta_0 \sqrt{1+z}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{21.3}$$

In his talk, *M. Ward* showed that if $r = 2$, $1 < p < 1 + \frac{4}{N}$, and $chi(z) = \frac{a}{b+cz}$, NLEP has a unique Hopf bifurcation. He also presented results on the Hopf bifurcation and oscillatory instability of multiple spikes in 1D. Some new instability are discovered: competition instabilities and synchronous oscillations. Dancer (2001) showed the existence of two positive unstable eigenvalues for τ large. However, a key open problem concerns proving a strict transversal crossing condition for χ of the form given above and for general values of r.

Question 11: uniqueness of Hopf bifurcation for other exponents and general $\chi(z)$. Can there exist several values of τ where there is a Hopf bifurcation? Is this bifurcation subcritical or supercritical?

In *Ward's* lecture it was shown that in the low-feed rate regime of the 1D Gray-Scott model where $A = O(\epsilon^{1/2})$ there is a spectral equivalence principle between the Gray-Scott and the Gierer-Meinhardt model in the sense that one can identify appropriate exponents (p, q, r, s) in the GM model to obtain the exact spectral problem associated with the Gray-Scott model in the low feed-rate regime. This implies that oscillatory and competition instabilities also occur for the Gray-Scott model.

The lecture of *Kolokolnikov* concerned the analysis of pulse-splitting behaviour of the 1D Gray-Scott model in the regime where $A = O(1)$, and where the finite domain places an important role. He introduced the following core problem that is connected to pulse splitting behaviour:

$$V^{''} - V + V^2 U = 0\,, \qquad 0 < y < \infty\,,$$
$$U^{''} = UV^2\,, \qquad 0 < y < \infty\,,$$
$$V^{'}(0) = U^{'}(0) = 0\,; \qquad V \to 0\,, \qquad U \sim By\,, \qquad \text{as} \quad y \to \infty\,,$$

where $U > 0$, $V > 0$, and

$$B \equiv \frac{A}{\coth(\theta_0/k)}\,, \qquad \theta_0 = D^{-1/2}\,. \tag{21.4}$$

As shown numerically in *Kolokolnikov's* lecture, there are two positive solutions to this core problem when $0 < B < 1.347$. He then showed formally that there are no k-spike equilibria in the pulse-splitting regime $A = O(1)$ when $A > A_{pk}$, where

$$A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right)\,.$$

In terms of this core problem, when τ is sufficiently small, *Kolokolnikov* predicted that a one-spike solution centred at the midpoint of a 1D domain will undergo 2^{m-1} spitting events, and that the final equilibrium state will have 2^m spikes where, for some smallest value of m, A lies in the interval

$$A_{p2^{m-1}} < A < A_{p2^m}\,.$$

He also analyzed the thresholds for the existence of travelling waves.

The core problem for V and U, without the effect (21.4) of the finite domain, was one focus of the lecture by *Kaper*. He gave specific results on the core problem and a bounding region that contains all solutions to the core problem.

Question 12: Give rigorous results on the global solution branches of the core problem. Estimate the fold point value 1.347 rigorously and describe the shape of the solutions on these branches.

The problem of the dynamics of multiple spikes is largely open: there are only scarce results (both formal and rigorous) on dynamics of spikes. Chen-Kowalzyk considered the dynamics of interior spikes shadow system case. D. Iron and M. Ward derived the dynamics of boundary spikes for the shadow systems and multiple spikes in 1D.

Kaper's lecture gave some results on the semi-strong dynamics of two spikes for the Gray-Scott model and for a generalized GM model. He derived a formula for the speed of separation of two pulses, and showed that blow-up solutions are possible for certain variants of the GM model.

Question 13: give a rigorous treatment of dynamics of multiple spike solutions spikes in 1D and 2D. Determine the stability of quasi-equilibrium solutions consisting of k spikes.

Nishiura gave an excellent survey of instabilities of particle-like solutions in reaction-diffusion systems. He showed self-replication and self-destruction phenomena in several systems, and showed different collision properties between travelling spots. In his other lecture, he showed that the notion of scattors is a very useful concept for understanding the input-output relation for the collision of two pulses. When pulses collide they can either annihilate, repel like fixed particles, repel and produce complicated oscillatory phenomena in their trailing edges, or produce a bound pair. *Nishiura* showed that special types of unstable steady or time-periodic solutions called scattors act as saddle points in phase space and are critical for determining the fate of colliding pulses. This concept was illustrated using the Complex Ginzburg Landau equation, the Gray-Scott model, and a three-component reaction diffusion model arising in gas-discharge phenomena.

Other Applications of Localization

There were several other talks of localization behaviour in other systems.

Ren discussed interface phenomena in non-local variational problems associated with di-block co-polymers. He gave a survey talk on the physics of the co-polymer problem and a sketch of the derivation of the nonlocal energy functional that determines the formation of interfaces. A basic summary of the set-up is that there are type A and type B monomers in a di-block co-polymer system which often form A-rich and B-rich microdomains. On a larger scale these phase domains give rise to a morphological pattern. The widely observed lamellar and wriggled lamellar patterns are modelled using the Ohta-Kawasaki model in which the free energy density field depends nonlocally on the monomer composition field. In his other lecture, *Ren* showed that in one dimension the Gamma-convergence technique can be used to reduce to this problem to a finite dimensional minimization problem. For each K there exists a 1D local minimizer with K+1 microdomains and K domain walls. Among these 1D local minimizers there is the 1D global minimizer that has optimal spacing between the domain walls. These 1D local minimizers are extended trivially to two dimensions to give solutions of lamellar patterns to the Euler-Lagrange equation. The stability of these solutions was studied and their spectra are found. A 1D local minimizer is stable in 2D only if it has sufficiently many domain walls. The 1D global minimizer is near the borderline of 2D stability. This interesting phenomenon was found to be related to the existence of wriggled lamellar solutions as seen from the bifurcation theory. The stability properties of the wriggled lamellar solutions was determined in the theory after some careful calculations.

The lecture of *Choksi* also concerned the dib-block co-polymer problem, but had a more global focus. He presented two physical problems in which pattern formation can be modelled via the minimization of a nonlocal free energy. In each case, he explained the origin and derivation of the free energy. He then used gamma convergence techniques to determine the scales and patterns for minimizers in several space dimensions.

The lecture of *Pearson* focused on reaction-diffusion and travelling wave patterns that are concentrated on a thin sheet, representing the cell boundary. In the *Xenopus laevis* oocyte, calcium ion channels are clustered in a thin shell near the outer cell wall. Motivated by this morphology he studied the effect of "sheet excitability" in an idealized reaction–diffusion system with a 2-dimensional sheet of sources embedded in 3-dimensional space. He found that waves undergo propagation failure with *increasing* diffusion coefficient and a scaling regime in which the wave speed is independent of the diffusion coefficient.

The lecture of *Matkowsky* focused on the dynamics of hot-spot solutions in a gasless combustion model, representing a solid sample where combustion occurs only on the surface of a cylinder of radius R. Different hot-spot behaviour was observed in their numerical computations as R was increased. For a fixed value of the Zeldovich number, if R is sufficiently small, slowly propagating planar pulsating flames are the only modes observed. As R is increased transitions to more complex modes of combustion occur, including (i) travelling waves (TWs), i.e., spin modes in which one or several symmetrically spaced hot spots (localized temperature maxima) rotate around the cylinder as the flame propagates along the cylindrical axis, thus following a helical path, (ii) counterpropagating (CP) modes, in which spots propagate in opposite angular directions around the cylinder, executing various types of dynamics, (iii) alternating spin CP modes (ASCP), where rotation of a spot around the cylinder is interrupted by periodic events in which a new spot is spontaneously created ahead of the rotating spot. The new spot splits into counterpropagating daughter spots, one of which collides with the original spot leading to their eventual mutual annihilation, while the other continues to spin. Other more complicated features were observed as R is increased. Since the hot-spots are localized it would be interesting to try to explain some of these different behaviours analytically using recent techniques developed for the Gierer-Meinhardt and Gray-Scott models.

Finally the lecture by *Kuske* concerned modulation theory of patterns. Modulation equations describe the behaviour of complex systems over long scales. However, their validity is often limited to near-criticality. A new multi-scale approach, combining energy arguments and balance of nonlinearities, yields modulation equations for localized buckling of a strut away from the critical load, where standard asymptotics and normal forms fail. Immediate connections to heterogeneous patterns in other applications are shown. Her approach was illustrated via simple one-dimensional models, motivated by numerics and experiments.

List of Participants

Choksi, Rustum (Simon Fraser University)
Cosner, Chris (University of Miami)
Dancer, Edward (University of Sydney)
Fife, Paul (University of Utah)
Guo, Jong-Sheng (National Taiwan Normal University)

Hillen, Thomas (University Alberta)
Horstmann, Dirk (Universitaet zu Koeln)
Kaper, Tasso (Boston University)
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Lee, Jung Min (University of Alberta)
Liu, Wenxiang (University of Alberta)
Matkowsky, Bernard (Northwestern University)
Nishiura, Yasumasa (Hokkaido University)
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Pearson, John (Los Alamos National Laboratory)
Ren, Xiaofeng (Utah State University)
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Chapter 22

Defects and Their Dynamics (03w5057)

August 9–16, 2003

Organizer(s): Peter Bates (Michigan State University), Lia Bronsard (McMaster University), Changfeng Gui (University of Connecticut)

Introduction

In material science, in biological systems, in chemical systems and in physics of macro and microscopic systems one encounters 'defects'. A defect might be described as a singularity or an abrupt change in a spatial parameter or field. Examples from material science include familiar structures such as phase boundaries between differing phases of a substance. Less well-known phenomena such as corkscrew defects in crystal structures are also very important. In biological systems, there is also a wide variety of defect-like structures of significant interest. Pattern formation in developmental biology is still important but new discoveries, such as pinwheels in the primary visual cortex provide a rich source of mathematical challenges. Similarly, the appearance and motion of vortices in superconducting materials, or the existence and evolution of black holes in astrophysics, lead to mathematical problems of great depth. These defects in general causes problems, for example in superconductors the motion of vortices represents loss of superconductivity, and the understanding of these defects is essential in developing methods to handle them. In recent years, tremendous progress has been made by mathematicians in understanding the existence, uniqueness, stability (or instability) and qualitative behaviour of solutions to mathematical models of some of the phenomena described above. However, there are still many open problems and areas where mathematical analysis can be brought successfully to bear and we hope that the methods being developed can prove useful in those studies.

This workshop brought together mathematicians and other scientists who are making strides in understanding defects and their motions in their particular areas of expertise, with the expectation that the exchange of ideas and techniques will benefit all. Below are some famous examples of the specific mathematical areas covered.

The Allen-Cahn equation and its various generalization

The Allen-Cahn equation is a well-known model for the mathematical study of bi-phase transition. It has received extensive study, and numerous important results have been obtained. For example, it is found that the interfacial surface moves by its mean curvature (in the limit as the diffusion coefficient approaches zero); the final shape of the interface is some minimal surface; the stationary shape of the interface has to be mainly smooth (i,e, the singular set of the interface must be of lower Hausdorff dimension), etc. However, there are still some very interesting open questions such as the uniqueness of the basic profile near the interface when it is not smooth, the rigidity of the interface under certain constraints, etc. These questions are also related to the

De Giorgi conjecture on the symmetry of certain stationary transition solutions to the Allen-Cahn equation. This is still open in higher dimensions even though proofs exist for low dimension and partial results exist for higher dimensions. More importantly, some more sophisticated phase transition problems may be modelled by variations of the Allen-Cahn equations, such as a non-local version where the interaction between material particles has long range. Discrete (e.g. lattice) and continuum models with non-local interaction are important for both material science and neuroscience, the latter situation being the modelling of large populations of neurons interconnected through dendrites and synapses. The vector-valued versions of the Allen-Cahn and Cahn-Hilliard equations are important for modelling transitions in multi-component materials. Some of these variations display interesting new phenomena and difficulties, such as pinning in the discrete model and triple junction motion in the multi-component case. Many problems are open and require expertise in geometry, PDEs, dynamical system, etc, and more importantly new ideas growing from the existing approaches.

Ginzburg-Landau and Nonlinear Shrödinger Equations

The modelling of vortices in superconductors with complex Ginzburg-Landau equations has been a very active area over the last decade. There are strong groups in France, Japan, PRC, and the US. Separately, they have made tremendous progress in showing the existence of vortex solutions, describing their stability and motion, and demonstrating the onset or nucleation of superconducting regions as the applied field is decreased past a certain critical value. We expect that the interaction of representatives of these groups through the workshop will allow greater progress to be made. We also expect that the interaction of these groups with those working in other areas will be beneficial to all. The motion of defects under the Nonlinear Shrödinger equation is also a very interesting and timely area of study. In this case, there are difficult problems associated with the spectrum of the linearized operator and also, the conserving nature of the flow shows that there is a lack of compactness. Still, many new ideas have been injected over the last five years or so. We hope that techniques for one system will have a bearing on the other.

Gray-Scott and Gierer-Meinhardt Equations

Recently there have been significant advances in the analysis of some well-known models of pattern formation such as the Gray-Scott and Gierer-Meinhardt Equations, arising in chemical reactor and biological systems. The appearance of stable spikes or spots and the onset of instability as spots subdivide are intriguing, both from a mathematical and biological point of view. The motion of several spots as they interact with each other and the boundary of the domain is not well-understood and yet the work on the Allen-Cahn equation and other systems seems to have a bearing here. Again, we bring together experts in both areas at this workshop and expect the interaction between scientists to be both lively and fruitful.

The workshop ran together with another closely related workshop “Localization Behavior in Reaction-Diffusion Systems and Applications to the Natural Sciences” organized by A. Bernoff (Harvey Mudd College), P. Fife (Univ. Utah), T. Hillen (Univ. Alberta), M. J. Ward (UBC) and J. Wei (Chinese Univ. Hong Kong). The talks were scheduled so that the participants of both workshops could go to all the talks and interact with each other.

Below are more detailed descriptions of the research topics which were presented in our workshop.

Defects and their dynamics: results and methods

Asymmetric triple phase transitions

When modelling materials with more than two phases, one must generalize the Allen-Cahn equation to a system where the order parameter is vector-valued. In the case of a ternary alloy for example, the typical potential W has three wells representing each of the three distinct phases. One then obtain interfaces involving triple junctions and to study the evolutions of these interfaces, one first needs to understand stationary solutions with triple junctions. There was some previous work by Bronsard-Gui-Schatzman in the case that the well is symmetric, corresponding to a isotropic material. In the case that the material is anisotropic, the potential will not be symmetric and the problem is much harder. The recent result that Dr. Schatzman presented is about the anisotropic case:

Let W be a potential defined on $\mathbf{R}^2$ which has exactly three non degenerate minima at points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$; W vanishes at each of these points. Under three extra conditions, namely (1) a weak coercivity assumption (2) a non-wetting condition (3) a (generic) non-degeneracy condition, it is proved that there exists a local minimizer of the Landau functional associated to the potential and appropriately renormalized; this minimizer is a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ which is approximately equal to $\mathbf{d}$ in an angular sector of angle $\theta_{\mathbf{d}}$, where $\mathbf{d}$ takes the value $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$. The angles $\theta_{\mathbf{a}}$, $\theta_{\mathbf{b}}$ and $\theta_{\mathbf{c}}$ satisfy Young's celebrated condition: if $e_{\mathbf{xy}}$ is the one-dimensional transition energy from $\mathbf{x}$ to $\mathbf{y}$, then

$$\frac{e_{\mathbf{ab}}}{\sin\theta_c} = \frac{e_{\mathbf{bc}}}{\sin\theta_a} = \frac{e_{\mathbf{ca}}}{\sin\theta_b}.$$

Moreover, the local minimizer of the Landau energy constructed here converges exponentially fast to its limits at infinity, along an appropriate set of directions.

This result as well as the approach presented will prove useful in the understanding of the evolution of triple junctions in anisotropic media. In particular, it will be fundamental in the rigorous derivation of the limit evolution problem for the limiting interfaces, which is still to this day an open problem.

Symmetry of constrained minimizers of Allen-Cahn energy

The famous mathematician De Giorgi made several conjecture in his lifetime. One of these relates to the Allen-Cahn equation in R^n and says that the only stable stationary solutions are the ones which are monotone in exactly one direction. In particular, this yields a better understanding of the behaviour of the solutions near the interfaces separating the different phases. This conjecture was recently solved in various space dimensions by Ghoussoub-Gui and Cabré. During the conference, Dr. C. Gui presented some new results and directions about the symmetry of minimizers of Allen- Cahn (Ginzburg-Landau) energy under some constraints: what is known and what we still do not know.

Interface dynamics for an anisotropic Allen-Cahn equation

In a set of groundbreaking papers in the early to mid 90's, it was shown mathematically that the interface dynamic for the Allen-Cahn equation is given by the mean curvature flow. Since then, there have been many difficult results obtained on more complicated but very important models. One such difficult and beautiful result was presented by Dr. Hilhort: One considers an anisotropic Allen-Cahn equation and replace the usual Euclidean metric by another one, changing in particular the shape of the unit circle with giving more weight to some spatial directions. The notions of normal to a surface and of mean curvature are also extended to this metric. The singular limit of the corresponding Neumann boundary value problem as the reaction term tends to infinity is then obtained. More precisely, after having shown the local in time existence and uniqueness of the classical solution of the limit problem, which involves anisotropic motion by mean curvature, both propagation and generation of interface properties is proved.

The type of approaches that were presented can be generalized to many other models in which singular perturbation is present. It was therefore very interesting to scientists present and will prove useful in the future.

Symmetries in anisotropic media

Imagine a carpet that is woven out of nonlinearly elastic strings whose deformation energy is given by $\sum_i \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p \, dx$ with $1 < p < \infty$. The corresponding differential operator $\sum_i \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right)$ is reminiscent of the p-Laplace operator $\sum_i \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right)$, but obviously different from it and it has no rotation invariance. Therefore some standard symmetry results like the radial symmetry of the first eigenfunction if Ω is a ball or the Faber Krahn inequality have to be modified for this operator.

In his lecture Dr. Kawohl from University of Cologne, explained to what extent such symmetry properties can be generalized and investigated also the limits $p \to \infty$ and $p \to 1$. Most of the results were obtained in cooperation with M. Belloni from Parma and V. Ferone from Napoli.

The idea of the method is as follows:
For any $p \in (1, \infty)$ there exists a first eigenfunction u_p which minimizes the corresponding Rayleigh quotient $\sum_i \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p \, dx / \int_\Omega |u|^p \, dx$ on the set $W_0^{1,p}(\Omega) \sim \{0\}$, it is of one sign (e.g. positive) and unique modulo scaling.

If Ω is convex, then u_p is logconcave, i.e. $\log u_p$ is concave.

Among all domains of given volume the first eigenvalue is minimized by an $\ell_{p'}$ ball with $p' = p/(p-1)$.

As $p \to \infty$, u_p tends to a viscosity solution of a complicated and very degenerate equation, see the paper with Belloni in ESAIM COCV, and as $p \to 1$ cum grano salis u_p tends to the characteristic function χ_ω of a subset ω of Ω. The eigenvalue tends to the so-called Cheeger constant, and ω infimizes perimeter over volume among all smooth compact subsets of Ω. Here perimeter has to be interpreted in the right metric.

The methods and results presented are very impressive and not only will be used in other problems but leave many nice open problems for scientists.

On the stable critical points of two-phase singular perturbation problems

One very powerful method used to study interfaces and critical points of functional is the method of Γ-convergence which relates to the deep Geometric Measure Theory. Dr. Y. Tonegawa from Sapporo, Japan, described some known results on the singular perturbation problem and Γ-convergence techniques initially studied by Sternberg, Modica, and others. He then presented some recent results on the non-minimal critical points as well as stable critical points via the method of geometric measure theory. Some results satisfied by stable minimal surfaces such as Schoen-Simon inequality were proved to be true for the limit sharp interfaces under some appropriate assumption. It is clear that such results can be used in other interfaces problems.

On first order corrections to the LSW theory for domain coarsening

The fourth order Cahn-Hilliard equation, in which the order parameter is conserved, also leads to very interesting interface dynamics and in particular is a model for the very important phenomenon known as spinodal decomposition and coarsening. Another approach to coarsening is via the classical theory by Lifshitz, Slyozov and Wagner which describes diffusion limited coarsening of particles in the limit of vanishing volume fraction. Recently there has been a large interest in identifying higher order correction terms due to some shortcomings of the LSW theory.

Dr. B. Niethammer, from University of Bonn, Germany, presented some joint work with A. Honig and F. Otto, in which a rigorous mathematical analysis in a stochastic setting is done which identifies the scaling of the first order correction to the LSW theory. In particular, the order of the relative deviation of the coarsening rate from the LSW theory shows a cross-over between 1/3 and 1/2 when screening effects become important.

She also discussed a self-consistent derivation of the expected growth rate of a particle under certain assumptions on the statistics of the system, and in particular she presented some result about the influence of correlations.

These type of methods are very new and promises to be extremely fruitful.

A new approach to modelling phase transition phenomena

The evolution of many inhomogeneous systems is often described by sharp-interface models, which are constructed from phenomenological descriptions of interfaces, or by phase-field models which can be constructed to obey explicitly the fundamental principles of statistical mechanics. Dr. Chmaj discussed possibly a third way: he showed that a nonlocal phase-field model leads to patterns with sharp interfaces. To be more precise, he considered the nonlocal double-well free energy functional, first suggested by van der Waals. After taking sign-changing interaction kernel with a special scaling, he used a combination of the Gamma-convergence method and convex envelope ideas, to show that the functional admits periodic L^2 local minimizers with thin but discontinuous layers. He then discussed the location of the interfaces (determined from a separate finite dimensional problem), which depends on the nonlocality in a sometimes counterintuitive way.

Patterns and travelling waves in materials with nonlocal and indefinite interaction

Dr. P. Bates presented some very impressive rigorous results related to general nonlocal phase-field models. More specifically, he spoke about patterns and travelling waves in materials with nonlocal and indefinite interaction of the form $u_t = d(J * u - u) + f(u)$. Here $d > 0$, f is bistable, either $u \in \ell^\infty$ in the discrete case or $u \in L^\infty$ in the continuum, and $*$ is convolution, discrete or continuous. The kernel J may change sign but has unit integral. He gave conditions under which stable stationary patterns exist and conditions under which traveling waves exist, even when $J(x)$ changes sign with x. Thus, the presence of both excitatory (ferromagnetic) and inhibitory (antiferromagnetic) couplings can lead to pattern formation or homogeneity depending on finer details in the connections.

The dynamics of patterns for reaction-diffusion systems in a cylindrical domain in 2D

Mathematically, variational structures are much easier to handle since we have a minimization procedure available for the energy. In his talk, Dr. S. Ei presented results for a system without this nice structure:

He considered solutions in 2D with the profile connecting stationary solutions in 1D. Then a structure like defect appears. According to the difference of energies as the stationary solutions in 1D, the movement of the defect is determined and in fact, it is theoretically obtained if the system has a variational structure. In this talk, he considered a reaction-diffusion system without variational structures and investigated the movement of the defect in the neighbourhood of Turing instability.

Asymptotic Spatial Patterns and Entire Solutions

In singularly perturbed reaction-diffusion equations on bounded domains, when the diffusion coefficient tends to zero, the behaviour of the steady state solutions depends on the qualitative properties of solutions of elliptic equations on the whole space or the half space. The bounded solutions of $u + f(u) = 0$ on the whole space or the half space (with certain boundary condition) determine the local asymptotic spatial behaviour of solutions to singularly perturbed problems. The bounded solutions on the whole space (entire solutions) is much more complicated than solutions on a half space. We will survey results on entire solutions from the classical Liouville theorem, radially symmetric solutions, to recent developments. In these earlier works, the nonexistence of patterns has shown for certain nonlinearities, and the typical patterns found are either radially symmetric or monotone. In the talk of Dr. J. Shi, periodic patterns and saddle solutions from the speaker's work were introduced in details, and related conjectures were also discussed at the end.

The Energetic Variational Approach in Complex Fluids

From the energetic point of view, most complicated hydrodynamical and rheological properties of the non-Newtonian complex fluids arise from the coupling and competing between the kinetic energy and different types of internal "elastic" energy. The examples include liquid crystal materials where the alignment of the molecule director contributes to the elastic energy; the Magneto-hydrodynamics (MHD) and Electro-hydrodynamics (EHD) where the magnetic and electrical fields are the source of the elasticity; different polymerical fluids; viscoelastical fluids; mixtures of different materials (where the heterogeneity is the reason of the elasticity) and fluids involving different surfactant materials. The coupling between the transport of these elastic effects by the flow field and the induced elastic stresses in the momentum equations assure the Hamiltonian (or dissipative) nature of the whole system. On the other hand, such coupling also reflect the influence of the micro-structure of the material to the hydrodynamical properties of the fluid and the vice versa.

In this talk, Dr. Liu presented an approach for some of these examples, where one employs a (classical) uniform energetic variational approach which takes into account all different micro effects in the energy functional. One applies the Hamilton's principle, with the special transport of the "elastic" variables, to derive the induced elastic stresses. This approach gives a coherent way to incorporate different mechanical effects, even those involving different scales. It guarantees the quasi-statical constitutive relation and the energy conservation relations. Moreover, it can take into account the multi-scale interactions in the materials.

When designing the numerical algorithms according to this energetic variational approach, the underlining energy law will assure the accuracy of the simulation results.

Many open questions in theoretical analysis, numerical simulations as well as modelling were discussed.

Locally Convex Hypersurfaces of Constant Curvature with Boundary

In this talk Dr. B Guan described results from joint work with Joel Spruck on the Plateau type problem of finding hypersurfaces of constant curvature with prescribed boundary. More precisely, the problem can be formulated as follows: given a smooth symmetric function f of n ($n \geq 2$) variables and a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of closed smooth embedded $(n-1)$ dimensional submanifolds of R^{n+1}, one asks whether there exist (immersed) hypersurfaces M in R^{n+1} of constant curvature $f(\kappa[M]) = K$ with boundary $\partial M = \Gamma$ for some constant K, where $\kappa[M] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of M. Important examples include the classical Plateau problem for minimal or constant mean curvature surfaces and the corresponding problem for Gauss curvature. In this and some earlier work they introduced two different approaches to the problem: the Perron method and the volume minimizing approximation. These methods are based on the solvability of the problem in the non-parametric setting (the Dirichlet problem) and an important uniform local graph representation property of locally convex hypersurfaces. It would be interesting to extend the methods to other classes of hypersurfaces.

Explicit kinetic relation from the first principles

Phase transitions occur in many instances, and Dr. A. Vainchtein, from University of Pittsburgh, presented a study of a fully inertial lattice model of a martensitic phase transition which takes into account interactions of first and second nearest neighbours. Although the model is Hamiltonian at the microscale, it generates a nontrivial macroscopic relation between the velocity of the martensitic phase boundary and the driving force of transformation. The apparent dissipation is due to the induced radiation of lattice waves carrying energy away from the front.

This study involves a lattice model, which should prove very useful since lattice models are a very natural approach to the study of materials as can be seen next.

Interfaces between orderings on lattices

Dr. J. Cahn is a very famous material scientists who has made the mathematical study of many important models in materials possible. We were very happy that he accepted to participate at our workshop. In his talk, Dr. J. Cahn presented his recent understanding and questions related to the study of interfaces between orderings on lattices:

Interfaces between different orderings of species on lattices have been much studied. A simple energy function displays minima for a variety of ordered domains and can be used for computing interfacial energies which are orientation dependent. Gradient systems for the time dependence give large systems of ODE. Interface motion is orientation dependent and planar interfaces exhibit propagation failure at low driving force.

Continuizing to obtain a related energy functional has to be done carefully to avoid ill-posed equations, and even then it affects results. Do the continuized equations give the same orientation dependence of the interfacial energy? Since the propagation failure is gone, the continuized motion results fail at low driving force, but do they converge to the discrete results for high driving forces?

There are many lattice specific results, which may be illustrated with four lattices; simple square on Z^2, face

$$fcc$$

and body

$$bcc$$

cubic (subsets of Z^3), and hexagonal close packed

$$hcp$$

(a lattice complex in 3D). Emphasis will be on fcc to introduce a subsequent talk by Nick Alikakos. Here a simple energy model reproduces 15 wells commonly observed, a single well for the disordered phase and three orderings, each with its set (4 + 4 + 6) of degenerate wells, and a catalogue of interfaces connecting them. Continuizing produces three order parameters and gradient terms, connected by the lattice symmetry, but not forming a vector.

Following this very nice set of results, Dr. N. Alikakos spoke in more detail on the phase transition between ordered and disordered states of Au-Cu alloy, constructing with mathematical rigour a wave profile found numerically by G. Tanoglu and discussed in the earlier talk by J. Cahn.

Finally, the last topic discussed in this summary is related to vortices in superconductors.

Periodic vortex lattices for the Lawrence-Doniach model of layered superconductors in a parallel field

High Temperature superconductors are layered materials which may be very anisotropic. One model used to study very anisotropic superconductors such as BISCO is the Lawrence-Doniach model. In her talk, Dr. L. Bronsard presented the Lawrence-Doniach model for layered superconductors, in which stacks of parallel superconducting planes are coupled via the Josephson effect. Assuming that the superconductor is placed in an external magnetic field oriented *parallel* to the superconducting planes, she presented joint work with Dr. S. Alama and Dr. J. Berlinski on their study of the periodic lattice configurations in the limit as the Josephson coupling parameter $r \to 0$. This limit leads to the "transparent state" discussed in the physics literature, which is observed in very anisotropic high-T_c superconductors at sufficiently high applied fields and below a critical temperature. Using a Lyapunov-Schmidt reduction they proved that energy minimization uniquely determines the geometry of the optimal vortex lattice: a period-2 (in the layers) array proposed by Bulaevskiĭ & Clem. Finally, she discussed the apparent conflict with previous results for finite-width samples, in which the minimizer in the small coupling regime takes the form of "vortex planes" (introduced by Theodorakis and Kuplevakhsky.)

The fact that they can describe rigorously the shape of the vortex lattice is very nice as this has not been done yet for the Ginzburg-Landau model for superconductors.

List of Participants

Alikakos, Nicholas (University of North Texas)
Bates, Peter (Brigham Young University)
Bronsard, Lia (McMaster University)
Cahn, John (National Institute of Standards and Technology)
Chen, Fengxin (University of Texas at San Antonio)
Chmaj, Adam (Self-supported)
Ei, Shin-Ichiro (Yokohama City University)
Freire, Alexandre (National Science Foundation)
Guan, Bo (Univeristy of Tennessee)
Gui, Changfeng (University of Connecticut)
Hilhorst, Danielle (Universite Paris-Sud)
Kawohl, Bernd (Universitat Koln)
Liang, Margaret (University of British Columbia)
Liu, Chun (Penn State University)
Niethammer, Barbara (Universitat Bonn)
Schatzman, Michelle (French National Centre for Scientific Research)
Shi, Junping (College of William and Mary)

Tonegawa, Yoshihiro (Hokkaido University)
Vainchtein, Anna (University of Pittsburgh)

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Chapter 23

Current Trends in Arithmetic Geometry and Number Theory (03w5032)

August 16–21, 2003

Organizer(s): Imin Chen (Simon Fraser University), Brian Conrad (University of Michigan, Ann Arbor), Eyal Goren (McGill University), Adrian Iovita (Concordia University), Chris Skinner (University of Michigan, Ann Arbor), Nike Vatsal (University of British Columbia)

Arithmetic geometry and number theory

The area of arithmetic geometry is motivated by studying questions in number theory through algebraic geometry and representation theory, a viewpoint which was hinted at in the 19th century and which has been brought to fruition very successfully this century. Due to the variety of the techniques and theory required, it is an area which maintains deep interconnections with other branches of mathematics such as algebra, analysis, and topology.

Arithmetic geometry has witnessed many significant results in past decades. Some of the better known examples include the proofs of the Mordell conjecture [5], Fermat's Last Theorem [10], and the modularity conjecture [3]. Other examples include progress towards the Birch and Swinnerton-Dyer conjecture (now a Clay Millenium Prize), the Langland's conjecture for GL_n over function fields [8] (work awarded a Fields medal in 2002), and results revolving around the Langlands' and Serre's conjectures.

p-adic methods

The field of p-adic numbers $\mathbb{Q}_p$ is obtained by completing the rationals with respect to a non-archimedean absolute value associated to a prime p which measures distance in terms of divisibility by p. It is analogue of the real numbers in terms of analysis (but with some important differences) and allows one to focus one's attention on the contribution of a single prime p from the point of view of arithmetic.

The use of p-adic methods in arithmetic geometry has been pervasive. Many recent developments in number theory have relied crucially on the use of p-adic methods. These arise in many forms, such as via p-adic representation theory, p-adic L-functions, and p-adic geometry. These have figured prominently in the recent progress on the Birch and Swinnerton-Dyer conjecture, the proof of Fermat's Last Theorem, and the modularity conjecture.

This workshop brought together both experts and newcomers to these areas of number theory. There were two components to the workshop:

- Three lectures per day on recent developments in the field (a total of 12 lectures), consisting of various mathematicians reporting on their research in the field.
- A series of instructional lectures on Φ-Γ modules, period rings, and their applications. These lectures were aimed at those who are not specialists in the field, and this series consisted of 2 lectures per day (a total of 9 lectures). The lectures were given by Brian Conrad, Adrian Iovita, Nathalie Wach, Pierre Colmez, Laurent Berger, and Kiran Kedlaya (in order of presentation).

The setting and support provided by BIRS greatly facilitated the aims of the scientific activities of the workshop. The travel expenses for junior participants were offset by support from MSRI and some of the conference organizers. The atmosphere was particularly conducive to extensive mathematical discussion among the participants outside of the time of lectures, both on the topics of the lectures as well as on their own current work.

p-adic Representation Theory

In number theory, much effort is devoted to the study of arithmetic properties of algebraic varieties over global fields (that is, finite extensions of the rational field or of a rational function field over a finite field) and over local fields (that is, finite extensions of the field $\mathbb{Q}_p$ or a formal Laurent series field over a finite field). A basic example is the problem of determining the rank of the group of rational points on an elliptic curve over a global field. The arithmetic properties of such algebraic varieties are often encoded in representation spaces (such as étale cohomology) for the absolute Galois group G_K of the global base field K. Somewhat abstracting the underlying structure of these basic examples, one is led to study the theory of *p-adic representations* of such Galois groups; these are continuous representations of the Galois group on finite-dimensional p-adic vector spaces, that is, vector spaces over $\mathbb{Q}_p$. Further development of the theory (inspired by the local Langlands conjectures) shows that it is equally important to study the representation theory of groups of the form $G(k)$ where k is a local field and G is a reductive algebraic group over k; historically such representations were on complex vector spaces, but in recent years there has developed a rich theory of such representations on (infinite-dimensional) p-adic vector spaces (where p is the residue characteristic of k).

Many examples of p-adic representations arise naturally from geometry in the sense that given a variety over a global field one can attach p-adic vector spaces with prescribed dimensions (arising from the étale cohomology groups of the variety) and on these the Galois group naturally acts in a continuous and linear way.

A conjecture of Fontaine and Mazur [7] gives local conditions which are necessary and sufficient to ensure that an irreducible p-adic representation arises from geometry. Roughly speaking, a p-adic representation is geometric if it is unramified outside a finite set S of places of K and its restriction to a decomposition group at a place of S is potentially semi-stable. By a theorem of Grothendieck, only the places $v \mid p$ impose an actual condition. The predicted dimensions of the variety are determined from the p-adic Hodge-Tate weights of the p-adic representation.

In the 1960's, Tate discovered that the p-adic étale cohomology of a smooth projective variety over a p-adic field should have a canonical structure analogous to the Hodge filtration on topological cohomology for smooth projective varieties over the complex numbers (and Tate constructed such a filtration in the case of degree-1 étale cohomology for an abelian variety with good reduction). Partly motivated by Tate's discovery, as well as Grothendieck's "mysterious functor" relating de Rham cohomology and crystalline cohomology in degree 1, Fontaine was led to develop the theory of so-called period rings (thereby leading to the creation of p-adic Hodge theory). These period rings and functors defined with them are the fundamental mechanism for distinguishing the "good" p-adic representations of local Galois groups G_{K_v} when $v|p$. The periods rings, typically denoted B, are topological K_v-algebras equipped with a continuous linear action of G_{K_v} as well as extra G_{K_v}-equivariant linear-algebra structure (such as a grading, or a filtration, etc). For each such B, one defines $D_B(V) = (B \otimes_{\mathbf{Q}_p} V)^{G_{K_v}}$; this is a finite-dimensional vector space over K_v with dimension at most the $\mathbf{Q}_p$-dimension of V and with auxiliary structure inherited from B. When equality holds then one says that V is *B-admissible*, and for various choices of B one gets various special classes of p-adic Galois representations (such as de Rham representations, crystalline representations, and semistable representations). "All" p-adic Galois representations arising from the étale cohomology of algebraic varieties over local fields are B-admissible for a suitable B, where the "best" choice of B depends on the geometry of the variety under consideration.

In many situations the functor D_B is fully faithful into a category of "semi-linear-algebra" structures, and V may be recovered from $D_B(V)$. This state of affairs inspires the problem of determining the essential image of D_B for various B (solved by Fontaine and Colmez in the case $B = B_{\mathrm{st}}$), and more generally it motivates the philosophy of translating hard problems concerning p-adic Galois representations into (hopefully more tractable) problems in a suitable "semi-linear algebra" category.

For example, in the early days of the subject, Fontaine gave a more tractable way to work with p-adic representations of a local Galois group G_{K_v} by identifying this with the category of étale (Φ, Γ)-modules. These are modules (over a certain ring) endowed with a semi-linear "Frobenius" endomorphism Φ and a semi-linear action of a certain "explicit" quotient Γ of G_{K_v} such that Φ commutes with the action of Γ and has slope 0.

Extending work by Cherbonnier and Colmez [4], Berger shows how to associate to every p-adic representation V an invariant $D^{\dagger}_{\mathrm{rig}}(V) := (B^{\dagger}_{\mathrm{rig}}(V) \otimes V)^G$ where $B^{\dagger}_{\mathrm{rig}}$ is the Robba ring . The Robba ring arises in the theory of p-adic differential equations in such a way that the invariant $D^{\dagger}_{\mathrm{rig}}(V)$ can be interpreted as a p-adic differential equation. Berger's construction uses Fontaine's theory of (Φ, Γ)-modules and allows one to recognize semi-stable and crystalline representations in the sense that $D_{\mathrm{st}}(V)$ and $D_{\mathrm{cris}}(V)$ can be constructed from $D^{\dagger}_{\mathrm{rig}}(V)$.

If V is de Rham, the associated p-adic differential equation has much better behaviour than one might have expected *a priori*. This allows one to obtain a classical p-adic differential equation, and using a recent theorem of André (also proved independently by Kedlaya and Mebkhout) this allows one to show that every de Rham representation is potentially semi-stable.

The research talks which touched upon recent progress in the area of p-adic representation theory were those by Jeremy Teitelbaum, Gebhard Böckle, Matt Emerton, and Kiran Kedlaya.

p-adic L-functions

L-functions are meromorphic functions of a complex variable, and they are obtained from algebraic objects by means of infinite products or sums. It is a mysterious phenomenon that L-functions capture arithmetic information about the algebraic object. The mechanism behind this is only well-understood in a few cases, but in general settings there are many conjectures which predict the precise relationship between the analytic properties of associated L-function and the arithmetic properties of the algebraic object.

The most celebrated example of an L-function is the Riemann zeta-function. This is known to capture information about the integers. Another well-known example is the L-function attached to an elliptic curve over $\mathbb{Q}$. The conjectural properties of this L-function encode the arithmetic of the elliptic curve, as is made precise by the Birch and Swinnerton-Dyer conjecture.

A natural point of view which has gained much attention is to try to construct and study L-functions which are meromorphic functions of a p-adic variable. One of the first attempts was by Kubota and Leopoldt. They constructed a p-adic zeta-function motivated by classical congruences. The construction of p-adic L-functions has been considerably generalized to broader contexts. Iwasawa formulated a Main Conjecture for his p-adic L-functions which was later proved by Mazur and Wiles. This conjecture became the paradigm for Main Conjectures in many other settings.

Several of the research talks reported recent progress on subjects related to p-adic L-functions and various Main Conjectures. These included the talks of Haruzo Hida, Eric Urban, Adebisi Agboola, and Robert Pollack.

p-adic Geometry

In the case of a p-adic representation arising from geometry, the geometrical properties of the variety in question are reflected in the properties of the representation and the associated automorphic form. Of particular interest are the geometrical properties of the variety as a p-adic analytic object. This is a viewpoint which gained prominence through work of Robert Coleman and Nicholas Katz on p-adic modular forms in the classical case.

The research talks in this and related areas were those given by Ehud de Shalit, Graham Herrick, Robert Coleman, Mak Trifkovic, and Thomas Zink.

Instructional lectures

The instructional component provided an introduction to the recent proof by Berger [2] of the p-adic Monodromy Conjecture of Fontaine described above. This conjecture states that every p-adic de Rham representation is potentially semi-stable. A good account of Berger's work can be found the featured review [9] by Adolfo Quirós.

1. **Brian Conrad** p-adic Hodge theory I
2. **Brian Conrad** p-adic Hodge Theory II
3. **Adrian Iovita** Introduction to Φ-Γ-modules I
4. **Adrian Iovita** Introduction to Φ-Γ-modules II
5. **Nathalie Wach** $\Phi - \Gamma$-modules of finite height
6. **Pierre Colmez** Overconvergent Φ-Γ-modules
7. **Laurent Berger** p-adic Galois representations and p-adic differential equations I
8. **Laurent Berger** p-adic Galois representations and p-adic differential equations II
9. **Kiran Kedlaya** Frobenius slope filtrations and Crew's conjecture

Titles and Abstracts of talks

Adebisi Agboola (UC Santa Barbara)

Title: Anticylotomic Main Conjectures for CM elliptic curves

Abstract: (joint with Ben Howard) This is a report on joint work with B. Howard. We shall discuss the Iwasawa theory of a CM elliptic curve E in the anticyclotomic Z_p extension of the CM field, where p is a prime of good, ordinary reduction for E. When the complex L-function of E vanishes to odd order, work of Greenberg shows that the Pontryagin dual of the p-power Selmer group over the anticyclotomic Z_p-extension is not a torsion Iwasawa module. We shall show that the dual of the Selmer group is a rank one Iwasawa module, and we shall prove one divisibility of an Iwasawa Main Conjecture for its torsion submodule.

Gebhard Böckle (ETH-Zurich)

Title: A conjecture of de Jong

Abstract: (joint with Chandrashekhar Khare) Let X be a smooth curve over a finite field k of characteristic p and $\overline{X}$ its base change to the algebraic closure $\overline{k}$ of k. In this context, one has the following conjecture of de Jong: Let $\rho : \pi_1(X) \to \mathrm{GL}_n(F_l[[T]])$ be continuous and $l \neq p$. Then $\rho(\pi_1(\overline{X}))$ is finite. In the first part of the talk I plan to present various reformulations and consequences of the conjecture of de Jong as well as known results. for instances, assuming the conjecture, one easily obtains an analogue of Serre's conjecture for function fields. In the second part, I will sketch an approach to proving the conjecture in many cases. It is based on the methods used in the modularity proofs by Wiles, Taylor, et al. as well as on the recent results of Lafforgue on the Langlands conjecture.

Robert Coleman (UC Berkeley)

Title: $X_0(125)$

Abstract: Suppose p is a prime and $(p, N) = 1$. The minimal model of $X_0(pN)$ over Z_p was determined by Deligne-Rapoport. It is also (semi-)stable. Edixhoven, Katz and Mazur determined the minimal model of $X_0(p^n N)$ and interpreted it moduli theoretically. It is not semi-stable in general. Edixhoven determined the stable model of $X_0(p^2 N)$ but gave no moduli-theoretic interpretation of his results. Results of McMurdy and myself interpret Edixhoven's model, compute the stable model of $X_0(125)$ and suggest that something very interesting is going on.

Matt Emerton (Northwestern University)

Title: On the ramification of Hecke algebras at Eisenstein primes

Abstract: (Joint with Frank Calagari) In his celebrated "Eisenstein ideal" paper, Mazur considers the completion of the Hecke algebra acting on weight two modular forms of prime level N at the so-called Eisenstein maximal ideals (the ideals of fusion between cuspforms and the weight two Eisenstein series). He shows that any such completion is finite, flat, monogenic Z_p-algebras (here p is the residue characteristic of the Eisenstein maximal ideal under consideration) and raises the question of computing its rank over Z_p. In this talk I will explain how to relate this rank to class field theoretic properties of the algebraic number field $Q((-N)^{1/p})$. When $p = 2$, we are able to determine the rank completely: it equals $2^{(m-1)} - 1$, where 2^m denotes the two-power part of the class number of $Q((-N)^{1/2})$. If p is odd, then our results are less definitive. However, when combined with earlier work of Merel, they do yield the following theorem: If $p \geq 5$, and $N \equiv 1 \pmod{p}$, then the p-part of the class group of $Q(N^{1/p})$ is cyclic only if $\prod_{\ell=1}^{(N-1)/2} \ell^\ell$ is not a pth power $\pmod{N}$. The method of proof depends on identifying the Eisenstein completions of the Hecke ring with certain universal deformation rings of Galois representations. This identification (which we prove following the method of Wiles) is interesting in itself, since it allows us to recover all of Mazur's structural results concerning these Eisenstein completions while avoiding any analysis of the arithmetic of the Jacobian $J_0(N)$.

Graham Herrick (Northwestern University)

Title: Cusp forms mod p and conjectural slope formulae

Abstract: This talk gave an explicit formula for the slopes of classical modular forms on $\Gamma_0(N)$ whose associated Galois representation is reducible when restricted to a decomposition group.

Haruzo Hida (UCLA)

Title: Anticyclotomic Main Conjecture

Abstract: Non-vanishing modulo p of almost all twisted Hecke L-values combined with the technique I invented with Tilouine in 1993 gives the divisibility of the anti-cyclotomic Iwasawa power series by the anticyclotomic p-adic Hecke L-function. Under the assumption of Taylor-Wiles-Fujiwara (which gives the identification of the Hilbert modular p-ordinary Hecke algebra with a Galois deformation ring), we can prove the reverse divisibility via the integrality theory of definite and indefinite theta series (an integral Jacquet-Langlands-Shimizu correspondence). I will give a sketch of the proof of the reverse divisibility.

Robert Pollack (University of Chicago)

Title: Relations between congruences of modular forms and the Main Conjecture

Abstract: (joint with Matthew Emerton, Tom Weston) This talk discussed a generalization to arbitrary weight of a result of Greenberg-Vatsal concerning μ and λ invariants in Hida families.

Ehud de Shalit (Hebrew University, Givát-Ram)

Title: The p-adic monodromy-weight conjecture for p-adically uniformized varieties

Abstract: (joint with Peter Schneider) The p-adic monodromy-weight conjecture asserts that the monodromy filtration and the weight filtration on the Hyodo-Kato cohomology of a smooth proper variety over Q_p with semistable reduction, coincide. It is a p-adic analogue of an l-adic conjecture due to Deligne. Both conjectures are in general open, but for p-adically uniformized varieties, they were recently proved by T. Ito (using l-adic cohomology) and, independently, by Gil Alon and the speaker (using p-adic cohomology). We shall explain the conjecture and the main ideas behind the p-adic proof. [In fact, as Ito observed, for p-adically uniformized varieties the p-adic and l-adic conjectures imply each other].

Jeremy Teiltelbaum (University of Illinois at Chicago)

Title: An update on p-adic analytic representation theory

Abstract: We will describe some results on the structure of p-adic analytic representations and relations with arithmetic, especially p-adic Galois representations.

Mak Trifkovic (McGill University)

Title: Elliptic curves over imaginary quadratic fields and p-adic constructions of rational points

Abstract: We will discuss a conjectural p-adic analytic construction of rational points on elliptic curves defined over an imaginary quadratic field K, using mixed period integrals a là Darmon. The main difference relative to curves over Q is the nature of the (merely conjectural) modularity: the form corresponding to E/K is a harmonic form on a suitable quotient of H^3, which is not an algebraic variety. We will present the computational evidence for some of the conjectures.

Eric Urban

Title: Deformations of Eisenstein series and applications

Abstract: (joint with Chris Skinner) This talk described work on divisibility of special values of various L-functions attached to automorphic forms on $GU(2,2)$ over K an imaginary quadratic field.

Thomas Zink (Universitat Bielefeld)

Title: Higher displays in the crystalline cohomology over an artinian ring

Abstract: We introduce the tensor category of higher displays over an artinian ring. The subcategory of nilpotent displays is equivalent to the category of formal p-divisible groups. The de Rham-Witt complex defines the structure of a display on the first crystalline cohomology group of a proper and smooth scheme. We show how a higher display structures should be defined on higher crystalline cohomology groups and obtain these structures for liftable schemes.

Agboola, Adebisi (University of California - Santa Barbara)
Berger, Laurent (Harvard University)
Boeckle, Gebhard (Institute for Experimental Mathematics)
Chang, Seunghwan (Brandeis University)
Chen, Imin (Simon Fraser University)
Chenevier, Gaëtan (Ecole Normale Superieure)
Coleman, Robert (University of California - Berkeley)
Colmez, Pierre (Université Pierre et Marie Curie)
Conrad, Brian (University of Michigan - Ann Arbor)
Cornut, Christophe (Jussieu)
de Shalit, Ehud (Hebrew University, Giv'at-Ram)
Diamond, Fred (Brandeis University)
Edixhoven, Bas (Mathematisch Instituut, Leiden)
Emerton, Matthew (Northwestern University)
Goren, Eyal (McGill University)
Greenberg, Ralph (University of Washington)
Herrick, Graham (Northwestern University)
Hida, Haruzo (University of California - Los Angeles)
Iovita, Adrian (University of Washington)
Kassaei, Payman (McGill University)
Kedlaya, Kiran S. (Massachusetts Institute of Technology)
Khare, Chandrashekhar (Tata Institute of Fundamental Research)
Kisin, Mark (Westfalische Wilhelms Universitat)
Mailhot, Jim (University of Washington)

Marshall, Susan (University of Texas)
Nichifor, Alexandra (University of Washington)
Niziol, Wieslawa (University of Utah)
Pollack, Robert (University of Chicago)
Ramakrishna, Ravi (McGill University)
Savitt, David (Institut des Hautes Études Scientifiques, Le Bois-Marie)
Schneider, Peter (Mathematisches Institut Muenster)
Skinner, Chris (University of Michigan, Ann Arbor)
Teitelbaum, Jeremy (University of Illinois at Chicago)
Trifkovic, Mak (Harvard University)
Urban, Eric (French National Center for Scientific Research)
Vatsal, Nike (University of British Columbia)
Virdol, Cristian (University of Southern California)
Wach, Nathalie (University of Strasbourg)
Wortmann, Sigrid (Mathematisches Institut Heidelberg)
Zink, Thomas (Universitat Bielefeld)

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Chapter 24

Computational Techniques for Moving Interfaces (03w5050)

August 23–28, 2003

Organizer(s): Randy LeVeque (University of Washington), Robert D. Russell (Simon Fraser University), Steven Ruuth (Simon Fraser University)

Introduction

Currently some of the most difficult problems in computational science involve moving interfaces between flowing or deforming media. Typically partial differential equations must be satisfied on each side of the interface (often different equations on each side) and these solutions coupled through relationships or jump conditions that must hold at the interface. These conditions may be in the form of differential equations on the lower-dimensional interface. Often the movement of the interface is unknown in advance and must be determined as part of the solution. The interface shape may be geometrically complex and may change topology with time. Particularly in three space dimensions, the ability to solve such problems accurately is limited. Exciting research is currently underway in the development of better algorithms, the analysis of the accuracy and stability of such algorithms, and the application of these techniques to specific scientific and engineering problems.

The primary goal of this Workshop was to bring together a number of researchers (mostly applied mathematicians) who are working on such methods, to foster interaction and the exchange of ideas. The focus was primarily on three broad classes of methods and the workshop started with introductory talks on these basic methodologies — *Moving Grid Methods* by **Mike Baines**, *Level Set Methods* by **Hong-Kai Zhao**, and *Fixed Grid / Moving Interface Methods* by **Randy LeVeque**. These overview talks were fleshed out over the course of the 5-day workshop in 24 additional talks on various mathematical aspects and applications. Several informal evening sessions were held to discuss technical details, recent progress, and directions for future research.

There was consensus that the Banff setting and informal workshop format created an ideal environment to foster openness and the surprisingly frank discussions. The organizers are extremely grateful to BIRS for having had the opportunity to hold this exciting workshop. Participants discussed the shortcomings as well as the successes of each approach and cross-fertilization between the groups led to interesting discussions and some new directions for research.

Moving Grid Methods

There are 3 types of methods to perform grid, or mesh, adaption — the so-called $h-$, $p-$, and $r-$ methods. The first two do static (fixed time) regridding, where the $h-$ method does coarsening or refining as needed and the $p-$ method takes higher or lower order approximations locally as needed. In contrast, the $r-$ methods, or *moving grid methods*, are designed in principle to move the grid in conjunction with the solutions to the time-dependent PDEs or corresponding interfaces. These $r-$ methods have received considerably less attention than the others; nevertheless, there have been some recent developments which clearly demonstrate their potential for problems such as those having moving interfaces. The intention at this workshop was to discuss implementation techniques and success at solving physical problems as well as progress in deriving an underlying theory for these methods.

Some general issues

The excellent overview of moving grid methods by **Mike Baines** set the stage for the talks to follow in this area. The grid generation problem can be equated to constructing a mapping $x(\xi, t)$ from computational space (with coordinate ξ) to physical space (with coordinate x). The two basic types of methods, location based and velocity based, generally involve respectively computing x by minimizing a variational form or computing the mesh velocity $v = x_t$ using a Lagrangian like formulation.

For the location based methods, several variational forms were reviewed. One common type of method involves solving the variational problem using a steepest descent method to introduce the time derivative for grid movement. Another is the classical moving finite element method of Miller. There is frequently no clear consensus on the relative merits of one of these approaches over another.

Several velocity based methods designed for general problems were then discussed. The ALE methods were described in a general framework, and then related to the Geometric Conservation Law, or GCL, method. The GCL method [1], which has undergone renewed interest and spawned investigation of related methods, was discussed in several talks.

Near the end of the workshop, a moving grid breakout session was held. It was attended by those working in the field as well as interested participants with expertise in the other workshop areas. Since moving grid methods are generally less developed and less well-known than the other adaptive techniques, there was felt to be a need to discuss what attempts should be made to see that recent advances make their way into the repertoire of more scientists and engineers. It was agreed that there were three key challenges (and partial solutions): (1) providing better reviews of the literature (e.g., through a SIAM Review article), (2) having more accessible software (aided by developing modular codes on a web site and compiling a list of test problems), and (3) developing a firmer theoretical foundation.

Overview of work presented

There was a diversity of talks varying from very theoretical talks to ones about the concrete solution of specific physical problems.

An underlying question with any adaptive method is how to interpret the various factors arising in anisotropic grid generation. **Weizhang Huang** discussed a general steady state error analysis for such grids. The three key grid features or qualities which play a role in determining the interpolation error (e.g., in a finite element analysis setting) are (1) geometric quality in physical space, (2) alignment quality in physical space or isotropy in computational space, and (3) adaptive quality or level of equidistribution. From the way that terms representing these three features arise in the interpolation error bound, it was shown how these terms tend to compensate for one another, but argued that generally the latter two are the most important. **Weiming Cao** performed a related analysis for anisotropic triangular grids. One additional feature was the role of two different types of stretched triangles – those with small and large angles – in the alignment. It was demonstrated why care must be taken to choose the former over the latter.

Chris Budd considered the class of PDEs for which scaling invariance and self similar solutions play a primary role. For them, moving grid methods are used which have the same scaling invariance built into the expanded physical PDE/moving grid PDE system. Numerical results for the resulting methods were shown for a number of challenging blowup problems in one space dimension. The argument was made that these

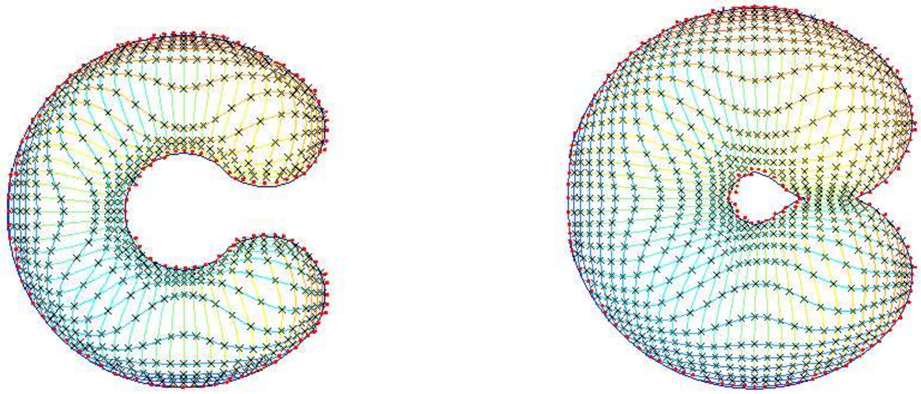

Figure 24.1: Starting from a uniform mesh, the mesh is adapted to put more grid points near the domain boundary (left). As the domain grows outwards with some constant normal velocity, a change in topology is automatically handled (right) [4].

moving grid methods are ideally suited for blowup problems because the grid naturally evolves on the proper space and time scales.

Jeff Williams discussed a method for doing adaptivity for time-dependent PDES based upon the Monge-Ampere equation. Specifically, relating the coordinate mapping $x(\xi, t)$ for mesh adaptivity to the mapping used for solving the classical Mass Transfer Problem (recast as a dynamic framework), the Monge-Ampere equation is solved for the gradient of $v = x_t$, allowing straightforward computation of x. The relation was given between this and the GCL method, another velocity based. Advantages and disadvantages of this promising method were presented, as well as computational results for two and three dimensional problems.

John Mackenzie discussed a number of features of his numerical implementation of a moving grid method for solving PDEs. An analysis of several choices of monitor functions was given and the importance of using a conservative form for the PDE in computational (quasi-Lagrangian) space discussed. Numerical examples of the method, including on a Stefan problem with grid moving with the interface, were given.

Ben Ong gave a talk bridging moving grid methods and level set methods. In his preliminary computational experiments, he demonstrated how the change in topology of a physical region, while captured well by level set methods when the boundary is fairly smooth, can lose resolution around areas of high curvature. Using a GCL approach, he was able to demonstrate the clear potential of combining these two areas. See Figure 24.1 for an illustration of his approach on an evolving domain with topological change.

Level Set Methods

Level set methods are numerical techniques introduced by Osher and Sethian in 1988 to track the motion of interfaces. Rather than evolving marker cells placed along the interface, these methods represent the interface as the zero contour of a function, ϕ, defined over the computational domain. The evolution of the interface is carried out according to the level set PDE,

$$\phi_t + v \cdot |\nabla \phi| = 0$$

where v is the normal velocity of the evolving interface. This interface velocity can depend on the geometry of the interface or on external physics defined by equations off the interface. It is commonplace to discretize level set equations on fixed, uniform grids, leading to relatively simple methods with good stability properties. These methods have the powerful advantage of automatically handling topological merger and breakage.

The generality and robustness of level set techniques have made them natural choices for a wide range of applications, including problems in fluid mechanics, computer graphics, manufacturing of computer chips, combustion and image processing.

A general issue

A particularly lively debate arose in the discussion following Hong-Kai Zhao's introductory talk. It was noted that while level set methods automatically handle topological shape changes, there will be situations where the corresponding evolutions are not physically correct. While the theory of viscosity solutions may provide some answers, it is clear that detailed modelling of the underlying physics will be needed in certain problems. No universal solution to this interesting problem was found during the workshop, however, discussions illuminated the need for careful design of level set algorithms in physical problems.

Overview of work presented

A broad spectrum of level set talks were given over the course of the workshop.

David Adalsteinsson gave a talk on his recent work on transport and diffusion of material quantities on propagating interfaces using level set methods. Material quantities defined on an interface are not easily handled by traditional level set methods. Adalsteinsson described an approach for extending the level set method to these problems.

Anne Bourlioux gave a talk on her research into multi-scale strategies for turbulent burning fronts. In many practical applications (for instance, engines), flame fronts are thin and can be viewed at the large scale as a zero-thickness interface that separates burnt and unburnt gases. Her talk described research on how the effective dynamics of the front at the large scales of interest are influenced by the small scale stirring by the turbulent flow.

Li-Tien Cheng discussed work that combines the level set method and the heterogeneous multiscale method for interface problems in multi-scale settings. Examples that were considered included flows related to the homogenization of Hamilton-Jacobi equations and to the phase-field model, all in the presence of highly oscillating or random data that introduce a small scale. His approach incorporated the advantages of both methods to produce a fast algorithm that handles the multiscale and topological aspects of the interface dynamics.

Oliver Dorn reported on his recent work on identifying, localizing and tracking penetrable objects. In this problem, an array of electromagnetic or acoustic sources is located at a certain position and emits waves which propagate through the environment and are scattered by objects. Dorn described approach for finding information on the location, trajectory, orientation and the shape of the moving object in a stable way using a variety of techniques including level set methods.

Isaac Klapper described recent work in the study of biofilm response to mechanical stress. Biofilms are dense ubiquitous aggregates of microorganisms. Klapper and collaborators are interested in characterizing physical properties of biofilm with the longterm aim of understanding phenomena such as mechanical failure. He described work on methods to treat these problems efficiently and accurately.

Ian Mitchell reported on some of his recent work on level set methods for control and verification. His talk described a recent method for warning air traffic controllers of of potential collisions. Drawing on results from optimal control, differential games and level set methods, he described methods for calculating the reachable sets corresponding to aircraft collisions. High dimensional systems are still difficult; however, one possible counter to Bellman's curse of dimensionality that was given is to compute an over approximation of a high dimensional reachable set as a collision of lower dimensional projections. See Figure 24.2. Recent related work on particle level set methods for increasing the accuracy of level set methods near high curvature regions was also described.

Jamie Sethian reported on a variety of recent work including the industrial simulation of evolving droplets in ink jet printers. His work on ink jet printers made use of recent level set techniques to robustly treat the evolving drops. He discussed how mass conservation is particularly relevant to the treatment of such problems and showed how practical methods can be designed by combining a 2nd order Godunov method, a finite element projection method and fast marching algorithms.

Peter Smereka described a Monte-Carlo method for surface growth simulations. His approach combined aspects of continuum mechanics and kinetic Monte-Carlo to achieve an efficient way to approximate models arising in epitaxial growth. The methods give a more realistic account of fluctuations in island shape than deterministic methods while still maintaining efficiency in problems where kinetic Monte-Carlo is impractically slow.

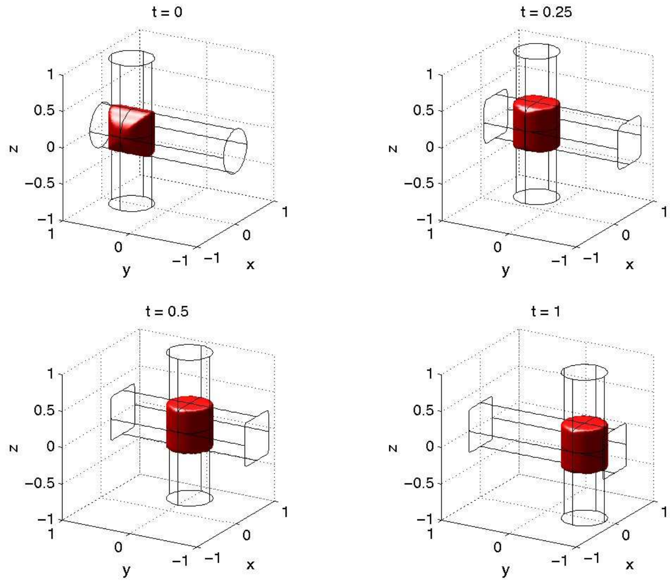

Figure 24.2: Overapproximating Reachable Sets by Hamilton-Jacobi Projections. From [3].

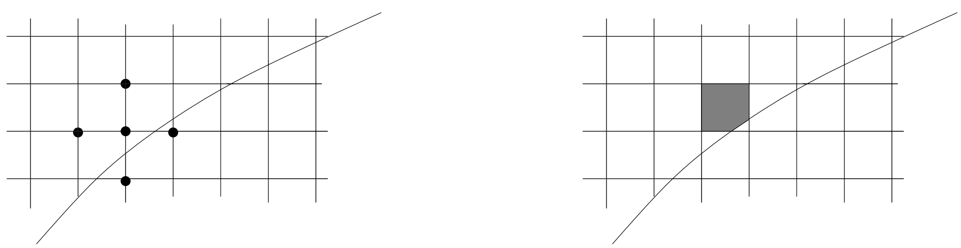

Figure 24.3: (a) A finite difference grid (b) A finite volume grid.

Fixed Grid / Moving Interface Methods

With this class of methods, a fixed computational grid is used over the global spatial domain that typically does not align with any internal interfaces. The interfaces are then represented as codimension 1 surfaces moving relative to the fixed grid. An advantage of this approach is that expensive grid generation and regridding is avoided each time step. Fast and accurate methods can often be used on the fixed grid, at least away from the interface, and special methods are needed only near the lower dimensional surface.

Some general issues

A number of issues arise that can each be addressed in many ways, leading to a multitude of methods of this type. Some of these issues will be broadly described and then some some specific methods and discussion points from the workshop will be mentioned.

What sort of global grid should be used and what is the underlying discretization method away from the interfaces? For example, a finite difference method might be applied to obtain pointwise approximations to the solution at discrete points as illustrated in Figure 24.3(a), or a finite volume method might be applied over grid cells as indicated in Figure 24.3(b). In each case an interface is also shown that cuts between grid points or through grid cells. A finite element method might be used with elements consisting of the distinct cells shown in Figure 24.3(b) or of the full rectangular cells with special basis functions that incorporate jump conditions at the interface.

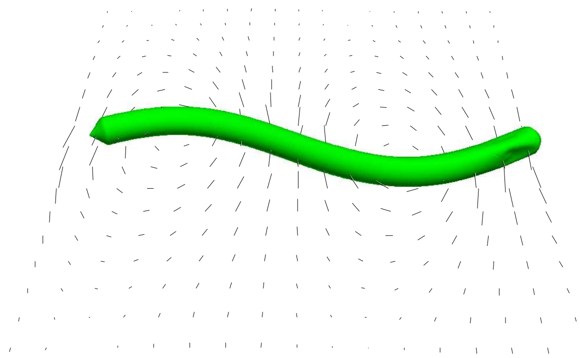

Figure 24.4: An immersed boundary computation of the motion of a nematode, depicting the fluid velocity field on the plane coinciding with the organism's centreline. From [2].

How are the equations discretized near the interface? Typically a PDE must be solved on each side with some coupling or jump conditions imposed at the interface. Depending on the application, it may be the same equation on each side or two very different equations. If the equation has the same form then it may be possible to difference across the interface with additional terms included from the interface, or it may be necessary to use one-sided approximations on each side with appropriate coupling.

How is the interface represented and moved? The interface might be specified by a set of marker points that are densely distributed on the interface (with spacing comparable to the mesh spacing) and the interface determined solely by these points. Alternatively, marker points may be spaced more widely and a curve or surface fit through these points at each time to determine the interface. For some problems it is better to have an entire spatial domain covered by marker points. In any case the marker points must be moved in an appropriate manner each time step. Another approach is to represent the interface location implicitly via a level set function. Then an evolution equation for this function must be developed that is compatible with the desired interface motion, and this equation coupled with the equations being solved.

Overview of work presented

A number of participants work on *immersed boundary methods*, an approach pioneered by Charles Peskin originally for modelling blood flow in the heart that has since been applied to many other problems, particularly in biofluid dynamics. See [5] for a recent review. In recent years a great deal of progress has been made in applying this method to full three dimensional simulations in biofluid dynamics. Figure 24.4 shows a sample computation of a swimming organism, from [2].

Robert Dillon reported on recent work modelling the movement of eucaryotic flagella and cilia. A three-dimensional model has been developed that represents the physiology of the axoneme in a detailed manner and links the elastic properties of this structure to the fluid dynamics via the immersed boundary method.

Gretar Tryggvason described recent work on simulating bubbly flow, where full three dimensional simulations are being performed with hundreds of bubbles in some cases. Figure 24.5 shows a portion of such a simulation. Recent work has focused on the use of methods for simulations of multifluid flows to understand the dynamics of large disperse systems and on the development of methods for systems where it is necessary

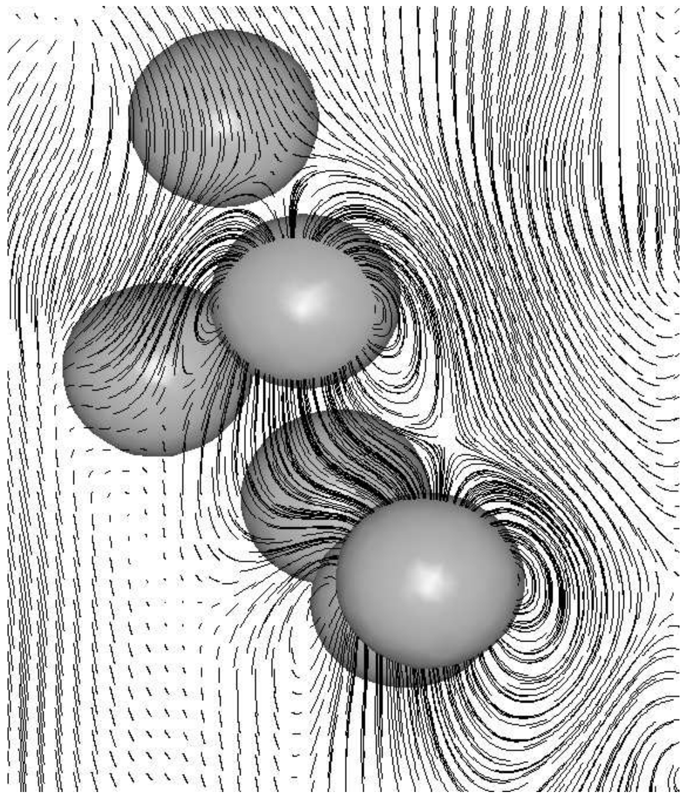

Figure 24.5: Flow field near rising bubbles. From [6].

to deal with complex physics, such as phase change in boiling processes.

Ricardo Cortez reported on *regularized Stokeslets*, a method for Stokes flow with immersed boundaries or obstacles that is based on smoothing a point force and explicitly calculating the resulting pressure and velocity. These can be superposed to determine the response to forces distributed along an interface. The velocity expression can be inverted to find the forces that impose a given velocity boundary condition. This allows the solution of flow problems past fixed obstacles as well as flexible boundaries.

Anna-Karin Tornberg described some recent research on the accuracy that can be achieved when using discrete delta functions distributed along an interface. This is an issue both in immersed boundary methods, where a tensor product of one-dimensional delta approximations is frequently used for a multidimensional delta function at a point, and in level set methods where a one-dimensional delta function based on the distance function comes into play.

John Stockie gave a talk on the stability of fluid flows containing immersed elastic boundaries, where the fluid-structure interaction is driven by periodic variations in the elastic properties of the solid material. This leads to parametric resonances that can be studied with Floquet theory and compared to direct numerical simulations of the fluid-structure interaction problem using the immersed boundary method.

Zhilin Li gave an overview talk on the *immersed interface method* and recent applications. This approach is based on the immersed boundary method, but instead of using a discretized version of a delta function or a dipole at an interface, the appropriate jump conditions for the partial differential equations are built into the discretization of the equations directly. When this can be done it typically results in sharper approximations of the the solution near the interface and perhaps higher order of accuracy overall. This has been applied to a number of moving interface problems in fluid dynamics, solidification, and elasticity. Some recent progress on finite element formulations was also described.

Xudong Liu described joint work with Songming Hou on a numerical method for solving variable coefficient elliptic equation with interfaces. In his talk a new 2nd order accurate numerical method on non-body-fitting grids was given for solving the variable coefficient elliptic equation. His method allowed the presence of interfaces where the variable coefficients, the source term, and hence the solution itself and its derivatives may be discontinuous.

Deborah Sulsky presented work on the material-point method, an extension of the particle-in-cell approach in fluid dynamics to problems in solid mechanics. Rather than tracking only the interfaces, particles within the solid are tracked. By tracking particles and their stress tensors during the deformation of the body, it is possible to represent large deformations without the problems of mesh tangling that could arise in other Lagrangian descriptions. Information from the particles is transferred to a background computational grid to solve the momentum equations and efficiently compute interactions. The method can be used to solve problems where elastic bodies come in contact, such as the problem shown in Figure 24.6.

Xiaolin Li discussed a front tracking approach to solving interface problems that is an integration of the original front tracking approach of Glimm and McBryan, the Eulerian level-set method by Osher and Sethian, and the marching cube method for computer graphics by Lorenson and Cline. The interface is described as a set of topologically connected marker points which follow the Lagrangian propagation based on the solution of Riemann problem (in fluid dynamics), and the use of the Riemann solution along an oblique boundary in space-time to compute the interface fluxes, making the method conservative.

Petri Fast talked about the use of moving overset grids to track interface dynamics. The key idea is to use thin, body-fitted grids that move and deform with moving boundaries, while using fixed Cartesian grids to cover most of the computational domain. This has the advantage of using a grid that conforms to the interface locally but without the need for global regridding. Fast discussed the Overture code developed at LLNL and applications to viscous fingering and to simulations of elastic boundaries in a flow.

List of Participants

Adalsteinsson, David (University of North Carolina -Chapel Hill)
Baines, Mike (University of Reading)
Bourlioux, Anne (Universite de Montreal)
Budd, Chris (University of Bath)
Calhoun, Donna (University of Washington)
Cao, Weiming (University of Texas at San Antonio)
Chen, Shaohua (University College of Cape Breton)
Cheng, Li-Tien (University of California - San Diego)
Cortez, Ricardo (Tulane University)
Dillon, Robert (Washington State University)
Dorn, Oliver (Universidad Carlos III de Madrid)
Fast, Petri (Lawrence Livermore National Laboratory)
Fauci, Lisa (Tulane University)
Huang, Huaxiong (York University)
Huang, Weizhang (University of Kansas)
Karni, Smadar (University of Michigan)
Klapper, Isaac (Montana State University)
Lai, Ming-Chih (Dept. of Applied Mathematics, National Chiao Tung University)
LeVeque, Randy (University of Washington)
Lee, Long (University of North Carolina - Chapel Hill)
Li, Xiaolin (State University of New York at Stony Brook)
Li, Zhilin (North Carolina State University)
Liang, Margaret (University of British Columbia)
Liu, Xu-Dong (University of California - Santa Barbara)
Mackenzie, John (University of Strathclyde)
Madzvammuse, Anotida (Auburn University)
Mitchell, Ian (University of California - Berkeley)
Ong, Ben (Simon Fraser University)
Russell, Robert (Simon Fraser University)
Ruuth, Steven (Simon Fraser University)
Sethian, Jamie (University of California)

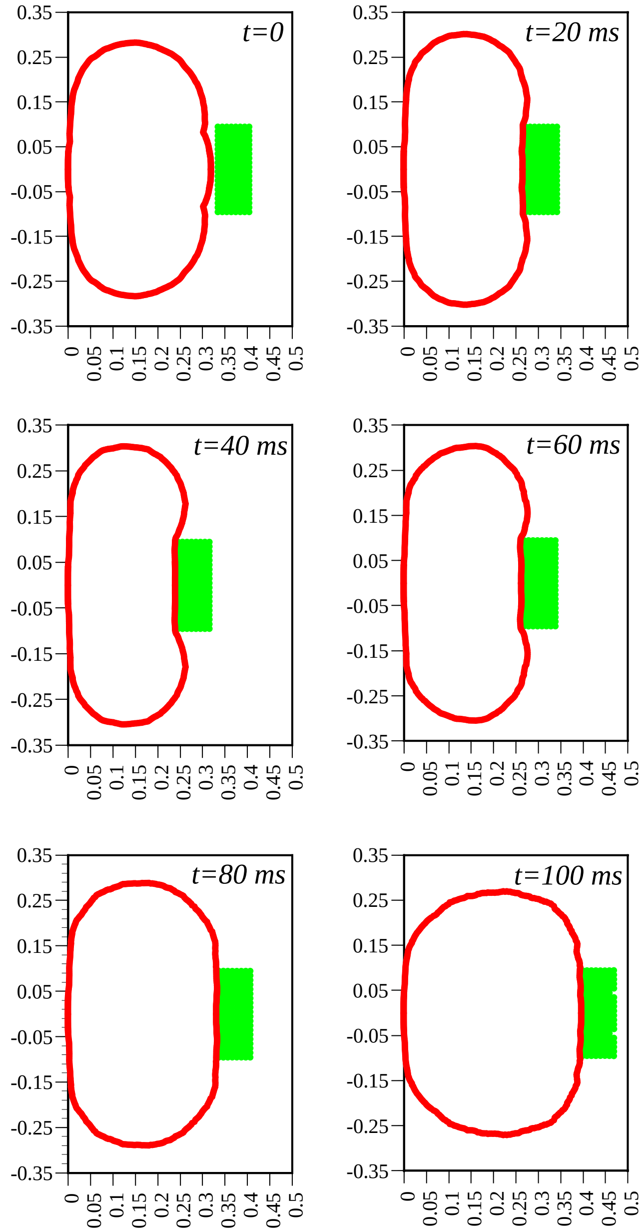

Figure 24.6: Axisymmetric calculation of a pre-inflated airbag being impacted by a solid, cylindrical probe, the deformation of the airbag as it is compressed and the subsequent rebound of the probe. From [7].

Smereka, Peter (University of Michigan)
Stockie, John (University of New Brunswick)
Sulsky, Deborah (University of New Mexico)
Tornberg, Anna-Karin (New York University)
Tryggvason, Gretar (Worcester Polytechnic Institute)
Vladimirova, Natalia (Art & Science Collaborations, Inc. Flash Center)
Williams, J. F. (University of Bath)
Zhang, Jianying (University of British Columbia)
Zhao, Hongkai (University of California - Irvine)

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Chapter 25

A Creative Writing Workshop at BIRS (03w5093)

August 31–September 4, 2003

Organizer(s): Chandler Davis (University of Toronto), Marjorie Senechal (Smith College)

The need to create a poetic-dramatic-narrative literature around mathematics is widely acknowledged by mathematicians and non-mathematicians alike. Yet creative writing about the content of mathematics is extremely rare, and creative writing about the act of mathematical creation even rarer. Despite the lively current interest in mathematics on the part of the non-scientific public, much creative writing in and about mathematics today reinforces the insider/outsider divide. The entrenched presupposition that artistic creation is unrelated to mathematical thought still endures, though some writers are struggling against this bias in various ways. Any effort to address these daunting challenges is necessarily experimental. We were encouraged by the statement on the Banff Centre's website, promising "freedom to experiment, with the knowledge that we learn from our failures as well as our successes. It is hoped that everyone will feel comfortable taking risks and challenging assumptions, creating new and unlikely alliances...."

For BIRS's first creative writing workshop, we brought together mathematicians, and non-mathematicians, all of whom are actively engaged in creative writing related to mathematics. In addition to professional mathematicians who also write, the group included non-mathematicians whose work concerns, or engages, mathematical thought: a playwright/director, a playwright/composer, a biographer, a poet, essayists on mathematical art, and a journalist.. Interests ranged from mathematical exposition to dramatization of the world of mathematics (both the people and the concepts) to exploring the relation between mathematics and poetical aesthetics.

In planning this workshop, we also deliberately mixed genres: exposition, biography, poetry, theatre, journalism, fiction and nonfiction, in the expectation that the mix would prove stimulating. (Indeed it did.) We expected all workshop participants, whether they identified themselves primarily as mathematicians, scientists, or writers, to be seriously engaged in a writing project consonant with our theme, willing to discuss some of their work-in-progress, and willing to subject their writing to others' reactions. We also asked them to circulate some of it (published or unpublished) well in advance of the meeting in Banff. (Most did, but not "well.")

As this was BIRS's first creative writing workshop and also our own, we decided not to plan the program in detail in advance. The mix of fifteen accomplished, creative, and (possibly) egotistical characters would, we thought, catalyze the most effective format. We did anticipate that part of each day would be given to group activities (modelled on writing workshop exercises), part would be presentations and discussion of work-in-progress, and part would be unscheduled.

As it turned out, most of the scheduled time slots were devoted to individual presentations. Briefly:

Marco Abate, a mathematician at the Universita di Pisa, spoke about his widely read comic adventures with mathematical themes; **Colin Adams**, a topologist-geometer well known in the American mathematical community for math-related satire, read several unpublished pieces; **Barry Cipra**, a versatile freelance writer on mathematics, both for mathematical and for general audiences, talked about ethical issues in mathematical exposition and journalism; **Florin Diacu**, author of several expositions on his field, celestial dynamics, and its history, read excerpts from a nonfiction book in progress;**Ivar Ekeland**, known for his overlapping accomplishments as a mathematician and a writer, read his forthcoming children's story, *Cat in Numberland*;**Claire Ferguson**, a writer on mathematical art, read some of her essays on the mathematical sculptures of her husband, Helaman Ferguson; **Emily Grosholz**, professor of philosophy at Penn State, poet, and Advisory Editor for *The Hudson Review*, led a discussion on aesthetics and read some of her poetry;**Paul Hoffman**, author of *The Man Who Loved Only Numbers*, the best- selling biography of Paul Erdos, read and discussed a chapter of that book, and also gave a public lecture on his more recent *Wings of Madness*;**Ellen Maddow**, composer and playwright (including a play featuring a woman mathematician) and her husband, playwright and director **Paul Zimet**, founders and leaders of the New York repertory company Talking Band, presented selections from videotapes of their play, *Star Messengers* (a music theatre production about Galileo and Kepler);**Bob Osserman**, a differential geometer and an advocate and practitioner of "humanistic mathematics" discussed an article-in-progress on Costa surfaces; **Osmo Pekonen**, a mathematician and prolific writer on math and other things and co-author of the first Finnish translation of *Beowulf*, discussed his film on early expeditions to measure the length of a degree of latitude;**Kameshwar Wali**, physicist and writer, author of *Chandra*, the biography of the astronomer Chandrasekhar, read selections from his book-in-progress on the history of violins;**Marjorie Senechal**, mathematician, writer, historian of science and technology, and workshop-co-organizer, discussed the "closet drama" she is writing about Louis Pasteur; **Chandler Davis**, mathematician, editor of *The Mathematical Intelligencer*, quondam science-fiction writer, poet, and workshop co-organizer did not take a time slot, as we had run out of them, but sparked the discussion throughout the four days.

The advance circulation has already produced one intriguing development: Ellen Maddow, inspired by Marjorie Senechal's article "The Symmetry Mystique," began writing a play based on symmetries (she presented a few early draft scenes at the workshop.) Marjorie persuaded her to extend "symmetries" to aperiodic tilings, such as Penrose's. Ellen has since applied for a partnership grant to continue this collaboration.

As we predicted, the value of the feedback was enhanced by the fact that many of the critics had backgrounds very different from the presenter's. As nobody was there in the role of teacher and nobody as student either, everyone was expected to come forward with criticisms; and every presenter had to be receptive to criticism and tolerant of criticism issuing from misunderstanding. The non-mathematicians were more used to this than the mathematicians (criticizing or being criticized for errors in mathematics is one thing, criticizing or being criticized for creative writing quite another), but we all improved in these respects as the workshop proceeded. Next time, we will emphasize this more explicitly, and also stress the fact that mathematicians must seek to learn what creative artists know (but may not know how to say) about artistic creation. At the same time, non?mathematicians must welcome being told when their productions miss the point mathematically. Next time we will be more insistent on our expectation that, at the workshop, participants present only work in progress, not work in press or already published. While published work can, and did, show us a wide breadth of possibilities and serve to introduce us to each other, both presenters and critics learn much more if the criticism might have an effect. Just as importantly, criticizing published work is a different enterprise, more like a newspaper reviewer or drama critic.

We agree that the workshop was a success and its objectives were achieved. The harmony among participants – who in many cases had had no previous contact and whose backgrounds were different – was extraordinary. Something was l earned about writing?cum? mathematics by everybody. We are planning the April 2004 workshop on the same general model, with some of the same participants (for an intended total of twenty rather than the fifteen), though modified a bit in accordance with our remarks above. We expect that the second workshop will be even more stimulating and conducive to future creative work.

We have not included a bibliography with this report, as technical mathematics was not the main focus of this workshop, but we will be glad to supply one if the Committee feels it would a useful addition. Although all of the workshops participants are well-published writers, there is, we agreed, a paucity of outlets for the kind of creative writing we are trying to encourage. We believe there is a public waiting to be engaged in mathematical ideas, but agents and publishers tend to be sceptical. One topics we will continue to explore in future workshops is ways of identifying and educating these essential middlemen, and also what contributions *The Mathematical Intelligencer* might make (for example, a participant suggested that the Intelligencer hold annual creative writing competitions in each of several genres).

A note on interaction with the Banff Centre: in our early planning, and in consultation with Carol Holmes, Director of the Banff Program in Writing and Publishing, we intended the 2003 workshop to run simultaneously with one in the Banff Writing Program and to interact with it (indeed, that is how the date was selected.) Unfortunately, this partner workshop had to be cancelled. We, and Carol, still believe strongly that both programs would be enhanced through intensive interaction with the other. To that end, we have applied for a joint BIRS/Writing Program workshop in 2005 with forty participants, twenty selected by each. Obviously, if this larger workshop takes place, we will have to modify our format somewhat: it would be impossible to achieve the intimacy of a group of twenty in a group double that size. However, as the participants will be selected in part on the basis of overlapping interests, we are confident that the program can be successful with a mix of simultaneous and plenary sessions.

List of Participants

Abate, Marco (Universita di Pisa)
Adams, Colin (Williams College)
Cipra, Barry
Davis, Chandler (University of Toronto)
Diacu, Florin (University of Victoria)
Ekeland, Ivar (Pacific Institute of Mathematical Sciences)
Ferguson, Claire
Grosholz, Emily (Pennsylvania State University)
Hoffman, Paul
Maddow, Ellen (New Dramatists)
Osserman, Robert (Math Sciences Research Institute)
Pekonen, Osmo (University of Jyvaskyla)
Senechal, Marjorie (Smith College)
Wali, Kameshwar C. (Syracuse University)
Zimet, Paul (Smith College)

Chapter 26

Locally Finite Lie Algebras (03w5095)

August 30–September 5, 2003

Organizer(s): Yuri Bahturin (Memorial University of Newfoundland), Georgia Benkart (University of Wisconsin, Madison), Ivan Penkov (University of California, Riverside), Helmut Strade (Hamburg University), Alexander Zalesskii (University of East Anglia)

The Motivation and the Goals of the Workshop

Being infinite objects, locally finite Lie algebras are very complicated in comparison with Lie algebras of finite dimension. Not only because the direct limits of finite-dimensional simple Lie algebras are much harder than their constituent algebras, but also since not every simple locally finite Lie algebras can be represented as a limit of semisimple finite dimensional algebras. Thus, in addition to "classical" methods of Lie Theory in order to successfully study locally finite Lie algebras, it is important to attract methods from other areas of mathematics. These ideas are essentially "in the air" because the other classes of locally finite algebras and the class of locally finite groups have been around for quite a long while.

The comparison with the theories of locally finite groups and associative algebras seems especially fruitful. The latter theory has close connections with C*-algebras, and we believe that the wealth of the methods of this latter theory can be applied to the study of locally finite Lie algebras. The classification result by Baranov-Zhilinski for diagonal embeddings of simple finite-dimensional Lie algebras provides just one example of a successful application of the associative methods to the theory of locally finite Lie algebras. But much more research is needed to get a better understanding of more general locally finite Lie algebras.

The theory of locally finite groups is also a very well developed subject with a number of excellent contributions by many distinguished mathematicians.The book by Kegel-Wehrfritz summarizes an early stage of development of this theory. A number of conferences have been held on this topic in the past, including a large conference in Turkey in 1996. The theory of locally finite groups is very rich with methods, but so far there have been very few attempts to apply them to locally finite Lie algebras. One of the most fruitful tools so far turned out to be the notion of an inductive system of representations introduced by Zalesskii.

Finally, it should be mentioned that in the last decade a number of important papers have been published on locally finite Lie algebras, devoted to various aspects, such as root systems, representation theory, structure theory, etc. It was very interesting to survey these results, estimate the progress made, formulate new conjectures and discuss the feasibility of their solution.

By holding a workshop on the topic, there seemed to be a very good chance of giving a boost to the subject. The goal of the meeting was to bring together people from all those different areas mentioned above, to compare the results and approaches, and to devise new strategies for studying locally finite Lie algebras.

It should be mentioned in conclusion that Canada is a country with excellent traditions in Lie theory, developed by a number of good researchers both in mathematics and physics. So it was only very appropriate to hold the conference suggested on the Canadian soil.

Finally we remark that a somewhat smaller workshop on locally finite Lie algebras was held as a part of the sequence of Californian Lie Theory Workshops early in 2001 at University of California-Los Angeles.

Preparation Period

The preparation work for the workshop "Locally Finite Lie Algebras" started in 2001 when the group of organizers was formed and the first tentative decisions had been made concerning the future participants. Then we applied with BIRS for a full 5-day workshop, with 40 participants. Later the same year we learnt about an approval of our application, although in the form of a half 5-day workshop, with 20 participants.

During the preparation period the main effort was directed to forming the team of participants, which should be optimal as concerns the goals of the workshop. At times, we encountered problems arising from the financing principles of BIRS, according to which only the stay at the venue of the workshop had to be covered. At the same time, we had a number of participants from Europe for whom travelling such a long distance could be quite costly. Here we have to acknowledge some financial help from the Department of Mathematics and Statistics of the Memorial University of Newfoundland.

The second problem was to try to attract top people from the areas of mathematics, which are not immediately connected to the topic of the workshop but yet could be very important for introducing new methods and comparing the extent of the development of similar subjects but in different areas. This turned out to be quite successful and we had such a group of well-known people from outside Lie Theory, with genuine interest in locally finite Lie algebras.

The organisational problems of this preparation period had been successfully solved, thanks to the most efficient work of the PIMS personnel.

Abstracts of Lectures/Talks

1. **Bruce Allison, University of Alberta, Canada**

   COVERING ALGEBRAS OF LIE ALGEBRAS

   In this talk, based on joint work with S. Berman and A. Pianzola, we will discuss loop algebras (also called covering algebras) that are constructed from a base algebra A and a finite order automorphism s of A. (When $s = 1$, the loop algebra is said to be untwisted.) In the case when the base algebra is a Lie algebra, loop algebras have been used to obtain realizations of affine Kac-Moody Lie algebras and, more generally, extended affine Lie algebras.

   In this talk, we describe some methods and results to decide when two loop algebras are isomorphic. These methods include an interpretation of loop algebras as forms of untwisted loop algebras. This interpretation allows us to apply tools from Galois cohomology in the study of the isomorphism problem.

2. **Yuri Bahturin, Memorial University of Newfoundland, Canada and Moscow State University, Russia**

   INTRODUCTION TO LOCALLY FINITE LIE ALGEBRAS

   The aim of this talk is to give basic definitions and formulate some general results in the theory of locally finite Lie algebras. We will exhibit several characteristic examples and describe connections with locally finite groups and associative algebras.

3. **Alexander Baranov, University of Leicester, UK**

   SIMPLE LOCALLY FINITE LIE ALGEBRAS

Simple locally finite Lie algebras are subdivided into three classes: finitary, diagonal, and non-diagonal. Several characterizations of these classes will be given and various properties of the corresponding algebras will be discussed.

4. **Georgia Benkart, University of Wisconsin-Madison, USA**

   LIFE ON THE WEDGE (AND SUPERWEDGE)

   This talk will survey various constructions of Lie algebras and Lie superalgebras using exterior powers. Recent work on Lie (super)algebras graded by finite root systems and locally finite simple Lie (super)algebras will be featured. If time allows, some connections with the Tits construction of exceptional simple Lie superalgebras will be presented.

5. **Roger Bryant, University of Manchester Institute of Science and Technology, UK**

   FREE LIE ALGEBRAS AS MODULES FOR GROUPS

   I shall describe some recent results concerned with the module structure of a free Lie algebra under the action of a group. Let G be a group, K a field and V a finite-dimensional KG-module. Let $L(V)$ be the free Lie algebra over K which has V as a subspace and every basis of V as a free generating set. The action of each element of G on V extends to a Lie algebra automorphism of $L(V)$. Thus $L(V)$ becomes a KG-module, and each homogeneous component $L^n(V)$ is a KG-submodule, called the nth Lie power of V. We consider the problem of determining the modules $L^n(V)$ up to isomorphism.

6. **Ivan Dimitrov, Queen's University, Canada**

   IRREDUCIBLE WEIGHT MODULES OF $gl(\infty)$

7. **George Elliott, University of Toronto, Canada**

   CLASSIFICATION FUNCTORS

   A classification functor is a functor, which distinguishes isomorphism classes. (Not too many seem to exist.) Abstract classification functors can sometimes be constructed by means of an intertwining argument—based on working with inner (or generalized inner) automorphisms. What is more difficult is to find a realization of an abstract classifying category in terms of a concrete category in which the morphisms are set maps—not just equivalence classes of such.

8. **Helge Glöckner, Technical University of Darmstadt, Germany**

   DIRECT LIMITS OF LIE GROUPS, TOPOLOGICAL GROUPS AND TOPOLOGICAL SPACES

   An ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional Lie groups is called *strict* if each G_n is closed in G_{n+1}, and equipped with the induced topology. In fundamental investigations by L. Natarajan, E. Rodríguez-Carrington and J.A. Wolf (1991-94), certain conditions on such directed systems had been described which ensure that the direct limit exponential map is sufficiently well-behaved in order to serve as a chart around the identity for an infinite-dimensional Lie group structure on the direct limit group $G = \bigcup_n G_n$. In the first part of the talk, I'll describe an alternative approach which always allows G to be turned into a Lie group, without any extra conditions on the directed system. Instead of using the exponential map, tubular neighbourhoods are used to create compatible families of charts. As an application, in many cases countable-dimensional locally finite Lie algebras can be integrated to Lie groups.

In the second part of the talk, I'll report on work in progress concerning direct limits of infinite-dimensional Lie groups. I'll consider three typical classes of infinite-dimensional Lie groups which, algebraically, are direct limits of infinite-dimensional Lie groups: 1. countable weak direct products of Lie groups; 2. test function groups; 3. groups of compactly supported diffeomorphisms. Extending earlier work by N. Tatsuuma, H. Shimomura and T. Hirai (1998), I'll discuss the direct limit property of such Lie groups (which may be satisfied or not) in the categories of topological spaces, topological groups, smooth manifolds, and in the category of Lie groups.

9. **Dimitar Grantcharov, University of California-Riverside, USA, and University of Alberta, Canada**

ON THE STRUCTURE AND CHARACTERS OF WEIGHT MODULES OF LIE ALGEBRAS AND SUPERALGEBRAS

In this talk we will present a method of studying weight modules of Lie superalgebras $\mathfrak{g}$ of type I, and $\mathfrak{g} = \mathbf{W}_n$. The method is based on Mathieu's result that every simple weight $\mathfrak{g}$-module M with finite weight multiplicities is obtained from a highest weight module $L(\lambda)$ by a composition Ψ of a twist and localization. We study the properties of the twisted localization Ψ and relate a Jordan-Hölder series of a highest weight module X with a Jordan-Hölder series of the module $\Psi(X)$. As a main application of the method we reduce the problems of finding a $\mathfrak{g}_0$-composition series and a character formula for all simple weight modules with central character χ to the same problems for simple highest weight modules with the same central character. Some of our results are new already in the case of a classical reductive Lie algebra $\mathfrak{g}$.

10. **David Handelman, University of Ottawa, Canada**

CLASSIFICATION OF LOCALLY SEMISIMPLE ALGEBRAS

This is a survey talk. All algebras are associative. A *locally semisimple* algebra is a union of an increasing family of (finite dimensional) algebras over a field. If the field is algebraically closed, then Elliott's theorem asserts that the naturally ordered Grothendieck group with an additional datum is a complete invariant. The structure of the invariant itself is fairly well understood, completely if the algebra is simple; this is a consequence of Choquet theory and a result of Effros, Handelman & Shen.

When the underlying field is real closed, the classification of locally semisimple algebras is via a triple, corresponding to the two finite dimensional division algebras. If A denotes the algebra and $\mathrm{K}_0(A)$ is ordered Grothendieck group, then a complete invariant is given by the triple $\mathrm{K}_0(A) \rightarrow \mathrm{K}_0(A \otimes \mathbf{C}) \rightarrow \mathrm{K}_0(A \otimes \mathbf{H})$ as homomorphisms of ordered Abelian groups, together with an additional datum. However, the structure for the invariant, that is, the classification theory for the invariant itself is not completely understood (except when the algebra is simple).

If the underlying field is arbitrary (but perfect), a complete invariant for the algebras is known, but far from well understood. It extends the invariant of the real case, and involves all the finite dimensional division algebras that "appear" in the algebra.

Returning to the case that the underlying field be the complexes, there are analogous classification results for locally semisimple algebras with an action of a group, usually a locally representable action of a compact group. The invariants that result have close connections to random walks and limit ratio results.

11. **Otto Kegel, Freiburg University, Germany**

HIGHLY TRANSITIVE PERMUTATION GROUPS

12. **Felix Leinen, Mainz University, Germany**

GROUP ALGEBRAS OF SIMPLE LOCALLY FINITE GROUPS (*joint work with Orazio Puglisi*)

After an introduction to the various types of simple locally finite groups G we shall discuss the ideal lattice of their group algebra $\mathbb{F}G$ over a field $\mathbb{F}$ of characteristic zero. It was shown by A. E. Zalesskiĭ, that the structure of such an ideal lattice is intimately related to the asymptotic behaviour of the $\mathbb{F}$-representations of the finite subgroups of G.

The ideal lattice turns out to be quite sparse in many cases. On the other hand, some tricky situations remain unsettled, especially when G is *approximated diagonally* by finite alternating groups.

Therefore we shall also discuss the convexly indecomposable normalized positive definite class functions $\mathbb{C}G \longrightarrow \mathbb{C}$ for certain direct limits G of finite alternating groups. These functions can be viewed as analogues of complex characters for G.

13. **Karl-Hermann Neeb, Technical University of Darmstadt, Germany**

APPROXIMATING INFINITE-DIMENSIONAL LIE GROUPS BY LOCALLY FINITE ONES

In the representation theory of infinite-dimensional Lie groups the following phenomenon occurs in many interesting situations: One is interested in an infinite-dimensional Lie group G containing a directed union of finite-dimensional subgroups which either is dense or at least "determines" in a certain sense the representations one is interested in. This establishes an interested link between certain direct limits of finite-dimensional groups and certain groups of operators on Hilbert spaces. The approximation by the finite-dimensional groups is crucial to determine the topology of the large group, its central extensions and, to some extent, also its representations

14. **Erhard Neher, University of Ottawa, Canada**

LOCALLY FINITE ROOT SYSTEMS

Locally finite root systems are defined in analogy to the definition of a finite root system, except that the finiteness condition is replaced by local finiteness, i.e., the intersection of the root system with every finite-dimensional subspace is finite. A theory of locally finite root systems has recently been developed by Ottmar Loos and the speaker (Weyl groups, bases, classification, parabolic subsets, positive systems, weights). In the talk, the basic structure theory will be presented. We will also consider Lie superalgebras graded by locally finite root systems, and give a characterization of the locally finite ones.

15. **Ivan Penkov, University of California-Riverside, USA**

RECENT ADVANCES IN LOCALLY FINITE LIE ALGEBRAS AND SUPERALGEBRAS AND SOME UNSOLVED PROBLEMS

I will describe a class of semisimple locally finite Lie superalgebras admitting a root decomposition (classically semisimple locally finite Lie superalgebras) which contains the class of root reductive locally finite Lie algebras, and in particular $gl(\infty)$. Then I will describe all Cartan subalgebras of $gl(\infty)$. Finally, I will describe the current state of the theory of weight modules over root-reductive locally finite Lie algebras. Throughout the talk, I will state open problems.

16. **Arturo Pianzola, University of Alberta, Canada**

LOOP ALGEBRAS.

The (twisted) loop algebras of finite dimensional simple complex Lie algebras were first considered by V. Kac to provide concrete realizations of the affine Kac-Moody algebras. I will begin by describing a procedure that allows one to view loop algebras in general in terms of (algebraic) principal homogeneous spaces. Several examples will be given to illustrate this point (which is very natural and geometric in nature).

The bulk of the talk will be centred around joint work with B. Allison and S. Berman on iterated loop algebras and related applications to the study of Extended Affine Lie Algebras.

If time permits, I will briefly talk about some applications of loop algebras to the classification of conformal algebras.

17. **Pavel Shumyatski, University of Brasilia, Brazil**

POSITIVE LAWS IN FIXED POINTS

Let A be a finite group acting coprimely on a finite group G. It is well-known that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has strong influence over the structure of G. The best illustration for this phenomenon is the fact that if G admits a fixed-point-free automorphism of prime order then G is nilpotent and the nilpotency class of G is bounded by a function depending only on the order of the automorphism (Higman-Thompson Theorem). Thus we see that in certain situations restrictions on centralizers of coprime automorphisms result in very specific identities that hold in G. An interesting problem is to describe as many such situations as possible. Powerful Lie-theoretic results of Zel'manov provide us with very effective tools for dealing with the problem. Those tools are employed to prove the following theorem.

THEOREM. *Let q be a prime. Let A be an elementary Abelian group of order q^3 acting on a finite q'-group G in such a manner that $C_G(a)$ satisfies a positive law of degree n for any $a \in A^{\#}$. Then the entire group G satisfies a positive law of degree bounded by a function of n and q only.*

The above theorem depends on the classification of finite simple groups which seems to be necessary to reduce the theorem to the case that G is nilpotent. Once this is done, Lie-theoric methods come into play and have crucial effect.

There are examples showing that the theorem fails if the group A has order q^2.

18. **Helmut Strade, Hamburg University, Germany**

LOCALLY FINITE LIE ALGEBRAS OVER FIELDS OF ARBITRARY CHARACTERISTIC.

Many constructions of locally finite Lie algebras use direct limits of simple finite dimensional Lie algebras. In positive characteristic the Block-Wilson-Strade-Premet classification determines these simple Lie algebras in characteristic $p > 3$. The additional families (besides the classical ones) allow many more types of embeddings, and therefore give rise to new families of locally finite Lie algebras. All these algebras have not yet been investigated at all.

We shall give an overview on these and some other typical characteristic p phenomena.

19. **Joe Wolf, University of California-Berkeley, USA**

    DOUBLE FIBRATION TRANSFORM

    The double fibration transforms considered here, carry cohomology of holomorphic vector bundles to spaces of holomorphic functions, in a manner equivariant for the action of a semisimple Lie group. The best-known example is the complex Penrose transform. In the last year there has been a lot of progress on the general theory for the double fibration transform from holomorphic vector bundles on a flag domain. This talk is an indication of the current state of the theory.

20. **Alexander Zalesskii, University of East Anglia, UK**

    CLASSIFICATION OF SIMPLE INFINITE DIMENSIONAL LIE SUBALGEBRAS OF LOCALLY FINITE ASSOCIATIVE ALGEBRAS

    This is a joint work with A. A. Baranov and Yu. A. Bahturin. The main result is a kind of classification of simple Lie subalgebras in locally finite associative algebras over complex number field. In fact, we reduce the problem to the classification of simple associative algebras with involution as follows.

    Let L be a simple Lie subalgebras (under the bracket multiplication) of locally finite associative algebra A. Then either L is of finite dimension or there exists an associative algebra B with involution $*$ such that L is isomorphic to the commutator subalgebra $[U,U]$ of the Lie algebra U of $*$-skew symmetric elements of B (that is, $U = \{b \in B : b^* = -b\}$. Additionally, B has no non-zero proper $*$-stable ideal. In general B is not an envelope of L in A.

The Scientifi c Outcome of the Workshop

The main outcome of the workshop was the mutual recognition of the fact that the theory of locally finite Lie algebras is an interesting topic in mathematics with a whole number of important difficult problems concerning both the structure and the representation theories and having already a number of serious achievements. It became even more apparent that using methods of adjacent areas of locally finite associative algebras, locally finite groups and infinite-dimensional Lie groups is vital for the further progress in this area.

One of the surprising discoveries learnt by the most of participants already during the workshop became the connection between simple Lie subalgebras of locally finite associative algebras and the root graded algebras. As noted by Bruce Allison, "It was interesting for me to learn of the connection between simple locally finite Lie algebras and root graded Lie algebras. I was surprised and interested to find out that there is a connection between the work of yourself [Bahturin], Zalesski and Baranov and the work of myself, Georgia and Yun Gao on BC graded algebras". Alexander Zalesskii writes: "I learned two things: connections between our work and Allison-Benkart-Gao, and connections between my work with I. Suprunenko on representations of algebraic groups in prime characteristic whose all weights are of dimension 1 and Benkart's work (with coauthors) on the similar problem for the highest weights infinite dimensional representations of Lie algebras in char 0 whose all weights are one dimensional. These links may lead to further interesting observations and research as I expect".

Another nice feature of the workshop was that in some talks a unification attempt was made for several theories mentioned above. This was especially noticeable in the talk of Otto Kegel who stood at the origin of many directions in Algebra. George Elliott, one of the key scholars in Operator Algebras, gave a talk about general classification principles, which have been or could be used in every theory, including locally finite associative and Lie algebras.

The results reported by the participants can be grouped as follows:

1. **Results on the classification of locally finite Lie algebras (and superalgebras)**.

The workshop clearly showed that the classification theory of locally finite simple associative and Lie algebras has reached a very mature state, and that many deep results on representations of locally finite simple algebras are available. The relevant talks are Baranov, Bahturin, Benkart, Penkov, Strade, Neher, Zalesskii. The classification results are the most exhaustive in the case of finitary Lie algebras of operators in infinite-dimensional space (Baranov, with Strade in the case of positive characteristic) and direct limits of simple finite-dimensional Lie algebras (Baranov-Zhilinskii). In the case of simple diagonal Lie algebras, which is the same as Lie subalgebras of locally finite associative algebras, the classification by Bahturin-Baranov-Zalesskii is actually a reduction to the classification of involution simple locally finite associative algebras. Some of the classification-like results in the the case of algebras over positive characteristic fields have been reported by H. Strade.

The root graded algebra approach to locally finite Lie algebras was elaborated in the talk of Georgia Benkart.

The results on the classification of locally finite root systems of Lie algebras and Lie superalgebras were described by Erhard Neher.

The main results of Penkov's was that any countably dimensional semisimple locally finite Lie superalgebra which admits a generalized root decomposition and is generated by the generalized root spaces is isomorphic to a direct sum of classical or exceptional simple Lie superalgebras and copies of $\mathfrak{sl}(m,\infty)$, $\mathfrak{sl}(\infty,\infty)$, $\mathfrak{osp}(m,\infty)$,$\mathfrak{osp}(\infty,\infty)$, $\mathfrak{osp}(\infty,2k)$, $\mathfrak{sp}(\infty)$ and $\mathfrak{sq}(\infty)$.

As already mentioned, a general approach to the classification problems was suggested in the lecture of George Elliott.

2. **Results on representations of locally finite Lie algebras and superalgebras as well as infinite dimensional Lie groups.**

   The relevant talks were roughly those by Penkov, Dimitrov, Benkart.

   The main result of Ivan Penkov's talk was a description of some invariants of weight modules over $\mathfrak{gl}(\infty)$. In particular I showed an integrable irreducible module with finite shadow and no highest weight.

   Ivan Dimitrov gave a classification of all irreducible weight modules of $\mathfrak{gl}(\infty)$ with finite dimensional weight spaces.

   Georgia Benkart has constructed some irreducible modules over diagonal Lie algebras using their presentation as root graded algebras.

   It was pointed out that for many important classes of locally finite Lie algebras only the most basic representation theory is available, including module of the highest weight. But these algebras very often have only roots but no root vectors, or even no root systems at all. The question was asked what can be the substitute of these notions in constructing the representation theory of such Lie algebras. The simplest example is a Lie algebra $L=\mathfrak{gl}(2^\infty)$ obtained as the direct limit of Lie algebras $L_n=\mathfrak{gl}(2^n)$, with the structure embeddings $X\to\mathrm{diag}\{X,X\}$, $X\in L_n$.

3. **Results on relations with associative algebras**

   Relevant talks are due to Baranov, Handelman, Zalesskii.

   Alexander Baranov presented some of his results on the diagonal locally finite Lie algebras, in particular his characterization of algebras, which satisfy "Ado's Theorem" about the possibility of embedding a locally finite Lie algebra into a locally finite associative algebra. It looks now that apart from locally finite Lie algebras, which have root decomposition, this is the best explored class with powerful methods and still a number of intriguing problems to solve. The diagonal Lie algebras have been touched upon also in the talks of Yuri Bahturin, Georgia Benkart, and Alexander Zalesskii.

   David Handelman gave a very illuminating talk, by invitation of the organizing committee, on the history and contemporary state of the classification theory of direct limits of semisimple finite-dimensional algebras, with or without additional structure, such as the action by groups. This talk is very important because, as mentioned, some classification problems of locally finite Lie algebras turned out to be equivalent to those of associative algebras with involution.

4. **Results on the limits of Lie groups**

   Relevant talks are due to Neeb and Gloeckner.

   As noted by Karl-Hermann Neeb, almost nothing is known, in a systematic fashion, on representations of infinite-dimensional Lie groups. There are many interesting Lie groups G whose Lie algebra $\mathfrak{g}$ contains a dense locally finite (simple) Lie algebra $\mathfrak{f}$, so that (unitary) representations of G automatically provide representations of $\mathfrak{f}$, in general by unbounded operators. It is an important problem to establish a link between the algebraic theory and the analytic (= Lie group) theory by determining systematically which representations of the Lie locally finite Lie algebra $\mathfrak{f}$ do arise from a representations of the global group G. Of course this depends heavily on the group G, so that one has to consider a subclass of representations of $\mathfrak{f}$ defined by the group G which should behave better than the "uncontrolled" representations of $\mathfrak{f}$.

   This method has been applied in the work of G.I. Olshanski in the context of topological groups G and in the context of representations of Lie groups by J. Wolf and his coauthors and by K.-H. Neeb to direct limits of highest weight representations of complex simple Lie groups. These special instances show that there often is a rich structure behind the interplay between the global group G and the Lie algebra $\mathfrak{f}$. Typical examples where these techniques have not yet explored and seem very promising are "classical" Lie subgroups of unit groups of AF C^*-algebras.

   Another interesting and very systematic talk on direct limits of finite or infinite-dimensional Lie groups was given by Helge Gloeckner, who showed a very important fact that in many cases countably dimensional locally finite-dimensional Lie algebras can be integrated into Lie groups. This of course immediately raises the questions of using this interaction to the benefit of both Lie groups and locally finite Lie algebras.

5. **Relations with non-locally finite Lie algebras**

   Relevant talks are due to Bruce Allison and Arturo Pianzola.

   These talks on loop algebras gave the participants a flavour of approaches and methods used in a neighbouring area of Infinite-Dimensional Lie algebras. Both authors mentioned that the intersection of these two classes is not very large but no doubt the cohomological methods suggested in their joint research are of great importance for locally finite Lie algebras.

6. **Relations with Group Theory**

   Relevant talks are due to Kegel, Leinen and Shumyatski.

   The lecture of Otto Kegel gave the participants a number of interesting examples of locally finite objects in groups, rings and Lie algebras and showed their connection to each other both mathematically and historically.

   The talk of Felix Leinen contained a lot of information about various classes of locally finite groups and their group rings. In the whole number of cases the lattice of ideals of such group algebras is extremely poor. Finding wide classes of locally finite Lie algebras whose universal algebras have the same property (no proper nonzero two sided ideals except the augmentation ideal) would be very interesting and the similarity of approaches via Zalesskii's inductive limit of representations gives hope that this can actually be done. The progress achieved in the theory of locally finite groups is very stimulating because this area famous by many difficult open problems.

   Pavel Shumyatski pointed out that using powerful results from Infinite-Dimensional Lie Theory has produced a number of interesting theorems in abstract group theory.

   In the talk of Roger Bryant the author discussed the Lie powers of representations of finite groups.

7. **Talks about phenomena in the finite dimensional case which can be useful in the direct limit case**

   These interesting talks have been delivered by Joe Wolf and Dimitar Grantcharov

General Comments

- " ...I enjoyed the conference very much. It gave a stimulating picture of a lively and developing area..."
- "...Overall, I was very pleased to learn about the field of locally finite Lie algebras. I was aware that there was activity in the field, but as a nonexpert this conference gave me an otherwise unavailable opportunity to find out the state of the area..."
- " ...Many thanks for a very nice, enjoyable and stimulating meeting! There is much I learned and much I'll have to look up.. ..."
- " ...Thank you so much for the wonderful conference. I got a lot out of it.. ..."

List of Participants

Allison, Bruce (University of Alberta)
Bahturin, Yuri (Memorial University)
Baranov, Alexander (University of Leicester)
Benkart, Georgia (University of Wisconsin - Madison)
Bryant, Roger (University of Manchester Institute of Science and Technology)
Dimitrov, Ivan (Queen's University)
Elliott, George (University of Toronto)
Glockner, Helge (Technische Universitat Darmstadt)
Grantcharov, Dimitar (University of Alberta)
Handelman, David (University of Ottawa)
Kegel, Otto (Freiburg University)
Leinen , Felix (Johannes Gutenberg-Universität Mainz)
Neeb, Karl-Hermann (Darmstadt University)
Neher, Erhard (University of Ottawa)
Penkov, Ivan (University of California Riverside)
Pianzola, Arturo (University of Alberta)
Shumyatsky, Pavel (University of Brasilia)
Strade, Helmut (University of Hamburg)
Wolf, Joseph (University of California - Berkeley)
Zalesskii, Alexander (University of East Anglia)

Chapter 27

Regularization in Statistics (03w5089)

September 6–11, 2003

Organizer(s): Ivan Mizera (University of Alberta), Roger Koenker (University Illinois-Urbana)

Overview

An annoying feature of the real world is that it often expects computational solutions to problems whose mathematical formulation clearly indicates that such efforts should be hopeless. Even worse, such situations seem to be the rule rather than the exception.

Applied mathematicians, especially those with interests in partial differential equations, recognized this fact early on. Following Hadamard's (1902) formulation, the idioms "ill-posed", followed by "inverse problem", have gradually become a staple of mathematical jargon. Not satisfied by simply naming the problem, a few intrepid souls began to look for approaches to deal with it. This initial phase culminated in the work of Tikhonov (1963) and his school, and independently with the work of Phillips (1962). Their numerous followers have continued to cultivate the regularization approach.

As often happens with problems urgently needing solutions, there were parallel efforts throughout the archipelago of mathematical thought. The recipe of Whittaker (1921) for smoothing mortality tables in actuarial science was noticed by Schoenberg (1964) and by Parzen (1961), and led subsequent authors toward nonparametric function estimation in reproducing kernel Hilbert spaces—allowing prior notions about the smoothness of target functions to be expressed as roughness penalties in the regularization framework. A broader perspective brings the recognition of the fundamental role regularization has played in modern statistics: Stein's (1956) work in decision theory on admissibility of least squares methods demonstrated the value of so-called shrinkage procedures. And Bayesian methods were quick to be extended and reinterpreted in this light. Recent computational advances have exerted a profound influence over the development of these ideas.

Thus it can be tentatively concluded that researchers interested in regularization recruit from three—at least—main groups: mathematicians with applied flavor, statisticians and similar individuals concerned with data analysis, and computing scientists. The crucial fact, however, is that, modulo technicalities and discipline-specific jargon, all of the above observe the fundamental principle. In the simplest instance the regularization paradigm seeks an acceptable solution to the linear system

$$Ax = b. \tag{27.1}$$

The inverse of A exists, but A is known to be ill-conditioned. Optimization of the penalized quadratic form,

$$\min_x \|Ax - b\|^2 + \lambda\phi(x). \tag{27.2}$$

may be considered. Again, in the simplest instance $\phi(x)$ may be taken to be $\|x\|^2$, leading to solutions of the form,

$$x = (A'A + \lambda I)^{-1} A'b. \tag{27.3}$$

It should to be stressed that many variations of this theme are possible. The entities may be infinite-dimensional, necessitating operators not matrices; the quadratic measure of lack-of-fit may be altered, and the penalty $\phi(x)$ may take many forms in an effort to accurately represent auxiliary information about solutions of the problem that was not explicitly incorporated into the original formulation. In the end, it is also crucial to choose some value of the regularization parameter λ and finally find efficient methods for solving (27.2). In statistics the leading problems of this type involve nonparametric regression and density estimation, solution of inverse problems leading to integral equations, and deconvolution, but a vast array of other problems from imaging, machine learning, and other regions of applied mathematics abound.

Emerging problems have led to novel modifications of the regularization approach. The scope has widened considerably, while some old problems still are not solved completely. In image processing, for example, edge detection and image segmentation objectives have stimulated work on total variation forms of the $\phi(x)$ regularization penalty. A common denominator with, say, classification methods used in machine learning, or nonparametric regression problems in statistics is nonlinearity; the latter brings challenges not only in computing, but above all in selection of regularization parameter and also in the subsequent theoretical interpretation. Advances in numerical analysis have played a crucial role in these developments.

These considerations taken together suggested that the time was ripe for a meeting of specialists from a broad range of fields to discuss recent developments in regularization. Our ambition was not to organize yet another meeting of a well established network of people whose work was already mutually familiar, but to bring together as diverse body of participants as possible, while maintaining a statistical core of the meeting—an understandable constraint given our professional orientation. An approach like this inevitably brings certain risks: rather than relying on people from our own circles, people we know well from regular meetings and have a fairly precise estimates what to expect of, we actively sought out new personalities—people from other circles of interests, with their own priorities and value systems. In particular, it was by no means clear whether the reward our meeting may have from them would be at least comparable to that it would have for us, statisticians—and therefore carried an inherent risk that we would be not be able to attract a wide enough circle of participants to realize our objectives.

The Workshop

The outcome, however, exceeded our most optimistic expectations. Despite some unlucky coincidences with other professional meetings, which could not be anticipated, we were fortunate to assemble—quoting from a recent email—"a fabulous collection of participants." The workshop brought together 34 participants from statistics and allied fields, including imaging, machine learning, numerical analysis, and applied mathematics. Departmental affiliations of the participants included mathematicians of both "pure" and "applied" flavor, computer scientists and electrical engineers, as well as a broad collection of statisticians from mathematics, statistics, biostatistics, economics, and psychology departments.

Inspired partly by Oberwolfach traditions, we decided to deal with program issues quite informally. The submission of abstracts and titles of talks was voluntary. Not everybody was automatically assumed to present a talk; this choice was left to participants themselves—and by no means could it be said that those opting for a discussant rôle remained silent. The decision to compose the program "on the fly" brought us some anxious moments at the very beginning of the workshop, but in the broader perspective it was instrumental in realizing our vision of a meeting in which talks are not only delivered but also listened to, a workshop as a community, not a railway station where people and trains come, stop, and go (we pursued this objective even at the cost of losing several desirable speakers).

Although some of the participants knew each other before—one could notice a group of statisticians with interests close to smoothing and functional data analysis, and a group of mathematicians working differential-equations approach to image processing—it was fascinating was to see people from different areas interact; speakers were often not addressing their peers, but rather the participants from other groups.

Day first: Sunday

The workshop started by a short opening address of Robert Moody, FRSC, the scientific director of the BIRS. This was immediately followed by the scientific program.

The opening talk by Rudy Beran "Regularized Bayes fits to incomplete unbalanced multi-way layouts" provided an ideal starting point for the workshop. Beran outlined how low-risk adaptive Bayes fits to large discrete multi-way ANOVA layouts provide fits whose risk converges to the minimum risk attainable over the candidate class, as the number of factor levels tends to infinity. Both ordinal and nominal factor levels, possibly unbalanced or incomplete, were considered. He also illustrated in a broader context how modern statistical computing provides a technological environment for advances in statistical thought beyond those supported by probability theory. Since the method employed are equivalent to regularization by penalized least squares, with smallest estimated risk used to select the penalty parameter, the talk brought everyone together to the core of the regularization problem.

The next talk "Learning with determined inputs and generalized Shannon-Nyquist sampling", by Stephen Smale, discussed extensions of classical Shannon-Nyquist sampling in the context of regularization and machine learning. It was, for most participants, a new perspective. Smale's traditional chalk-and-blackboard style created an immediate rapport with the predominantly statistical audience, and a vigorous discussion ensued. We were delighted to see how quickly participants adapted to this informal style, which set a good "workshop" atmosphere for the subsequent sessions.

The morning session concluded with the talk "Regularization in statistics: metamorphoses and leftovers" of one of organizers, Ivan Mizera. The talk essentially aimed at outlining some issues that were to follow: trying, in the simplest regularization setting of nonparametric regression to indicate similarities and differences between various possible approaches. Topics addressed included the difference between dense (functional, gridded) and scattered (point) data formulation; various motivations for nonlinearity and how they relate to linear problems; certain bivariate aspects - the role of the domain and subsequent implications for the numerical developments; the classical desiderata for smoothing versus emerging alternative objectives.

The afternoon session was devoted to several areas of application. Enno Mammen in his talk "Optimal estimation in additive regression models" discussed optimal estimation of a additive nonparametric components. He discussed the additive regression model and showed that up to the first order an additive component can be estimated as if the other components were known, proving this claim for kernel smoothers, local polynomials, smoothing splines and orthogonal series estimators.

A glimpse into econometrics was brought by Joel Horowitz in his talk on "Nonparametric estimation in the presence of instrumental variables". He suggested two nonparametric approaches, based on kernel methods and orthogonal series, respectively, to estimating regression functions involving instrumental variables. For the first time in this class of problems optimal convergence rates were derived, and showed that they are attained by particular estimators.

Paul Speckman's talk on "Adaptive function estimation using smooth stochastic variability priors" considered Bayesian nonparametric regression, with priors on the unknown function corresponding to smoothing with L-splines, where the penalty term also includes a weight function that is optimized to give a locally adaptive smoother. He showed that appropriate priors can be constructed that are similar to stochastic volatility models in finance indicating their usefulness in nonparametric regression and also density estimation.

Finally, C. Samuel Kou in his talk "Statistical analysis of single molecule experiments in chemistry" spoke about statistical challenges connected with the recent technological advances allowing scientists to follow biochemical process on a single molecule basis.

Day second: Monday

The second day began with the talk of Emmanuel Candés on "Chirplets: Multiscale detection and recovery of chirps". He considered the problem of detecting and recovering chirps—signals that are neither smoothly varying nor stationary but rather exhibit rapid oscillations and rapid changes in their frequency content. Such behaviour is quite different than the traditionally notions of smoothness and homogeneity for noisy data. Building on recent advances in computational harmonic analysis, he designed libraries of multiscale chirplets, and introduced detection strategies that are more sensitive than existing feature detectors. Structured algorithms that exploit information in the chirplet dictionary are used to chain chirplets together adaptively so as to form chirps with polygonal instantaneous frequency; these structured algorithms are so sensitive that they allow to detect signals whenever their strength makes them detectable by any method, no matter how intractable. An applications to the detection of gravitational waves was discussed.

The two other morning talks were given by mathematicians who have made significant contributions to the rigorous analysis of total-variation methods for imaging. Antonin Chambolle in "Total variation minimization–an algorithm for total variation minimization and applications" presented an algorithm for minimizing total variation under a quadratic constraint, based on the dual formulation and discussed its applications to image processing.

Otmar Scherzer in "Tube methods for bounded variation regularization" used statistical modelling for developing regularization models for de-noising, de-jittering, and de-blurring applications in image processing. Inspired by connections to the taut-string algorithm for total variation optimization in one dimension, analogous methods were proposed for image processing in dimension two. This provided a nice link to more statistically motivated discussion of taut string methods by Arne Kovac and Laurie Davies later in the conference.

In the afternoon, the second of the organizers, Roger Koenker spoke about "Total variation regularization for noisy, scattered data." He described two variants of total variation regularization for bivariate function estimation: piecewise constant functions on Voronoi tessellations, or voronograms, and piecewise linear functions on Delaunay tessellations, or triograms. The accent was on efficient computing using primal-dual interior point methods and the sparsity of the underlying linear algebra enabling quite large problems to be solved. This theme was carried forward by the computational character of the other afternoon talks.

Arne Kovac spoke on "Taut strings and modality", and addressed the vital question of alternative objectives. The usual goal in nonparametric regression and density estimation is to specify a function f that adequately represent the data, but do not contain spurious local extremes. Multiresolution versions of the taut string method were proposed to generate a sequence of functions with increasing number of local extreme values; the close connection to total variation methods was also emphasized.

Robert Nowak spoke on "Near minimax optimal learning with dyadic classification trees." He reported on a family of computationally practical classifiers that converge to the optimal (Bayes error) classifier at near-minimax optimal rates for a variety of distributions. The classifiers are based on dyadic classification trees, which involve adaptively pruned partitions of the feature space, whose key aspect is their spatial adaptivity, enabling local (rather than global) fitting of the decision boundary. Their risk analysis involves a spatial decomposition of the usual concentration inequalities, leading to a spatially adaptive, data-dependent pruning criterion. The classifiers are practical and the same time provide rate of convergence within a logarithmic factor of the minimax optimal rate.

Finally, Sylvain Sardy spoke about "L_1 penalized likelihood nonparametric function estimation" and discussed the relationship between Laplace Markov random fields and total variation regularization for regression, density estimation and linear inverse problems. He proposed a relaxation algorithm to solve a dual optimization problem and considered ways of choosing the regularization parameter λ—contributing thus to the final characterization of Tuesday as a "total-variation day".

On Monday evening, we decided to hold Round-table discussion on λ selection. How salient this topic is is illustrated by the fact that the question "How do you select λ?" appeared in the discussion to nearly every talk at the workshop. The discussion was led in a very exquisite manner by Grace Wahba. Since we consider this event one of the high moments of the meeting, we decided to capture its elusive character on the video recording. Once available on web, we believe that there will be quite demand for this *coup de theatre*.

Day third: Tuesday

Chong Gu in his talk "Model Diagnostics for Penalized Likelihood Estimates" spoke about functional ANOVA decompositions that can be incorporated into multivariate function estimation through penalized likelihood methods. He presented some simple diagnostics for the "testing" of selected model terms in the decomposition; the elimination of practically insignificant terms generally enhances the interpretability of the estimates, and sometimes may also have inferential implications. The diagnostics were illustrated in the settings of regression, density estimation, and hazard model estimation.

Dennis Cox's talk "Functional-data ANOVA via randomization based multiple comparisons procedure" considered an approach to a functional-data ANOVA using univariate ANOVA methods pointwise on a grid and applying the randomization based multiple comparisons method of Westfall and Young to obtain a desired significance level. This method is relatively straightforward and gives very interpretable results (regions in the space of the independent variable where there are significant differences), but there have been concerns that there are potential problems. Cox conjectured that the Westfall and Young procedure in fact overcomes this problem and as the grid becomes finer, the corrected p-values converge to an approximately continuous function and gave some motivation and numerical examples in support of this conjecture.

"Bayesian logic regression" by Charles Kooperberg addressed fitting regression models of single nucleiotide polymorphism data: using many, essentially binary, covariates. MCMC is used on the class of regression models of interest, and rather than summarizing all models by, say, a mean all the MCMC models are summarized in a qualitative way.

Finally, in the talk "From Data to Differential Equations Differential equations" Jim Ramsay argued that regularization penalties could be adapted to differential equations representing the underlying processes generating observed functional data, and as such offer a number of potential advantages over conventional parametric or basis expansion models. They explicitly model the behaviour of derivatives, and derivative estimates based on them are often superior to those derived from conventional data smoothers; they have the capacity to model curve-to-curve variation as well as within subject variability and the known structural features can be built into them more easily than is usually the case for conventional functional models. Finally they offer a wider range of ways to introduce stochastic behaviour into models. Some illustrations of the performance of these methods were given from process control in chemical engineering and for medical data on treatment regimes for lupus.

Tuesday morning underscored the remarkable achievements of the classical, quadratic, Hilbert-space based approach to regularization in statistics; Tuesday afternoon was originally planned to be free. Adverse weather, however, altered this plan, and the program was adaptively restructured to shift the free afternoon on Wednesday, in the hope of improved weather.

The afternoon session began with Selim Esedoglu's talk on "Decomposition of images by the anisotropic Rudin-Osher-Fatemi model." He reported on generalizations of total variation based image de-noising model of Rudin, Osher, and Fatemi, designed to privilege certain edge directions. He consider the resulting anisotropic energies and studied properties of their minimizers.

Frits Ruymgaart's talk "Improving regression function estimators in indirect models" explained that the traditional estimators of linear functionals of the regression function in general, and of their Fourier coefficients in particular, are not asymptotically efficient in the sense of LeCam-Hájek-van der Vaart, except, possibly, when the error distribution is normal. In particular, when repeated measurements are available, an improvement procedure can be carried out for error distributions with heavy tails, starting with a preliminary estimator based on linear combinations of the order statistics in the subsamples.

Finally, Curt Vogel spoke on "A Wavefront reconstruction problem in adaptive optics." He reviewed the basic concepts behind the adaptive optics, which can dramatically improve the resolution of ground-based optical telescopes. He described in some detail the computation of the wavefront reconstructor, the mapping from sensor measurements to mirror deformations, and highlighted its connection to statistical regularization.

The evening was devoted to a social program event—the song recital of a mezzosoprano Kathryn Whitney, a native of Calgary recently elected Creative Arts Fellow in Music at Wolfson College of the University of Oxford. The location of BIRS within Banff Centre greatly facilitates the organization of events like this. BIRS manager Andrea Lundquist was very instrumental in making the event possible; we owe the Department of Music our deepest thanks for providing the venue and technical support. We were glad to invite to this event the members of the focused research group on the "Arithmetic of fundamental groups ", with whom we shared BIRS, and were delighted to see that they enjoyed the concert with us.

Day fourth: Wednesday

The morning session started with an illuminating overview "The method of moments and statistical computation", by Gene Golub. The talk brought out many stimulating ideas related to the fundamental issues of implementation of every single method mentioned in the workshop—computational linear algebra.

Werner Stuetzle then spoke about "Spline smoothing on surfaces", presenting a method for estimating functions on topologically and/or geometrically complex surfaces from possibly noisy observations. An extension of spline smoothing, using a finite element method. The results of an experiment comparing finite element approximations to exact smoothing splines on the sphere were shown, and examples suggesting that generalized cross-validation is an effective way of determining the optimal degree of smoothing for function estimation on surfaces were given.

The rest of the morning belonged to another two representatives of the vital UCLA group. Jianhong Jackie Shen spoke about "A young mathematician's reflection on vision: Illposedness and regularizations" giving an overview of the topological, geometric, and statistical/Bayesian regularization techniques of recent work on visual perception.

Finally, Tony Chan in "Geometric and total variation regularization in inverse problems" reviewed recent advances in the area of inverse problems in which one wishes to recover functions that are piecewise smooth separated by lower dimensional interfaces. For these problems, it is important to choose a regularization technique that will respect and preserve the discontinuities of the function values, as well as control the geometric regularity of the interfaces. Notable examples of this class of regularization include minimizing the total variation of the function and minimizing the surface area of the interfaces; applications include image restoration and segmentation, elliptic inverse problems, and medical image tomography problems.

The afternoon was free—and fortunately the weather improved, so participants could also enjoy the marvellous Rocky Mountain surrounding of the Banff Centre.

Day fifth and last: Thursday

In the first morning talk on "Oracle inequalities for regularized estimators", Sara van de Geer emphasized the special role of the L_1 penalty. She investigated least squares estimators with general penalties, using smoothing splines as special case and examined which penalties achieve an oracle inequality for the estimator. The latter results showed that the L_1 penalty adapts to both the smoothness as well as to possibly non-quadratic margin behaviour.

Laurie Davies talk "Approximating data" considered the basic idea that a model can be taken as an adequate approximation for a data set if typical samples generated under the model share the essential features of the data set. This somewhat loose formulation was made precise and related problems of topologies in data analysis and inference were discussed. Examples of the idea of data approximation involve nonparametric regression, densities and financial data.

The workshop concluded by the talk of Grace Wahba on "The multicategory support vector machine and the polychotomous penalized likelihood estimate". She described two modern methods for statistical model building and classification, penalized likelihood methods and support vector machines, both of which are obtained as solutions to optimization problems in reproducing kernel Hilbert spaces. Two category support vector machines are very well known, while the multi-category ones were introduced in the talk: they include modifications for unequal misclassification costs and unrepresentative training sets, is new. The applications of each method were described, in a demographic study and meteorology.

Final thoughts

The talks spanned a broad spectrum of both theoretical and applied work on regularization. The enthusiastic response of the participants, including, but not limited to their emails after the conference, fully justified our initial confidence in the timeliness of the workshop and its success. In particular, we are confident that the workshop accomplished its main objective of cross-fertilization. Some representatives of the L_2 approach expressed their interest by L_1 techniques, and conversely, some achievements in the classical setting provided a great inspiration for new ramifications. Problems arising in applications provided a better focus for theoreticians, and, on the other hand, applied people assured the former that theoretical insights are much needed and appreciated. Everyone learnt valuable lessons about numerical methods, about relevant mathematical techniques, and about areas of application.

To better document the workshop in a publicly accessible place, and also for the future reference, we constructed the follow-up webpages at

`http://www.stat.ualberta.ca/ mizera/confer.html#banff`

containing documentation about program, abstracts, social events, and a fairly complete collection of transparencies for talks.

A full evaluation of the success of the workshop will certainly be only possible after some time. But even given our current perspective, it may be of some interest to note that under the auspices of the special semester devoted to inverse problems, at the Institute for Pure and Applied Mathematics of the University of California, Los Angeles, a special, previously unplanned, two-day workshop on statistical aspects of regularization was organized in November 2003. We would like to believe that the inspiration came partly from our workshop.

We are indebted to all participants; from those who helped us beyond the usual standard and thus deserve special thanks, we would like to especially thank Rudy Beran, Tony Chan, Andrew Conn, Steve Portnoy, Jim Ramsay, and Grace Wahba.

We can hardly express our thanks to the BIRS for the opportunity to organize the workshop. The scientific director, Robert Moody, was very supportive from the very beginning; but also substantially helped us with technicalities on the site, which included his introduction to the BIRS facilities on the eve of the workshop. Amanda Kanuka flawlessly handled the correspondence, and Brent Kearney assured that the excellent technical facilities worked smoothly. Last but not least, it was a great pleasure to work with the BIRS manager Andrea Lundquist.

List of Participants

Beran, Rudolf (University of California, Davis)
Candes, Emmanuel (California Institute of Technology)
Chambolle , Antonin (Universite de Paris-Dauphine)
Chan, Tony (University of California Los Angeles)
Conn, Andrew R. (Thomas J. Watson Research Center)
Cox, Dennis (Rice University)
Davies, P. Laurie (University of Essen)
Esedoglu, Selim (University of California - Los Angeles)
Golub, Gene (Stanford University)
Gu, Chong (Purdue University)
He, Xuming (University of Illinois at Urbana-Champaign)
Horowitz, Joel L. (Northwestern University)
Koenker, Roger (University of Illinois, Urbana-Champaign)
Kooperberg, Charles (Fred Hutchinson Cancer Research Center)
Kou, Samuel (Harvard University)
Kovac, Arne (University of Essen)
Mammen, Enno (Universitat Heidelberg)
Meise, Monika (University of Essen)
Mizera, Ivan (University of Alberta)
Nowak, Robert (University of Wisconsin-Madison)
Oh, Hee-Seok (University of Alberta)
Portnoy, Steve (University of Illinois at Urbana-Champaign)
Ramsay, Jim (McGill University)
Ramsay, Tim (University of Ottawa)
Ruymgaart, Frits (Texas Tech University)
Sardy, Sylvain (Swiss Federal Institute of Technology)
Scherzer, Otmar (University of Innsbruck)
Shen, Jianhong (University of Minnesota)
Smale, Stephen (Toyota Technological Institute at Chicago)
Speckman, Paul (University of Missouri)
Stuetzle, Werner (University of Washington)

Vogel, Curtis (Montana State University)
Wahba, Grace (University of Wisconsin-Madison)
van de Geer, Sara (University of Leiden)

Chapter 28

Topology in and Around Dimension Three (03w5068)

September 13–18, 2003

Organizer(s): Steven Boyer (Universit´e du Qu´ebec à Montr´eal), Martin Scharlemann (University of California, Santa Barbara), Abigail Thompson (University of California, Davis)

Goals of the workshop

The study of 3-dimensional manifolds from a topological and geometrical point of view is a subject rich in technique, application, and connections with other areas of mathematics. Research is driven by various open problems of a foundational nature. The Poincaré conjecture and the more general geometrization conjecture of Thurston are the best known. Many individuals and groups have worked on these problems and their approaches have tended to be highly specific. Distinct sub-fields have arisen within 3-manifold theory distinguished not only by techniques and strategies, but also by the background mathematical culture needed to assimilate their methods. Thus, though at one level the goal of the workshop was to provide a forum for the examination of the current state of the field, at another, it was to bring together researchers from the various subfields to share their particular expertise with the other participants. Also included were a handful of researchers whose work focused on other areas, but had a relevance for 3-manifold topology.

An overview of the areas covered in the workshop

One of the main concerns of 3-manifold topology is the development of useful ways to describe them. For instance as geometric objects, as Dehn fillings, or by Heegaard splittings. This was reflected in the topics broached at the workshop.

Geometric structures on 3-manifolds

For over a quarter century, Thurston's vision of 3-manifolds as geometric objects has pervaded much of the thinking about these spaces. His geometrization conjecture [Th1], [Th2] provides an explicit description of the building blocks for compact 3-manifolds and, more generally, compact 3-orbifolds. Its verification has been a major focus of research and has spurred the enhancement of existing methods as well as the development new ones. Beyond that, the geometric viewpoint has led to the introduction of arithmetic, analytic, algebro-geometric, and other techniques into the subject (e.g. [MR], [P1,2,3], [CS1]).

The geometrization conjecture for manifolds contends that a compact 3-manifold admits a geometric decomposition. This means that it can be cut open along an essentially canonical family of surfaces of non-negative Euler characteristic in such a way that each of the remaining pieces admits a complete, locally homogeneous Riemannian metric. Thus the pieces admit geometric structures based on one of the eight 3-dimensional geometries [Sc] of which the spherical, Euclidean and hyperbolic geometries are the best known. To date, the conjecture has been verified in many cases and we can be quite specific in describing a connected, irreducible manifold for which it is not yet known. It is closed, connected, orientable, contains no essential surfaces, and its fundamental group does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The conjecture contends that such a manifold is hyperbolic if it has an infinite fundamental group with trivial centre, otherwise it is the quotient of the 3-sphere by a finite group of isometries which acts freely.

Quite recently, G. Perelman has made important advances in theory of Ricci flow on 3-manifolds [P1], [P2], [P3] which he claims leads to a proof of Thurston's conjecture. This has yet to be verified, but it is clear that his methods will have a significant impact on 3-dimensional topology. At the workshop, Ian Agol showed how they could be used to improve estimates on the minimal volume problem for hyperbolic 3-manifolds. Many of the other talks were related to geometrization in one way or other, as will be detailed in the following sections.

Virtual properties of 3-manifolds

A 3-manifold is called *Haken* if it is irreducible and contains a 2-sided incompressible surface. Waldhausen's *virtual Haken conjecture* states that a compact, connected, orientable, irreducible 3-manifold M with infinite fundamental group is finitely covered by a Haken manifold. Stronger versions state that a finite cover of M with positive first Betti number can be found (the *virtually positive first Betti number conjecture*) and in most cases, the cover can be taken to have an arbitrarily large first Betti number (the *virtually infinite first Betti number conjecture*). It is known that a positive solution to Waldhausen's conjecture would imply that the geometrization conjecture holds for irreducible 3-manifolds except, perhaps, for those with a finite fundamental group (note that these possible exceptions include homotopy 3-spheres) [GMT], [Ga1], [Ga2], [CJ]. A positive solution to Waldhausen's conjecture would have other important consequences in 3-manifold topology as well: homotopy equivalences between manifolds to which the conjecture applies are homotopic to homeomorphisms, homotopic homeomorphisms of such manifolds are isotopic, the residual finiteness of the fundamental groups of 3-manifolds, etc.

These conjectures have received an increasing amount of attention in recent years and various approaches have been developed to examine them. This fact was reflected in the four talks at the workshop which dealt with them. Nathan Dunfield described his joint work with W. Thurston on random 3-manifolds. Given a (closed, connected, orientable) 3-manifold M and a finite (simple) group G, they obtain results on the probability that M admits a finite regular cover with group G, and the probability that there is such a cover with positive first Betti number (http://www.its.caltech.edu/ dunfield/). Marc Lackenby discussed the relationship between the virtual Haken conjecture and the growth rates of the Heegaard genus of the finite covers of a closed, connected, orientable 3-manifold [La1], [La2] and in particular has found has found sufficient conditions for a positive solution to the the conjectures. X. Zhang described arithmetic conditions on the trace fields of the representations $\pi_1(M) \to PSL_2(\mathbb{C})$ which are sufficient for a positive solution to the the conjectures [Zh]. Finally, Genevieve Walsh showed that a significant proportion of the manifolds obtained by Dehn surgery on a 2-bridge knot are virtually Haken [Wa].

Dehn filling

The problem of understanding the geometric decompositions of the Dehn fillings of a hyperbolic manifold M with a torus boundary has attracted a lot of attention over the last twenty years. Thurston developed a theory of hyperbolic Dehn filling [Th1], [Th2] which formed the basis of the proof of his orbifold theorem [BPL], [CHK]. Peter Shalen spoke in the workshop of the programme he and Marc Culler have developed to prove the Poincaré conjecture from a Dehn filling point of view (see e.g. [CS2]). The programme is based on certain Seifert fibred space recognition theorems (using $SL_2(\mathbb{C})$-character variety methods), complemented by a method for producing certain types of knots in homotopy 3-spheres.

Over the years, a fairly accurate picture of the finite set of *exceptional fillings* of M, i.e. those which yield non-hyperbolic manifolds, has emerged [Go], though a precise description has yet to be found, and there is a lot of work to be done on the problem of describing the topology of M when it admits more than one exceptional filling. Ying-Qing Wu spoke in the workshop on his recent work classifying the exceptional Dehn fillings of hyperbolic arborescent knot exteriors.

If the virtual Haken conjecture is true, then existing results [GLu], [BZ2] can be used to show that at most five Dehn fillings of a compact, connected, orientable hyperbolic 3-manifold M with torus boundary can be non virtually Haken. To date, it is not even known whether such a manifold admits only finitely many non virtually Haken Dehn filling, though partial results are known [CL], [BZ1], [DT]. Genevieve Walsh's work mentioned in §28 is a contribution to this problem.

Heegaard structures on 3-manifolds and applications

Heegaard splittings of 3-manifolds are among the most natural decompositions one can hope for, yet they remain difficult to analyze and to work with. Part of the difficulty lies in the construction of interesting examples. "Interesting" is the key word here; examples of manifolds with Heegaard splittings are rife, it is the sorting-out of the examples and the distinguishing among them that is the problem. Despite the hazards, approaching central 3-manifold questions via Heegaard splittings continues to be attractive, first because the the decomposition into handlebodies seems so simple, and second because they arise naturally in many different instances. They also transparently connect the topology of the manifold with the algebra of its fundamental group.

Schultens and Kobayashi both gave talks on particular examples of splittings in which there is a discrepancy between the anticipated and the actual Heegaard genus. Schultens (in joint work with R. Weidmann) described the structure of Heegaard splittings of graph manifolds, and used this to produce examples in which the Heegaard genus is arbitrarily larger than the rank of the fundamental group of the manifold. In the previously known examples of this phenomena the difference was one. In the other direction, Kobayashi examined the asymptotic behaviour of the tunnel number of a knot in the 3-sphere under the connect-sum operation, and proved that it "usually" grows more slowly than one would expect, that is, the Heegaard genus of the complement of the summed knots is smaller than one would expect. The simplest example of an application of his theorem is to tunnel number one knots which are not 1-bridge on an unknotted torus. These knots have also appeared crucially as examples to show that the tunnel number is super-additive under connect sum [MSY]; thus they appear to somehow raise the Heegaard genus artificially when a few are added together, and to lower it artificially in the limit.

Schleimer discussed properties related to the distance of a Heegaard splitting as defined by Hempel [He]. Reducible Heegaard splittings have distance zero; weakly reducible splittings have distance at most one, and splittings of toroidal manifolds are known to have distance at most two. Thus examining properties of manifolds with Heegaard splittings with distance at most three is closely related to the Geometrization Conjecture; Schleimer discussed the relation between the distance of a Heegaard splitting for a manifold and its possible geometric structure. A more classical idea is to try to determine whether a 3-manifold is simply-connected by discerning properties of a Heegaard splitting. In this spirit Rolfsen showed that a Heegaard diagram for a homology 3-sphere, interpreted as a collection of curves on the boundary of a handlebody, can be embedded in the 3-sphere in such a way that each curve of the diagram bounds a surface with interior disjoint from the handlebody. Lackenby and Dunfield's work is discussed elsewhere, but both dealt with general properties of 3-manifolds approached via Heegaard splittings. One could argue that Heegaard splittings can provide a unifying theme for approaching some of the central problems in 3-manifolds, including many of those those discussed at this conference.

In another direction, Rob Kirby described his work with Paul Melvin on finding a combinatorial description of the chain complexes which arise in Oszvath-Szabo Floer homology for 3-manifolds. This theory is based on a method, pioneered by Andrew Casson, which uses a Heegaard splitting to "coordinatize" a 3-manifold. The usefulness and power of this use of Heegaard splittings can be seen by the important applications of Casson's work [AM], and, more recently, that of Kronheimer, Wrowka, Oszvath and Szabo [KMOS], to our understanding of 3-manifolds and surgery theory.

Knots and braids

A background theme of the workshop was the emerging connections between questions in 3-manifold theory and related questions in dimensions 2 and 4. Several talks on the general subjects of knots and braids were illustrative of this connection.

A central problem of classical knot theory is to develop criteria that determine if two knots in 3-space are isotopic; that is, to determine whether or not there is a smooth level-preserving imbedding $S^1 \times I \to S^3 \times I$ which restricts to the two knots at each end. Expressed in this way, it appears to be almost a 4-dimensional question and, indeed, it becomes one if one simply drops the requirement that the embedding be level-preserving. So, two knots are *concordant* if there is an embedding $S^1 \times I \to S^3 \times I$ that restricts to each knot on each end.

This simple extension of knot isotopy gives rise to a group structure on classical knots, the so called knot concordance group, and understanding the group has been an important theme in geometric topology for some time, in higher dimension as well as dimensions 3 and 4, cf [Le]. Classical work of Casson and Gordon [CG] showed that the mere algebraic criteria that work in higher dimension fail in the classical dimension, and this has given rise to the question of exactly how complicated the classical knot cobordism groups can be.

In coordinated talks, Cochran and Teichner demonstrate that this concordance group is very, very complicated. Indeed, there is a geometric construction, called a *grope*, for producing such concordances, and an associated numerical level of complexity called the height of the grope. This gives a sort of filtration of the concordance group by the integers and they are able to show that each stage is non-trivial. Their methods are wide-ranging and deep, from hands-on Kirby calculus, to homological algebra over noncommutative rings, to von Neumann rho invariants of Cheeger-Gromov. In addition, there is a related version of the work which can be thought of as a completely 3-dimensional program.

Shelley Harvey described a related approach to a different collection of geometric problems, including the problems of determining whether a 3-manifold fibers over S^1, whether it is a Seifert fibered space, and whether its product with S^1 admits a symplectic structure. There are sequences of integral invariants r_n, d_n (the latter related to the size of the successive quotients of the derived series of the fundamental group) which are computable obstructions to the 3-manifold having the geometric structures just noted.

Harvey's question about symplectic structure on certain 4-manifolds points out another topic that interrelates topology in dimensions 3 and 4. A symplectic structure on a bounded 4-manifold gives rise to what is called a contact structure on the 3-manifold boundary. Both Etnyre and Menasco separately discussed connections between the standard contact structure on S^3 and classical knots in S^3. Etnyre described how to associate a Legendrian torus to a knot, which seems to give rise to useful invariants based on the contact homology of the torus, invariants related to earlier combinatorial invariants studied by Lenhard Ng [Ng1, Ng2]. Menasco described the construction of knots with the same transversal invariants in the standard contact structure on S^3 yet the knots are not transversally isotopic. The construction makes use of Menasco's joint work with Joan Birman on the Markov Theorem without stabilization.

Finally, in a wide-ranging talk, Stephen Bigelow discussed connections between braids and the representation theory of Iwahori-Hecke algebras.

Dimensions 2, 4 and other matters

A general relationship between 3-dimensional hyperbolic geometry, based on the Lie group $PSL_2(\mathbb{C})$, and certain aspects of topological quantum field theory, related to the quantum group $U_q(sl_2(\mathbb{C}))$, seems to be emerging, although still in a very imprecise way. At the workshop, Francis Bonahon discussed his joint work with Xiaobo Liu which gives an example of such a connection, describing quantum hyperbolic invariants of surface diffeomorphisms based on the Chekhov-Fock quantization of Teichmuller space. The main point was that finite dimensional representations of the Chekhov-Fock algebra (a completely algebraic object) are controlled by the same data as pleated surfaces in hyperbolic 3-manifolds.

Danny Ruberman talked about his joint work with Nikolai Saveliev on a topic in "3.5-dimensional" topology: the relation between Rohlin's invariant of 3-manifolds and 4-dimensional gauge theory [RS1], [RS2]. Taubes had shown how the Casson invariant for homology 3-spheres could be defined using gauge theory. For 4-manifolds with the $\mathbb{Z}[\mathbb{Z}]$-homology of $S^1 \times S^3$, a Casson-type invariant can also be defined via gauge theory, as well as a Rohlin invariant. Ruberman discussed the natural conjecture that these are the same, modulo 2, in analogy to the 3-dimensional case, and outlined a proof for the case when the manifold fibers over the circle with finite order monodromy. The conjecture has some interesting implications about the homology cobordism group and other classical problems.

Ian Hambleton discussed his joint work with Mihail Tanase on finite group actions on definite 4-manifolds [HT]. Equivariant Yang-Mills moduli space to investigate the relation between the singular set, isotropy representations at fixed points, and permutation modules realized by the induced action on homology for smooth group actions on certain 4-manifolds.

List of Participants

Agol, Ian (University of Illinois at Chicago)
Bigelow, Stephen (University of California - Santa Barbara)
Bonahon, Francis (University of Southern California)
Boyer, Steve (University of Quebec)
Calegari, Daniel (California Institute of Technology)
Cochran, Tim (Rice University)
Coffey, James (University of Melbourne)
Collin, Olivier (Université du Québec a Montreal)
Cooper, Daryl (University of California - Santa Barbara)
Culler, Marc (University of Illinois at Chicago)
Dunfield, Nathan (Harvard University)
Etnyre, John (University of Pennsylvania)
Eudave-Munoz, Mario (University Nacional Autonoma de Mexico)
Frohman, Charles (University of Iowa)
Hambleton, Ian (McMaster University)
Harvey, Shelley (University of California - San Diego)
Indurskis, Gabriel (Universite du Quebec a Montreal)
Johnson, Jesse (University of California - Davis)
Kerckhoff, Steven (Stanford University)
Kirby, Rob (University of California - Berkeley)
Klaff, Benjamin (University of Illinois at Chicago)
Klodginski, Elizabeth (University of Michigan)
Kobayashi, Tsuyoshi (Nara Women's University)
Krebes, David (University of Calgary)
Lackenby, Marc (University of Oxford)
Long, Darren (University of California - Santa Barbara)
Menasco, William (University at Buffalo)
Rolfsen, Dale (University of British Columbia)
Ruberman, Daniel (Brandeis University)
Rubinstein, J. Hyam (University of Melbourne)
Safnuk, Brad (University of California - Davis)
Scharlemann, Martin (University of California - Santa Barbara)
Schleimer, Saul (University of Illinois at Chicago)
Schultens, Jennifer (Emory University)
Shalen, Peter (University of Illinois at Chicago)
Teichner, Peter (University of California - San Diego)
Thompson, Abigail (University of California - Davis)
Tillmann, Stephan (Université du Québec a Montreal)
Walsh, Genevieve (University of California - Davis)
Wu, Ying-Qing (University of Iowa)
Zhang, Xingru (University at Buffalo)

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Chapter 29

Structural and Probabilistic Approaches to Graph Colouring (03w5059)

September 20–25, 2003

Organizer(s): Bruce Reed (McGill University), Paul Seymour (Princeton University)

A graph consists of a set of objects called nodes and a set of links between the nodes called edges (formally, each edge is an unordered pair of nodes). We colour a graph by assigning integers to its nodes so that vertices joined by an edge receive different integers.

Graph colouring is one of the oldest and most fundamental areas of graph theory and indeed of discrete mathematics. The first graph theory problem posed was the celebrated four colour conjecture which states that the regions of any planar map can be four coloured in such a way that every pair of regions which share a border receive different colours. This conjecture was posed in 1854 by Guthrie and answered in the affirmative by Appel and Haken in 1976.

Graph colouring arises naturally in many areas of research, both theoretical and applied. For example frequency assignment for mobile telephone networks can be modelled as a graph colouring problem, and the study of colouring perfect graphs is related to fundamental problems about the relationship between integer and linear programming. Hadwiger's conjecture, a generalization of the four colour conjecture, is an important open problem which has generated a considerable body of work

Objectives

The workshop will focus specifically on two approaches to graph colouring, probabilistic and structural. We intend to bring together the communities who attack graph colouring problems using these two kinds of tools. One aim of the workshop is simply to foster interaction within these two communities. A more ambitious aim is to foster collaborations which involve the joint application of both techniques.

There has been considerable recent progress on structural approaches for graph colouring and in particular their application to the Strong Perfect Graph Conjecture. Important progress has also been made recently using probabilistic techniques. Thus, it seems an appropriate time to bring together researchers from these two communities to discuss their current research.

List of Participants

Addario-Berry, Louigi (McGill University)
Bondy, Adrian (French National Centre for Scientific Research)
Chudnovsky, Maria (Princeton University)
Chvatal, Vasek (Rutgers University)

Cornuejols, Gerard (Carnegie Mellon University)
DeVos, Matt (Princeton University)
Fountoulakis, Nikolaos (McGill University)
Goddyn, Luis (Simon Fraser University)
Hare, Donovan (Okanagan University College)
Havet, Frederic (l'Institut National de Recherche en Informatique et en Automatique Sophia-Antipolis)
Hayward, Ryan (University of Alberta)
Hoang, Chinh (Wilfred Laurier University)
Kahn, Jeff (Rutgers University)
Kawarabayashi, Kenichi (Princeton University)
Keevash, Peter (Princeton University)
Kochol, Martin (Slovak Academy of Sciences)
Lohman, Mike (Princeton University)
McCuaig, William (University of Toronto)
McDiarmid, Colin (Oxford University)
Meagher, Conor (McGill University)
Mohar, Bojan (University of Ljubljana)
Molloy, Michael (University of Toronto)
Oum, Sang-il (Princeton University)
Reed, Bruce (McGill University)
Robertson, G. Neil (Ohio State University)
Seymour, Paul (Princeton University)
Sudakov, Benny (Princeton University)
Thomas, Robin (Georgia Institute of Technology)
Vu, Van (University of California, San Diego)
Welsh, Dominic (University of Oxford)

Chapter 30

Stochastic Partial Differential Equations (03w5079)

September 28–October 2, 2003

Organizer(s): Martin T. Barlow (University of British Columbia), Robert Dalang (ETH Lausane), Edwin A. Perkins (University of British Columbia)

The conference was attended by 41 participants, including Ph.D. students, postdoctoral fellows, young researchers and international leaders in stochastic PDEs and related fields. An attempt was made to introduce new ideas from deterministic PDE (Souganidis' talk on homogenization in PDE) and statistical physics (Thomas' lecture on a stochastic wave equation approach to non-equilibrium heat flow). It was evident that the level of the talks throughout the meeting, both with respect to the science and level of exposition was unusually high. What follows is an attempt to focus on some of the key topics discussed at the meeting both in the lectures and in the informal meeting rooms.

Stochastic Differential Equations and Partial Differential Equations

Nick Krylov discussed uniqueness problems for the finite dimensional stochastic differential equation (SDE)

$$dX_t = a(X_t)dW_t, \qquad X_0 = x \in \mathbb{R}^d; \tag{30.1}$$

here W is a d-dimensional Brownian motion, and $a(x)$, $x \in \mathbb{R}^d$ is bounded measurable family of $d \times d$ matrices. There are two important types of uniqueness: strong, which means that the solution X is a function of the driving Brownian motion W, and weak, which means uniqueness of the probability law of the solution X. For Lipschitz a Itô [11] proved strong uniqueness, and for continuous uniformly elliptic a Stroock and Varadhan [21] proved weak uniqueness.

Suppose that a is uniformly elliptic. A counterexample of Barlow [1] shows that strong uniqueness may fail even if $d = 1$, while Nadirashvili [19] proved that if $d \geq 3$ then weak uniqueness need not hold.

Krylov then described some recent and very interesting results on the set of initial x for which weak uniqueness holds. His first main result was that if weak uniqueness holds for all $x \in D - \{x_0\}$, where x_0 is an interior point of the domain D, then weak uniqueness holds for all $x \in D$. By iteratively applying this result the exceptional set can be enlarged to sets with a cluster point, infinitely many cluster points, etc. By combining this theorem with an older (and largely overlooked result) of Lorenzi [14] which allows discontinuity on hyperplanes, one can derive a number of uniqueness results including those of Bass and Pardoux in [2].

Rich Bass spoke of ongoing work of Athreya and Perkins, which develops an idea of Walsh and uses the Fourier transform to study parabolic SPDEs on a compact set via a related class of infinite dimensional

Ornstein-Uhlenbeck SDEs. Essentially one has

$$dX_t = a(X_t)dW_t - b(X_t)dt, \tag{30.2}$$

where W is an infinite dimensional Brownian motion, and $a(\cdot)$ and $b(\cdot)$ are Hölder continuous, and a is uniformly elliptic. For this problem the interesting question is weak uniqueness.

The analysis of (2) assumes only Hölder continuity of the coefficients and uses a perturbation technique to obtain Schauder type estimates on the derivatives of the resolvent in terms of a norm associated with the semigroup from which you are perturbing. A second part of the analysis uses infinite dimensional localization methods to reduce to the perturbative setting. The norm used here turns out to coincide with that used by Bass and Perkins in [3] to handle a class of singular SDEs arising in the theory of interactive superprocesses in both the finite and countable dimension cases. Hence the norm, used already in work of Canarsa and DaPrato [6] in a different context, would appear to apply more widely than originally thought, and is being used by Dawson and Perkins to analyze another singular SDE arising in the renormalization analysis of Dawson, Greven et al. It was also interesting to compare these techniques with those of DaPrato and his co-authors, thanks to the presence at the workshop of Lorenzo Zambotti.

P. Souganidis discussed a different kind of interaction between randomness and differential equations in a survey of new methods in the theory of homogenization. He considered the stochastic homogenization of the Hamilton-Jacobi equation (1) $u_t^\epsilon + H(x/\epsilon, Du^\epsilon, \omega) = 0$ in $\mathbf{R}^N \times (0,\infty)$, $u^\epsilon = u_0$ on $\mathbf{R}^N \times \{0\}$, and assumed that the Hamiltonian is superlinear and convex with respect to the spatial variable. By a generalization of the Lax-Oleĭnik formula, he showed that the solution u of (1) can be expressed in terms of a "fundamental" solution. Then the behaviour of the latter as $\epsilon \to 0$ can be analyzed by Γ-convergence type arguments and the ergodic theorem. He proved convergence toward the unique viscosity solution $u \in \mathrm{BUC}(\mathbf{R}^N \times [0,\infty))$ of $u_t + \overline{H}(Du) = 0$ in $\mathbf{R}^N \times (0,\infty)$, $u = u_0$ on $\mathbf{R}^N \times \{0\}$, where $\overline{H}$ is a convex function.

Superprocesses

Superprocesses arise as solutions to a particularly delicate parabolic SPDE driven by white noise. The basic example is the equation for super-Brownian motion:

$$\frac{\partial u}{\partial t} = \Delta u + u^{1/2}\dot{W}. \tag{30.3}$$

Here $\dot{W}$ is space time white noise, and $u = u(x,t)(\omega)$, $x \in D \subset \mathbb{R}^d$, $t \in [0,\infty)$. This equation is delicate because if $d \geq 2$ solutions must be interpreted in a generalized sense, and the equation is nonlinear. Weak uniqueness of non-negative solutions has been known for some time, but strong uniqueness (that is, whether u is measurable with respect to the white noise $\dot{W}$) remains open.

Nevertheless, the special nature of the equation allows for a remarkable array of exact calculations. Leonid Mytnik reported on his work with J.-F. Le Gall and Ed Perkins ([18]). This includes the resolution of a well-known problem on the existence of a continuous density for stable branching in 1 dimension but not in higher dimensions. One of the areas of application of superprocesses is in the study of a class of nonlinear boundary value problems of the form

$$\Delta u = u^{1+\beta} \text{ on } D.$$

The existence of a continuous exit measure density allows for a general probabilistic representation of u in terms of this exit density from D, the boundary behaviour of u at its well-behaved points and the set of points at which it explodes badly. This is one refinement of a far-reaching program of Dynkin, Kuznetsov, Le Gall and Veron.

One exciting recent development of superprocesses is its emergence as universal limit for stochastic spatial systems near criticality above the critical dimension. Remco van der Hofstad reported on his result with Gordon Slade [10] on critical oriented percolation in $\mathbb{Z}_+ \times \mathbb{Z}^d$. They have proved that if $d \geq 4$ then the finite dimensional distributions converge to those of the canonical measure of super-Brownian motion. His presentation included a nice explanation of the lace expansion, a combinatorial method introduced by Brydges and Spencer [5]. Super-Brownian motion has emerged as a universal scaling limit for a variety of interacting particle systems, percolation models and combinatorial structures over the past eight years. This field started with a 1987 conjecture of Rick Durrett that the one dimensional long range contact process when rescaled converges to super-Brownian motion with a killing term proportional to the density. This was confirmed in 1995 by Mueller and Tribe [16] and the higher dimensional analogues were proved by Durrett and Perkins [8]. In 1993 David Aldous conjectured that rescaled lattice trees above 8 dimensions should converge to Integrated Super-excursion (ISE), that is the integral of a super-Brownian cluster conditioned to have mass one. This was confirmed by Derbez and Slade [9] using the lace expansion. A number of results, some using the lace expansion and others using martingale methods have been proved since. A good survey of these results appears in [20]. One interesting development mentioned by van der Hofstad was his ongoing work with Akira Sakai extending the above results of Durrett and Perkins to short range critical contact processes above 4 dimensions.

A key unsolved problem for oriented percolation are the precise asymptotics for the survival probability of the critical cluster as time gets large. One expects it to be that of critical branching random walk with parameters given by their general limit theorem. This is the analogue of the classical Bramson-Griffeath result voter model, but the absence of a useful dual makes the question much harder. A second hard unresolved problem, discussed at the meeting is the extension of van der Hofstad's convergence theorem to the level of processes. Tightness for these self-repellent spatial processes appears to be a hard problem. If we are to find a way of combining martingale methods with the lace expansion, the Markovian setting of oriented percolation would seem to be the best setting for doing so. Mark Holmes, a Ph.D. student at UBC is currently working on this with van der Hofstad, Gord Slade and Ed Perkins.

Parabolic SPDE

An important outstanding problem in superprocesses is the question of pathwise uniqueness of the SPDE (3), giving the density of super-Brownian motion in one dimension. Lorenzo Zambotti presented a fascinating calculation in the context of solutions to the reflecting parabolic stochastic PDE of Nualart-Pardoux on $[0, 1]$:

$$\frac{\partial u}{\partial t} = \Delta u + \dot{W} + \eta,\ u \geq 0,\ u(t, 0) = u(t, 1) = 0,$$

$$\eta(\cdot, x) \text{ increasing with support contained in } \{t : u(t, x) = 0\}.$$

Solutions to this SPDE are Hölder of index 1/4 in time and so for a fixed x the process $u(\cdot, x)$ is not a semimartingale. In spite of this, he was able to use chaos expansions to derive an Itô type formula for smooth functions of $u(t, x)$. These kind of calculations have been tried, mainly without success, to extend standard pathwise uniqueness arguments for SDEs to SPDEs. Many people in the room found this to be familiar territory, except that Zambotti was able to see it through in this context. Mytnik, Barlow and Perkins have a program for pathwise uniqueness involving signed solutions to the SPDE for the density of super-Brownian motion and a competing species model in which particles of two types annihilate when they collide. Showing equivalence between these models is one step in this program and requires a calculation like that of Zambotti's but for a convex function. It is safe to say this will lead to a future work among these participants.

Zambotti also described a number of explicit calculations he could do with solutions of this equation using reversibility with respect to the law of the 3-dimensional Bessel bridge. Especially interesting was his joint work with Mueller and Dalang on the maximum number of zeros of the solutions to Bessel type SPDEs. The answer is shown to be 3 or 4 for an appropriate range of dimensions.

This can of course be thought of as potential theoretic results for a class of infinite dimensional solutions to parabolic SPDEs. It is related to the results presented by Eulalia Nualart for hyperbolic SPDEs: see Section 4 below.

Although the connection with parabolic SPDE is not immediately apparent, Jeremy Quastel's lecture on the superdiffusivity of the asymmetric exclusion process generated considerable interest here. He considered an exclusion process on $\mathbb{Z}^d$ with initial law independent coin tossing with mean $1/2$, say. Particles move according to a kernel with non-zero mean provided the selected site is vacant. It is known that the asymptotic mean square displacement of a tagged particle at time t is diffusive (ie. grows linearly in t) in dimensions 3 or more but the behaviour in 1 or 2 dimensions was open. A quick analysis of the generator shows that this may be viewed as a discretization of the stochastic Burger's equation

$$\frac{\partial u}{\partial t} = \nabla u^2 + \Delta u + \nabla \dot{W}.$$

Quastel outlined the proof for $d = 1$ of a result with H.T. Yau, Salmhofer and Landim which showed that the system is super-diffusive in one or two dimensions as the mean square displacement grows at least as fast as $t^{5/4}$ if $d = 1$ and $t(\log t)^{2/3}$ if $d = 2$. The conjectured rates are $t^{4/3}$ and $t(\log t)^{1/2}$, respectively. The result reduces to studying the asymptotics of the resolvent acting on a particular second order polynomial. They consider the generator and hence resolvent as an asymmetric perturbation of the symmetric exclusion process. The latter maps degree n polynomials to degree n polynomials but the symmetric operator increases the degree. The standard resolvent equation is therefore not closed, a familiar dilemma for a variety of SPDEs. Nonetheless they are able to truncate these equations and show the resulting iterated solutions provide upper (even iterations) and lower (odd iterations) bounds for the quantity of interest. A particular special inequality on degree two polynomials then leads to the above bound. For $d = 2$ H.T. Yau was able to analyze higher order iterations to get the conjectured result. Although one feels such special inequalities may be rare, one certainly wants to return to other settings where moment equations are not closed in case a similar trick can give some information here.

Not unrelated to the above was Boris Rozovsky's lecture on stochastic Navier-Stokes equations in two spatial dimensions and his proposed method of calculating moments of the solutions. The onset of turbulence in fluid mechanics is often associated with some random forcing terms in classical Navier-Stokes. Again one finds the moment equations associated with these stochastic equations are not closed and hence computations are problematic. He showed that 2-d stochastic NS has a unique strong solution which could be represented as the conditional expectation (given the driving white noise) of the flow of solutions of a related SDE. The latter involves non-Lipschitz coefficients and hence may be non-unique (this idea also appears in work of Le Jan and Raimond [13]). The chaos expansion of the strong solution is then used to effectively estimate the moments.

Hyperbolic SPDEs

Eulalia Nualart gave a systematic account of the problem of characterizing polar sets, mainly in the context of systems of hyperbolic SPDEs, using the Malliavin calculus to get estimates on the actual densities of the conditional increments of the solutions. In particular, she showed that a compact set K in Euclidean space is polar for the solution to a non-linear hyperbolic system of d SPDEs in two coordinates if and only if $\mathrm{Cap}_{d-4}(K) = 0$. Here $\mathrm{Cap}_{d-4}(\cdot)$ is the Newtonian capacity. This extends results of Khoshnevisan and Shi [12] for the Brownian sheet, and appears to be the first potential-theoretic result for non-linear SPDEs. The parabolic case is somewhat more difficult due to the different structure of the underlying filtrations. The partial result obtained to date (joint with R. Dalang) for a d-dimensional system of non-linear heat equations in one spatial dimension extends the results from the linear case obtained by Mueller and Tribe [17]. For the moment, the results here are less complete but lead to the interesting conjecture that for d-dimensional parabolic SPDEs driven by space-time white noise, the critical dimension for polarity for the range is $d - 6$. The sufficiency of this condition (dimension less that $d - 6$ implies polarity) has been proved.

Olivier Lévèque presented some results concerning non-linear hyperbolic SPDEs in spatial dimensions greater than one. One central difficulty is that the Green's function is generally not a function but a measure or a (Schwarz) distribution. For instance, for the wave equation, the Green's function is an unbounded function in dimension 2, a positive measure in dimension 3, and a distribution but not a measure in dimensions 4 or more. Therefore, J.B. Walsh's notion of (worthy) martingale measure (see [22]) must be extended in order to give an integral formulation of the SPDE. In the linear case, for the equation on the whole space, O. Lévèque presented a necessary and sufficient condition for the existence of a function-valued solution, and a related but slightly stronger condition for the existence of a random field solution (the main difference between the two notions of solution is that the latter is defined for every point in space-time and is continuous in mean-square). Under the stronger condition, he established the existence of a random field solution to non-linear forms of the equation.

Olivier Lévèque also considered the case of hyperbolic equations in a ball with isotropic driving noise concentrated on a sphere inside the ball. In this case, he presented a sufficient condition for existence of the solution in the linear case, and showed that the condition is necessary if the sphere is the boundary of the ball.

Larry Thomas showed how a stochastic wave equation could be used to model non-equilibrium dynamics of heat flow. This involved a stochastic wave equation driven by Gaussian noise that is modelled by Brownian motions. He presented a global existence theorem for the solution of the equation.

The Brownian sheet

R. Dalang and T. Mountford presented recent results concerning the Hausdorff dimension of the boundary of individual excursions of the Brownian sheet. A central step is the determination of gambler's ruin probabilities for additive Brownian motion, which was the subject of R. Dalang's lecture. Given that the value at the origin of an additive Brownian motion is $x \in (0, 1)$, what is the probability $\mathcal{E}(x)$ that there exists a path in the plane, starting at the origin, along which the additive Brownian motion hits 1 before 0? The solution to this problem relies on an algorithm developed by Dalang and Walsh [7], which makes it possible to express the problem as an absorption probability for a discrete-time continuous state space Markov chain of order 2. An exact explicit formula has been obtained, which implies in particular that as $x \downarrow 0$, $\mathcal{E}(x) \simeq x^{\lambda_1}$, where

$$\lambda_1 = \frac{1}{2}\left(5 - \sqrt{13 + 4\sqrt{5}}\right).$$

In his lecture, T. Mountford explained how to use this result to show that the Hausdorff dimension of the boundary of individual excursions of the Brownian sheet is $\frac{3}{2} - \frac{\lambda_1}{2}$. Indeed, the first issue is to characterize those points which are on the boundary of an individual excursion of height at least one. Such points must satisfy two conditions: they are on the level set, and from these points, there is a curve along which the Brownian sheet hits 1 before 0. These two events are nearly independent, and because the Brownian sheet can be locally approximated by additive Brownian motion, the gambler's ruin probability mentioned above comes into play. However, additional ingredients are needed, already for an upper bound, because there is an error term in the approximation: a modification of the Dalang-Walsh algorithm is needed which is stable under continuous perturbations of the sample paths. The lower bound is more delicate, since this involves looking at the probability of escaping simultaneously from two distinct points and getting bounds on the probability of this event. These bounds have been obtained and establish the result mentioned above.

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Chapter 31

Quadratic Forms, Algebraic Groups, and Galois Cohomology (03w5029)

October 4–9, 2003

Organizer(s): Richard Elman (University of California, Los Angeles), Alexander Merkurjev (University of California, Los Angeles), Jan Min´ač (University of Western Ontario), Carl Riehm (McMaster University)

The algebraic theory of quadratic forms began with the seminal paper of E. Witt [31] in 1937, where what are now called "Witt's Theorem" and the "Witt ring" first appeared. But it was not until a remarkable series of papers [19] by A.Pfister in the mid-1960s that the theory was transformed into a significant field in its own right. The period 1965 to 1980 can be considered the "first phase" of the algebraic theory of quadratic forms and is well documented in the books of T.-Y.Lam [13] (1973) and W.Scharlau [25] (1985). Lam's book itself came at a critical time and greatly influenced the development and popularity of the field.

In 1981 another phase began with the first use of sophisticated techniques from outside the field—in this case algebraic geometry—by A.S.Merkurjev, who proved a long-standing conjecture of A.A.Albert on a presentation for the exponent two subgroup of the Brauer group. This answered the first open case of the Milnor conjecture of 1970 [17] on the relationship between algebraic K-theory, the Brauer group and the Witt ring. As the preeminent open problem for many years in the algebraic theory of quadratic forms, the Milnor conjecture exerted a profound influence on the subject. This work was extended shortly thereafter (1982) in a paper by Merkurjev and A.A.Suslin [15], showing that when the field F contains the n^{th} roots of unity, the n-torsion in the Brauer group of F is isomorphic to the algebraic K-group $\mathrm{K}_2(F)/n\mathrm{K}_2(F)$.

During this time, quadratic form theory also widened its involvement with other areas of mathematics. T.A.Springer's use of Galois cohomology in 1959 [26] in recasting some of the classical invariants of quadratic forms in terms of the Galois cohomology of the orthogonal group was one of the first applications of Galois cohomology in algebraic groups, following A.Weil's classification of algebraic groups of classical type. And the strong approximation theorem of M.Kneser [10] for orthogonal groups led to a generalization of that theorem in the theory of semisimple algebraic groups. Siegel's work [24] on the representations of one quadratic form by another led to the use of adèles in algebraic groups by T.Tamagawa [28], in particular to the notion of Tamagawa numbers in this wider context.

This relationship of Galois cohomology and algebraic groups to quadratic form theory continues to grow in importance as this workshop has strikingly demonstrated. These fields have become increasingly sophisticated in recent years, mainly through further incursions by algebraic geometry and, in particular, by motivic methods. Voevodsky's work, for which he received the Fields Medal in 2002 [7], has been very influential, not least because of his complete (positive) solution of Milnor's Conjecture. A highlight of our workshop was his proof of the final link in the confirmation of the Bloch-Kato Conjecture, which can be viewed as the most general form of Milnor's Conjecture.

Thus the primary emphasis in the meeting dealt with the impact of motivic methods on the subjects of the workshop. Many new and startling results in the algebraic theory of quadratic forms have been proved by these methods. There were as well many interesting and very important talks on a variety of other topics which do not fit within a coherent group or groupings.

Therefore we begin this report with a description of the talks on motivic methods, followed by a section on "miscellaneous" results.

Motivic Methods

V. Voevodsky. In the late nineties V. Voevodsky developed an algebraic homotopy theory in algebraic geometry similar to that in algebraic topology. He defined the (stable) motivic homotopy category and certain spectra that give rise to interesting cohomology theories such as motivic cohomology, K-theory and algebraic cobordism. Note that before the work of Voevodsky, the only comprehensive cohomology theory available in algebra was the algebraic K-theory (defined by D. Quillen by means of algebraic topology). In the eighties, A. Beilinson [3] and S. Lichtenbaum [14] predicted the existence of motivic cohomology and conjectured relationships between it and étale motivic cohomology theories. This conjecture (known as the Beilinson-Lichtenbaum Conjecture) has been one of the central conjectures in algebraic geometry. A particular case of the Beilinson-Lichtenbaum Conjecture, the Bloch-Kato Conjecture [4], asserts that the norm residue homomorphism

$$\mathrm{K}_n(F)/p\mathrm{K}_n(F) \to H^n_{\text{ét}}(F, \mu_p^{\otimes n})$$

is an isomorphism for every field F, positive integer n and prime $p \neq \operatorname{char} F$. But A. Suslin and V. Voevodsky [27] proved that in fact the Bloch-Kato Conjecture is equivalent to the Beilinson-Lichtenbaum Conjecture.

The Bloch-Kato Conjecture has its origins in Milnor's Conjecture, which is the case of $p = 2$. As mentioned earlier in the introduction, the first steps in the proof of that special case were made in a paper of Merkurjev who verified it for $n = 2$, and then shortly thereafter, Merkurjev and Suslin gave a proof the general case of $n = 2$, and then a few years ago, V. Voevodsky provided a proof for the general case of Milnor's Conjecture.

The highlight of this workshop was the announcement by Voevodsky in his conference talk of a full solution of the Bloch-Kato Conjecture [30]. Thus, the Beilinson-Lichtenbaum Conjecture is now proven in full generality!

Ph. Gille. Another look at the Bloch-Kato Conjecture was given by Ph. Gille in his talk. He considered an "elementary" approach involving Severi-Brauer varieties instead of general splitting varieties, and Tate's continuous Galois cohomology instead of motivic cohomology. Gille gave an equivalent reformulation of the Bloch-Kato Conjecture in this setting.

A. Vishik. There has been striking progress in the algebraic theory of quadratic forms since Voevodsky introduced his motivic methods. In particular the long-standing problem on the relation between Milnor's K-theory and the graded Witt ring was solved by D. Orlov, A. Vishik and V. Voevodsky [18]. More precisely, they proved that the canonical homomorphism

$$\mathrm{K}_n/2\mathrm{K}_n(F) \to I^nF/I^{n+1}F,$$

where IF is the fundamental ideal of the Witt ring of a field F, is an isomorphism. Another important conjecture on the description of the first Witt index of quadratic forms has been solved by N. Karpenko. In the proof he used the algebraic Steenrod operations invented by Voevodsky and described (on the Chow groups of algebraic varieties) by P. Brosnan. In his talk Vishik discussed other operations that arise on the level of the cobordism groups of algebraic varieties. He is now using these operations in an attempt to describe the so-called generic discrete invariant of quadratic forms. This invariant includes all discrete invariants such as splitting patterns of quadratic forms and dimensions of quadratic forms in I^nF.

N. Karpenko. Karpenko announced a solution of the following problem: If q is an anisotropic quadratic form in I^n of dimension less than 2^{n+1}, then

$$\dim q = 2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^k$$

for some $k = 1, 2, \ldots, n$. In the proof he uses the whole spectrum of modern "elementary" techniques—the Steenrod operations of P. Brosnan and the motivic decomposition of quadratic forms in the category of Chow motives (developed in works of A.Vishik [29]). A solution of this problem had been announced earlier by Vishik, who used different and more involved techniques.

P. Brosnan. Brosnan presented an alternative "elementary" proof of Rost's Nilpotence Theorem [22]:

Let X be a projective quadric over a field F, with motive $M(X_F)$ in the category of Chow motives. Then for every field extension L/F the kernel of the canonical ring homomorphism

$$\operatorname{End} M(X_F) \to \operatorname{End} M(X_L)$$

consists of nilpotent elements.

The Nilpotence Theorem is an essential ingredient of the basis of the motivic theory of quadratic forms. Brosnan's proof avoids the use theory of cyclic modules involved in the original proof of M. Rost.

V. Chernousov. Chernousov reported on the generalization of Rost's Nilpotence Theorem to the whole class of projective homogeneous varieties. The main ingredient of the proof is a motivic decomposition of isotropic projective homogeneous varieties into a direct sum of motives of twisted anisotropic projective homogeneous varieties.

F. Morel. Morel discussed the construction and computation of stable cohomology operations in the cohomology theory on simplicial smooth schemes $\mathcal{X}$ given by $H^*_{Nis}(\mathcal{X}; k_*)$, the Nisnevich cohomology of $\mathcal{X}$ with coefficients in the unramified mod ℓ Milnor K-theory sheaves k_* (where ℓ is a prime different from $\operatorname{char} k$). He showed how the Bloch-Kato conjecture at ℓ predicts the structure of the algebra of all stable cohomology operations and that conversely, the knowledge of that algebra almost implies formally the Bloch-Kato conjecture. This explains the difficulty of computing that algebra, as opposed to the computation of the Steenrod algebra in mod ℓ-motivic cohomology which is "easy" by comparison.

He also explained how just the existence of some operations have non-trivial consequences for Milnor K-theory; for example in the case $\ell = 2$ the existence of the operation Sq^2 is "close" to proving the Milnor conjecture on quadratic forms, and an example of the construction of an explicit "extended power operation", which might prove useful in this regard, was given.

Miscellaneous Talks

J. Arason, B. Jacob. The study of quadratic forms over fields in characteristic two has a different flavor than that in other characteristics. Results are also often different. One outstanding example has been the computation of the Witt group of quadratic forms over a rational function field when the base field has characteristic two. Two talks were given on this topic, each independently solving this problem. The basis of such a computation is the local case.

Arason provided a presentation for the Witt group in characteristic two and used careful computations among the generators and relations to determine the Witt group of a Laurent series field. He then showed how to determine the Witt group of a rational function field (in one variable) over a field of characteristic two as a corollary.

Jacob, in collaboration with R. Aravire, also determined the local case and hence the rational function field case. They also obtained a reciprocity law, at least if the base field is perfect (a condition they are currently attempting to remove).

P. Balmer. Balmer lectured on his joint work with R. Preeti on odd indexed Witt groups of semi-local rings. The setting is an appropriate triangulated category with duality [2] and arose from studies in L-theory ([20], [16]). This category has nice cohomological and topological properties and agrees with M. Knebusch's Witt groups on an algebraic variety [9] and M. Karoubi's Witt group of an exact category with duality [8]. The main study is that of the odd indexed components W^{2i+1} of the total Witt group over (not necessarily commutative) local and semi-local rings. The factors in the total Witt group have periodicity four. In the general case, Witt cancellation does not hold, and it is conjectured that the odd indexed Witt groups encode information about this failure of Witt cancellation when the ring has an involution. For example, over commutative semi-local rings with involution the identity, cancellation holds and these odd indexed groups are indeed trivial. Balmer and Preeti also determine a decomposition of the odd Witt groups over semi-simple rings with involution, which depends only on the simple factors with involutions of the first kind. They then relate the third Witt group with maximal ideals in the commutative semi-local case. In particular, they show that if R is semi-local and commutative, $W^1 = 0$ and $W^3 = (\mathbf{Z}/2\mathbf{Z})^m$ where m is a computable integer bounded by the number of maximal ideals in R.

G. Berhuy. A most interesting and natural invariant called the *essential dimension* introduced by Z. Reichstein [21] measures the number of parameters needed to describe a given structure up to isomorphism. For example, to describe all quadratic forms of rank n over a field one needs at least n parameters, and n is in fact needed in general. Such definitive answers are rare, but upper and lower bounds have been determined in several interesting cases. In his talk, Berhuy presented his work with G. Favi on cubics over a field, sketching a proof that the essential dimension for the set of cubics in three variables is precisely three (assuming the field contains a cube root of unity and has characteristic not two or three).

S. Gille. One of the primary computations in quadratic form theory is that of the Witt ring of a Laurent series ring, based on work of M. Karoubi and A. Ranicki using L-theory. Geometrically, this can be viewed as the Witt ring of the product of an affine scheme and a punctured affine line. Gille and P. Balmer generalized this result by computing the total Witt ring of the product of a regular finite dimensional scheme with a union of punctured affine spaces (when two is a unit). The usual methodology of reducing to the affine case and the localization sequence is followed by using Koszul complexes instead of coherent Witt theory. This allows globalization of a certain key element of the theorem whose construction they show to be independent of the constructions introduced in the proof. As an application they apply the theorem to an affine hyperbolic space over a regular ring, retrieving a theorem of Karoubi in the case of a field.

D. Hoffmann. Hoffmann presented his work on an algebraic introduction to p-forms. This study mimics known quadratic form theoretic techniques applied to the case that F is a field of positive characteristic p and a p-form is an additive form on a finite dimensional vector space where scalars pull out to the p^{th} power, in other words $a_1X_1^p + a_2X_2^p + \cdots + a_nX_n^p$. Because of the binomial theorem, many of the analogues of quadratic form theory—such as Pfister forms, the Cassels-Pfister theorem, the subform theorem, the Knebusch-Norm theorem—follow. The main reason for this is the fact that the coefficients $a_1, a_2, \ldots, a_n$ generate a field over F^p reflecting important properties of the form.

M. Knus. A classical result of Hurwitz gives the complete list of quadratic composition algebras with identity over a given field F: the field itself, quadratic extensions, quaternions or octonions over F. Thus such a composition is only possible in dimension $1, 2, 4$ or 8. Rost gave a purely tensor categorical proof of this result about the possible dimensions by considering the vector algebra of pure elements inside such a composition algebra. The universal tensorial object associated with a vector algebra can be interpreted as a category of graphs and graph manipulations lead to the equation

$$d(d-1)(d-3)(d-7) = 0$$

for the dimension d, which occurs as a numerical invariant of the category. A complete list of cubic compositions was given by Schafer, using structure theory. According to Rost, graph theoretical computations can also be applied to such compositions: the pure elements (i.e. the elements orthogonal to 1) admit the structure of a “symmetric” composition. Two numerical invariants d and e can be attached to the graph category of a symmetric composition; d is the dimension and e is associated to a “Casimir” element. The possible values are $(d, e) = (0, 0)$, $(1, 1)$, $(2, 0)$, $(4, 4)$, $(8, 0)$ and $(8, 36)$. This gives the possible dimensions 1, 2, 3, 5 and (twice) 9 for the cubic composition. Simple exceptional Jordan algebras of dimension 27 satisfy a generalized notion of cubic composition. In a dissertation in progress (L. Cadorin), a graph theoretical approach is developed to include the case of exceptional Jordan algebras (type D_4).

D. Lewis. Lewis presented his joint work with T. Unger and J. van Geel on the Hasse Principle for Hermitian forms over a quaternion algebra (with the standard involution) over a number field. The Hasse Principle is known to fail in the case of skew hermitian forms. It was unknown whether the Principle fails if weakened by replacing equivalence by similarity (when the form is of odd dimension). Lewis, Unger, and van Geel prove this also fails. In fact they show that there exist locally equivalent forms which are not globally similar. The proof involves defining an invariant (equivalent to Bartle's invariant) that can detect such a counterexample.

Discussion afterwards indicated this yields a explicit computation for the Tate-Šafarevič group for the projective unitary group.

R. Parimala. The determination of conditions under which a quasi-projective variety having a zero cycle of degree one has a rational point is a problem of long standing. This is, of course, true for conics and elliptic curves. It is clear that certain restrictions should be assumed. The most reasonable varieties to consider, where a positive answer may occur, are homogeneous spaces of a connected linear algebraic group. The answer is known to be positive for torsors of some groups by work of E. Bayer and H.W. Lenstra Jr., M. Rost, V. Chernousov, and S. Garibaldi, and over number fields by the work of J.-J. Sansuc. Parimala discussed this problem and indicated a possible way to construct a counterexample for the non-projective case over a Laurent series field over a p-adic field.

This led to much discussion after the lecture as to whether this approach would produce such a counterexample. It was determined that it could not without modification.

A. Pfister. As many of the talks demonstrated, new and sophisticated methodologies are developing to attack problems in quadratic form theory. The question always arises of whether one can obtain some of the results by more elementary means. For example, Karpenko simplified some of Vishik's proofs and talked about this at the conference. One of the first results one proves in quadratic form theory is to determine when an element is a norm from a quadratic extension. Although the usual proof of this is not deep, Pfister presented a proof that is completely elementary, and independent of the Brauer group or cohomology. The motivation for doing this is to enable Merkurjev's theorem in Milnor K-theory to be presented in an introductory course in quadratic forms.

Z. Reichstein. Reichstein lectured on his joint work with N. Lemire and V. Popov on Cayley groups. The exponential map is a crucial tool in studying Lie groups, but suffers from the fact that it is not algebraic. One would like to find an algebraic analogue even in the case of characteristic zero. By analogy with the classical Cayley map for the orthogonal group, a natural candidate would be the following: Let G be an algebraic group over a field k. Does there exist a G-equivariant birational isomorphism $\mathrm{Lie}\, G \dashrightarrow G$? If such a birational map exists, call G a *Cayley* group. Suppose that k is algebraically closed of characteristic zero. Luna asked in 1980 which G are Cayley? For such a field k, Reichstein determines which simple groups are Cayley and also which are *stably Cayley*, i.e., $G \times (k^*)^n$ is Cayley for some n.

M. Rost. Morley's Theorem states that the (appropriately chosen) three points of intersection of trisectors of the angles of a triangle form an equilateral triangle. (cf. that the bisectors of angles of a triangle meet in a point.) All known proofs eventually rest on computation using the Euclidean metric, the most recent by A. Connes [6]. Rost talked on this theorem and Connes's proof. He also indicated how one could formulate it in group cohomology independent of any Euclidean structure.

D. Saltman. The notion of "trialitarian algebra" was introduced in [11]. The underlying structure is a central simple algebra with an orthogonal involution, of degree 8 over a cubic étale algebra. The trialitarian condition relates this algebra to its Clifford algebra. M.-A. Knus, R. Parimala and R. Sridharan constructed a generic trialitarian algebra, defined using the invariants of the group $T = PG0_8^+ \rtimes S_3$ where $PG0_8^+$ is a group of projective proper similitudes [12]. This theory is parallel to the theory of central simple algebras but instead of PGL_n, T is used. In his lecture, Saltman described the center of the generic trialitarian algebra as the field of multiplicative invariants of the Weyl group $((S_2)^3 \rtimes S_4) \rtimes S_3$. He also constructed analogues of Azumaya algebras and Brauer factor sets and described trialitarian algebras in terms of Brauer factor sets, in a way similar to the classical Brauer construction of central simple algebras. In so doing, he defined a new type of cocycle attached to a pair of groups $H \subset G$, which he named $G - H$ cocycles, and which are used to describe their associated Azumaya crossed products.

A. Schultz, J. Swallow. Let $K^\times$ be the multiplicative group of the field K. Already in 1947, I.R. Šafarevič in his influential paper on p-extensions [23] realized that the growth of $\dim_{\mathbb{F}_p} K^\times/K^{\times p}$ where K runs over the finite Galois p-extensions of F can yield important information about the Galois group $G_F(p)$ of the maximal p-extension of a field F. In fact Šafarevič was able to show that if F is a local field not containing a primitive p^{th} root of unity then $G_F(p)$ is a free pro-p-group using information about $\dim_{\mathbb{F}_p} K^\times/K^{\times p}$ as above.

In the 1960s D.K. Fadeev and Z.I. Borevič [5] succeeded in classifying possible $\mathrm{Gal}(K/F)$-modules $K^\times/K^{\times p}$ for cyclic extensions of local fields of degree p^n.

In recent work, Mináč, Schultz and Swallow classified all Galois $G(K/F)$-modules $K^\times/K^{\times p}$ for cyclic extensions K/F of degree p^n where char $F \neq p$. Their description relies upon arithmetical invariants associated with K/F.

This work and other related work have already been used by Mináč and Swallow for finding conditions for the solution of specific Galois embedding problems and providing explicit solutions when they exist. They also determined which arithmetic invariants attached to cyclic extensions K/F of degree p, which are used for the classification of a Galois module $K^\times/K^{\times p}$, are actually realizable for a suitable extension K/F.

These investigations are closely related to previous investigations of Galois modules attached to fields with Galois groups of exponent 2 by A. Adem, W. Gao, D. Karagueuzian and J. Mináč [1]. These results can also possibly be useful in determining the Galois $G(K/F)$-module $\mathrm{K}^*(K)/p\mathrm{K}^*(K)$ (where $\mathrm{K}^*(K)$ is the Milnor ring of the field K).

Conclusion

During the last decade the revolutionary methods of motivic homotopy theory have intervened in the algebraic theory of quadratic forms. Many long-standing conjectures have been solved, as evidenced by this conference. These new methods affirm that even in a subject as well worked-over as the algebraic theory of quadratic forms, significant progress on interesting problems, often in unexpected directions, is still possible, and provide convincing evidence of continuing progress in the future. Of course these methods are producing striking results in many other fields as well, and it may be that those in quadratic forms will, as they have often done in the past, foreshadow similar and analogous progress in fields such as algebraic groups and Galois cohomology.

Nevertheless many important open problems remain in the algebraic theory of quadratic forms—for example, description of the generic discrete invariant of a quadratic form—which will be attacked by means of motivic methods as well as by more traditional techniques.

The conference provided a marvellous venue for exchange of ideas and the establishment of collaboration. One of the participants told us that he had never before come away from a conference with so many new ideas for his research. The BIRS facilities were greatly appreciated by all of us, and we extend our sincere gratitude to the management and staff of the institute for enabling us to have an extraordinarily successful conference.

List of Participants

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Bhandari, Ganesh (University of Western Ontario)
Brosnan, Patrick (University of California - Los Angeles)

Brussel, Eric (Emory University)
Chernousov, Vladimir (University of Alberta)
Elman, Richard (University of California - Los Angeles)
Garibaldi, Skip (Emory University)
Gille, Philippe (Universite Paris-Sud)
Gille, Stefan (Universitaet Muenster)
Hoffmann, Detlev (Université de Franche-Comté)
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Chapter 32

Banff Credit Risk Conference 2003 (03w5023)

October 11–16, 2003

Organizer(s): Tom Åstebro (University of Toronto), Peter A. Beling (University of Virginia), David Hand (Imperial College), Robert Oliver (University of California, Berkeley), Lyn C. Thomas (Southampton University)

Workshop Objectives

The objective of this workshop was to bring together a small group of people interested in the foundations and underlying theory of the mathematical and statistical prediction and decision-making models in Retail Credit Risk, an area whose most important components include the behaviours, actions and preferences of individuals for financial products. We were particularly interested in attempting to understand the similarities, differences, and interactions between retail credit risk and corporate financial risk. Modern portfolio theory, which now has an enormous scientific literature (both theoretical and experimental), plays a central role in investments, trading and valuation of assets through option pricing formulae and arbitrage models. It is worth noting that the Merton [8] and Black-Scholes [2] papers on option pricing and asset valuation are now some of the most widely cited scientific papers and have served as the foundation for the development of a rich theory of corporate risk. Although there are many similarities between retail and corporate risk, the differences are greater still - as the atomic building block of retail credit risk appears to be the account of a single individual with behavioural preferences, whereas in the corporate world the building block for a large portfolio of assets is comprised of shares of stocks or bonds of publicly traded and priced corporations.

The meetings and workshops allowed us to identify and frame the most important unsolved problems, discover linkages with modern financial theory and attempt to show where the similarities with corporate finance are meaningful and realistic. About half of the conference was devoted to four special interest workshops whose topics and special problems are reported in greater detail in the body of this report. In the first three days, a small number of papers were presented that posed questions and issues; in the afternoon of the first meeting day we organized a preliminary taxonomy of unsolved problems and asked each participant to add to or modify the list. The topics of interest were then subdivided into four major workshops, which from that point forward met during breaks and periods when no papers were presented. In several cases the workshops broke down into even smaller sub-groups and by the third and fourth day of the conference individuals and groups were making presentations of preliminary findings and recommendations to all participants. Having small individual meeting rooms with whiteboards and projectors was an invaluable asset. The findings of the workshops are reported in the sections that follow.

Default Models for Portfolios of Consumer Loans

The problem we addressed was whether there exist mathematical or statistical models describing the credit risk of portfolios of consumer loans. This is an important topic because the Basel New Accord assumes specific formulae for this which are taken directly from Merton's models of credit risk in corporate loans [8]. One also wants to develop such loan models for other reasons, such as portfolio management and securitization of loans. Thus we considered problems in three related areas:

- *New Models:* Can we develop new models that describe credit risk of consumer loans and portfolios of consumer loans ?
- *Ties to Corporate Risk:* Can we construct models of portfolio credit risk for consumer loans that link directly with the corporate credit risk models?
- *Existing Models:* Are there models of the credit risk of consumer loans and portfolios of consumer loans that lead to the formulae appropriate for current and future versions of the Basel regulations?

New Models

Three possible approaches were considered – models that segment on the dynamics and level of the default losses; models exploiting conditional independence once economic and age-related factors had been removed; and models which sought to mimic the reduced form and survival analysis approaches to credit risk for corporate loans.

The most advanced 'dynamics of losses segmentation model' for the credit risk of portfolios of consumer loans was that presented by Wedling [13]. This is an empirically based approach which requires considerable data so is probably most appropriately developed by credit bureaus or regulators. It begins by defining a large number of segments on socio-demographic variables (say more than 300) which are the basic building blocks of the portfolio. These are used to estimate a loss index function, such as actual payments compared with expected payments. One then merges segments, creating larger groupings with homogeneous dynamics of loss and similar correlations between loss indices (calendar time is used by Wedling, so this gives the response to economic effects, but one could use duration since lending start as well which would pick up financial naivety and fraud). Typically there will be around five such segments. One then does a regression of loss over given time interval against applicant and loan variables and segments according to expected loss levels. One chooses only a few, say three (high, medium and low), such segments. The resulting subsegments which come from combining the dynamics and the levels of loss (15 segments for the indicative numbers given)are then considered separately and a loss distribution is built on each by first deciding empirically on distribution function and then fitting. Correlations between the subsegments are obtained by using correlation coefficients in segment definitions and a chosen copula. One then does Monte Carlo simulation to get final results for loss distribution. It would seem very unlikely that one could obtain an analytic expression for this final distribution.

The independence approach derives from the belief that a model of loan loss should be developed which is based on the structure of conditional independence of default losses given uncertain economic factors such as interest rate, unemployment etc. Marginalization of these economic factors would induce a mathematical structure to the observed correlations between loan losses which may be fundamental to understanding the underlying dependencies, and may, in addition, help distinguish the retail credit process from the corporate one. A theory developed along these lines would not only make direct use of available data on defaults and default losses but would also give a theoretical rationale for the underlying common cause correlations between individual accounts.

The last class of models is considered in Section 2.2.

Ties to Corporate Risk

We identified six approaches to modelling the credit risk of portfolios of corporate loans:

1. Creditmetrics (J.P.Morgan 1997, see [3] a mark-to-market, ratings-based approach.

2. KMV (Kealhofer, McQuown, Vasicek, see [5] a mark-to market Merton-style model.
3. CreditRisk+ (Credit Suisse 1997, see [4] which is a default mode actuarial style model.
4. Credit Portfolio View (Wilson 1999, see [14] which is logistic regression model using lagged and correlated macroeconomic variables.
5. Markov Chain Reduced Form Models (Jarrow Lando Turnbull 1997, see [6] where default is exogenous but one estimates transition between ratings.
6. Intensity based reduced form models (Lando 1998, see [7] which use survival analysis ideas to estimate directly time to default as a function of macro-economic variables.

As was argued above there are problems with translating the models based on the Merton approach to the consumer loans unless they incorporate jump effects and even then there are problems in identifying what is a default level of a consumer's assets or his credit worthiness let alone how to estimate the correlation in these. Thus it would seem that if one wishes to try and relate models for the credit risk in both corporate and consumer loans the following are the better options.

CreditRisk+ and Related Models

This is a default model with two states – default, not default - not unlike the approach in Section 2.3. It calculates capital requirement based on actuarial approaches found in the property insurance literature. It has minimal data input but only gives loss rates, not loan value changes. It assumes each loan has a small probability of default that is independent of default of other loans. So the distribution is Binomial; one usually takes the Poisson approximation to get analytic expressions. The severities of losses are put into bands; combining frequency of default and severity of losses gives distribution of losses for each exposure band which are then summed across exposure bands. Most of these results can be applied to consumer loans, but it was pointed out the model proves difficult if default probabilities are high (above 4%) since the Poisson approximation is no longer valid and one would need to simulate using the Binomial distribution.

Markov chain reduced form type models

These are mark-to-market models (so there are several states the loan can be in). The state space is a ratings agencies rating of the bonds. Default occurs when the rating hits level D. One can build either continuous or discrete time Markov process of the change in the bonds rating. The transition matrices are estimated as a mixture of the historical process and some limiting risk processes (i.e. with transition matrices I, $p(j,j) = 1$ so no default or $p(j,D) = 1$ so all default) to get correlations. To deal with economic cycles one can let the transition matrices $p(j,k) = f($ macro variables, shock factors $)$ where the former are obtained from the data and the latter are simulated.

In a consumer loan related model one could envisage the ratings being behavioural score buckets plus a bucket for default. One could use historical transition matrices (roll out rates) and then follow the rest of the Markov chain reduced form approach. There would be lots of parameters to estimate but it has the advantage that as one gets ratings distributions at each period, one can check early and often that the model is tracking reality.

Intensity based reduced form type models

These are default mode models and are similar to survival analysis in which we estimate the hazard rate (intensity function) as a function of economic variables and loan dependent variables)

$$h(t) = \exp\{a + b\,I(t) + cW(t)\}, \qquad E[I(t)W(t)] \neq 0$$

under the assumption that $(I(t), W(t))$ is multivariate normal with assumed correlation structure derived theoretically or experimentally from the joint distributions of I and W.

Lando (Lando 1998 [7]) uses a Cox process whose structure depends on the identification of different state variables.

A consumer version of the model could be based on the survival analysis approach to behavioural scoring developed by Stepanova et al. [11].

Existing Models

There is no real clarity in how the Basel formulae were developed save that they are based on a model for the credit risk of a portfolio of corporate loans in which the portfolio consists of infinitely granular one year loans with one risk factor and value of borrowers' assets being log normally distributed. Although it is difficult to obtain mathematical derivations and references in the published literature, such models are usually attributed to proprietary models developed by the companies Credit Metrics, KMV, and CreditRisk. The assumption is that companies default if debts exceed assets and that the correlation between companies' share prices describes the correlation between their asset movements.

It was suggested that the following simple common factor model can be used to derive the Basel formulae (see, e.g., [10]). In this it is assumed that the distribution of n defaults in a portfolio of N firms is given by the well-known Binomial probability mass function

$$\mathrm{P}\left\{n \text{ defaults in } (0,T)\right\} = \binom{N}{n} p^n (1-p)^{N-n} \qquad 0 \le n \le N,$$

where p is the (common) probability of default of one firm. For large N, this distribution is approximated by the normal density. Under specialized assumptions for the losses of the firm given default, it is usually a straightforward exercise to assess the distribution of losses (risk) to the portfolio under independence assumptions. If we assume that $V_j(t)$ denotes the value of firm j at time t, common factor models assume that default of the firm occurs when this value drops below a pre-specified barrier, say K. The distribution of losses to a portfolio composed of shares of these firms are usually assumed to be proportional to the product of a fractional loss given default with the credit exposure of the firm.

The next assumption that is usually made is that the value of the firm is composed of a term with a common cause factor and a noise term structured in such a way that, given the common factor, firm defaults are independent of one another but correlation between firms exists because of the common cause factor. It is a straightforward but often difficult probabilistic calculation to determine the effect of removing the condition of the common cause factor. In general, if there are more than two common cause factors, analytical solutions are not easy to obtain. Under restrictive assumptions it is sometimes possible to derive analytic results.

A common assumption is that the $V_n(T)$ are jointly normally distributed random variables with covariance matrix Σ, i.e., the firm values are jointly dependent (the Credit metric model), and the barrier is determined by the probability of default $K = \Phi^{-1}(p)$. For example, if

$$V_n(T) = \sqrt{\rho}\, Y + \sqrt{1-\rho}\, \epsilon_n \qquad n = 1, 2, \ldots, N$$

and one assumes that Y and ϵ_n are i.i.d. standard normal random variables, then the probability that the firm's value $V_n(T)$ falls below the barrier K, given that the common factor Y takes on the value y, is

$$p(y) = \Phi\left(\frac{K - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right),$$

which means that the probability of m or more defaults is given by

$$\sum_{n=m}^{N} \binom{N}{n} \int_{-\infty}^{+\infty} \left(\Phi\left(\frac{K-\sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)\right)^n \left(1 - \Phi\left(\frac{K-\sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)\right)^{N-n} \phi(y)dy.$$

It can be shown that as the number of firms in the portfolio goes to infinity the continuous distribution function for the number of defaults exceeding x is $1 - F(x)$ where

$$F(x) = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, \Phi^{-1}(x) - \Phi^{-1}(p)\right)\right)$$

Generalizations have also been made to include cases where each firm has a probability of default p_n rather than a common p as described above, to include assumptions of multiple factors, relaxation of the i.i.d. normality assumption for Y and noise terms as well as the introduction of stochastic time-dependent volatilities.

If one seeks to consider this in the consumer loan context there are a number of problems. Do consumers default when the value of their assets fall below a prespecified barrier? More fundamentally, do consumers even know the value of their assets and if they do can they realize them? What would be a suitable K for consumers and how does one estimate the covariance matrix Σ in the consumer case when there is no equivalent of share price? These are definitional problems but there is also a structural one, as we shall discuss below.

Suppose we develop a model for distribution of loan losses of a consumer loan inspired by the corporate models of Merton [8] and Vasicek [12]. This would lead to

$$\frac{dV}{V} = Adt + BdW + CdY$$

$$\text{Percentage Change in Value} = \text{Drift} + \text{Gaussian Diffusion} + \text{Stochastic Jump Process}$$

a diffusion equation describes the change of the value of a single firm (the borrower) over time. Debt of the firm is assumed to be constant over time. When the asset value drops below the debt line (or some function of it) default is assumed to occur. Under strong assumptions distributions of loan losses for portfolio of such loans can be derived as in the way described above. More recently Zhou [15] extended the model by adding jump processes (also, see [9]).

The analogy for retail credit is that if we define a new variable, say U, to be the creditworthiness of a single consumer, an analogous diffusion equation can be derived in which the consumer (borrower) has a put option on his credit worthiness with a fixed strike price, R. Parameters are denoted by A', B', C'', R and a starting point U_0. The A term may be zero or can take into account such factors as the ageing of the consumer. R is analogous to debt. Some felt that behavioural score might be a proxy (a noisy signal) for U and one could apply the model on behavioural score.

$$\begin{aligned} \frac{dU}{U} &= A'dt + B'dW + C''dY \\ \frac{dU}{U} &= \text{Drift + Diffusion(Normal) + Jump Income Shocks(Poisson)} \end{aligned}$$

The parameters of the diffusion equation can possibly be estimated from microeconomic data such as credit bureau data. There seems to be some preliminary evidence that over short time periods credit scores may follow a simple geometric Brownian motion.

The problem is that it is not possible to obtain the Basel formulae if one has the jump term in the model and yet it was felt by all workshop participants that the jump was possibly the most important feature for a good model of the retail credit process – the reason being that events such as divorce, termination of employment, etc., must be accounted for.

Improved Models

This workshop addressed three main problems:

1. Reject Inference
2. Bayesian Marginalization, and
3. Dropout/Withdrawal Inference.

Reject inference

Financial institutions build models to predict creditworthiness, u, from variables, x, available at the time of application. Such models are based on a retrospective database of customers for whom the x variables are known. The outcome, y, which for convenience we will take to be a binary variable (1=good outcome, such as ‘repay the loan’; 0=bad outcome, such as ‘default’), will also be known for those customers who were

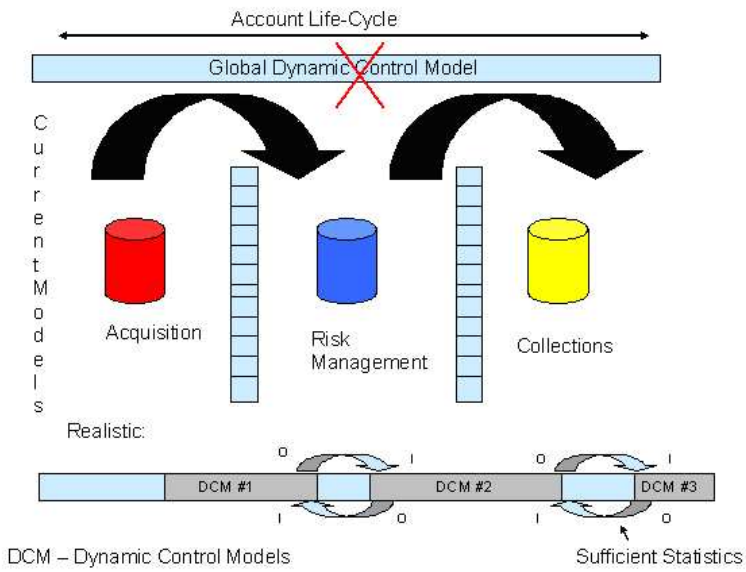

Figure 32.1: State-space modeling for retail credit.

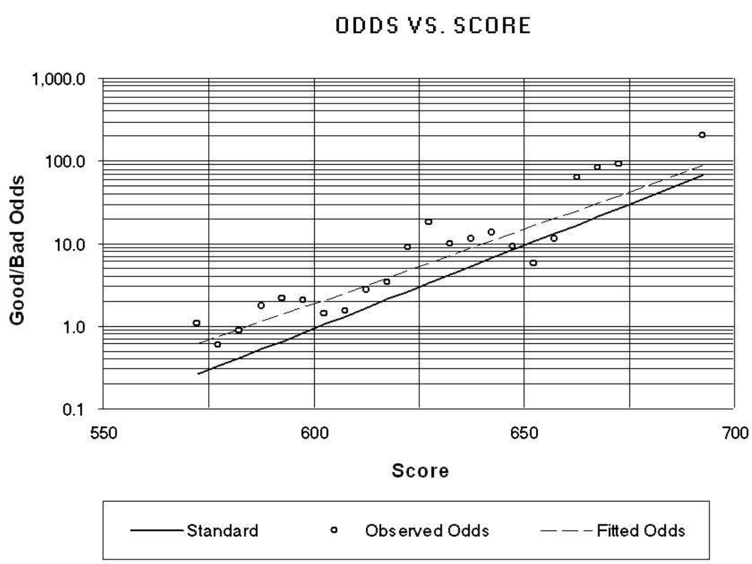

Figure 32.2: Odds versus score.

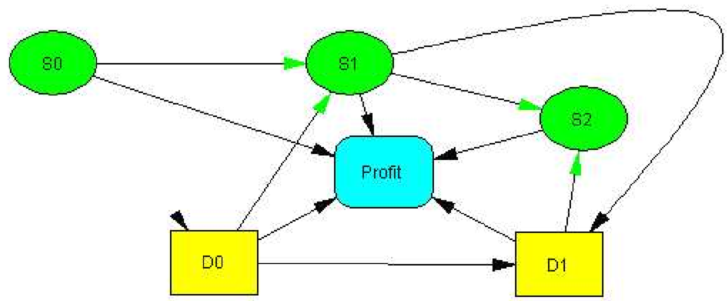

Figure 32.3: Influence Diagram.

previously accepted, but it is meaningless to speak of a good/bad outcome variable for customers who were rejected. Let a be a random variable taking the value 1 if a customer was accepted, and 0 if they were rejected.

Our aim, then, is to estimate $f(u \mid x)$, the distribution of creditworthiness, u, given the variables, x, available at the time of application. Future accept/reject decisions will be based on a comparison of some summary of our estimate of $f(u \mid x)$ with a threshold, with this threshold being determined from operational considerations. For example, we might compare the median of $f(u \mid x)$ with a threshold t, accepting an applicant with vector x if this median is above t, since we estimate that more than 50% of such people have a creditworthiness exceeding t. A difficulty arises, however, because $f(u \mid x)$ is the distribution across *applicants*, whereas we have outcome information y (related to u as described below) only for those *accepted*: we have a biased sample. *Reject inference* is the term used for strategies aimed at overcoming this problem. The problem is a universal one in the retail credit industry, and has been the focus of much interest. Several papers at the workshop described particular issues of and approaches to reject inference, and a workshop was devoted to it.

We have the decomposition

$$f(u \mid x) = f(u \mid x, a = 0)P(a = 0 \mid x) + f(u \mid x, a = 1)P(a = 1 \mid x)$$

and we can immediately distinguish between two situations.

Case 1: when the accept/reject decision was based solely on variables included amongst those now available in x, so that $a = a(x)$. That is, x may include extra variables which were not used in the accept/reject decision, but certainly includes all variables which were used in that decision. In this case, we have outcome information, y, on all of the applicants in region $A = \{x : a(x) = 1\}$ and outcome information, y, on none of the applicants in region $R = \{x : a(x) = 0\}$, where R is the complement of A. In such a situation, the only possible strategy is to build a model for the data in region A and extrapolate it over region R. The model in region A will be unbiased by the rejection process (assuming it is a properly specified model). If the extrapolation is not into parts of R far from the surface separating A from R then one might expect the model to perform reasonably well.

Case 2: when information, v, additional to that in x, was used in making the accept/reject decision, so that $a = a(x, v)$. This is the case if policy overrides were used (though it may then be difficult to articulate v explicitly). It is also the case if variables are no longer collected. In this case, the distribution of creditworthiness, u, amongst those accepted, is likely to differ from the distribution of creditworthiness amongst those rejected, for a given x.

A simple model often used in the industry assumes that

$$f(u \mid x, a = 0) = f(u \mid x, a = 1),$$

so that

$$P(a = 0 \mid x, u) = P(a = 0 \mid x).$$

That is, this model assumes that the creditworthiness distribution is the same (at given x) for applicants who are accepted and applicants who are rejected. This is the missing-at-random assumption. It assumes that, conditional on x, the actual value of the creditworthiness, u, does not influence the accept/reject decision.

In Case 1, since $a = a(x)$, we see that this applies, so that this case involves data which are missing at random. In Case 1 the values of u are missing at random. Unfortunately, since complete outcome data are observed in region A and no outcome data in region R, this does not expedite the analysis: extrapolation is necessary.

In Case 2, since the additional information in v is likely to be related to the creditworthiness variable u (else why was v used?), u is non-ignorably missing.

Of course, we only observe y, not u. The variables u and y are related. A simple model takes

$$\begin{cases} y = 1 & if\ u > s \\ y = 0 & otherwise \end{cases}$$

where s is a threshold. A more sophisticated model acknowledges that there are additional random influences on outcome, even given someone's creditworthiness, and takes $y = 1$ if $u + \delta > s$, where δ is a random variable, but we will not describe this in this summary. The simple model assumes that 'creditworthiness' is the sole determinant of outcome. Hence

$$P(y = 1 \mid x) = \int_s^\infty f(u \mid x)du$$

$$= P(a = 0 \mid x) \int_s^\infty f(u \mid x, a = 0)du + P(a = 1 \mid x) \int_s^\infty f(u \mid x, a = 1)du$$

$$= P(a = 0 \mid x) \int_s^\infty f(u \mid x, a = 0)du + P(a = 1 \mid x)P(y = 1 \mid x, a = 1)$$

In the right hand side of this expression, the probabilities $P(a = 0 \mid x)$ and $P(a = 1 \mid x)$ can be estimated immediately from the retrospective database as the proportions of customers rejected and accepted. Similarly, the probability $P(y = 1 \mid x, a = 1)$ can be estimated immediately from the retrospective database as the proportion of accepted customers who have good outcomes. Unfortunately, as explained above, we cannot estimate $\int_s^\infty f(u \mid x, a = 0)$ from the data. Reject inference describes attempts to infer the creditworthiness status of the rejected applicants, so that $\int_s^\infty f(u \mid x, a = 0)$ and hence $P(y = 1 \mid x)$ may be estimated.

In order to tackle Case 2, it is necessary to obtain extra information. This can take various forms including (i) assumptions about the forms of the distributions involved, and (ii) information from other suppliers on outcomes of rejected applicants.

There are two distinct estimation strategies for the non-ignorably missing case. The *selection model* postulates an explicit model for the missing data probabilities, so that

$$f(u, a \mid x, \theta, \phi) = f(u \mid x, \theta)P(a \mid x, u, \phi).$$

In contrast, the *pattern-mixture model* describes the marginal distribution of u as a mixture over the missing data patterns:

$$f(u, a \mid x, \phi, \pi) = f(u \mid x, a, \phi)P(a \mid x, \phi),$$

where θ and ϕ are parameters of the respective models.

The most famous selection model is that due to Heckman. This is based on making assumptions about the forms of the distributions, in particular, explaining the influence of u on the probability that $a = 0$ via the relationship between u and an unobserved variable v by assuming that u and v have a bivariate normal distribution:

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim N(\begin{pmatrix} x^T\beta_u \\ x^T\beta_v \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix})$$

with

$$P(a = 0 \mid x, u, v, \psi) = \begin{cases} 1 & v > 0 \\ 0 & v \leq 0 \end{cases}$$

From this it follows that

$$P(a = 0 \mid x, u, \psi) = \Phi(\frac{x^T\beta_v + \rho(u - x^T\beta_u)/\sigma}{\sqrt{1 - \rho^2}})$$

We see immediately from this that, in the special case of no correlation between u and v, the probability that u will be missing is independent of u, given x - we have the MAR case.

In general, we then have

$$f(u \mid x, \theta, \psi) = \frac{f(u \mid x, a = 1, \theta) P(a = 1 \mid x)}{1 - P(a = 0 \mid x, u, \psi)},$$

with all of the components on the right hand side being estimable. From this, of course, $P(y = 1)$ is estimated by integration.

Unfortunately, it seems that selection models are sensitive to the assumptions made.

Bayesian Marginalization

In designing retail credit risk models there appears to be widespread belief that models should recognize major changes or disruptions in 'lifestyle' variables - examples include such events as divorce, termination of employment, heart attack. In many cases, the lifestyle data is only available in small samples at critical points of time and may be buried or hidden from view in the data on characteristics of individuals that is normally available. Nevertheless, the effects of lifestyle data may influence some of the characteristics in the standard datasets. Given a dataset D_n we want to make the prediction

$$p(y_{n+1}|D_n) \qquad D_n = \{y_i, \mathbf{x}_i\} \qquad 1 \leq i \leq n$$

by using information from a 'latent' or unobservable variable θ, and model parameter estimates, β. Note that the index on the y (say default, fraud, "creditworthiness") to be predicted is $(n + 1)$ whereas the dataset has a subscript of n. Assuming that posterior densities can be obtained for all parameters the predictive density for y given the data is

$$p(y_{n+1}|D_n) = \int p(y_{n+1}|D_n, \beta, \theta) p(\beta, \theta|D_n) d\beta d\theta.$$

The first term inside the integral is independent of the data D_n given the parameters β and θ, i.e.

$$p(y_{n+1}|D_n, \beta, \theta) = p(y_{n+1}|\beta, \theta);$$

thus, the conditioning on the dataset used to develop the model has been removed. The second term is the posterior distribution of the unobservable parameters and latent variable conditioned on D_n. We have

$$p(\beta, \theta \mid D_n) = p(\theta \mid \beta, D_n) p(\beta \mid D_n),$$

in which $p(\theta \mid \beta, D_n) = p(\theta \mid \beta)$, the parametric distribution adopted for θ. For example, $p(\theta \mid \beta) = \lambda(\beta) \mathrm{e}^{-\lambda(\beta)\theta}$, where $\lambda(\beta)$ is a given function of β.

In practice, accurate estimates of the βs are derived from the data D_n: we denote these estimates by $\hat{\beta} = \hat{\beta}(D_n)$ and assume that the posterior $p(\beta \mid D_n)$ is effectively concentrated at $\hat{\beta}$. Then, the predictive equation can be replaced by the approximation

$$p(y_{n+1}|D_n) \approx \int p(y_{n+1} \mid \hat{\beta}, \theta) p(\theta \mid \hat{\beta}) d\theta.$$

Use of the method rests on trying out various models for the effect of the latent variable, assessing the effect of the specified conditional density function $p(\theta \mid \beta)$.

It is possible that the latent variable can be observed and measured in a separate experiment unrelated to the data used for deriving the parameter estimates of β. To obtain $p(y_{n+1}|D_n)$ it is suggested to use a two-stage procedure where the form of $p(\theta|\beta, z)$ is first estimated from a small-scale survey where information on θ can be obtained from additional observations of z at some significant cost. (In general, z can also be a vector although, for simplicity, we only consider a single variable). Then one can obtain a new (approximate) predictive density

$$p(y_{n+1}|D_n, z) = \int p(y_{n+1}|\hat{\beta}, \theta) p(\theta|\hat{\beta}, z)$$

in which we have assumed that

$$p(y \mid \beta, \theta, D_n, z) = p(y \mid \beta, \theta) \text{ and } p(\beta, \theta \mid D_n, z) = p(\theta \mid \beta, z)p(\beta \mid D_n).$$

The steps recommended to obtain $p(y_{n+1}|D_n, z)$ are:
1. In a small-scale (possibly expensive) survey, relate θ to z;
2. In that survey test that y does not significantly depend on z;
3. Infer estimate of θ or distribution of θ from β and z;
4. Calculate the predictive distribution of y_{n+1} given D_n and z.

Steps 1 through 4 are similar to those performed in 2-stage least squares estimation with latent variables (see, e.g., [21]) except that here we switch from one (small-scale) data set when estimating θ, another when estimating $p(y|\beta, \theta)$.

Another approach that may be more efficient is to use maximum likelihood estimation where the first $p(\theta|\beta, z)$ and second $p(y|\beta, \theta)$ stage equations are estimated simultaneously (iteratively back and forth) using for example the EM algorithm developed by Dempster et al [23]. The point of a simultaneous estimation is that the first-pass estimate of stage 1 in the 2-stage approach above might not be the most efficient. The simultaneous estimation approach poses a technical challenge as one is toggling between two different data-sets to converge on global parameter estimates.

Dropout/Withdrawal Inference

This problem can be formulated in a similar way to the reject inference problem where we define

$$a = \begin{cases} 1 & \text{if applicant declines the offer made} \\ 0 & \text{otherwise} \end{cases}$$
$$y = \begin{cases} 1 & \text{if applicant is creditworthy} \\ 0 & \text{otherwise,} \end{cases}$$

and where x is a vector of applicant descriptors.

In this case we know even less about the selection mechanism than in the reject inference case. This means there are likely to be even more unobserved variables in the selection equation than in the reject inference case. This makes the bias from using the drop-in sample even more likely. The same analysis as above holds with MNAR more likely. The magnitude of the bias depends crucially on how able we are to specify the true selection equation. Our hypothesis, following Åstebro and Bernhard (2003, see [17]) is that the better credit worthy applicants are likely to self-select out from the pool of applicant given a single price offer. Indicators of that self-selection bias are possibly, education, work experience and other variables that makes the applicant go to suppliers of credit that are more sensitive to this information, typically friends family and angel investors. In a credit market where other lenders are also not able to evaluate human capital, and in addition there is little or no informal market for credit, then this problem diminishes. In the U.S the informal market is rather large and so the problem is probably of some significance.

Selection will also depend on offers made by competing credit suppliers. These offers will be dependent on applicants characteristics and on anticipated offers by other suppliers.

Decision models and dynamic control actions

Issue

Credit management can be seen as a control problem. The aim of the exercise is for managers to take actions to achieve business objectives. The process by which actions are chosen is the focus this workshop. The decision process can be thought of as a function mapping from states describing the lender, the borrower, and any relevant context to the actions. What form should such a function take?

The motivation behind this workshop was the belief that, currently, decision functions constitute a very small and specialized subset of the space of all possible decision functions. The concern is that the subset which is commonly used, is used for historical or other accidental reasons, rather than fitness to purpose. Current decision functions typically do not choose actions to optimise some objective function; they usually only consider the effect of the current action and not possible future actions; they usually do not take into account the sophisticated analysis of the interplay between the lender and the borrower; and they typically do not consider the possibility of a wide range of actions at one time. Are there classes of decision functions which are particularly well-suited to the types of problems facing the credit manager?

Often, there exists the well-known stove-pipe situation, where different analytical models in the life of an account do not effectively communicate in terms of integrating decision models. Acquisition models, for example, often do not allow for the fact that there are numerous actions that can be taken by risk management to control accounts and improve profitability. Thus, an average case is assumed for the behaviour of the account. It is unlikely that a global dynamic control model, i.e., one that covers the entire life-cycle of an account or portfolio of accounts, can or will be constructed in the near future. This is a result of the complexity and data requirements for such a model. It is our belief that dynamic control models can be built in the near future that cross the individual traditional stovepipes, for example, from acquisition to risk management.

Aims

The aim of this work was to provide a conceptual framework for modelling individually and collectively the decisions involved in the consumer lending process. This would go from prospect mailing, at one end, to longevity bonuses, attrition modelling, collection scoring, and rebranding at the other. One would not expect to have one model to cover the whole process, but one could expect coherence between the models where coherence would mean consistency of the objective functions and descriptions of the answers that are consistent when the models of different phases of the process overlap. This would allow the scoring community to identify the issues that need to be modelled: risk-based pricing, adaptive adjustments in product features at the individual customer level, appropriate segmenting of the population, and overall customer-profit optimisation.

A dynamic programming approach

The generalised approach that was beginning to be formulated during the workshop has much in common with a dynamic programming or optimal control approach. This is not really surprising, since the idea of using this approach on a very simple consumer credit model was suggested forty years ago by Cyert and Davidson (1962) and Liebman [1]. What is more surprising is that the ideas have not been taken up for practical implementation by the industry. We speculate that this is partly because of the division into the distinct stovepipes mentioned above: those concerned with acquisitions have different optimality criteria from those concerned with risk, and there is relatively little communication between them.

The problem has various critical dimensions which would need to be taken account of in any complete model, including: the number of periods of time, the number of players (a single lender; a lender and a borrower; a lender and two types of borrower; more). The model can be made as complex as one wants, and the trick, as in all scientific modelling, will be to establish a model which is simple enough to be implementable in practice, but sophisticated enough to be useful. It was too much to hope that such a model could be developed at the meeting itself.

Scorecard Alignments

Problem:

Model Updating. During the development phase much effort is made to ensure that the scorecard is as predictive as possible on the holdout sample. However, the performance of the scorecard soon begins to deviate from the theoretical performance when applied to new data in the live environment.

Questions:

What level of performance deterioration warrants a scorecard rebuild?
When to realign versus when to rebuild?

In the case of deteriorating performance three options are considered:

1. Scorecard realignment

   It is common for a scorecard to undergo realignment at regular periods (usually 12 months). The process involves assessing previously scored data for which the full outcome period is observed. The observed score to odds relationship will be determined using linear regression and adjustments made to the slope and intercept of the regression so that the scores will be aligned with the standard score to odds relationship as shown in Fig. 6.

2. Scorecard weights re-estimation

   The cost associated with compiling a data set would be the same whether a full re-development or a weight re-estimation were being carried out. Therefore this approach is quite uncommon and would only happen if other system issues were pertinent.

3. Redevelop scorecard

   Due to the desirability of a stable live system and the prohibitive cost of a new rebuild, scorecards often are used for long periods of time. Regular model developments do not happen.

When to rebuild

During the development phase much effort is expended to achieve high predictability of a new scorecard on the training sample. Much of this effort goes towards defining suitable x-variables derived from the application information. However, the performance of a scorecard deteriorates as it continues to be applied to new applications in the live environment. This loss of discrimination carries a cost. Thus, a decision has to be made on when to renew the scorecard, i.e. to find the optimal balance between the costs of renewal and non-renewal.

An applicant for credit is required to supply various details such as age, employment status, residence status, etc. On the basis of this information, summarized as a vector $x = (x_1, ..., x_p)$ of scores, a decision is made as to whether to issue credit or not. A common vehicle for this decision is a logistic regression function that has been developed on a training sample. Thus, the training data have been used to estimate the parameters $\beta = (\beta_1, ..., \beta_p)$ in the model

$$\mathrm{P}(G \mid x) = 1/(1 + \mathrm{e}^{-x^{\mathrm{T}}\beta}),$$

where $\mathrm{P}(G \mid x)$ denotes the probability that an applicant with score vector x will be a 'Good' (defined as one who will always repay a loan). The model may be re-expressed in the form

$$\log[\mathrm{P}(G \mid x)/\{1 - \mathrm{P}(G \mid x)\}] = x^{\mathrm{T}}\beta.$$

The left-hand side here is the *log-odds*, odds being the ratio of Good to Bad probabilities, and the right-hand side is the score combination. In practice, the score combination is subjected to a linear transformation, say as $a + b(x^{\mathrm{T}}\beta)$, in order to make the resulting *score* take values in a standardized range. The standard straight-line relationship of log-odds to score is shown as the bold line in Fig. 1. A score threshold is set and only applicants with a score above the threshold will be accepted, i.e offered credit.

There are various levels of renewal of a scorecard, each incurring a significant cost. However, we are here concerned only with the cost of non-renewal, and for this we need some notation. Let $\gamma_G(t_0, t)$ be the probability of accepting a Good at time t using a scorecard developed at time t_0, where $t_0 \leq t$. Likewise, let $\gamma_B(t_0, t)$ be the probability of accepting a Bad. One common practice is to set the score threshold to achieve a given ratio of accepted Goods and Bads; for illustration, we will take this ratio to be 15/1.

Among applicants accepted at time t, using a scorecard developed at time t_0, the odds of Goods to Bads is

$$\begin{aligned}\phi(t_0,t) &= \mathrm{P}(Good \mid accepted)/\mathrm{P}(Bad \mid accepted)\\ &= \mathrm{P}(accepted \mid Good)\mathrm{P}(Good)/\mathrm{P}(accepted \mid Bad)\mathrm{P}(Bad)\\ &= \{\gamma_G(t_0,t)\pi_G\}/\{\gamma_B(t_0,t)\pi_B\},\end{aligned}$$

where $\pi_G = 1 - \pi_B$ is the overall proportion of Good applicants.

(For simplicity, it is assumed here that π_G is constant over time.)

For illustration, suppose that, at time t, there are N_t applicants for a loan of 5000 (pounds) at interest rate r, so the amount to be repaid is $5000(1+r)$. On average, there will be $\pi_G N_t$ Good applicants, of whom $\gamma_G(t_0,t)\pi_G N_t$ are accepted on the basis of the t_0-vintage scorecard. Likewise, the expected number of Bads accepted is $\gamma_B(t_0,t)\pi_B N_t$. Each accepted Good will yield a profit of $5000r$ and each accepted Bad will lead to a loss of 5000. (For simplicity, we assume that the whole amount is lost to a Bad, otherwise a specified fraction of 5000 is to be applied.) Thus, the expected net profit from the N_t applicants is

$$5000r\gamma_G(t_0,t)\pi_G N_t - 5000\gamma_B(t_0,t)\pi_B N_t = 5000N_t\gamma_B(t_0,t)\pi_B\{r\phi(t_0,t)-1\}.$$

Thus, the expected profit difference between using an up-to-date scorecard and one developed at time $t_0 < t$, is

$$5000N_t\pi_B[\gamma_B(t,t)\{r\phi(t,t)-1\} - \gamma_B(t_0,t)\{r\phi(t_0,t)-1\}].$$

Suppose, for example, that $\gamma_G(t,t) = 0.90$, $\gamma_B(t,t) = 0.06$, $\gamma_G(t_0,t) = 0.84$ and $\gamma_B(t_0,t) = 0.07$; then $\phi(t,t) = 15$ and $\phi(t_0,t) = 12$. Also, suppose that $r = 0.1$. Then, the expected difference is

$$5000N_t\pi_B[0.06\{1.5-1\} - 0.07\{1.2-1\}] = 80N_t\pi_B.$$

This cost can be set against that of renewing the scorecard and thereby contribute to the decision process.

How to determine when a rebuild is needed

When the cut-off score is chosen in the scorecard development process this is defining $\gamma_G(t_0,t_0)$ and $\gamma_B(t_0,t_0)$ - they are the coordinates of the point on the ROC curve chosen as the cut-off. When one starts implementing the scorecard one is actually calculating $\gamma_G(t_0,t_0+d)$ and $\gamma_B(t_0,t_0+d)$, where d is the lead time between the development sample observation point and the scorecard implementation point. It seems reasonable to assume that, if a scorecard was left for ever, it would lose all its discrimination, i.e. $\gamma_G(t_0,\infty) = \gamma_B(t_0,\infty)$, which is a point on a diagonal ROC curve. If nothing is done to the scorecard, and if one assumes that the deterioration is the same for goods as bads then it moves on the straight line on the ROC curve between these two points. If there are regular recalibrations, so that one keeps the same accept rate, then one moves on a line where $a(t) = \gamma_G(t_0,t)(1-\pi_B) + \gamma_B(t_0,t)\pi_B$ is a constant and ends up at a different point on the ROC curve diagonal. In either case one also has to estimate the speed of movement along these curves. The simplest reasonable movement would be negative exponential so that, after time t, if the length of the curve is a, the scorecard would have moved $a(1-\mathrm{e}^{-bt})$ along it. One can estimate b, and also check the validity of the negative exponential assumption by looking at the delinquency reports, which are segmented by start date and duration of account. When one has these estimates of γ_G and γ_B one can apply the decision structure described in Section 5.3 to determine when a rebuild is likely to be advantageous, but this has yet to be done in practice.

Summary and Conclusions

The BIRS meetings and workshops gave attendees the opportunity to informally discuss and describe some of the underlying mathematical and statistical risk problems associated with retail credit. We concluded that the published literature on these types of problems at both the account and portfolio level is sparse.

One needs to build on the success of consumer credit scorings ability to make assessments of relative likelihoods of default and other risk outcomes to make good time-dependent probability and categorical forecasts. There is considerable confusion in the credit literature between probability forecasts and categorical forecasts and how the former, in conjunction with business and financial assumptions lead to decision rules (e.g. an Accept/Reject rule) that can then be used to derive the latter.

There appear to be important difficulties in understanding the theoretical and experimental ways in which retail risk scores can be transformed and converted into absolute default probabilities largely because traditional scoring has been used to establish relative improvement in simple statistical and business measures that depend on the probability of default. To the best of our knowledge there are no documented academic or industry attempts to incorporate multiple economic factors into the population odds of default or into the stochastic process describing the evolution of future scores.

We discussed the important behavioural risks associated with consumer lending, including the formulation of the consumer as an active participant in the borrowing/lending negotiation process. These negotiations depend on the preference functions of both parties. We presented a preliminary borrowing/lending model that incorporated many aspects of negotiation and exchange between lender and borrower and concluded that experiments to evaluate preferences are a prerequisite to developing models and strategies for customization of financial products.

Simulation techniques in part depend on relevant performance data but it was by no means clear that we have common agreement on probability models that capture the essential features of the prediction and decision-making structure as an integral component of the lending process.

There was some discussion as to how artificial neural networks could and could not be used to predict (on the basis of past data fits) the outcomes of random quantities that depend on future controls and actions not recorded in the historical fitting exercises.

Work was undertaken to develop models based on the monitoring tools in consumer credit risk systems to identify when such systems needed recalibration, re-estimation and replacement.

A part of one workshop discussed risk-based pricing models that included the propensity of the customer to take a loan offer at a specified price and term; there was considerable debate as to the form of the elasticity of response to price of risk. Some specific models were discussed and there was general agreement that this important area needed considerable future study and experimentation.

There was unanimous agreement that there are enormous differences between retail and wholesale credit financial markets because retail risk is affected by social behaviour as well as by business cycles and economic factors. A further complication is that there is scant pricing information available for the purchase and sale of retail loan portfolios in either a primary or secondary market. These additional complexities offer important research challenges to academics and practitioners alike.

There appeared to be near-unanimous agreement that the parameters and models that are widely used in measuring, assessing and predicting default risk in wholesale commercial loan markets cannot be applied to retail loan portfolios.

In one workshop we discussed an approach that could be used to cope with the highly dimensional models that are traditionally used to study correlated returns (which are used in the standard value at risk models for liquid corporate securities). The approach would be to correctly formulate a portfolio loan model that, at its core, has a strong conditional independence structure in which correlations are induced by marginalizing over one or more common-cause factors.

A set of problems and models that attracted considerable interest was the inclusion of the proposed Basel II capital accords in setting required levels of regulatory and equity capital. These requirements would recognize the risk contributions of different loan types as well as the composition of individual/behavioural risk profiles. Several models and theoretical studies were proposed.

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Chapter 33

MITACS Theme and Consortia Meetings (03w5301)

October 18–25, 2003

Organizer(s): MITACS, Peter Borwein (Simon Fraser University), Evangelos Kranakis (Carleton University), Brian Alspach (University of Regina), Michael Mackey (McGill University)

Outline

Mathematics of Computer Algebra and Analysis—Consortium Meeting
Project Leader: Peter Borwein

IT Theme Meeting
Team Leader: Evangelos Kranakis

Communication and Information Networks Research Consortium Meeting
Project Leader: Brian Alspach

Biomedical Theme Meeting
Team Leader: Michael Mackey

List of Participants

Ahokas, Graeme (University of Calgary)
Alspach, Brian (University of Regina)
Amiraslani, Amir (University of Western Ontario)
Bajaj, Naresh (University of Calgary)
Barbeau, Michel (Carleton University)
Beauchemin, Catherine (University of Alberta)
Borodin, Allan (Dalhousie University)
Borwein, Peter (Simon Fraser University)
Botting, Brad (Waterloo University)
Bub, Gil (McGill University)
Bull, Shelley (University of Toronto)
Carpenter, Eric (University of Alberta)
Chul, Lee Hyun (Dalhousie University)

Cleve, Richard (University of Calgary)
Coombs, Dan (University of British Columbia)
Corbeil-Letourneau, Simon (McGill University)
Corey, Mary (University of Toronto)
Corless, Rob (University of Western Ontario)
Dyke, Cheryl (University of British Columbia)
Eberly, Ron (University of Calgary)
Fee, Greg (Simon Fraser University)
Ferguson, Ron (Simon Fraser University)
Gerhard, Jurgen (Maplesoft)
Ghavam, Amir (Carleton University)
Giesbrecht, Mark (University of Waterloo)
Glass, Leon (McGill University)
Gomez-Acevedo, Horacio (University of Alberta)
Graham, Jinko (Simon Fraser University)
Hahn, Gena (University of Montreal)
Hall, Jeyanthi (Carleton University)
Hamdy, Safuat (University of Calgary)
Hatfield, Adam (Government of Canada)
Hill, Alan
Hovinen, Bradford (University of Waterloo)
Hu, Nikki (University of Alberta)
Janssen, Jeannette (Dalhousie University)
Jaumard, Brigitte (Ecole Polytechnique de Montreal)
Kalyaniwalla, Nauzer (Dalhousie University)
Keshet, Leah (University of British Columbia)
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Knauer, Josh (Simon Fraser University)
Kranakis, Evangelos (Carleton University)
Kublik, Richard (University of British Columbia)
LI, Xiangwen (University of Regina)
Labahn, Georg (Waterloo University)
Larribe, Fabrice (McGill University)
Lee, Sophia (University of Toronto)
Leon, Josh (University of Calgary)
Leung, Henry (University of Calgary)
Li, Michael (University of Alberta)
Li, Zheyin (Grace) (Carleton University)
Lin, Bo (Dalhousie University)
Liu, Hongyu (Dalhousie University)
Liu, R.
Long, Hongwei (University of Alberta)
Loredo-Osti, J C. (McGill University)
Luchko, Tyler (University of Alberta)
Luo, Honghui (Carleton University)
M'lan, Cyr Emile (Hospital for Sick Children)
Mackey, Michael (McGill University)
Maree, Stan (University of British Columbia)
Marsh, Rebeccah (University of Alberta)
Milios, Evangelos (Dalhousie University)
Mirea, Lucia (University of Toronto)
Monagan, Mike (Simon Fraser University)
Morgan, Ken (Montreal University)
Mueller, Siguna (University of Calgary)

Ng, Raymond (University of British Columbia)
Nowakowski, Richard (Dalhousie University)
Pike, David (Memorial University of Newfoundland)
Pujo-Menjouet, Laurent (McGill University)
Ramanampanoharana, Tantely (University of Montreal)
Roche, Austin (Simon Fraser University)
Sajna, Mateja (University of Regina)
Schuurmans, Dale (University of Waterloo)
Shin, Jean (Simon Fraser University)
Singh, Karen
Smith, Bruce (Halifax University)
Srinivasan, Raj (University of Saskatchewan)
Storjohann, Arne (University of Waterloo)
Sun, Wei
Syed, Zainab (University of Calgary)
Tang, Helen (Carleton University)
Tritchler, David (University of Toronto)
Tsoi, Vincent (University of Calgary)
Tuszynski, J (University of Alberta)
Tyson, Rebecca (Okanagan University College)
Vigmond, Edward (University of Calgary)
Wan, Tao (Carleton University)
Wan, Xiaomeng (Dalhousie University)
Wang, Changping (Dalhousie University)
Watmough, James (University of New Brunswick)
Whitesides, Sue (McGill University)
Williams, Hugh (University of Calgary)
Wu, Jianhong (York University)
Xu, Dashun (University of New Brunswick)
Xu, Wei (University of Toronto)
Xu, Zhen (Carleton University)
Zeng, QingLing (York University)
Zhang, Richard (Simon Fraser University)
Zhao, Xingqui
Zhu, Huaiping (York University)

Chapter 34

Current Trends in Representation Theory of Finite Groups (03w5099)

October 25–30 2003

Organizer(s): Jonathan L. Alperin (University of Chicago), Gerald Cliff (University of Alberta), Michel Brou´e (Institut Henri Poincar´e and Universit´e Paris VII)

Introduction

It is over a century since Frobenius initiated the study of group representations. One feature of the subject in his day was the interplay between the general theory and the study of the important special groups (for Frobenius, the symmetric groups and $PSL(2, p)$ for example) and this connection has continued to be a central theme of the field ever since, with ideas, questions and motivation flowing two ways.

The conference exhibited this connection in many ways and the program displays the current work on general theories and the study of the special groups, mainly the reflection groups, finite groups of Lie type, and other related groups. This latter work divides into the study of the representation theory for the natural characteristic and the cross characteristic case, and we shall organize the report along these lines.

General theory

There are several interesting conjectures, none of which are proved for all finite groups, dealing with modular representations. Most prominent are conjectures of Brauer, Alperin, Alperin-McKay, Broué, and Dade. Broué's conjectures, refined by Rickard, have been studied intensively in the last 10 years. A reference is [8]. Broué and Rickard, at least conjecturally, introduced the theory of derived categories into representation theory of finite groups. In the case that G is a finite reductive group, there is a strong with cohomology of Deligne-Lusztig varieties, as will be discussed in section 4.

Reporting on work with Raphael Rouquier, Joe Chuang spoke on filtered derived equivalences. The authors had proved the conjecture of Rickard establishing derived equivalences between blocks using complexes suggested by Rickard, a very notable recent contribution. These equivalences have remarkable properties giving rise to filtered derived equivalences, a new idea, which was introduced in the talk and then developed during the lecture. This is an example of work on special groups, the symmetric groups, leading to new general results.

Jon Carlson spoke on joint work with Jacques Thévenaz [4,5]. They have proved that relative syzygies of Alperin generate the torsion-free part of the group of endo-trivial modules and give another set of such generators by a homological construction. Group cohomology is used heavily in the arguments. Serge Bouc surveyed his work on biset functors and their connection with endo-permutation modules. In particular, he discussed ideas, which if carried through, would lead to the classification of all such modules. The work of Carlson and Thévenaz, recent work by Mazza and these idea of Bouc give credence to optimism about progress in this area.

Lluis Puig's talk was entitled "The $k*$-localizing functor" and deals with the open question, of interest to homotopy theorists as well as group theorists, whether certain local structure of groups can be "put together" in a construction that leads to a new algebra. Puig lectured on his work and considerable progress. One of the key tools are the endopermutation modules discussed above, in particular to the work of Carlson and Thevenaz.

Radha Kessar spoke on the related topic of fusion systems and whether certain specific ones could arise in the study of non-principal blocks. Using the structure of the representation theory of simple groups, she showed that no such examples could exist, another example of connections between the parts of representation theory being very relevant.

There is a well-known and important character correspondence, due to Isaacs and Glauberman. Watanabe and Horimoto discovered a correspondence of blocks playing a role. Koshitani spoke about his work following up on the work of Watanabe constructing associated Morita equivalences.

Geoff Robinson reported on his work on the number of irreducible characters in blocks, an old topic with striking results, like the Brauer-Feit theorem and important conjectures. He has developed some striking new estimates.

Gabrielle Nebe spoke on integral representations of finite groups and, in particular, on the radical idealizer chain of symmetric orders. Detailed results were given for the case of cyclic blocks but much remains unknown and intriguing.

Equal characteristic: the general linear and symmetric groups

There has been interest for over a hundred years in studying representations of the symmetric group. The irreducible ones at characteristic 0 are well known; forms of these over the integers can be reduced modulo a prime p, and the problem is to decompose these into irreducibles at characteristic p. This problem of finding these decomposition numbers has attracted much attention. Although quite a lot is known, there is no general answer and no conjecture.

Great progress has been made by Gordon James and Alexander Kleshchev (two of our speakers.) The Iwahori-Hecke algebra $\mathcal{H}_q$ of type A is a deformation of the group algebra of the symmetric group S_n; when the parameter q is a p-th root of unity where p is prime, the decomposition numbers for the representations of $\mathcal{H}_q$ are related, but not equal, to the decomposition numbers for representations of the symmetric group at characteristic p. In recent years a remarkable conjecture, partly inspired by Kleshchev's work, was made by A. Lascoux, Bernard Leclerc (one of our speakers), and R. Thibon [7], relating decomposition numbers for $\mathcal{H}_q$, where q is a p-th root of unity, to canonical bases of certain Fock spaces, which are representations of quantized enveloping algebras of affine Kac-Moody Lie algebras of type $A_{p-1}^{(1)}$. This conjecture was proved by Ariki [1] and Grojnowski (both participants in this workshop.)

The representation theory of the symmetric groups has long been known to be related to the representation theory of general linear groups $GL(n, K)$. At characteristic 0 this was discovered by Frobenius and Schur in the early 1900's; there is still much interest at characteristic p. There are several ways of looking at this relation. The general linear group $GL(V)$ acts on the r-fold tensor power $\otimes^r V$, and the symmetric group S_r also acts by permuting the ordering of the the vectors in a tensor. It is known that these actions are mutual centralizers of each other. This is called Schur-Weyl duality. There is also a finite dimensional algebra, called the Schur algebra $S(n, r)$ whose representation theory is a portion of the theory for $GL(n, K)$, and also is related to representations of S_r. There is also a quantized version of this, the q-Schur algebra, closely related to the Hecke algebra $\mathcal{H}_q$.

Gordon James has a conjecture that in certain cases, the decomposition numbers for representations of the symmetric group at characteristic p are equal to the decomposition numbers for $\mathcal{H}_q$ at a p-th root of unity, which are now known by Lascoux-Leclerc-Thibon-Ariki-Grojnowski. James spoke at our meeting on an interesting new conjecture which equates some decomposition numbers for the symmetric group at different primes. He also gave some intriguing consequences of this conjecture. In the smallest case where his conjecture is not known to hold, if it were false then his earlier conjecture on decomposition numbers for

H_q would also fail. He also discussed a way of calculating some of the decomposition numbers of certain blocks of the symmetric group.

Anne Henke spoke on comparing the representation theory of Schur algebras $S(n, r)$ for different values of n and r. A method was given involving the use of exterior powers of the natural representation V of $GL(V)$ to compare theories for different Schur algebras.

Jon Brundan spoke on Kazhdan-Lusztig theories in type A. The original Kazhdan-Lusztig conjecture for $gl_n(C)$ describes the composition multiplicities of Verma modules in terms of certain Kazhdan-Lusztig polynomials arising from the Hecke algebra of type A. Thanks to Schur-Weyl duality these polynomials can be defined in a completely different way – replacing the Hecke algebra with the quantized enveloping algebra $U_q(gl_\infty)$ and defining a canonical basis in its natural tensor space using Lusztig's quasi-R-matrix. This leads to a reformulation of the Kazhdan-Lusztig conjecture peculiar to type A which admits several remarkable generalizations – for example, the well known Lascoux-Leclerc-Thibon conjecture proved by Ariki and Grojnowski is one such. Brundan spoke about the finite dimensional representation theory of the supergroups $GL(m|n)$ and $Q(n)$ which fit neatly into this philosophy.

A. Kleshchev spoke on joint work with Brundan. To every nilpotent class e in a semisimple complex Lie algebra, Premet, in 2002, associated a certain associative algebra W. Roughly speaking, W is the endomorphism algebra of a generalized Gelfand-Graev representation corresponding to e. Little is known about the structure and representation theory of W. He explained the importance of W.

Hyohe Miyachi spoke on some results on canonical bases of Fock space; he elaborated on some of his joint work with Bernard Leclerc. The interest stems from the Lascoux-Leclerc-Thibon conjecture. There are also connections with conjectures of Broué. In general the canonical bases are not explicitly computed, but in certain cases one can give explicit closed formulae for them.

Bernard Leclerc spoke on Lusztig's semicanonical bases. In 2000, Lusztig introduced a new basis of the positive part of the enveloping algebra of a Kac-Moody algebra : the semicanonical basis. He spoke on a comparison between the canonical and the semicanonical basis and multiplicative properties of the dual semicanonical basis

Christine Bessenrodt spoke on invariants such as the determinant or the Smith normal form of certain integral matrices which come from the character tables of the symmetric groups and its double covers. As a consequence, she has a new proof of a strengthened version of a conjecture of Mathas on the determinant of the Cartan matrix of a Hecke algebra $\mathcal{H}_q$ of type A, at a primitive p-th root of unity.

A natural question is whether there is a version of Schur-Weyl duality for groups other than $GL(V)$, for instance for symplectic or orthogonal groups. Richard Dipper spoke on this. Tensor space V^r is replaced by a mixed tensor space, defined as r-fold tensor product of V tensored with the s-fold tensor product of the dual space of V. There is an action of the Brauer algebra on tensor space and it is known by a theorem of Brauer, that Schur-Weyl duality holds, if K has characteristic zero. In case of mixed tensor space, the same holds, replacing the Brauer algebra by a certain subalgebra, called walled Brauer algebra, by a theorem of Bankart et al. Dipper considered the case that K has finite characteristic.

Alison Parker spoke on determining all the homomorphisms between Weyl modules for $SL_3(K)$ where K is an algebraically closed field of characteristic at least five. As a corollary of this result she obtained all the homomorphisms between Specht modules for the symmetric group when the partitions have at most three parts and the prime is at least five. She also found that the Hom spaces are always at moat one dimensional in both cases.

Karin Erdmann spoke on certain modules for the Schur algebra $S(n, r)$, called tilting modules. for Schur algebras $S(3, r)$ over characteristic two The Schur algebras $S(n, r)$ are quasi-hereditary, where the standard modules are the usual Weyl modules, and the costandard modules are the duals of the Weyl modules, The 'tilting modules' which are the modules which have a filtration by Weyl modules and also a filtration by dual Weyl modules, play an important role for the understanding of decomposition numbers for general linear groups, and also for symmetric groups. When $p = 2$ and $n = 3$, the only tilting modules missing to get an induction going are the ones labelled by the partition with one part. We have a new parametrization of these tilting modules (and also of some classes of Weyl modules), in terms of their restriction to the finite groups $GL(3, 2)$. This uses finite-group technology and alsio Auslander-Reiten theory.

Andrew Mathas spoke on elementary divisors of Specht modules. Let $\mathcal{H}_q(\mathfrak{S}_n)$ be the Iwahori–Hecke algebra of the symmetric group. This algebra is semisimple over the rational function field $\mathbb{Q}(q)$, where q is an indeterminate, and its irreducible representations over this field are q-analogues $S_q(\lambda)$ of the Specht modules of the symmetric groups. The q-Specht modules have an "integral form" which is defined over the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ and they come equipped with a natural bilinear form with values in this ring. Now $\mathbb{Z}[q, q^{-1}]$ is not a principal ideal domain. Nonetheless, one can try to compute the elementary divisors of the Gram matrix of the bilinear form on $S_q(\lambda)$. When they are defined, one gets a precise relationship between the elementary divisors of the Specht module $S_q(\lambda)$ and $S_q(\lambda')$, where λ' is the conjugate partition. Also, the elementary divisors can be computed when λ is a hook partition. There are examples to show that in general the elementary divisors do not exist.

Representations of finite reductive groups, cross characteristic

There is a large amount of work on representations of finite groups of Lie type, such as $SL(n, k)$, where k is a field of characteristic p, where the groups act on vector spaces over a field K of characteristic 0 or characteristic $\ell \neq p$. Enormous progress was made by Deligne and Lusztig, who found representations coming from ℓ-adic cohomology of certain algebraic varieties.

There has been great interest in block theory for groups of Lie type at cross characteristic. This started with important work of Fong, Srinivasan, and Broué, all of whom were at this workshop. Complex reflection groups are increasingly playing an important role in this theory.

Broué's conjectures predict that the ℓ-adic cohomology complex of a Deligne-Lusztig variety is a tilting complex for a derived equivalence between the principal ℓ-block of G and that of the normalizer of the ℓ-Sylow subgroup of G.

Ongoing work on block theory at cross-characteristic by Broué, Fong, and Srinivasan was described by Bhama Srinivasan. They have a conjecture giving explicit bijections of characters which would imply the Dade's ordinary conjecture, including the Isaacs-Navarro conjecture, for unipotent blocks of finite reductive groups. The Isaacs-Navarro conjecture [6] is a very interesting refinement of conjectures of McKay and Alperin on correspondences between numbers of characters of certain degrees in a group, or a block, and those of a Sylow subgroup, or a defect group. In addition, the conjecture extends the work of Broué, Malle, and Michel [2] to the case of nonabelian defect group. The bijection conjecture has been checked for $GL(n, q)$.

Character values of finite reductive groups G can be calculated, at least in principle, using Lusztig's geometric theory of character sheaves on G. Toshiaki Shoji described what still has to be done to obtain all the character values. First, one must establish a conjecture of Lusztig that certain known class functions are given in terms of so-called almost characters. Secondly, there are certain constants which turn up in these theories which must be explicitly calculate. Finally, one must know the scalars involved in generalized Green functions. Shoji's work shows that Lusztig's conjecture is true, for certain primes p, if $G(\overline{k})$ has connected centre, where k is the algebraic closure of the finite field k. The case of $SL(n, k)$ has long been complicated, in part because $SL(n, \overline{k})$ it has disconnected centre. Shoji explained how all the character values of $SL(n, k)$ can be computed, if the characteristic p of k is large enough. Before Shoji's lecture there was a long discussion, on Tunnel mountain, among Broué, Cliff, Michel, and Shoji, about whether or not one can actually write down all the irreducible character values of $SL(n, k)$, using Lusztig's character sheaves. It was decided that, given Shoji's work, this perhaps would be a suitable project for a Ph.D. student.

Gunter Malle spoke on joint work with Raphael Rouquier (also attending) on the determination of families of irreducibles characters of certain finite complex reflection groups. These families generalize the notion of two-sided (Kazhdan-Lusztig) cells of finite Weyl groups, and they share many of the properties of these two-sided cells. Also, as for Weyl groups, they have a connection to the hypothetical representation theory of spetses associated to spetsial complex reflection groups. This completes earlier work by Broué and Kim for imprimitive complex reflection groups.

Gerhard Hiss spoke on unitary designs, representations of finite unitary Groups, and the function fields of the Fermat curves. Fermat curves over for special degrees and fields yield unitary designs with high symmetry (containing the 3-dimensional projective unitary group.) An analogous construction gives a family of unitary designs, called Ree unitals, with the Ree groups as automorphism groups. Here, the Fermat curves are replaced by 1-dimensional Deligne-Lusztig varieties. Again, the representation theory of the Ree groups and properties of the function fields of these Deligne-Lusztig varieties yield information about the elementary divisors of the incidence matrices of the Ree unitals.

There has been much recent work on Schur indices of irreducible characters of finite groups of Lie type.

There is a conjecture that the Schur index of an irreducible character of any finite quasi-simple group G is one or two. Due to the classification of finite simple groups, one is led to case that that G is a finite reductive group. Through the work of Gow, Lusztig, Ohmori and Geck, the Schur indices of all unipotent characters of finite groups of Lie type are now known. Geck spoke on the last open case which was only settled recently: the cuspidal unipotent characters of $E_7(q)$ have Schur index 2, but only if q is an even power of a prime congruent to 1 modulo 4 (a rare case of non-generic behaviour !)

List of Participants

Alperin, Jonathan (University of Chicago)
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Bessenrodt, Christine (Universität Hannover)
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Brundan, Jonathan (University of Oregon)
Carlson, Jon (University of Georgia)
Chuang, Joseph (University of Bristol)
Cliff, Gerald (University of Alberta)
Corran, Ruth (Institut Henri Poincare)
Dipper, Richard (Universitaet Stuttgart)
Erdmann, Karin (Mathematical Institute)
Fong, Paul (University of Illinois at Chicago)
Geck, Meinolf (Université Lyon 1)
Grojnowski, Ian (University of Cambridge)
Henke, Anne (University of Leicester)
Hertweck, Martin (Universitat Stuttgart)
Hiss, Gerhard (The Aachen University of Technology)
James, Gordon (Imperial College)
Kessar, Radha (Ohio State University)
Kim, Sungsoon (Universite Paris 7 (et d'Amiens))
Kleshchev, Alexander (University of Oregon)
Koshitani, Shigeo (Chiba University)
Kuelshammer, Burkhard (University of Jena)
Kunugi, Naoko (Aichi University of Education)
Leclerc, Bernard (Universite de Caen)
Malle, Gunter (Universitaet Kassel)
Mathas, Andrew (University of Sydney)
McNeilly, David (University of Alberta)
Michel, Jean (Universite Paris 7)
Miyachi, Hyohe (Institut des Hautes Etudes Scientifiques)
Nebe, Gabriele (Universitaet Ulm)
Parker, Alison (University of Oxford)
Puig, Lluis (French National Centre for Scientific Research)
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Robinson, Geoffrey (University of Birmingham)
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Shoji, Toshiaki (Nagoya University)
Srinivasan, Bhama (University of Illinois at Chicago)

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Chapter 35

Galaxy Formation: A Herculean Challenge (03w5302)

November 1–6, 2003

Organizer(s): Arif Babul (University of Victoria), Julio F. Navarro (University of Victoria), Thomas R. Quinn (University of Washington), George Lake (Washington State University), Jeremiah P. Ostriker (University of Cambridge & Princeton University)

Scientific Stage

The nearly perfect isotropy of the cosmic microwave background (CMB) tells us that the early universe was, to a high degree, smooth and homogeneous. Explaining the transition from these smooth beginnings to today's highly organized universe consisting of structures that span a range of scales that exceed 10 billion in mass, from sub-galactic systems to clusters of galaxies, from superclusters to giant sheets of galaxies enveloping vast voids millions of light years in size, is one of the Grand Challenges of modern theoretical astrophysics.

Of all the pieces of this fascinating cosmic puzzle, perhaps none is as intriguing as the formation of galaxies like our own Milky Way. The building blocks of cosmic structures, galaxies exhibit a perplexing morphological variety that bears witness to the intricate paths of their formation. As dramatic advances in detector and telescope technologies make it possible to identify galaxies so distant that they are seen at a fraction of the age of nearby systems, they also bring a sense of urgency to questions that only a few years ago seemed confined to the territory of theoretical speculation. How do galaxies form and evolve? How do they acquire their observed structure?

According to current understanding, galaxies we see today started out as the tiniest flecks in the froth of the very early universe. By virtue of gravitational instability, these flecks—actually, small amplitude fluctuations in the early Universe density field—grew first into small-scale systems floating in a complex of filaments and planes. Galaxies, to first order, are the result of the subsequent merging and accretion of these smaller systems.

However, whereas gravity reigns supreme on scales of superclusters and voids, and magneto-hydro-electro-dynamics is responsible for the evolution of structures on the scale of stars, all of these non-linear, highly coupled physical processes are thought to be of critical importance during the formation of a galaxy. In short, galaxy formation is a highly non-linear problem involving the three-dimensional evolution of self-gravitating collisional fluids (which are subject to gas and hydro dynamical effects), non-collisional fluids (which interact via gravity only), and their coupled radiation fields in an expanding Universe. And while analytic studies of galaxy formation are useful for purposes of gaining specific insights, they can be applied only under highly constrained circumstances. Numerical approaches offer the only option viable means of

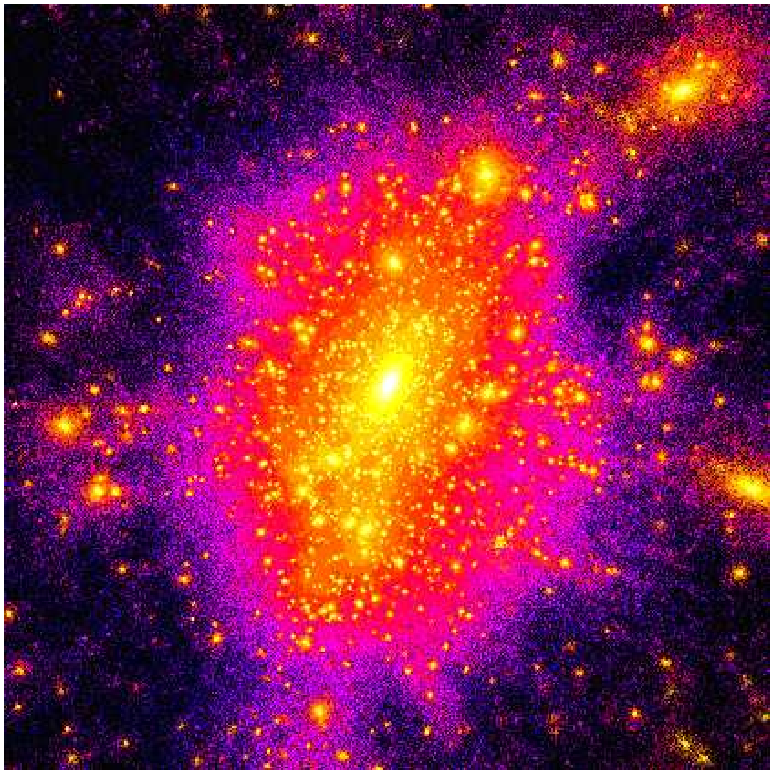

Figure 35.1: . . . *if matter was evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses scattered at great distances from one to another. . .* —**Isaac Newton** in a letter to Bishop Bentley 1692

constructing a realistic unconstrained description of the galaxy formation process and in the process, understand how confluence of gravitational, hydrodynamic and radiative interactions give rise to the phenomenal diversity in the observed galaxy population.

Computing the Universe

The numerical approach is a relatively recent innovation, having developed after the widespread proliferation of computing resources within the academic environment. The field of computational astrophysics has developed rapidly, with the sophistication and the diversity of computational methods growing alongside the available computing power. The approach, though extremely powerful, has its limitations. Fundamentally, numerical simulations depend much on such features as the computation scheme being used, nature of the approximations in the initial conditions, the use of appropriate boundary conditions, etc. In addition, one has to worry about the effects of round-off errors, which can be particularly troublesome in strongly chaotic conditions, on the end result. And then, there is the natural limitation that numerical simulations cannot explicitly treat all of the physical processes at play. To add to this, to correctly model all the important physical processes suspected of playing a role in shaping a galaxy requires numerical simulations that span a dynamic range in excess of 107 in spatial and temporal scales.

While fully realistic simulations appear to be beyond the realm of feasibility at the present, there is a glimmer of hope. The seemingly inexorable increase in raw computing power and the rise of inexpensive massively parallel computing platforms are already making an impact. Beyond these, attempts to address the limitations have resulted in tremendous technological, algorithmic and coding innovation. These range from the development of dedicated special-purpose computers, such as GRAPE, and the development of a variety of schemes, such as adaptive mesh refinement (AMR), total variation diminishing method (TVD), N-body tree codes, and smoothed particle hydrodynamics (SPH) and its offshoots, etc., for solving gravity equations as well as the hyperbolic system of gas dynamical conservation equations, to approximate methods for handling poorly understood, often unresolved processes such as star formation, turbulent cascades, turbulent energy dissipation, conduction, narrow shock fronts, dynamical and hydrodynamic instabilities, radiative transfer, etc.

Nonetheless, uncertainties abound: What is the basis for determining which physical processes to include in the simulations and which to ignore? Are there aspects of the various coding methodologies? Which of the schemes is the best? Which areas of computational and algorithmic developments ought to be the focus of increased research and development?

Moreover, there are a whole host of questions regarding the modelling of the "sub-grid" physics. A number of different approximations that purport to treat these processes and deliver the results at the simulation level. At present, however, there is little consensus on what processes need to be modelled, how they should be approximated, and how the resulting modules ought to be incorporated into the cosmological simulation codes.

At present, the above issues are best addressed through a three-pronged approach:

(1) A rigorous testing of each individual code under a broad diversity of conditions

(2) Careful comparisons of the results from different codes under controlled circumstances.

(3) Detailed comparison of simulation results against observations, followed by equally detailed analyses of the simulation code and the built-in assumptions in order to identify the origins of any differences.

The Workshop

The BIRS "hot topics" workshop brought together some of the most active and prominent researchers in the area of computational cosmology, participants such as George Lake, the founding head of the N-body shop at the University of Washington, T. Quinn, the present head of the N-Body Shop, world-renowned cosmologists such as J.P. Ostriker (Cambridge/Princeton), J. Primack (UCSC) and M. Steinmetz (AIP, Potsdam), and rising stars such as T. Abel (Penn State), R. Davé (Arizona) and V. Springel (Harvard/Max Planck-Garching), with the aim of discussing the above-mentioned challenges.

The aim was to enable the participants to engage each other in a stimulating yet relaxed and low-pressure setting in order discuss freely and openly problems with the existing approaches as well as possible solutions. The meeting schedule was purposely structured to facilitate such discussions. The key themes of discussion were:

(a) Assessing existing simulation codes: successes and failures.

(b) New computational and algorithmic developments: where does the future lie?

(c) Innovative approaches to modelling sub-grid physics: truth or consequences?

Recognizing that comparison against observations is an important way of testing the simulation codes, a handful of leading young observational cosmologists were also invited to discuss their findings and pose challenges for the computational cosmologists.

The full list of the participants is included below.

Workshop Summary

Cosmological numerical simulations have proven remarkably successful at accounting for the cosmic structure formation, including the formation of individual galaxies and larger scale structure, in the broad-brush sense. Among the successes is the understanding of the origins of, and the successful reproduction—in a quantitative sense—of the large-scale distribution of galaxies, a distribution that resembles a three-dimensional web spun out of chains of galaxies, occasionally punctuated by massive dynamic swarms of up to a thousand bright galaxies held together by their mutual gravity, all woven around giant voids millions of light-years across. Additionally, observations of galaxies reaching back across almost 12 billion years of cosmic time offer strong support to the hierarchical build-up of structure driven primarily by gravitational instability. In spite of this, a number of disagreements between simulation results and observations have emerged, disagreements that were strongly reinforced as simulation results were confronted by observations over the course of the workshop.

Of these, the most serious and consequently, the one that came to dominate the workshop discussions is the so-called cooling catastrophe. This catastrophe manifests when gas is allowed to cool radiatively. In conjunction with standard star formation prescriptions, this catastrophe leads to higher than expected fraction of gas condensing out of the hot phase, resulting in bigger stellar systems—in comparisons with the observations—at the centres of dark matter halos across the entire halo mass spectrum. For the purposes of discussion, this overgrowth of stellar systems will be dubbed "gigantism". Comparisons of Fardal's simulation results against observations showed that in the standard implementation of SPH, gigantism in the simulations exists at all epochs accessible to observational study.

Discussions of the gigantism, and of the cooling catastrophe more generally, took a number of different turns. Davé noted that the standard implementation of SPH has a limited ability to resolve steep density gradients because these formulations are based on the assumption that the density gradient across the SPH smoothing kernel of a particle is small. This assumption, however, breaks down when cooling is allowed. In galaxy halos, for example, cooling gives rise to dense regions of cold gas embedded in a hot medium. Under such circumstances, density can change by several orders of magnitude within the smoothing length of a hot gas particle in the boundary region. The consequence is an overestimate in the density of the hot particles and hence, an overestimate of the cooling rate—cooling scales as the second power of density.

Springel further noted that the choice of either the thermal energy or the entropy as an independent variable also makes a difference. His experiments have shown that in cases where concentrated energy deposition takes place (e.g. energy input due to supernova explosions) or when gas is being compressed in the presence of steep gradients, the implementation of SPH that relies on the thermal energy as an independent quantity gives rise to unphysical conditions. For example, in the case of adiabatic compression of gas, there is leakage of entropy and hence, an artificial cooling of the gas. One consequence is, again, excessive cooling of the gas when cooling is effective.

Implementations that attempt to address the above deficits, such as the explicit entropy and energy conserving SPH scheme of Springel and collaborators, appear promising in that the rate of cooling of gas in regions defined by less than a few thousand SPH particles is reduced by a factor of ~ 2. This reduction, however, is not sufficient to completely eliminate the galaxy gigantism problem.

One possible explanation of the above is improper modelling of sub-grid physics, especially those processes that generate energy feedback. Feedback can potentially stall cooling by offsetting radiative losses and even replenishing the energy reservoir. Several of the participants (Somerville, Benson, Thacker, Stinson, Cox) discussed different schemes for modelling sub-grid physics. And while some of the schemes appeared promising in that the resulting simulations were able to produce isolated disk galaxies with properties comparable to those of the Milky Way within halos comparable in size to that in which the Milky Way resides, and limited excessive cooling in smaller halos, all approaches failed to eliminate the creep towards gigantism in larger systems. Estimates suggested that under the present schemes, gigantism in the large halos is averted only if feedback efficiencies exceed 100%, a result that led to several bouts of discussion on whether the simulations were capturing all the relevant physical processes or whether there were important processes that SPH failed to model correctly.

Wadsley discussed the failure of SPH to model turbulent cascade and mixing correctly. The problem is potentially related to the cooling catastrophe in that energy input into the gas is often a combination of turbulent kinetic energy and thermal energy. In cases where the radiative timescale is short, the temporal lifetime of this energy input is determined by the rate at which the turbulence dissipates into heat. In present implementations, the transformation of kinetic energy into thermal energy occurs rapidly, the latter then being susceptible to being radiated away. Wadsley discussed an experimental approach that he has been exploring to ameliorate this problem. One problem with the proposed scheme is that it does not converge as the number of particles is increased.

A number of participants argued that thermal conduction is a potentially important effect that ought to be included as part of the basic physics package in the simulations. McCarthy discussed his recent analytic and semi-analytic investigations of heating and cooling in galaxy clusters, concluding that while conduction may further help reduce cooling, it is not the panacea to the cooling crisis. Additionally, several of the numericists described attempts to include conduction in their simulations, noting that the finite-difference approximations, if not handled with great care, can lead to the emergence of unphysical states and instabilities.

Ostriker summarized the current state of affairs in a session titled “Galaxy Formation: More Questions than Answers”. He advocated investing increased research and development of alternate approaches, such as the TVD scheme. The TVD scheme follows both the collisionless dark matter and baryons. The baryon fluid is represented by a fixed grid, while the dark matter component is represented by particles that move with respect to the grid. The force of gravity is computed using the Particle-Mesh (PM) method while the hyperbolic system of conservation equations (conservation of mass, momentum, and energy or pseudo-entropy) that govern the evolution of the gas component are solved using an explicit, second-order Eulerian finite difference scheme. Unlike SPH, the TVD scheme is able to capture shocks with the need to introduce artificial viscosity in the equations of motion. The current avatar of TVD incorporates a number of inefficiencies (with respect to the current generation of SPH codes). Among these are (1) it is difficult to make the PM gravity solver highly parallel as opposed to the tree-based gravity engines that form the foundation of the SPH codes; (2) the fixed grid constrains how well regions of high density can be resolved, a problem that could potentially be addressed through the implementation of an adaptive grid.

Finally, while the discussion of the cooling catastrophe and its impact on the formation of cosmic structure dominated the meeting, a number of other long-standing problems were also considered. These ranged from discussions of biases inherent in the construction of initial conditions (Fardal and Thacker), limitations to proper modelling of dynamical processes and resonance capture due to discreet particle noise in N-body simulations (Babul, Benson and Holley-Bockelmann), and the problem of artificial dissolution of substructure in high density regions (Taylor and Lake), to whether current constraints on the dynamic range that numerical simulations can efficiently span seriously constrained studies of galaxy formation (Navarro, Cox, Stern, Stinson). In addition, Abel showed some amazing simulations of the very first structures—the first massive, isolated stars—made using AMR techniques. That such simulations are becoming possible is exciting because it means that the end of the “cosmic dark ages” is now amenable to study.

All in all, the workshop was a tremendous success. The sentiment was echoed by all of the participants, most of who felt that the BIRS Workshop on Computational Cosmology should become a regular event. The participants were very impressed by the location, the organization and by the excellent facilities. The relaxed setting facilitated open and frank discussions of the challenges in modelling the formation of cosmic structure, particularly galaxies. Ample unstructured time gave participants opportunities to engage in constructive dialogues and to formulate plans for collaborations and code comparisons. Towards the end of the meeting, at least five new collaborative efforts had emerged.

List of Participants

Abadi, Mario (University of Victoria)
Abel, Tom (Penn State University)
Babul, Arif (University of Victoria)
Balogh, Michael (University of Durham)
Benson, Andrew (California Institute of Technology)
Chapman, Scott (California Institute of Technology)
Couchman, Hugh (McMaster University)
Cox, T. J. (University of California - Santa Cruz)
Dave, Romeel (University of Arizona)
Ellison, Sara (Pontificia Universidad Catolica de Chile)
Fardal, Mark (University of Victoria)
Ferguson, Harry (Space Telescope Science Institute)
GuhaThakurta, Raja (University of California - Santa Cruz)
Hayashi, Eric (University of Victoria)
Hearn, Nathan C. (Washington State University)
Hoekstra, Henk (University of Toronto)
Holley-Bockelmann, Kelly (University of Masachusetts)

Idzi, Rafal (Johns Hopkins University)
Lake, George (Washington State University)
McCarthy, Ian (University of Victoria)
Moustakas, Leonidas (Space Telescope Science Institute)
Navarro, Julio (University of Victoria)
Ostriker, Jeremiah (University of Cambridge)
Pogosyan, Dmitri (University of Alberta)
Primack, Joel (University of California - Santa Cruz)
Quinn, Tom (University of Washington)
Rix, Hans-Walter (Max-Planck-Institute for Astronomy)
Sawicki, Marcin (National Research Council)
Scott, Douglas (University of British Columbia)
Shapley, Alice (California Institute of Technology)
Siebert, Arnaud (Steward Observatory)
Somerville, Rachel (Space Telescope Science Institute)
Springel, Volker (Max-Planck-Institute for Astrophysics)
Steinmetz, Matthias (Astrophysikalisches Institut Potsdam)
Stern, Luke (University of Victoria)
Stinson, Greg (University of Washington)
Taylor, James (University of Oxford)
Thacker, Robert (McMaster University)
Thanjavur, Karun (University of Victoria)
Wadsley, James (McMaster University)

Chapter 36

MSRI Hot Topic: Floer Homology for 3-manifolds (03w5303)

November 8–13, 2003

Organizer(s): Yakov Eliashberg (Stanford University), Robion Kirby (Chair) (University of California, Berkeley), Peter Kronheimer (Harvard University), with Tomasz Mrowka (Massachusetts Institute of Technology), Peter Ozsv´ath (Columbia University), Zolt´an Szab´o (Princeton University)

Introduction

This was a lively and productive conference, as befits a "Hot Topic". Perhaps the highlight, described in more detail below, was the discovery (through discussions among the participants) that Eliashberg could prove a theorem on capping off a symplectic 4-manifold with convex boundary, which was then used by Kronheimer and Mrowka to prove Property P for knots, and was also used by Ozsváth and Szabó to determine the genus of a knot by its Floer homology.

Property P refers to the 40 year old conjecture that Dehn surgery on a knot in S^3 never gives a homotopy 3-sphere unless the knot is the unknot. It was already known that Dehn surgery did not give S^3 [15], so the Kronheimer-Mrowka result would also follow from Perelman's work once it is approved.

The genus of a knot K in S^3 is the minimal genus of a spanning "Seifert" surface $F \in S^3$, $\partial F = K$. Arguably this has been the most important invariant of a knot for 80+ year, but is has been very difficult to calculate. Now it is determined by the "highest" spin$_C$ structure for which the Heegaard Floer homology is non-trivial; this is reasonably calculable.

History and Heegaard Floer Homology

This accounts begins with the historical background to the use of gauge theoretic methods in solving topological problems about low-dimensional manifolds.

In spring 1982, Simon Donaldson announced his spectacular applications of gauge theory to the differential topology of 4-manifolds [3]. Using work of Taubes and Uhlenbeck, Donaldson showed that the moduli space of almost self dual connections on a certain C^2 bundle over a smooth 4-manifold X^4 provided invariants which ruled out the existence of 4-manifolds with definite intersection forms. Together with Freedman's classification of simply connected topological 4-manifolds [10], this showed the existence of exotic, smooth structures on R^4.

The subject was difficult and results came only after considerable work. Donaldson discovered what are now called Donaldson polynomials, and later *basic classes* were found.

In fall 1994 Nathan Seiberg and Edward Witten announced a new set of partial differential equations for a connection on the bundle of spinors on S^4. The equations were technically much simpler than in Donaldson's case, and applications were quick to follow, e.g. the Thom Conjecture [17].

Beginning in 2001, Peter Ozsváth and Zoltán Szabó introduced a new version of Floer homology – the *Heegaard Floer homology* – based on Heegaard splittings of genus g of an oriented 3-manifold Y^3. The methods are almost purely combinatorial except for the crucial use of pseudoholomorphic discs in the g-fold symmetric product of a Heegaard surface for Y.

More precisely, Y is presented by a Heegaard diagram (Σ, α, β) where Σ is an oriented two-manifold and $\alpha = \{\alpha_1, ..., \alpha_g\}$ and $\beta = \{\beta_1, ..., \beta_g\}$ are attaching circles for two handlebodies which bound Σ. A choice of complex structure on Σ induces one on its g-fold symmetric product. Moreover, the products

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g$$

are tori embedded in $\mathrm{Sym}^g(\Sigma)$, which are totally real with respect to the induced complex structure on $\mathrm{Sym}^g(\Sigma)$. One can now set up a variant of Lagrangian Floer homology [8] in this setting – that is, the homology of a chain complex whose generators are intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and whose boundary operator counts pseudo-holomorphic disks in $\mathrm{Sym}^g(\Sigma)$ whose boundary lies in $\mathbb{T}_\alpha \cup \mathbb{T}_\beta$. Indeed, in order to get non-trivial information about the three-manifold, we need another piece of data, a choice of reference point $z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ...\beta_g$. The data $(\Sigma, \alpha, \beta, z)$ is called a *pointed Heegaard diagram*. This point z induces a subvariety $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$, and various variants of Heegaard Floer homology are obtained by using this subvariety in various ways. For example, the simplest non-trivial version of Heegaard Floer homology, $\widehat{\mathrm{H}F}(Y)$, counts pseudo-holomorphic disks in $\mathrm{Sym}^g(\Sigma)$ which are disjoint from $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$. In all, there are four versions of this Floer homology $\widehat{\mathrm{H}F}(Y)$, $\mathrm{H}F^-(Y)$, $\mathrm{H}F^+(Y)$, and $\mathrm{H}F^\infty(Y)$.

Although the definition of these groups depends on a great deal of auxiliary information – a Heegaard diagram for Y, a choice of complex structure on Σ (and indeed a small perturbation of the induced almost-complex structure on $\mathrm{Sym}^g(\Sigma)$) – it is proved in [22] that the homology of the complex is in fact a topological invariant of Y. Indeed, in [23], it is shown that Heegaard Floer homology is natural under cobordisms between three-manifolds; i.e. if W is a cobordism from Y_1 to Y_2, there is an induced map on the four variants of Floer homology, which is a diffeomorphism invariant of W. These maps are then used in [23] to construct a four-manifold invariant whose formal properties suggest a close connection to the Seiberg-Witten invariant for four-manifolds. Indeed, based on this, and an overwhelming amount of calculational evidence, it is conjectured in [22] that the two theories are isomorphic.

But each theory has its own advantages. Heegaard Floer homology is more combinatorial in flavor than Seiberg-Witten theory. For example, the generators of the Heegaard Floer complex are purely combinatorial. Thanks to this concrete nature, several technical devices to facilitate the calculation of Heegaard Floer homology groups were obtained in [24]. A key device is a surgery *exact triangle* which relates the Floer homology groups of three-manifolds which are related by certain Dehn surgeries. (Surgery exact triangles first appeared in the work of Andreas Floer for his version of instanton Floer homology [9].)

Another device is provided by a Heegaard Floer invariant for knots which is not difficult to construct once the Heegaard Floer package is constructed, see [25] and also [30]. Specifically, there is an invariant associated to a knot $K \subset S^3$ (or more generally an oriented link) which is a bigraded Abelian group $\widehat{\mathrm{HFK}}_d(K, s)$, with $d, s \in \mathbb{Z}$ (we call d the Maslov grading and s the Spin^c grading; in the case where the oriented link has an even number of components, $d \in \frac{1}{2} + \mathbb{Z}$). The Euler characteristic in the d direction gives the Alexander polynomial of K, i.e. if T is a formal variable, then the sum

$$\sum_{s \in \mathbb{Z}} \chi(\widehat{\mathrm{HFK}}_*(K, s)) \cdot T^s$$

calculates the symmetrized Alexander polynomial of K. This invariant satisfies a skein exact sequence, where the three terms appearing in the sequence are the invariants associated to the three links obtained by changing any given crossing or alternatively forming the oriented resolution of that crossing. As such, it provides a theory which is similar in spirit to work of Khovanov [19], who constructs a bigraded homology theory associated to links, whose Euler characteristic is the Jones polynomial. It should be noted, though, that unlike the knot invariant from Heegaard Floer homology, the differentials for Khovanov's complex are purely combinatorial in their definition. However, knot Floer homology can be calculated for a large family of knots, and it is related to the Heegaard Floer homology of the three-manifolds obtained by surgeries on the knot. Like the Alexander polynomial, Heegaard Floer homology gives a lower bound on the Seifert genus of a knot,

$$\max\{s \in \mathbb{Z} \big| \widehat{\mathrm{HFK}}_*(K,s)\} \leq g(K). \tag{36.1}$$

Generators for the knot Floer complex have a concise Morse-theoretic interpretation. Fix a knot $K \subset S^3$. A perfect Morse function is said to be *compatible with* K, if K is realized as a union of two of the flows which connect the index three and zero critical points (for some choice of generic Riemannian metric μ on S^3). Thus, the knot K is specified by a Heegaard diagram for S^3, equipped with two distinguished points w and z where the knot K meets the Heegaard surface. In this case, a *simultaneous trajectory* is a collection $\mathbf{x}$ of gradient flowlines for the Morse function which connect all the remaining (index two and one) critical points of f. From the point of view of Heegaard diagrams, a simultaneous trajectory is an intersection point in the g-fold symmetric product of $\Sigma \ \mathrm{Sym}^g(\Sigma)$ (where g is the genus of Σ) of two g-dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g,$$

where here $\{\alpha_i\}_{i=1}^g$ resp. $\{\beta_i\}_{i=1}^g$ denote the attaching circles the two handlebodies. Let $X = X(f,\mu)$ denote the set of simultaneous trajectories. Any two simultaneous trajectories differ by a one-cycle in the knot complement M and hence, if we fix an identification $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, we obtain a difference map

$$\epsilon\colon X \times X \longrightarrow \mathbb{Z}.$$

The Spin^c grading of a simltaneous trajectory is determined as follows. There is a unique map (defined up to an overall sign)

$$s\colon X \longrightarrow \mathbb{Z}$$

with the property that $s(\mathbf{x}) - s(\mathbf{y}) = \epsilon(\mathbf{x},\mathbf{y})$, which satisfies the property that $\sum_{\mathbf{x}\in X} T^{s(\mathbf{x})}$ is symmetric, as a Laurent polynomial in $\mathbb{Z}/2\mathbb{Z}[T,T^{-1}]$.

Simultaneous trajectories can be viewed as a generalization of some very familiar objects from knot theory. To this end, note that a knot projection, together with a distinguished edge, induces in a natural way a compatible Heegaard diagram. The simultaneous trajectories for this Heegaard diagram can be identified with the "Kauffman states" for the knot projection; see [18] for an account of Kauffman states, and [26] for their relationship with simultaneous trajectories.

On the other hand, Seiberg-Witten theory has some advantages over Heegaard Floer homology. Foremost amongst these advantages is its close connection with the geometry (as opposed to the topology) of the underlying manifold. For example, in [32], Taubes shows that a symplectic four-manifold has non-vanishing Seiberg-Witten invariant, by using the symplectic form as a perturbation for the equations.

In principle, the shortcomings of the two theories can be bridged. Short of proving that the two theories are isomorphic, one could either come up with more combinatorial methods for calculating Seiberg-Witten invariants, or alternatively, one could try to translate geometric input on a manifold into more combinatorial data which are reflected in Heegaard diagrams and can be detected by Heegaard Floer homology. For example, seminal work of Donaldson [4] gives a nearly combinatorial formulation of the symplectic condition, showing that a symplectic manifold always admits a compatible Lefschetz fibration. The induced two-handle decomposition of the four-manifold can then be used to prove that Heegaard Floer invariant of a symplectic four-manifold is non-trivial [27].

And building such bridges is clearly important for topological applications. For example, in [12], Gordon conjectured that if a knot $K \subset S^3$ has the property that $r = p/q$ Dehn surgery on K is orientation-preserving diffeomorphic to p/q Dehn surgery on the unknot, then K is the unknot. To illustrate the role of orientations here, note that $+5$ surgery on the right-handed-trefoil is a lens space which is orientation-preserving diffeomorphic to -5 surgery on the unknot, i.e. it is orientation reversing diffeomorphic to $+5$ surgery on the unknot. The case where $r = 2$ provides an example where this orientation issue becomes irrelevant. In this case, one obtains a simpler statement of the conjecture (c.f. [13]) that no surgery on a non-trivial knot in S^3 gives real projective three-space.

Many cases of Gordon's conjecture have been known for some time. For example, the case where $p = 0$, the conjecture is known to hold by celebrated work of Gabai [11]. The case where $q \neq 1$, the theorem is confirmed by the cyclic surgery theorem of Culler-Gordon-Luecke-Shalen [14]. In the case where $p/q = \pm 1$, the theorem was established by Gordon and Luecke [15]. But the case where r is integral (and $|r| > 1$) – the case with the most immediate four-dimensional interpretation (since Y is obtained as an integral surgery on a knot in S^3 if and only if it bounds a four-manifold which admits a Morse function with exactly two critical points: one of index zero, the other of index one) – this case remained open until this year.

It was proved in [28] using the surgery long exact sequence for Heegaard Floer homology (which had been missing from Seiberg-Witten theory) that if a knot in S^3 has the property that $S^3_p(K) = S^3_p(U)$ (where U is the unknot), then the Heegaard Floer homology of $S^3_0(K)$ is isomorphic to that of $S^3_0(U)$. But the construction of Kronheimer and Mrowka [21] (building upon work of Gabai [11], Eliashberg and Thurston [5]) shows that the Seiberg-Witten Floer homology distinguishes $S^3_0(K)$ from $S^3_0(U)$ (a result which had been missing from Heegaard Floer homology). In sum: the remaining part of Gordon's conjecture could be proved either by establishing a surgery exact triangle for Seiberg-Witten Floer homology, or by proving the analogues of Kronheimer and Mrowka's genus bounds in Heegaard Floer homology.

In the Fall of 2003 (shortly before the conference), Kronheimer, Mrowka, Ozsváth, and Szabó verified Gordon's conjecture, by establishing a surgery long exact sequence for the Seiberg-Witten monopole equations.

History and Seiberg-Witten Floer Homology

Beginning with seminal works of Mikhail Gromov (see [16]) and Daniel Bennequin (see [1]), the symplectic topology of 4-manifolds and the contact topology of 3-manifolds have firmly established themselves as an integral part of low-dimensional topology. The theory of J-holomorphic curves developed by Gromov in [16] was linked with Seiberg-Witten (SW) theory by Clifford Taubes (see [32, 33]) who proved that for symplectic manifolds SW-invariants coincide with certain kinds of Gromov enumerative invariants for holomorphic curves. Together with Taubes' non-vanishing theorem for SW-invariants of symplectic manifolds this for the first time showed the existence of J-holomorphic curves in certain closed symplectic 4-manifolds.

It turns out that for the extension of these results to 3-dimensional topology it is important to understand the interaction between contact 3-manifolds which bound symplectic 4-manifolds. Note that although a symplectic structure on a 4-manifold does not induce any contact structure on its boundary, it is useful to consider certain compatibility conditions between symplectic and contact structures. Suppose we are given a symplectic manifold (W, ω) with boundary $V = \partial W$ which carries a contact structure ξ. First of all, both the contact structure ξ on V and the symplectic structure ω on W define an orientation of V. These two orientations may coincide: in this case the boundary is called *convex*, or be opposite in the concave case. Second, we may ask if there exists a contact form α such that $\omega|_V = d\alpha$, or at least $\omega|_\xi = d\alpha|_\xi$. In the first case (W, ω) is called a *strong symplectic filling*, in the second case a *weak symplectic filling*. One can consider even a stronger filling condition which requires that α extends to the whole W and that the Liouville vector field X defined by the equation $i(X)\omega = \alpha$ is gradient-like for a Morse function on W which is constant on the boundary. In this case (W, ω) is called a *Stein filling* of (V, ξ).

In their paper [21] Kronheimer and Mrowka developed a relative version of SW-theory and defined an invariant of a contact structure on a 3-manifold which takes its value in SW-Floer homology of the manifold. As an analogue of Taubes' non-vanishing theorem from [34] they proved that for weakly fillable (or even a seemingly weaker notion of weakly semi-fillable) contact structures their invariant does not vanish.

On the other hand, Ozsváth and Szabó defined in [27] a similar contact invariant in the context of Heegaard Floer homology theory. Their invariant was seemingly easier to compute but the analogue of the Kronheimer-Mrowka non-vanishing result was established only for Stein-fillable contact structures.

During the conference there was an active discussion of how the non-vanishing result in Heegaard Floer homology can be proven in the same generality as the the corresponding result in SW Floer homology. One of the questions asked by Olga Plamenevskaya led Eliashberg to realize that the answer depends on a problem of finding a symplectic manifold bounding a 3-manifold fibered over S^1 with symplectic fibers (which arises as a 0-surgery on the binding of an open book decomposition associated with the contact structure). He then realized that the answer to this question was essentially known and filled in the details of the argument. During the continuing discussion Ozsváth and Szabó realized that the same argument allowed several other advances in Heegard homology theory. Furthermore Peter Kronheimer pointed out that Eliashberg's observation together with the recent work of Feehan and Leness (see [7]) about the relation between Donaldson and SW-invariants, allowed him to complete his joint program with Tom Mrowka for proving Property P.

Symplectic Fillings

Here are more mathematical details of Eliashberg's argument.

Theorem 1 *Let (V, ξ) be a contact manifold and ω a closed 2-form on V such that $\omega|_\xi > 0$. Suppose that we are given an open book decomposition of V with a binding B. Let V' be obtained from V by a Morse surgery along B with a canonical 0-framing, so that V' is fibered over S^1. Let W be the corresponding cobordism, $\partial W = (-V) \cup V'$. Then W admits a symplectic form Ω such that $\Omega|_V = \omega$ and Ω is positive on fibers of the fibration $V' \to S^1$.*

Theorem 2 *Let (V, ξ) and ω be as in Theorem 1. Then there exists a symplectic manifold (W', Ω') such that $\partial W' = -V$ and $\Omega'|_V = \omega$. Moreover, one can arrange that (W', Ω') contains the symplectic cobordism (W, Ω) constructed in Theorem 1 as a subdomain, adjacent to the boundary. In particular, any symplectic manifold which weakly fills the contact manifold (V, ξ), can be symplectically embedded as a subdomain into a closed symplectic manifold.*

Eliashberg's theorem, together with previously known properties of Heegaard Floer homology (see especially [27]) now lead quickly to the non-vanishing theorem for the Heegaard Floer invariant of a symplectically semi-fillable contact structure. Specifically, suppose that W_1 is a symplectic filling of a three-manifold Y, Eliashberg constructs a four-manifold W_2 with the property that $V = W_1 \cup_Y W_2$ is symplectic. By [27], we know that the invariant of V is non-trivial, and hence the Heegaard Floer homology of Y must be non-trivial, as well.

Thus, we have the missing piece required to verify Gordon's conjecture purely within the framework of Heegaard Floer homology. But there are applications of this non-vanishing theorem in Heegaard Floer homology which go beyond reproofs of gauge theoretic results. It now follows [29] that if g is a knot in S^3, then $\widehat{\mathrm{HFK}}(K, g)$ is always non-trivial, i.e. the lower bound from Inequality (36.1) is sharp. In turn, this result admits a restatement which is independent of Heegaard Floer homology:

Corollary 3 *The Seifert genus of a knot K is the minimum over all compatible Heegaard diagrams for K of the maximum of $s(\mathbf{x})$ over all the simultaneous trajectories.*

Other Lectures and Results

One of the subjects which was actively discussed during the conference is the relation of Heegaard Floer homology theory of Ozsváth-Szabó, periodic Floer homology theory of Michael Hutchings and Michael Thaddeus, and the project of embedded contact homology theory which is under construction by Michael Hutchings, Yakov Eliashberg, Michael Sullivan and others. Several students are also involved in this project. In particular, Stanford student Robert Lipshitz gave an informal talk during one of the evening sessions where he sketched the construction of Heegard Floer homology using holomorphic curves in 4-manifolds, rather than g-folded symmetric products of surfaces.

David Gay spoke on his work (with Kirby) in which they show how to construct harmonic 2-forms on 4-manifolds in terms of handlebody decompositions. The 2-forms vanish transversely along a collection of circles and are symplectic in the complement of these circles. He discussed the extent to which he can prescribe spin^C structures and J-holomorphicity of certain surfaces and he worked through some explicit examples.

Mikhail Khovanov talked about his construction of a bigraded homology theory of links with the quantum $sl(3)$ invariant as the Euler characteristic.

Ciprian Manolescu described an associated suspension spectrum $SWF(Y)$ whose homology is the Seiberg-Witten Floer homology starting with a 3-manifold Y with $b_1(Y) = 0$ and a spin^C structure on Y. Given a cobordism between 3-manifolds, there is an associated morphism between their spectra, and a gluing theorem holds: composition of cobordisms corresponds to composition of morphisms. For 3-manifolds with $b_1 > 0$, assuming the vanishing of a certain K-theoretic obstruction, there is a pro-spectrum analogue of SWF.

Andras Nemethi spoke about calculating the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds with a given spin^C-structure starting with the plumbing graph. He gave precise combinatorial algorithms for cases including all rational and weakly elliptic singularities, which in the case of Seifert fibered 3-manifolds is expressed in terms of the Seifert invariants.

Brendan Owens and Saso Strle discussed an application of Ozsváth and Szabó's Froyshov-type invariant to cobordisms of rational homology spheres. Computations in the case of Seifert fibered spaces were used to obtain bounds on the 4-ball genus of Montesinos links.

Jake Rasmussen described a strange correlation between the Khovanov and knot Floer homologies which works in a striking number of cases.

Andras Stipsicz uses Heegard Floer Homology theory to distinguish contact structures and gives examples of tight non-fillable contact 3-manifolds.

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Chapter 37

The Interaction of Finite Type and Gromov-Witten Invariants (03w5020)

November 15–20, 2003

Organizer(s): Dave Auckly (Kansas State University), Jim Bryan (University of British Columbia)

This is the final report on the workshop titled "The interaction of finite type and Gromov-Witten invariants." This was a five-day workshop held at the Banff International Research station from November 15 - November 20, 2003.

Before five years ago, there was no interaction between these two distinct fields. In 1998 Gopakumar and Vafa suggested a relation between the GW invariants and certain (integer) counts of BPS states in M-theory [14]. Shortly thereafter work appeared showing that open string theory on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is equivalent to Chern-Simons theory [1, 29, 24, 23]. This is a duality between Chern-Simons theory and open string theory built on the conjecture of t'Hooft relating large N gauge theories and string theories. These ideas developed very quickly in the theoretical physics literature. The resulting papers suggest strong links between these distinct mathematical areas. The mathematical communities working on finite-type invariants and Gromov-Witten invariants were and still are largely disjoint. The goal of the workshop was to bring together people working on finite-type invariants, people working on Gromov-Witten invariants and physicists who could explain the recent results in physics linking these two areas.

The workshop was an overwhelming success, and is sure to influence the development of mathematics in these two areas. We did succeed in bringing together mathematicians from these two areas and a physicist to tell us why we should interact. We predict that in another five years, the two mathematical disciplines will no longer be separate, but will have many close mathematical connections. Many of the participants at this workshop will contribute to this area of mathematical development, and the mathematical interaction started at this workshop will not just be limited to the participants. We are in the process of preparing a workshop proceedings that will communicate some of the exciting developments in this area to the world-wide mathematical community.

We will now give a brief technical overview of the subject matter covered at the workshop. The first area represented at the workshop was finite-type invariants and Chern-Simons theory. The theory of finite type invariants has roots in mathematical physics, statistical mechanics, operator algebras, topology and singularity theory.

The physical intuition for these invariants was first explained by E. Witten as a Topological quantum field theory based on the Chern-Simons invariant [39]. The invariant of a link in a 3-manifold is described as the path integral,

$$Z_k(M, L) = \int_{\mathcal{A}/\mathcal{G}} e^{\frac{i}{2g_s}\mathsf{CS}(A)} \prod \mathrm{Tr}_{\rho_j}(\mathrm{hol}_A(L_j)) DA. \tag{1}$$

Here $g_s = \frac{2\pi}{k+N}$ is the string coupling constant. Motivated by properties that formally follow from the path integral, N. Reshetikhin and V. Turaev gave a mathematical definition of a 3-manifold invariant satisfying the same formal axioms [33]. A slightly weaker invariant was given by V. Turaev and O. Viro [37]. These invariants have been extended and studied see [20].

At the same time that the Witten-Reshetikhin-Turaev invariants were being developed, V. Vassiliev began the theory of finite type invariants by applying techniques from singularity theory to the space of knots [38]. The fundamental concept is the order of an invariant. An invariant, I, is said to have order less than n if

$$\sum_{\sigma}(-1)^{\sum_{k=1}^{n}\sigma_k} I(X_{\sigma_1,\sigma_2\ldots,\sigma_n}) = 0.$$

Here each σ_k is either 0 or 1, and $X_{\sigma_1\ldots\sigma_n}$ is an object obtained from X by making modifications to X at each of the locations labelled with a non-trivial σ. For example, if K is a knot projection with two labelled crossings K_{01} represents the knot obtained by changing the second crossing. The study of finite type invariants is tractable because the space of all order n invariants forms a finite dimensional space. The theory of finite type knot and link invariants was the further developed by J.Birman , X. Lin, and D. Bar-Natan. [5], [3].

Since the path integral of (1) is reminiscent of a Fourier integral operator, it is not surprising that the stationary phase approximation led to important new discoveries. This approximation was studied in [7], [34] and [21]. Higher order terms in the stationary phase approximation were similar to finite type invariants.

The second area represented at this workshop is Gromov-Witten theory. Gromov-Witten (GW) invariants are invariants of a complex projective manifold X (more generally a symplectic manifold) that are invariant under deformations of the Kahler (more generally, almost-Kahler) structure of X. GW invariants are certain integrals over a moduli space of holomorphic maps of Riemann surfaces into X and can consequently often be related to the enumerative geometry of X (“counting curves” on X). They arose in physics as correlators in a certain topological string theory.

The mathematically rigorous foundations of Gromov-Witten theory have been developed over the last 15 years, drawing on techniques from algebraic geometry, geometric analysis, algebra, and topology (see for example [8] and the references therein). However, the ideas from physics have continued to be extremely influential, notably the idea of mirror symmetry. In physics, this is the statement that type IIA string theory compactified on a Calabi-Yau 3-fold M is equivalent to type IIB string theory compactified on a different Calabi-Yau 3-fold W (the “mirror manifold”). This leads to very surprising predictions for the number of rational curves on M in terms of the variation of Hodge structure on W. This aspect of the “mirror conjecture” has been recently proven for a wide class of Calabi-Yau 3-folds using mathematical localization techniques (consequently giving an understanding of the mirror symmetry phenomenon that is very different from the physics “proofs”) [13, 28, 4].

The physically conjectured relationship between Chern-Simons theory and Gromov-Witten theory may be expressed using the free energy. For simplicity we restrict to the case of S^3. The Chern-Simons free energy is by definition,

$$F^{cs}(N, g_s) = \ln Z = \sum F_\Gamma W_{U(N)}(\Gamma) g_s^n(\Gamma).$$

Here Z is the partition function defined by the path integral (1), and the sum on the right arises from the stationary phase expansion of Z. The sum is taken over a collection of trivalent graphs, f_Γ are rational numbers that depend on the graph, $W_{U(N)}(\Gamma)$ is the $U(N)$ weight system applied to the graph and $n(\Gamma)$ is half the number of vertices of the graph. The weight system may be expressed as a sum of terms depending on the number of boundary components of surfaces obtained by thickening the graphs. These surfaces with boundary may be interpreted as open strings, however there is a way to combine the open strings into closed strings. Introduce a formal parameter, $t = g_s N$. Using this parameter the free energy may be expressed as $F^{cs}(t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}$. This has the form of the free energy of a string theory. The relevant string theory is constructed via the “Conifold transition.” The complex structure on the cotangent bundle of S^3 has a certain deformation whose limit is a complex 3-fold having an isolated singularity; this is the so-called “conifold”. This singularity has a crepant resolution which is isomorphic to the total space of the bundle $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$. Chern-Simons theory on S^3 is equivalent to Gromov-Witten theory on X. The moduli space of holomorphic maps of genus g in homology class β has a zero dimensional $\mathbb{Q}$-virtual class. The degree of this class a rational number, $N_{g,\beta}$. The Gromov-Witten free energy is then given by, $F^{GW}(t) = \sum_{g,\beta} N_{g,\beta} e^{t\beta} g_s^{2g-2}$. The most basic prediction of Chern-Simons String duality is that $F^{cs}(t) = F^{GW}(t)$.

The core of our workshop consisted of 23 invited 45-minute scientific talks and 16 15-minute question answer sessions.

Marcos Marino is a theoretical physicist in the high-energy theory department at CERN. He gave a series of four lectures on the physical arguments for Chern Simons string duality. His lectures were titled: CS invariants in the $1/N$ expansion, CS invariants on S^3 knot theory and enumerative geometry, Extension to other 3-manifolds, and Gromov-Witten theory of CY manifolds and CS theory.

Hans Wenzl is a mathematician at UC San Diego. He gave a pair of lectures titled: Knot invariants from Quantum groups. In his talks, he reviewed the construction of topological invariants from modular tensor categories, and the construction of such categories from quantum groups.

Chiu-Chu (Melissa) Liu is a mathematician at Harvard. She gave three lectures. Her lectures were titled Open Gromov-Witten theory, Virtual localization, and Formulae of one-partition and two partition Hodge integrals. Her talks described how to define open GW invariants in the S^1-equivariant case, how to apply localization to compute GW invariants, and provided a proof of conjectured formula of Marino-Vafa and Zhou that provide further evidence for CS string duality.

Justin Sawon is a mathematician at SUNY at Stony Brook. He gave a talk titled: Perturbative expansion of Chern-Simons theory. His talk covered the step from path integrals to Feynman diagrams and trivalent graphs.

Jun Li is a mathematician at Stanford. He gave a pair of lectures. His lectures described relative Gromov-Witten invariants.

Dror Bar-Natan is a mathematician at the University of Toronto. He gave a pair of lectures titled: Introduction to Perturbative Chern-Simons theory, and Introduction to Khovanov Homology. His first talk covered the role of trivalent graphs in three related areas - Lie theory, perturbative CS theory, and Vassiliev theory. His second talk introduced Khovanov homology for tangles. The q-Euler characteristic of Khovanov homology is the Jones polynomial. One surprise of this workshop was the physical conjecture of a new interpretation of Khovanov homology that Marino gave in his lectures.

Conan Leung is a mathematician at the University of Minnesota. He gave a lecture on Branes and Instantons for vector cross products.

Stavros Garoufalidis is a mathematician at the Georgia Institute of Technology. He gave a lecture titled BPS invariants of links and a conjecture of Labastida-Marino-Ooguri-Vafa. Chern-Simons string duality predicts a relation between Link invariants and Gromov-Witten invariants. The resulting Gromov-Witten invariants should be related to BPS invariants (integer counts of embedded holomorphic curves) according to the Gopakumar-Vafa conjecture. The resulting integrability predictions for link invariants were expressed as the LMOV conjecture. Garoufalidis gave a proof of this conjecture.

Michael Hutchings is a mathematician at UC Berkeley. He gave a talk titled: The embedded contact homology of T^3.

Jozef Przytycki is a mathematician at George Washington University. He gave a talk titled Khovanov homology of tangles and I-bundles over surfaces.

Justin Roberts is a mathematician at UC San Diego. He gave a talk on Rozanski-Witten invariants.

Takashi Kimura is a mathematician at Boston University. He gave a talk titled: Admissible covers, equivariant topological field theories, and orbifolding.

Jacob Shapiro is a mathematician at the University of British Columbia. He gave a talk titled: On the Gopakumar-Vafa conjecture for a local elliptic K3 surface.

Lenny Ng is a mathematician at AIM. He gave a talk on contact homology of Ooguri-Vafa Lagrangians.

The participants at the workshop had many favorable things to say about the Banff International Research Station and the workshop.

Dagan Karp learned exciting new mathematics that is not available elsewhere, and began a new project or collaboration investigating one object from the point of view of Gromov-Witten theory, open Gromov-Witten theory, and Chern-Simons theory.

David Gay came away with two new research projects. He intends to study the Lagrangians in $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ corresponding to various knots. He also intends on studying the relationship between G2 geometry and the degeneration of closed 2-forms along circles.

Hans Boden and Chris Herald proved a new theorem related to their ongoing project on Casson invariants.

Justin Roberts thought the timing of the meeting was perfect. He also thought having just one physicist was the right thing to do because it forced the physicist to interact with the mathematicians.

Kai Behrend said "[The workshop] is excellent. I'm getting lots of new ideas.

Melissa Liu commented that the subject of the workshop was directly related to her research and that she got a lot out of it.

Everyone commented about how spectacular the center was. Setting up a mathematics institute at an art center is a great way to encourage creativity in the mathematical sciences.

List of Participants

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Chapter 38

Theory and Numerics of Matrix Eigenvalue Problems (03w5008)

November 22–27, 2003

Organizer(s): James W. Demmel (University of California, Berkeley), Nicholas J. Higham (University of Manchester), Peter Lancaster (University of Calgary)

Matrix eigenvalue problems arise in many applications in science and engineering, ranging from the dynamical analysis of structural systems such as bridges and buildings to theories of elementary particles in atomic physics. Many current engineering design processes depend on the reliable computation of eigenvalues of matrices or matrix polynomials of possibly huge dimensions and often having particular structures. Underlying these computations are theory and numerical algorithms developed over the last 50 years, since the introduction of digital computers.

The aim of this workshop was to bring together researchers in the theory and numerical solution of eigenvalue problems with a view to surveying the state of the art, promoting collaboration, and making progress on the many challenging problems in this area. Also invited were users of these algorithms in applications. A unique feature of the workshop was that researchers from all parts of the spectrum from core linear algebra to numerical linear algebra to applications were able to interact and work together intensively for the duration of the workshop.

The workshop had 36 attendees. One confirmed participant had to withdraw the week before the meeting due to a family illness. The organizers encouraged everyone to fully participate in the workshop by giving a talk, and indeed every attendee gave a 30-minute presentation.

The attendees spanned the range from a PhD student and a recent PhD of just 2 weeks, to emeritus professors. Four participants were women. About one third of the attendees were from Europe, the rest from North America.

Several themes from the workshop are now discussed. We have not attempted to mention every speaker.

Structured Eigenvalue Problems

Eigenvalue problems with structure are increasingly being encountered in applications such as control theory and dynamical systems. Here the matrices may be, for example, (point or block) Toeplitz, Hankel or Hamiltonian, perhaps combined with other properties such as symmetry or skew-symmetry, or may possess group or algebraic structures. The aim is to develop theory and numerical methods that respect the structure and spectral properties of the problem, because this can lead to significantly faster and/or more accurate solutions. For example, an algorithm for solving a real Hamiltonian eigenvalue problem should ideally iterate exclusively with Hamiltonian matrices and should produce computed eigenvalues with symmetry about the real and imaginary axes. On the theoretical side, it is necessary to develop canonical forms, perturbation theory that measures how the eigensystems change when the parameters in the matrix are perturbed, and information about the possible eigensystems.

Bini (Pisa) and Gu (Berkeley) talked about work by their respective groups on efficient implementation of the QR algorithm for companion matrices. A companion matrix is the result of expressing a polynomial root finding problem as a matrix eigenvalue problem (this is how MATLAB finds polynomial roots), and is also of independent interest. The idea is to adapt the QR algorithm to exploit the structure in the Hessenberg iterates that results from the fact that a companion matrix is a low rank perturbation of a matrix with displacement structure. By exploiting the structure it is possible to implement the entire QR algorithm in $O(n^2)$ operations, as opposed to the usual $O(n^3)$ operations for a full matrix. Achieving this order has been a long-standing open problem and this new work may well result in software packages such as MATLAB adopting the new QR implementation.

A different form of structure was studied in the talks by N. Mackey (Kalamazoo), Tisseur (Manchester) and Higham (Manchester), namely, automorphism group and Lie and Jordan algebra structure associated with a scalar product on $\mathbb{R}^n$ or $\mathbb{C}^n$. Mackey described Givens- and Householder-like transformations in an automorphism group for introducing zeros into vectors, which are essential tools in structured matrix computations. She also discussed to what extent when A itself has structure the factors in standard decompositions (polar decomposition, matrix square roots, matrix sign decomposition, singular value decomposition) inherit this structure. Tisseur investigated the structured mapping problem of characterizing all structured matrices A such that $Ax = b$ for given x and b; this problem arises in computing structured backward errors and condition numbers, for example. Higham derived conditions on a matrix function f for it to preserve automorphism group structure. The principal matrix square root was shown to be one such function and he developed methods for computing the square root that exploit the group structure.

Polynomial Eigenvalue Problems

Polynomial eigenvalue problems (PEPs) have been attracting growing interest in recent years. The special case of the quadratic eigenvalue problem (QEP) $(\lambda^2 A + \lambda B + C)x = 0$ is fundamental to vibration problems with damping.

Both Bai (U.C. Davis—talk given by Demmel in his absence) and Meerbergen (Free Field Technologies) described Krylov subspace methods for the QEP that exploit the structure of the problem and are more efficient in storage and computation than the application of standard techniques to the generalized eigenvalue problem (GEP) that results when the QEP is linearized. Bai's approach culminates in projecting the QEP to a lower dimensional QEP that retains symmetry or skew-symmetry in the coefficient matrices. These new techniques are expected to have a major impact on the numerical solution of PEPs as well as on model reduction.

Related to the problem of solving the QEP is that of finding a solvent of the matrix polynomial $AX^2 + BX + C = 0$. Guo (Regina) examined a particular structured QEP modelling a gyroscopic system. After a Cayley transform, a solvent can be obtained, by solving a much-studied matrix equation of the form $X + A^T X^{-1} A = Q$. The method of cyclic reduction provides an efficient iterative method for the solution of this equation.

Lancaster (Calgary) discussed an inverse quadratic eigenvalue problem: Given complete data on eigenvalues and eigenvectors for an underdamped system of size n, determine the three real $n \times n$ coefficient matrices. Three approaches were compared using spectral theory, structure preserving similarities, and factorization theory.

S. Mackey (Manchester) discussed "palindromic matrix polynomials", which in the quadratic case have the form $\lambda^2 A + \lambda B + A^T$, where $B = B^T$. These problems arise in some recent applications in SAW filters and in modelling the German railway system. Mackey showed how to construct linearizations that are themselves palindromic and which enable this structure to be exploited numerically.

Mehrmann (Berlin) discussed numerical solution of parametric eigenvalue problems in robust control, emphasizing the importance of H_∞ norm computations and showing how to avoid the use of Riccati equations. Embedding into matrix pencils and using structured eigensolvers were emphasized.

Koev (MIT) described how to accurately compute the eigensystem of a totally nonnegative matrix. Working exclusively with the representation of the matrix into bidiagonal factors, he showed how the matrix can be reduced to (nonsymmetric) tridiagonal form, after which a symmetrization converts to a bidiagonal SVD problem that is solved by the standard QR iteration. This guarantees that all eigenvalues are computed to high relative accuracy, whereas conventional algorithms guarantee no relative accuracy at all.

Watkins (Pullman) presented a unifying viewpoint on GR algorithms for products of matrices. He showed that several methods previously thought of as extensions of the QR algorithm, including an algorithm for computing the periodic Schur form, are in fact special cases of the QR algorithm.

Applications

Several speakers described applications of eigenvalue problems in science and engineering.

Liu (SLAC, Stanford) described work in the DOE TOPS SciDAC project on large eigenvalue problems in cavity design for particle accelerators. One important program for solving these problems is Omega3P, a parallel distributed-memory finite-element code for electromagnetic modeling of complex 3D structures. A variety of algorithms for solving the resulting large sparse eigenproblems in parallel were presented and compared.

Frommer (Wuppertal) discussed Krylov subspace methods for computing $\mathrm{sign}(Q)b$, where Q is a very large, sparse, complex matrix arising in lattice chromodynamics computations in particle physics. He compared the use of rational approximations to the sign function developed over the last 15 years by numerical analysts, an old rational approximation of Zolotarev that requires the spectrum to lie in $[-b,-a] \cup [a,b]$ with a and b known, and more recently developed Krylov techniques. He concluded that Lanczos based projection techniques are often close to optimal.

Dhillon (Austin) described inverse eigenvalue problems and matrix nearness problems arising in the timely problem of wireless communications, comparing finite-step methods with alternating projection methods. He introduced the concept of tight frames and explained its relevance.

Carrington (Montreal) discussed large eigenvalue problems in quantum dynamics. He described the chemistry background to the eigenvalue problems he deals with and explained how chemists currently solve these problems.

Theory

Most of the talks included theory to a greater or lesser extent. In this section we include talks whose main purpose was to develop new theoretical results or techniques, and which have not already been mentioned in earlier sections.

Byers (U. Kansas) showed that an arithmetic can be defined for matrix pencils $A - \lambda B$, which can lead to more concise and elegant derivations of algorithms.

Zhou (Calgary) considered the eigenvalue problem for non-selfadjoint analytic matrix functions of two complex variables. She showed that, by using Newton diagram methods it is possible to identify derivatives of eigenvalues branching from a multiple semisimple eigenvalue.

Li (William and Mary) described eigenvalue and singular value inequalities and the relation with Littlewood-Richardson coefficients.

Tsatsomeros (U. Washington) gave a theorem on spectrum localization for almost skew-symmetric matrices.

Rodman (William and Mary) described a class of robustness problems in matrix analysis. For the algebraic Riccati equation and the generalized polar decomposition he investigated the stability of the solution/factors under perturbations, according to two different definitions of stability.

Sparse Eigenvalue Computations

In many practical applications the matrices of interest are huge, but sparse: relatively few entries are nonzero. Moreover, typically only a few eigenpairs are required, so that much computation time and storage can be saved. The operations that can be performed are limited to matrix-vector products and sparse direct or preconditioned iterative solvers.

Sorensen (Rice) gave a theory of convergence of polynomial restarted Krylov methods for eigenvalue computations, motivated by trying to understand an earlier paper by Beattie, Embree and Rossi. The main questions were to understand how nonnormality and the distribution of the starting vector with respect to the desired subspace affected convergence. By using results from complex approximation theory, asymptotic convergence rates (if not the actual error bounds) as a function of iteration number are accurately predicted.

Lehoucq (Sandia, Albuquerque) gave a numerical comparison of algorithms for computing a large number of eigenvectors of a generalized eigenvalue problem arising from a modal analysis of elastic structures using preconditioned iterative methods. In particular, he examined algebraic-multigrid-based alternatives to shift-invert Lanczos with a direct solver. On a set of difficult numerical examples, Locally Optimal Block Preconditioned Conjugate Gradient method (LOBPCG) emerged as the best alternative, provided a sufficiently large block was used.

Knyazev (Denver, Colorado), asked “Is there life after the Lanczos method?” and concluded positively by describing his LOBPCG method (also discussed by Lehoucq). The basic idea (of the unblocked method) is to minimize the Rayleigh Quotient on the subspace spanned by the current approximation, the current residual, and the previous approximation. Numerical tests indicate that it has the same linear (but not super-linear) convergence speed as Lanczos.

Spence (Bath) discussed the effect of inexact solves on the convergence of inverse iteration, showing that a quadratic rate of convergence can be obtained if residuals of the solves are suitably bounded.

Conclusions

The workshop provided an excellent atmosphere and facilities for interaction between attendees and for research collaboration. The ample supply of meeting rooms, computer terminals in every bedroom, ready access to printers, and well-stocked 24-hour lounge and kitchen, made working during the workshop easy and pleasant.

It is a pleasure to thank the staff at BIRS for the excellent facilities and support.

Existing collaborations were continued, and new ones initiated.

One of us (JWD) formulated plans for a new release of the Fortran linear algebra library LAPACK through discussions with participants during the workshop. Another of us (NJH) is involved in two grant proposals joint with other participants begun during the workshop and now submitted. The third of us (PL) initiated plans for collaboration on a book on matrix analysis in indefinite scalar product spaces, and for a BIRS application on a similar topic.

The participants appreciated the chance to interact with people they would not normally meet at other conferences. We feel that our aim of encouraging interaction between researchers across the spectrum from core linear algebra to numerical linear algebra to applications was fully justified and we are confident that in due course, as ideas and collaborations begun at the workshop reach fruition, the workshop will prove to have achieved its aims.

List of Participants

Bai, Zhaojun (University of California - Davis)
Barlow, Jesse (Penn State University)
Bini, Dario (Universita’ di Pisa)
Boulton, Lyonell (University of Calgary)
Byers, Ralph (University of Kansas)
Carrington, Tucker (Universite de Montreal)
Demmel, J.W. (University of California - Berkeley)
Dhillon, Inderjit (University of Texas at Austin)
Driessel, Kenneth R. (Colorado State University)
Drmac, Zlatko (University of Zagreb)
Frommer, Andreas (Bergische Universitat Wuppertal)
Golub, Gene (Stanford University)
Gu, Ming (University of California - Berkeley)

Guo, Chun-Hua (University of Regina)
Higham, N.J. (University of Manchester)
Ipsen, Ilse (North Carolina State University)
Kagstrom, Bo (Umea University)
Kirkland, Steve (University of Regina)
Knyazev, Andrew (University of Colorado at Denver)
Koev, Plamen (Massachusetts Institute of Technology)
Lancaster, Peter (University of Calgary)
Lehoucq, Richard (Sandia National Laboratories)
Li, Chi-Kwong (College of William and Mary)
Liu, Yifan (Stanford University)
Mackey, Niloufer (Western Michigan University)
Mackey, Steve (Western Michigan University)
Mathias, Roy (College of William and Mary)
Meerbergen, Karl (Free Field Technologies)
Mehrmann, Volker (Technische Universitat Berlin)
Parlett, Beresford (University of California - Berkeley)
Plestenjak, Bor (University of Ljubljana)
Rodman, Leiba (College of William and Mary)
Sorensen, Danny (Rice University)
Spence, Alastair (University of Bath)
Tisseur, Francoise (University of Manchester)
Tsatsomeros, Michael (Washington State University)
Watkins, David (Washington State University)
Zhou, Fei (University of Calgary)

Chapter 39

Nonlinear Dynamics of Thin Films and Fluid Interfaces (03w5021)

November 30–December 4, 2003

Organizer(s): Robert Almgren (University of Toronto), Robert P. Behringer (Duke University), Andrea L. Bertozzi (University of California, Los Angeles and Duke University), Mary C. Pugh (University of Toronto), Michael Shearer (North Carolina State University), Thomas P. Witelski (Duke University)

Abstract

This five-day multidisciplinary workshop focused on the mathematics of free-surface fluid flow. Building on theoretical and experimental developments of the last few years, the workshop brought together mathematicians and physicists at the forefront of research in experiments, analysis, computation, and modelling. The workshop yielded a rich exchange of ideas in the areas of thin liquid films, dynamic contact lines, slender jets, Hele-Shaw flow, and fluid interfaces.

Overview

The mathematics of thin liquid films and fluid interfaces has developed substantially in the last ten years, and is now being pursued by different groups of mathematicians, physicists, and engineers across Canada, Europe, and the United States. The workshop brought together experimentalists, analysts, modelers, and computational fluid dynamics experts for a meeting that yielded a vigorous assessment of the field from many different points of view.

The clear air and an early winter snow blanketing the spectacular surrounding mountains, coupled with the superb facilities of the Banff Institute, provided an ideal setting for the workshop. The meeting was notable for the many informal discussions, lively interaction during talks and generally friendly and cooperative atmosphere. While many of the participants have met before, there were also many new contacts established, and lots of resolve to continue to provide forums for continued dialogue between groups separated by large distances.

The mechanics of fluids with free surfaces or interfaces rests on the largely unexplored area of nonlinear fourth-order partial differential equations. Fourth-order derivatives arise from surface tension, which appears in the equations as gradients of the curvature of the free surface. Fourth-order equations have applications to several active fields of scientific research, including materials science, nanotechnology, and biology. The interaction of physicists and engineers with applied mathematicians and analysts has provided tremendous motivation for the mathematics; the mathematics in turn has contributed substantially to understanding of the wide variety of interesting physical phenomena in this area.

The workshop was linked to and partially supported by an NSF Focused Research Group (FRG) Grant (NSF-DMS #0073841/0074049) on the dynamics of thin films. This grant is directed by four of the workshop organizers: Behringer, Bertozzi, Shearer, and Witelski.

We encourage interested readers to do a literature search on speakers of interest and look at their articles for references. A conference proceedings is planned, to appear as a special volume of the journal Physica D.

Free-surface flows: Modelling and Dimensional Reductions

Free-surface flows are a challenging area of research in the physical sciences [1, 2, 3]. They are important for industrial and technological processes such as coating flows, film drainage, fluid jetting and droplet formation. Physical experiments with novel and intricate dynamics have pushed classic engineering models to their physical limits. The mathematical modelling and analysis of such dynamics has produced new generations of problems and results in free boundary problems for nonlinear partial differential equations [4, 5].

The mathematical description of a physical free-surface flow is composed of a number of elements. Assume the fluid is located in a time-dependent region $D_t \subset \mathbb{R}^3$. Initial conditions for D_0 and the velocity $\vec{u}_0(\vec{x})$ for all points in the domain are required. Each fluid particle $\vec{x}$ in this region moves with a velocity $\vec{u}(\vec{x}, t)$ determined by dynamical conservation laws for the conservation of mass and momentum. For example, if the fluid is an incompressible Newtonian liquid, the governing equations are the Navier-Stokes equations. The system has boundary conditions at ∂D_t. The conditions on ∂D_t can be mixed if portions of the boundary are liquid/gas, liquid/liquid, or liquid/solid interfaces with physical models dictating different conditions on each type of interface. Finally, there is a kinematic boundary condition at ∂D_t which determines the velocity $\vec{u}(\vec{x}, t)$ for points on the boundary. The subsequent evolution of the flow will be influenced by any forces that may be acting on the interior or the surface of the domain.

Direct numerical simulation or analysis of the full system of equations is intractable in most cases. And even when possible, large-scale numerical simulations do not necessarily provide insight into the fundamental physical mechanisms at work. Valuable understanding has been gained from studies of asymptotic models that are applicable in limiting cases. These models take more tractable forms, yet retain much of the rich dynamics of the experiments; their study spurs advances in analysis and computation. The use of such models was a major theme of the workshop, with many striking results of analysis and numerical simulation coupled to beautiful and instructive physical experimental findings.

In deriving an asymptotic model, one often makes an approximation of the region D_t, an approximation of the velocity $\vec{u}(\vec{x}, t)$, and approximations of the boundary conditions. This leads to an "asymptotic model" of the full flow. Asymptotic models of free-surface flows yield systems of lower dimension and simpler structure. *Lubrication models* give a single partial differential equation for the motion of the fluid interface, while *long wave models* give coupled evolution equations for the dynamics of the fluid interface and average flow velocity. These modelling techniques have produced striking advances in several areas:

- Thin viscous films on solid surfaces: Lubrication models for low Reynolds number flow of a thin viscous film on a solid surface take the form of a nonlinear fourth-order degenerate PDE for the height of the film's free surface, $h(x, y, t)$,

$$\frac{\partial h}{\partial t} + \nabla \cdot \Big(M(h)\, \nabla \big[\nabla^2 h + P(h) \big] \ + \ F(h)\, \vec{e}_1 \Big) \ = \ R(h). \tag{39.1}$$

  Here the liquid/solid interface is assumed to be at $z = 0$ and the air/liquid interface at time t is located at $z = h(x, y, t)$ and so D_t is approximated by $\mathbb{R}^2 \times [0, h(x, y, t)]$. The full velocity $\vec{u}(\vec{x}, t)$ is approximated by a laminar velocity $(v_1(x, y, t), v_2(x, y, t), 0)$ which is fully determined in terms of $h(x, y, t)$. Various physical effects including system geometry, gravity, surface tension, thermodynamic effects, chemical kinetics, materials properties and constitutive equations of state are all incorporated into the functions for $F(h)$, $M(h)$, $P(h)$, and $R(h)$. Under different conditions, complicated dynamics including rupture singularities and non-classical shocks occur in this model.

- The Hele-Shaw cell: In a Hele-Shaw cell, a fluid is placed between two closely-spaced parallel plates. The fluid moves in response to pressure gradients arising from surface tension and externally imposed forces. The equations describing this problem are the same as those that describe dendritic solidification in an important limiting case, and are also very similar to those describing flow through a porous medium. In the absence of externally imposed forces, the asymptotic model is

$$\begin{cases} \Delta p = 0 & \text{in } \Omega_t \\ p = \tau\kappa & \text{on } \partial\Omega_t \\ \vec{v} = -\nabla p & \text{on } \partial\Omega_t. \end{cases} \tag{39.2}$$

If the parallel plates are a distance 2δ apart, the fluid is in the region $\Omega_t \times [-\delta, \delta]$ at time t. The region Ω_t is the fluid region that one would observe by viewing the Hele-Shaw cell from above (if the two glass plates were on a flat table). The fluid in Ω_t moves with a velocity that is proportional to the gradient of the pressure p. (That is, the full three dimensional domain D_t has been approximated by $\Omega_t \times [-\delta, \delta]$ and the full velocity $\vec{u}(\vec{x}, t)$ has been approximated by a two-dimensional velocity $(-p_x(x, y, t), -p_y(x, y, t), 0)$.) The pressure jump at a point on the air/liquid interface $\partial\Omega_t$ is proportional to the curvature of the interface at that point times the surface tension parameter. The system (39.2) is elliptic: the velocity at a point $x \in \Omega_t$ is influenced by the entire domain. If the region Ω_t has a thin neck (if Ω_t were shaped like a barbell, for example, with a neck located at $(x, \pm h(x, t))$ for $x \in [-x_0, x_0]$) a further dimensional reduction reduces the nonlocal system (39.2) to a fourth-order PDE,

$$h_t + (hh_{xxx})_x = 0, \tag{39.3}$$

that holds for $x \in [-x_0, x_0]$. This PDE has interesting dynamics and remarkably complex singularity behavior [6].

- Instabilities of slender laminar jets of fluid. Consider a dripping faucet. As a pendant drop begins to form, there is a strand of fluid that connects the drop to the faucet's mouth. This strand of fluid is an example of a jet; it begins to develop instabilities and ultimately breaks and the drop falls. If the full three-dimensional domain D_t is approximately axisymmetric and with the jet's air/liquid interface located at $(h(z, t)\cos(\theta), h(z, t)\sin(\theta), z)$ with gravity is acting in the $-z$ direction, the long-wave model for this system yields a system of partial differential equations for the evolution of the jet radius and of the radially averaged velocity

$$\begin{cases} v_t &= -vv_z - \frac{p_z}{\rho} + 3\nu\frac{(h^2 v_z)_z}{h^2} - g \\ h_t &= -vh_z - \frac{1}{2}v_z h \\ p &= \gamma\left[\frac{1}{h(1+h_z^2)^{1/2}} - \frac{h_{zz}}{h(1+h_z^2)^{3/2}}\right]. \end{cases} \tag{39.4}$$

Here the full velocity $\vec{u}(\vec{x}, t)$ has been approximated by an axial velocity $(0, 0, v(z, t))$, p is the pressure, ν is the kinematic viscosity, ρ is the density, γ is the surface tension, and g is the gravitational constant. These equations capture the interaction of viscosity, inertia, and surface tension [1].

Experiments show that surface tension has important, yet often subtle, effects on interface dynamics. In the modelling equations, surface tension effects are produced by the curvature of the fluid interface, and by gradients of related properties on the interface. Consequently all of these physical problems are modelled by strongly nonlinear, sometimes nonlocal, higher-order nonlinear partial differential equations. These models have provided fundamental and important mathematical challenges. Interesting free boundary problems arise and much is being learnt about their solutions.

Important changes in the qualitative structure of the physical systems generally appear as finite-time singularities in the corresponding simplified model: formation of dry spots in a thin film, pinch-off of a thin neck in a Hele-Shaw cell, or pinch-off of a thin jet as in a dripping faucet. A great deal is now known computationally, mathematically and experimentally about the structure of singularities in the model systems. This knowledge is providing insight into the behavior of the full systems.

Analysis of thin film equations

One of the fundamental mathematical challenges involved in the study of equation (39.1) is that the the mobility coefficient, $M(h)$, vanishes at $h = 0$: $M(h) = O(h^n)$, $n > 0$. Assume that the mass is conserved

($R(h) = 0$) and there is no external driving ($F(h) = 0$). Further, consider the case of a film which is uniform in one direction. Equation (39.1) can then be written as

$$h_t + (hU)_x = 0 \tag{39.5}$$

where $U(x,t)$ is the average velocity across the cross-section of height $h(x,t)$ located at x at time t. It follows that in order for the thin film to form a dry spot in finite time,

$$h_{\min}(t) := h(x_{\min}(t), t) \to 0 \tag{39.6}$$

as $t \to T$, there must be a divergence of $U_x(x_{\min}(t), t)$ as $t \to T$. This loss of regularity prevents nonnegative h from going negative at a point (and becoming "unphysical"). Also, if there is a contact line at $x = a(t)$ (i.e. $h(x,t) > 0$ for $x \lesssim a(t)$ and $\lim_{x\to a(t)} h(x,t) = 0$) then this contact line cannot move unless $\lim_{x\to a(t)} U(x,t) = \pm\infty$.

These are heuristic explanations for how the interplay between the mobility $M(h)$ and the regularity of the solution h is crucial for the model to reflect the correct physics: the film thickness should be nonnegative at all times and the contact line should move with finite speed. Proving such behaviors analytically is nontrivial because equation (39.1) is fourth-order and the comparison-method-based techniques used for second-order degenerate equations do not hold. The development of new analytical techniques is an active area of research; six of the speakers presented analytical work on such thin film equations.

The study of exact solutions such as steady states, travelling waves, and self-similar solutions has proven to be very helpful in understanding general solution properties. In this direction, Mark Bowen presented a collection of exact solutions to the thin film equation

$$h_t + (h^n h_{xxx})_x = 0. \tag{39.7}$$

These included "dam break" self-similar solutions, separable solutions, and self-similar solutions that had a combination of fixed and free boundary conditions. His studies included how the value of n affected the solutions and their properties [7, 8, 9].

Continuing in the theme of this particular equation, Lorenzo Giacomelli presented work in which he rigorously derived the lubrication equation [10]

$$h_t + (hh_{xxx})_x = 0 \tag{39.8}$$

directly from the related Hele-Shaw problem. The derivation was done by finding the limiting behavior of solutions of the Hele-Shaw problem and the equation they obey, rather than performing asymptotics on the evolution equation. This work is especially exciting because it provides a new bridge between PDE analysts and experts in asymptotics.

Günther Grün presented new interpolation inequalities for

$$h_t + \nabla \cdot (h^n \nabla \Delta h) = 0 \tag{39.9}$$

and its analogues. He considered a higher-dimensional case than Bowen and Giacomelli: $\vec{x} \in \mathbb{R}^d$ with $d \geq 2$. The interpolation inequalities allow him to prove the existence of weak nonnegative solutions, finite speed of propagation for compactly supported weak solutions, as well as waiting time results [11, 12]. And as a bonus, he presented numerical simulations of complex structures being formed in thin films on silicon wafers that are being driven by molecular forces at the liquid/solid interface [13].

Continuing in the direction of liquid film equations with non-advective instabilities, Mary Pugh presented results concerning a critical-exponent long-wave unstable thin film equation [14],

$$h_t + (h^n h_{xxx})_x + (h^{n+2} h_x)_x = 0. \tag{39.10}$$

This included self-similar solutions that that blow up in finite time in a focusing manner and a study of how these solutions are related to the steady states of the evolution. Dejan Slepcev studied a special case of this critical-exponent equation ($n = 1$) and presented linear stability results for steady droplets, self-similar source-type solutions, and self-similar blow-up solutions [15].

Another system in which there can be surprising instabilities was studied by Sandra Wieland. She presented her thesis work on thin liquid films which have surfactants on the air/liquid surface. This leads to a coupled system involving a lubrication equation for the film thickness and an advection diffusion equation for the surfactant concentration. The system has hyperbolic aspects to it and its analysis is nontrivial and illuminating. A related talk by Barry Edmonstone focused on new phenomena and their realization in numerical experiments.

Filament Instabilities and Moving Contact Lines

Physical experiments are uncovering a broad array of dynamics in thin liquid films and fluid interfaces. The connections to industrial processes, such as spin coating in the manufacture of microprocessors, are also at an early stage of development. Some examples of recent progress in this area include scaling laws for jets and drops, a problem of relevance to inkjet printers, to electrospinning of fibers, and to nanotechnology printing devices.

Two of the speakers presented research on filaments and jets:

Linda Smolka presented her work on liquid filaments in extensional flows [16]. She considered exact, cylindrical solutions in which the radius is time-dependent and used these in a study of various viscoelastic constitutive models. This work is of definite industrial and scientific interest as it suggests ways to use extensional flows to measure rheological properties of nonstandard materials.

Marco Fontelos also considered axisymmetric structures in non-Newtonian flows, specifically in a model where the rheology's constitutive law is Oldroyd-B. He used asymptotics and numerics in an elegant study of the dynamics of droplets in a "beads on a string" configuration in which one has a collection of droplets connected by thin filaments; the filaments allow the droplets to interact and fuse [17].

Another area of intense experimental activity is that of moving contact lines, specifically how the microscopic physics at the contact line can influence large-scale properties of the flow. To date, most of the analysis of driven films has addressed the question of linear instability and wavelength selection. Understanding nonlinear dynamics requires cooperation between the experimentalists, modelers, and analysts in the following ways; (i) from experiments we need to understand basic nonlinear structures that emerge in this problem, including fingers, teeth-like patterns, and capillary ridge instabilities, (ii) once these patterns are well identified, modelers should work to isolate the essential physical mechanisms present and provide simplified models that are tractable for analysts, (iii) a combination of analysis and computation of the models will lead to a general understanding of the problem that can then be compared with experiments and guide further experiments. Ten of the speakers contributed to this program by reporting on their studies involving driven films and contact line instabilities of various types:

Stephen Wilson presented his work on steady rivulet flows down inclined planes. This is an extensive combination of analytical and computational work that considers both constant and temperature-dependent surface tension [18, 19].

Michael Schatz presented his experimental work in which he optically heats regions near the contact line, inducing a surface-tension driven flow [20]. This allowed a direct study of transient amplification near the contact line.

Roman Grigoriev presented careful computational studies of the transient amplification observed by Schatz and others, with the goal of understanding the widely differing predictions made by theorists. In addition, he presented striking experiments in which he used optical heating in a feedback control system to suppress contact line instabilities [20, 21].

Len Schwartz presented a collection of numerical studies of thin film flows with different types of driving forces and substrate geometries [22]. His studies included models with effects such as drying, the effects of surfactants, and the response to local contaminants on the air/liquid surface. At the end of his talk, he engaged the audience in a lively discussion of the "correct" boundary condition to place at the free boundary represented by the triple junction of fluid, surface and air, when the contact line is moving. His point of view is that a disjoining pressure should be introduced to avoid the contact line and thus avoid the well-known stress singularity that seemingly prevents the contact line from moving if the usual no-slip boundary condition is assumed between fluid and solid substrate [23]. This intriguing issue is a famous outstanding problem in the field of dynamic contact lines; it was alluded to repeatedly during lectures and in discussion.

Barbara Wagner presented her work on dewetting films. Here, one has a thin film which is rupturing in regions. In near a rupture region, the bulk of the film is receding from an area where the film has either ruptured or become very thin. Experiments show intricate transient dynamics and long-time patterns. Wagner presented numerical studies in which she finds that the receding front has a ridge that is linearly unstable in the spanwise direction, thus allowing additional structures to form.

Lou Kondic presented computational studies of gravitationally driven thin films on heterogeneous surfaces [24, 25, 26]. His studies focused especially on the dynamics of the contact line and how it responds to nearby perturbations. Of particular interest was the influence of the spatial structure of the perturbations.

Javier Diez presented experiments studying thin films flowing down a vertical plate. He studied both the constant volume and the constant flux flows and compared the experimental results to numerical simulations, in a study of how the thickness of the precursor film assumed in the simulations affects the relevance of the simulations to the experiments.

Barry Edmonstone presented numerical simulations studying surfactant-ladened thin films flowing down an inclined plane, as modelled by a coupled system for the film thickness and the surfactant concentration. He considered both fixed volume and fixed flux boundary conditions and studied instabilities at the contact line and within the surfactant monolayer concentration.

Martine Ben Amar presented her work on small droplets sliding down inclined planes [27]. Experiments show that there is a critical angle of inclination (or critical sliding speed) at which the the shape of the trailing contact line develops a corner. She presented modelling and asymptotics concerning this phenomenon, which is especially difficult to study because the droplets are fully three-dimensional.

Undercompressive shocks in driven films

The discovery of undercompressive shocks in driven films is an example of the interdisciplinary nature of studies on thin liquid films. This discovery arose from a three-way confluence of analytical and computational work on thin liquid films, of analytical studies of hyperbolic PDEs, and of experiments on thermally driven thin liquid films. Early experiments of Cazabat and Fanton [28] concerning a thin film that is climbing an inclined plate found that very thin films produced the characteristic fingering instability, while slightly thicker films showed a strong tendency not to form fingers. The latter behavior was observed to be associated with a broadening of the capillary ridge behind the contact line.

The experiments prompted Bertozzi, Münch, and Shearer to work on a mathematical model for this problem [29, 30]. The relevant form of equation (39.1) for the experiment has $M(h) = h^3$ and $P(h) = R(h) = 0$. What was unusual about the experiment is that the thermal gradient acts against the pull of gravity, producing a non-convex flux of the form $F(h) = h^2 - h^3$. In this case, equation (39.1) is a hyperbolic conservation law with degenerate fourth-order diffusion. Travelling wave solutions, correspond to either classical shocks, satisfying the well-known entropy condition, in which characteristics enter the shock from both sides, or to undercompressive shocks for which characteristics pass through. The undercompressive shocks help to explain the lack of fingering in thicker films, and the associated broadening of the capillary ridge. This comparison between theory and experiment is the first clear evidence of the existence of scalar law undercompressive shocks in a physical experiment. Further work on this problem has drawn on different branches of applied mathematics. For example, dynamical systems methods and Lyapunov functions were used to prove existence of undercompressive waves; Evans functions methods resulted in a connection between the topology of the travelling wave phase space and the stability of the waves.

Bob Behringer presented recent experimental and computational results of Jeanman Sur in this context [31]. The film climbs the plate driven by a temperature gradient that induces variation in the surface tension. Using a version of dip coating, undercompressive shocks were induced at the trailing edge of the film, in contrast to the original experiments in which the undercompressive wave appears at the leading edge. In addition, lasers are used to perturb the film at a chosen wavelength, thus experimentally studying the linear stability of the system. Andreas Münch presented simulations of these so-called reverse undercompressive fronts in thin film flow, including numerical stability studies [32].

Michael Shearer presented his work on Lax shocks and undercompressive shocks in the context of driven thin liquid films [33, 34]. He demonstrated how the structure of waves observed in experiments and numerical simulations can be explained using only the hyperbolic conservation law, together with an appropriate kinetic relation (that selects the undercompressive shocks) and nucleation criterion (that distinguishes between classical and nonclassical wave structures).

Peter Howard presented his work on nonlinear stability for viscous shock waves arising in conservation laws with high order viscosity [35, 36, 37]. He presented an overview of the pointwise Green's function approach for analyzing stability and discussed what this approach can and cannot do in the case of high order viscosity.

Burt Tilley presented modelling and computational work in which he studied flow of two incompressible immiscible viscous fluids in an inclined channel [38, 39]. In this system, he studied Lax shocks, undercompressive shocks, and rarefaction waves. Numerical simulations help identify parameter regimes for the different phenomena.

General problems on interface motion

Alexander Golovin studied the self-organization of quantum dots in thin solid films [40]. This included a weakly nonlinear stability analysis of spatially regular patterns in the presence of both epitaxial stress and anisotropic surface tension.

Amy Novick-Cohen studied a problem in solid materials in which a grain boundary that couples an exterior surface which is evolving under the influence of surface diffusion to an interior material which is antiphase. The problem is fully nonlinear and no assumption has been made about the interface's being the graph of a function. She presented a family of exact travelling wave solutions for the problem.

John Bush performed aquabatics. He presented experiments in which a jet of water impacts on a flat surface, or two jets impact on each other. Both situations produced structures with surprising symmetries and great temporal stability [41]. Analysis of the flows help to explain the observed structures. In addition, he discussed various types of insects that walk on water, including a robotic insect that his research group had built out of a soda can [42]. John's presentation was further distinguished by his use of artistic graphics and colourful photos of experiments.

Dan Lathrop presented experiments involving topological changes of Newtonian thin jets as well as experiments in turbulence and possible connections to singularities in inviscid flow. He presented an impressive collection of power-law behaviors in real fluids, as well as a novel way to "measure" turbulence [43, 44].

John King presented his studies of a thin fluid film in which the air/liquid free interface has been replaced with a rubber sheet, resulting in a sixth–order modelling equation:

$$h_t = (h^n h_{xxxxx})_x. \tag{39.11}$$

He demonstrated the power of local asymptotics, finding a large collection of special solutions and studying their dependence on the exponent n [45].

Linda Cummings presented thin film models for nematic liquid crystals, including different types of anchoring for the rods at one or both of the surfaces (air/liquid and liquid/solid). Lubrication models were found for the film thickness, allowing for stability studies [46].

Andy Bernoff presented experiments and modelling for polymer monolayers resting on the flat air/liquid interface of a sub fluid, like an oil slick on water. Experiments show a different dynamic than that of oil on water — when mixed, the droplets of polymer do not break apart, instead they return to their droplet shape however extremely they have been sheared. Bernoff proposed a clever model that blends two and three dimensional behavior; he compared his model to the classical (two dimensional) Hele Shaw problem.

Mike Miksis presented extensive level set simulations of air bubbles rising in inclined channels as part of his continuing studies of air bubbles in blood vessels [47, 48]. Varying the Reynolds number and the inclination angle of the channel, he observed steady bubbles, time-periodic bubbles, and bubbles that rupture onto the wall of the channel.

Brian Wetton presented a new numerical method for elliptic problems in which the domain is separated into two regions by a free boundary, on which mixed linear Dirichlet-Neumann conditions are specified. He presented an iterative approach in which a "generalised Stefan velocity" is computed and used to decrease the residual.

List of Participants

Behringer, Robert (Duke University)
Ben Amar, Martine (Ecole Normale Supérieure)
Bernoff, Andrew (Harvey Mudd College)
Bowen, Mark (Nottingham University)
Bush, John (Massachusetts Institute of Technology)
Cummings, Linda (University of Nottingham)
Diez, Javier Alberto (Instituto de Fisica Arroyo Seco)
Edmonstone, Barry (Imperial College London)
Fontelos, Marco (Universidad Rey Juan Carlos)
Giacomelli, Lorenzo (Università di Roma)
Golovin, Alexander A. (Northwestern University)
Grigoriev, Roman (Georgia Institute of Technology)
Gruen, Guenther (University of Bonn)
Haskett, Ryan (Duke University)
Howard, Peter (Texas A&M University)
King, John R. (University of Nottingham)
Kondic, Lou (New Jersey Institute of Technology)
Lathrop, Daniel (University of Maryland)
Levy, Rachel (North Carolina State University)
Miksis, Michael J. (Northwestern University)
Muench, Andreas (Humboldt-Universitaet Berlin)
Novick-Cohen, Amy (Israel Institute of Technology)
Novruzi, Arian (University of Ottawa)
Pugh, Mary (University of Toronto)
Schatz, Michael (Georgia Institute of Technology)
Schwartz, Leonard W. (University of Delaware)
Shearer, Michael (North Carolina State University)
Slepcev, Dejan (University of Toronto)
Smolka, Linda (Duke University)
Tilley, Burt (Franklin W. Olin College of Engineering)
Wagner, Barbara (Weierstrauss Institute for Applied Analysis and Stochastics)
Wetton, Brian (University of British Columbia)
Wieland, Sandra (Institut für Angewandte Mathematik)
Wilson, Stephen (University of Strathclyde)
Witelski, Thomas (Duke University)

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Chapter 40

Calabi-Yau Varieties and Mirror Symmetry (03w5061)

December 6–11, 2003

Organizer(s): Victor Batyrev (University of Tubingen), Shinobu Hosono (University of Tokyo), James D. Lewis (University of Alberta), Bong H. Lian (Brandeis Univ.), Shing-Tung Yau (Harvard University), Noriko Yui (Queen's University), Don Zagier (Max Planck Institut)

This is a report for the five day workshop at the Banff International Research Station (BIRS) on "Calabi–Yau Varieties and Mirror Symmetry" held from December 6 to 11, 2003. There were 38 participants. The workshop was a huge success. This workshop was the second in the series, following the first one held at the Fields Institute in the summer of 2001. We have witnessed enormous progress since the last one, and we are hoping to have the third workshop in two years time elsewhere.

We are planning to publish the Proceedings of this workshop. In fact, the contract for the Proceedings has already been signed with International Press/American Mathematical Society. The editors will be James Lewis, Shing-Tung Yau and Noriko Yui. All papers will be refereed rigorously. The Proceedings is open to all participants, not only to the presenters of talks. The deadline for submitting papers is tentatively set at the end of August 2004.

Here is a sample response from participants.

Dear Noriko,
First let me congratulate you on the success of the conference, it was really unique and beautiful. Even John McKay told me that it was probably one of the best conferences he's ever attended, especially for the variety of the themes that all come together (and I think the food also was a factor).
Best regards,
Abdellah Sebbar

Dear Noriko,
As promised, some CY-problems as TeX-file. Best, and thanks again for organizing such a nice meeting,
Eckart Viehweg

Dear Noriko,
yes, it was a nice workshop. I will try to send the problem list as soon as possible.
Best, Duco van Straten

Dear Noriko,

The BIRS workshop definitely was very enjoyable and also very useful. I enjoyed talking to you and a number of other participants, in particular Jan and Helena, and also Eckart Viehweg, Klaus Hulek and Duco van Straten. Duco has great stamina even when the time is close to midnight!

I'll write up the problems over the weekend and send it to you early next week.

Thanks again for the invitation to such a nice workshop!

Best,

Rolf Schimmrigk

Dear Noriko,

Thanks a lot for your email. The workshop was really a fun. I will send you my problems very soon. Merry Christmas and Happy New Year!!!

Best

Andrey Todorov

Dear Noriko Yui,

Thank you again for the wonderful workshop. As I told you, for the first family, $x+1/x+y+1/y+z+1/z-k$ and $k=18,30,102,198$, you get a tau quadratic satisfying respectively $6x^2+5=0$, $6x^2+7=0$, $6x^2+13=0$, $6x^2+17=0$ defining number fields with class number 4.

Best wishes and Happy New Year.

Marie Jose BERTIN

The main themes

1. Arithmetic of Calabi–Yau varieties and mirror symmetry: Arithmetic of elliptic curves, $K3$ surfaces, Calabi–Yau threefolds, and higher dimensional Calabi–Yau varieties in connection with mirror symmetry. These will include the following topics and problems: Interpretation of mirror symmetry phenomena of Calabi–Yau varieties in terms of zeta-functions and L-series of the varieties in question, the modularity conjectures for Calabi–Yau varieties, the conjectures of Birch and Swinnerton-Dyer for elliptic curves and abelian varieties, the conjectures of Beilinson–Bloch on special values of L-series and algebraic cycles. Calabi–Yau varieties of CM (complex multiplication) type and their possible connections to rational conformal field theories.

2. Algebraic cycles, (classical and p-adic) Hodge theory, K-theory, Quantum cohomology theory for Calabi–Yau varieties. Of particular interest here are the regulators of [higher] algebraic cycles, and some evidence that the Calabi–Yau varieties provide the "most interesting" examples of regulator calculations.

3. Moduli theory for Calabi–Yau manifolds. Moduli of abelian varieties, $K3$ surfaces and Calabi–Yau threefolds. These will lead to classification problems of Calabi–Yau varieties, e.g., computations of period maps and period domains for $K3$ surfaces, classification of rigid Calabi–Yau threefolds.

4. Mirror symmetry for Calabi–Yau varieties and modular forms. Characterization of mirror maps in connection with mirror moonshine phenomenon. Rigorous definition of D-branes, and geometry behind D-branes. Borcherds product formula and mirror symmetry. Modular forms in mirror symmetry.

Background

A Calabi–Yau variety of dimension d is a complex manifold with trivial canonical bundle and vanishing Hodge numbers $h^{i,0}$ for $0<i<d$. For instance, a dimension 1 Calabi–Yau variety is an elliptic curve, a dimension 2 Calabi–Yau variety is a K3 surface, and a dimension 3 one is a Calabi–Yau threefold.

(A) One of the most significant developments in the last decade in Theoretical Physics (High Energy) is, arguably, *string theory* and *mirror symmetry*. String theory proposes a model for the physical world which purports its fundamental constituents as 1-dimensional mathematical objects "strings" rather than 0-dimensional objects "points". Mirror symmetry is a conjecture in string theory that certain "mirror pairs" of

Calabi–Yau manifolds give rise to isomorphic physical theories. Calabi–Yau manifolds appear in the theory because in passing from the 10-dimensional space time to a physically realistic description in four dimension, string theory requires that the additional 6-dimensional space is to be a Calabi–Yau manifold.

Though the idea of mirror symmetry has originated in physics, in recent years, the field of mirror symmetry has exploded onto the mathematical scene. It has inspired many new developments in algebraic geometry, toric geometry, Riemann surfaces theory, infinite dimensional Lie algebras, among others. For instance, the mirror symmetry has been used to tackle the problem of counting number of rational curves on Calabi–Yau threefolds.

In the course of mirror symmetry, it has become more and more apparent that Calabi–Yau varieties enjoy tremendously rich arithmetic properties. For instance, arithmetic objects such as: modular forms, and modular functions of one and more variables, algebraic cycles, L-functions of Calabi–Yau varieties, have popped up onto the scene. Also special classes of Calabi–Yau manifolds, e.g., of Fermat type, or their deformations, offer promising testing grounds for physical predictions as well as rigorous mathematical analysis and computations.

(B) One of the most significant developments in the last decade in Arithmetic Geometry and Number Theory is the proof of the Taniyama-Shimura-Weil conjecture on the so-called *modularity of elliptic curves defined over the field* $\mathbf{Q}$ by A. Wiles and his disciples. Wiles' idea is to exploit 2–dimensional Galois representations arising from elliptic curves and modular curves, and establish their equivalence. His method ought to be applied to explore arithmetic of Calabi–Yau threefolds. In particular, rigid Calabi–Yau threefolds defined over the field of rational numbers are equipped with 2–dimensional Galois representations which are conjecturally equivalent to modular forms of one variable of weight 4 on some congruence subgroups of $\mathrm{PSL}(2,\mathbf{Z})$. This might be regarded as concrete realizations of the conjecture of Fontaine and Mazur that every odd irreducible 2–dimensional Galois representations arising from geometry should be modular. Recently the modularity of odd mod 7 2–dimensional Galois representations has been established. It is one of our aims to understand this result in the modularity conjecture for rigid Calabi–Yau threefolds over $\mathbf{Q}$. For not necessarily *rigid* Calabi–Yau threefolds over $\mathbf{Q}$, the Langlands Program predicts that there should be some automorphic forms attached to them. We plan to test the so-called *modularity conjectures* for Calabi–Yau varieties over $\mathbf{Q}$ or more generally over number fields, first trying to understand them for some special classes of Calabi–Yau threefolds, e.g., those mentioned in (A) and more generally for motives arising from Calabi–Yau threefolds.

The determination of zeta-functions and L–series of Calabi–Yau will undoubtedly be one of the central themes in this endeavour. Recent works of Candelas, de la Ossa and Villegas have brought in "semi-periods" and the GKZ hypergeometric systems in the determination of zeta-functions of one-parameter deformations of Calabi–Yau threefolds. One of our goals is to understand their findings and their consequences on physics, arithmetic and geometry, and also the work of J. Stienstra on zeta-functions of ordinary Calabi–Yau manifolds via Dwork theory presented at the pilot workshop in July 2001 ought to be analyzed with vigor.

(C) There are a number of intriguing developments in the theory of algebraic cycles in the past 25 years, that not surprisingly, should open the door to an infusion of new techniques in the study of Calabi-Yau manifolds and mirror symmetry. The impact of classical Hodge theory as well as the p-adic Hodge theory, is clearly evident. On the algebraic side, there is the relationship of algebraic K-theory and Chow groups, leading to the Bloch-Quillen-Gersten resolution description of Chow groups. There is also the more recent relationship of Bloch's higher Chows groups and higher K-theory (a higher Riemann-Roch theorem), and a conjectured "arithmetic index theorem". The influence of the work of Bloch and Beilinson on the subject of algebraic cycles is profound. For instance there are the fascinating Bloch-Beilinson conjectures on the existence of a natural filtration on the Chow groups, whose graded pieces can be described in terms of extension data, and their conjectures about injectivity of certain regulators of cycle groups of varieties over number fields. There is also the work of others on how conjecturally this filtration can be explained in terms of kernels of higher regulators and arithmetic Hodge structures. The Calabi-Yau manifolds present an ideal testing ground for some of these conjectures.

In particular, the theory of D-branes ought to be pursued providing rigorous mathematical definition, and its connection to algebraic cycles, etc.

Objectives

The recent progresses mentioned above (A), (B) and (C), based on so many interactions with so many areas of mathematics and physics, have contributed to a considerable degree of inaccessibility to mathematicians and physicists working in their respective fields, not to mention, graduate students. Perhaps one of the greatest obstacles facing mathematicians and physicists is that each camp has its own language. Mathematicians have had difficulties isolating mathematical ideas in physics literatures, and vice versa for physicists. At the pilot workshop of this program at the Fields Institute in July 2001, we have witnessed firsthand how these barriers have started melting away. We hope that our proposed half year program at the Fields Institute would follow up the ground breaking efforts of the pilot workshop to the full fruition.

Geometry around mirror symmetry and string theory has been pursued by many mathematicians (complex geometers, toric geometers, and others), and great progress has been witnessed in understanding geometric aspects of the problem. In fact, recently a number of excellent books and survey articles have been published explaining complex geometric aspects of mirror symmetry on Calabi–Yau threefolds as well as on K3 surfaces.

Further, in the past two decades, a number of people who have studied that part of algebraic geometry dealing with Hodge theory and algebraic cycles, have found applications of their work in Quantum Cohomology, Mirror Symmetry and Calabi-Yau manifolds. One anticipates that these interactions between the various "schools" will blossom in the near future.

Arithmetic aspects on Calabi–Yau varieties and mirror symmetry, however, are yet to be explored vigorously. For instance, Wiles' method may be used to establish the *modularity for rigid Calabi–Yau threefolds defined over the field of rational numbers* a la Fontaine and Mazur. Also, investigation on the intermediate Jacobians of Calabi–Yau threefolds ought to be pursued using, for instance, p-adic Hodge theory, modular symbols. Again the recent paper of Candelas, et. al on the computation of the zeta-functions of Calabi–Yau manifolds over finite fields reveal a surprising connection of mirror symmetry to p-adic L-functions (which are the essential ingredients in Iwasawa theory). Further investigation on p-adic analysis in physics pertinent to mirror symmetry is proposed in this program. The construction of algebraic cycles on Calabi–Yau threefolds (generalizing the method of Bloch), investigation of L-functions of Calabi–Yau threefolds a la the conjectures of Beilinson and Bloch, among others, ought to be pursued with more rigor and intensity. In fact, the pilot workshop in July 2001 was mostly concentrated on arithmetic aspects of Calabi–Yau Varieties and Mirror Symmetry. One of the outcomes of the pilot workshop is that we have begun to understand the mirror symmetry phenomenon for a mirror pair of quintic hypersurfaces in $\mathbf{P}^4$ at the level of local zeta-functions.

Moduli theory of Calabi–Yau manifolds has gained maturity in recent years. For instance, modular functions, McKay–Thompson series, Borcherds product formula, are coming into the centre stage. Investigation on moduli theory of Calabi–Yau manifolds will certainly be one of the main themes pursued in this program.

Our goal is to bring together experts working in physics, geometry and arithmetic around Calabi–Yau varieties and mirror symmetry, and to exchange ideas and learn the subjects first-hand, mingling with researchers with different expertise. We expect these interactions to lead to progress in solving open problems in mathematics and physics as well as to pave ways to new developments.

Abstracts of Talks

Jim Bryan (University of British Columbia)

Topological Quantum Field Theory and Gromov–Witten Invariants of Curves in Calabi–Yau Threefolds

Topological Quantum Field Theory, as formulated by Atiyah, provided a general frame-work for understanding invariants of manifolds. The structure of TQFTs in dimension $1+1$ (i.e. surfaces with boundaries) is completely understood by elementary means yet they can still yield surprising results. For each positive integer d, we define a one-parameter family of $(1+1)$-dimensional TQFTs $Z_d(t)$ which specializes at $t=0$ to the famous Witten-Dijgraaf-Freed-Quinn TQFT for gauge theory with finite gauge group S_d (the d-th symmetric group). Our family of TQFTs completely encodes all the degree d local Gromov-Witten invariants of a curve (of arbitrary genus) in a Calabi-Yau threefold. This provides us with a "structure theorem" for these local invariants (a.k.a. multiple cover formulae). Using these ideas we completely determine the local invariants for $d < 7$.

Kentaro Hori (University of Toronto, Physics and Mathematics)

Calabi–Yau Orientifolds

I introduce orientifolds to mathematicians and discuss some applications. Orientifolds are associated with unoriented strings. Real algebraic geometry plays an important role, just as symplectic geometry and algebraic geometry did for oriented strings.

Xi Chen and James D. Lewis (University of Alberta)

Hodge D-Conjecture for $K3$ and Abelian Surfaces

We prove that the real regulator map is surjective on a general K3 or Abelian surface. The heart of the proof involves the use of rational curves on K3 and a degeneration argument. It is closely related to the recent progress on the enumerative geometry on K3. This is joint work with James Lewis.

Chuck Doran (Columbia University)

Integral Structures, Toric Geometry, and Homological Mirror Symmetry

We establish the isomorphisms over **Z** of cohomology/K-theory, global monodromy, and invariant symplectic forms predicted by Kontsevich's Homological Mirror Symmetry Conjecture for certain one dimensional families of Calabi-Yau threefolds with $h^{2,1} = 1$. These families arise as hypersurfaces or complete intersections in Gorenstein toric Fano varieties, and their mirrors are described by the Batyrev-Borisov construction. Our method involves (1) classifying all rank four integral variations of Hodge structure over $\mathbf{P}^1 \sim \{0, 1, \infty\}$ with maximal unipotent local monodromy about 0 and local monodromy about 1 unipotent of rank 1, and (2) checking, using properties of nef partitions of reflexive polytopes, that the **Z**-VHS of our Calabi-Yau families match those picked out by the K-theory of their mirrors via the HMS Conjecture. This is joint work with John Morgan.

Andrey Todorov (University of California Santa Cruz)

Higher Dimensional Analogues of Dedekind Eta Function

It is a well known fact that the Kronecker limit formula gives an explicit formula for regularized determinants of flat metrics on elliptic curves. It established the relation between the regularized determinant of the flat metrics on elliptic curves and their discriminants. This relation can be interpreted as follows; There exists a holomorphic section (multivalued) of the dual of the determinant line bundle such that its L^2 norm is equal to the regularized determinant of the Laplacian acting on $(0, 1)$ forms. Kronocker limit formula established that the holomorphic section constructed from the determinant line bundle is the Dedekind eta function.

In the talk we discuss the existence of the analogue of the Dedekind eta function for K3 surfaces and CY manifolds. The construction of the generalized Dedekind eta function is based on the variational formulae for the determinants of the Laplacians of a Calabi-Yau metric acting on functions and forms of type $(0, 1)$ on CY manifolds, K3 surfaces and Enriques surfaces. Based on the variational formulae we will establish the existence of a holomorphic section of some power N of the dual of determinant line bundle on the moduli space of odd dimensional CY manifolds whose L^2 norm is the N^{th} power of the regularized determinant of the Laplacian acting on $(0, 1)$. This holomorphic section of the determinant line bundle is the analogue of the Dedekind eta function for odd dimensional CY manifolds. In case of even dimensional CY manifold and we will show the existence of a holomorphic section of the relative dualizing sheaf of the moduli space.

In case of K3 surfaces the construction of the Dedekind Eta function is done on the moduli of Kähler-Einstein-Calabi-Yau metrics and then projected to the moduli of polarized algebraic K3 surfaces.

We will discuss also that the L^2 norm on the relative dualizing sheaf is a good metric in the sense of Mumford. This implies that the Weil-Petersson volumes of the moduli spaces of CY manifolds are rational numbers. When M is a CY threefold we will outline how to prove that the regularized determinant of the

Laplacian acting on $(0,1)$ forms is bounded and that the section η^N vanishes on the discriminant locus.

Brian Forbes (University of California Los Angeles)

Open String Mirror Maps from Picard–Fuchs Equations on Relative Cohomology

A method for computing the open string mirror map and superpotential, using an extended set of Picard-Fuchs equations, is presented. This is based on techniques used by Lerche, Mayr and Warner. For X a toric hypersurface and Y a hypersurface in X, the mirror map and superpotential are written down explicitly. As an example, the case of K_{p^2} is worked out and shown to agree with the literature.

Eckart Viehweg (University of Essen)

Complex Multiplication, Griffiths–Yukawa Couplings, and Rigidity for Families of Hypersurfaces

Reporting on joint work with Kang Zuo: math.AG/0307398.

Let $M(d,n)$ be the moduli stack of hypersurfaces of degree $d > n$ in the complex projective n-space, and let $M(d,n;1)$ be the sub-stack, parameterizing hypersurfaces obtained as a d-fold cyclic covering of the projective $n-1$-space, ramified over a hypersurface of degree d. Iterating this construction, one obtains $M(d,n;r)$. The substack $M(d,n;1)$ is rigid in $M(d,n)$, although the Griffiths-Yukawa coupling degenerates for $d < 2n$, hence in particular for Calabi-Yau hypersurfaces. On the other hand, for all $d > n$ the sub-stack $M(d,n;2)$ deforms. One can calculate the exact length of the Griffiths-Yukawa coupling over $M(d,n;r)$. As a byproduct one finds a rigid family of quintic hypersurfaces over some 4-dimensional subvariety M of the moduli stack, and a dense set of points in M, where the fibres have complex multiplication.

Yi Zhang (Zhejiang University, China)

Some Results on Families of Calabi–Yau Varieties

The aim of the lecture is to show some new results related to the families of projective Calabi-Yau manifolds.

First, the author introduces concisely the results of rigid problem related to Shafarevich conjecture of Calabi-Yau which are included in the author preprint "The Rigidity of Families of Projective Calabi-Yau manifolds, Math.AG/0308034", i.e. he shows that some important families of Calabi-Yau manifolds are rigid,for examples:

(I) Lefschetz pencils of odd dimensional Calabi-Yau manifolds are rigid;

(II) Strong degenerate families (in some sense, an not need to be CY manifolds) are rigid;

(III) Families of CY manifolds admitting a degeneration with maximal unipotent monodromy must be rigid.

The main methods of the author to attack the problems is that the degenerate theory of variation of Hodge Structure, Yang-Mills theory of Higgs bundle and Deligne-Katz's theory on monodromy,etc.

Initiated by the results of rigidity problems,the author study the important and interesting object in family geometry: Mumford-Tate group. The author wants to understand how Mumford-Tate groups control the families,especially the families of Calabi-Yau threefolds? Are there necessary relations between Mumford-Tate groups and rigidity problems? For example,he shows some relations between the global monodromy group and Mumford-Tate group.

Rolf Schimmrigk (Kennesaw State University)

Complex Multiplication of Calabi–Yau Varieties and String Theory

Abelian varieties with complex multiplication can be identified as the basic cohomological building blocks of certain types of Calabi-Yau manifolds. It is therefore possible to define the notion of complex multiplication for Calabi-Yau spaces via the complex multiplication type of these abelian varieties. The aim of this talk is to show how this symmetry illuminates the exactly solvable conformal field theoretic nature of Calabi-Yau varieties.

4:45pm–5:45pm: Wei-Dong Ruan (University of Illinois at Chicago)

Generalized Special Lagrangian Torus Fibration for Calabi–Yau Manifolds

In light of SYZ conjecture, special Lagrangian torus fibration play important role in mirror symmetry. In this talk, we will discuss new examples of special Lagrangian submanifolds and the construction of global generalized special Lagrangian torus fibration for Calabi-Yau manifolds.

Jan Stienstra (University of Utrecht)

Between Bloch–Beilinson and Seiburg–Witten: Informal subtitle: Six Examples Everybody Thinks (S)He Knows

The same regulator image for elliptic curves in $C * xC*$ turns out to give the relation between Mahler measure, L-function values, modular forms in the articles by Deninger and Rodriguez Villegas and calculations of Gromow-Witten invariants (or instanton numbers) in physics papers by Klemm-Mayr-Vafa and Lerche-Mayr-Warner. The q-parameters in the mathematics and physics papers are inverse functions of each other. This is concretely illustrated for six families of elliptic curves, which have also appeared in many other contexts. There are very interesting connections with recent work of Kenyon, Okounkov, Vafa et al. on dimer models and melting crystals.

Matt Kerr (University of California Los Angeles)

Geometric View of Regulators and Higher Chow Groups

We collect together some techniques and results due to A. Collino, J. Lewis, S. Muller-Stach, S. Saito and others; the main object of study is the indecomposable part of $CH^p(X, n)$ in the cases $3 \geq p \geq n \geq 0$, where X is a Calabi-Yau or Abelian 3-fold or surface. Our aim is to discuss regulator formulae (including those developed in our work), connectivity results, degeneration techniques, and differential equations satisfied by regulator "periods" in families. We will also indicate some interesting open problems.

Shi-shyr Roan (Academia Sinica, Taipei, Taiwan)

Rational Curves in Rigid Calabi–Yau Threefold

We determine all the Kummer-surface-type Calabi-Yau (CY) 3-folds, i.e., those $\widehat{T/G}$ obtained by resolution of a 3-torus-orbifold T/G with only isolated singularities. There are only two such CY spaces: one with $G = Z_3$, and the other with $G = Z_7$. These CY 3-folds $\widehat{T/G}$ are all rigid, hence no complex structure deformation for the varieties. We further investigate problems of rational curves in $\widehat{T/G}$ not contained in exceptional divisors, by considering the counting number d of points in a rational curve C meeting exceptional divisors in a certain manner. We have obtained the constraint on d. With the smallest number d, the complete solution of C in $\widehat{T/G}$ is obtained for both cases. In the case $G = Z_3$, we have derived an effective method of constructing C in $\widehat{T/G}$, and obtained the explicit forms of rational curves for some other d by the method.

Belazs Szendroi (University of Utrecht)

Calabi–Yau Threefolds in Weighted Homogeneous Varieties

Reporting on joint work with Anita Buckley.

Let (X, D) be a Calabi-Yau threefold with quotient singularities, polarized by an ample Q-Cartier divisor. We prove a formula expressing the dimension of the vector space $H^0(X, nD)$ in terms of global numerical invariants of (X, D) and local invariants of D at the quotient singularities of X. Based on this formula, we construct several new families of Calabi-Yau threefolds in weighted homogeneous varieties, generalizations of weighted projective spaces introduced by Corti and Reid. In some cases, we show how to compute Hodge numbers of (smooth Calabi-Yau models of) these threefolds using birational geometry.

Shinobu Hosono (University of Tokyo)

GKZ Hypergeometric Series, Mirror Symmetry, and Singularity Theory

In the last workshop at Fields Institute(July, 2001), I talked about GKZ hypergeometric series taking values in the cohomology group of a Calabi-Yau manifold, and made a conjecture on the period integrals of the mirror Calabi-Yau manifold. In the case of two dimensional toric (non-compact) Calabi-Yau manifolds, I will verify the conjecture by relating the hypergeometric series to the integral solutions of K. Saito's differential equation in singularity theory. I will also present some three dimensional examples, and try to refine the conjecture.

Marie José Bertin (Université Pierre et Marie Curie (Paris 6))

Mahler's Measure and L-Series of $K3$ Hypersurfaces

We express in terms of Eisenstein-Kronecker series the Mahler's measure of two families of polynomials defining $K3$ hypersurfaces. For some of these polynomials we relate their Mahler's measure with the L-series of the corresponding $K3$-surface.

Klaus Hulek (University of Hannover)

Examples of Non-Rigid Modular Calabi–Yau Manifolds

Reporting on joint work with Helena Verrill.

In this talk we want to present some examples of non-rigid Calabi-Yau varieties whose L-series is modular. These examples are constructed by considering nodal Calabi-Yau varieties in the toric variety associated to the A_4 root lattice.

Kenichiro Kimura (University of Tsukuba)

K_1 of a self-product of a curve

Beilinson's conjectures on special values of L-functions predicts the existence of interesting higher Chow cycles on varieties over number fields. I will explain about the attempts to create such cycles mainly in the case of a self- product of a curve.

Keiji Oguiso (University of Tokyo)

Simple Groups, Solvable Groups and $K3$ surfaces

We characterize the following three particular K3 surfaces, among all the complex K3 surfaces, by means of finite group symmetries:

(1) the Fermat quartic K3 surface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\,.$$

(2) the Klein-Mukai quartic K3 surface

$$x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4 = 0\,,$$

i.e. the cyclic covering of degree 4 of projective plane branched along the Klein quartic curve. This is a joint work with D.-Q. Zhang.

(3) the minimal resolution of

$$s^2(x^3 + y^3 + z^3) - 3(s^2 + t^2)xyz = 0$$

in $\mathbf{P}^1 \times \mathbf{P}^2$, i.e. the minimal resolution of the double cover, branched along two singular fibres, of the (rational) elliptic modular surface with level 3 structure. This is a joint work with J.H. Keum and D.-Q. Zhang.

These three are all singular K3 surfaces in the sense of Shioda. The surface (1) is uniquely characterized as the K3 surface admitting either the solvable finite group action of maximum order or the nilpotent finite group action of maximum order. The surface (2) (resp. (3)) is uniquely characterized as the K3 surface admitting an action of the maximal possible finite extension of the simple group $L_2(7)$ (resp. $L_2(9) \simeq A_6$). In each case, we also show the uniqueness of the groups and their actions.

By a result of Mukai, the finite simple (non-commutative) groups which can act on some K3 surfaces are only $L_2(7)$, $L_2(9)$ and A_5, the first three groups in ATLAS. Among these three, the first two are the maximal simple groups (with respect to the inclusion of groups) which can acts on K3 surfaces. If possible, I would like to discuss about non-maximal A_5 case, too.

John McKay (Concordia University)

About Everything; Subtitle: Three Sporadic Groups and Affine Lie Data

I promote two conjectures - one I discovered 25 years ago - and the other just this year. A deep connection exists between affine $E6, E7$, and $E8$ data, and certain Fischer involutions of $F24'$, B, and M. The groups of 27 lines on a 3-ic, and 28 bitangents on a 4-ic have a large significant literature but the 120 tritangent planes on a 6-ic curve of genus 4 do not. The fundamental groups of type $E6, E7, E8$ are the Schur multipliers of the corresponding sporadic groups. The second conjecture is the appearance of the class number, 194, of M as a Picard number in “Numerical Oddities” of hep-th/0002012 by Aspinwall, Katz and Morrison.

Slava Archava (McMaster University)

Hodge Cycles of Milnor Fibers

In this talk I will describe our joint project with Hossein Movasati in which we attempt to study the space of Hodge cycles on a Milnor fiber of a non-composite polynomial (and more generally on an affine hypersurface complement) using the characterization of Hodge cycles by vanishing of appropriate periods. To carry out this program we need a description (as explicit as possible) of the mixed Hodge structure on the cohomology of the variety under investigation and an explicit basis for its homology. In the case of a quasi-homogeneous polynomial we use Steenbrink's description of the Hodge filtration on the cohomology of the Milnor fiber as order of the pole filtration, generalizing classical results of Griffiths for the cohomology of a smooth hypersurface in projective space.

Albrecht Klemm had to cancel his participation at last minute. He was scheduled to speak with the following title.

Albrecht Klemm (University of Wisconsin, Physics)

The Topological Vertex

Authors: Mina Aganagic, Albrecht Klemm, Marcos Marino, Cumrun Vafa

We construct a cubic field theory which provides all genus amplitudes of the topological A-model for all non-compact Calabi-Yau toric threefolds. The topology of a given Feynman diagram encodes the topology of a fixed Calabi-Yau, with Schwinger parameters playing the role of Kahler classes of Calabi-Yau. We interpret this result as an operatorial computation of the amplitudes in the B-model mirror which is the Kodaira-Spencer quantum theory. The only degree of freedom of this theory is an unconventional chiral scalar on a Riemann surface. In this setup we identify the B-branes on the mirror Riemann surface as fermions related to the chiral boson by bosonization.

Problems

During the problem session and informal discussions, a number of problems were proposed, which are collected here for the sake of future discussions.

Problem 1. (Eckart Viehweg, Kang Zuo)

Let M_h be the moduli scheme of polarized Calabi-Yau n-folds. Does there exists a compact curve C and a non trivial morphism $\varphi : C \to M_h$, which is induced by a smooth family $f : X \to C$, such that C is a rigid

Shimura curve, controlling the VHS $R^n f_* \mathbf{Q}_X$.

By taking the quotient of a special family of Abelian surfaces by the involution one finds examples for $n = 2$, i.e. for $K3$-surfaces. In "Families over curves with a strictly maximal Higgs field" we show that there are no such families, for n odd, and that in general, the VHS $R^n f_* \mathbf{Q}_X$ is isometric to one, build up by the weight one VHS of the family of Abelian varieties, parameterized by C. So in a vague way, this problem is related to the question, whether such an isometry has some geometric meaning.

Problem 2. (Eckart Viehweg, Kang Zuo) Find ball quotients and Shimura varieties in the moduli scheme of polarized Calabi-Yau manifolds.

Problem 3. (Eckart Viehweg, Kang Zuo) Let $g : Z \to S \times S'$ be a smooth family of Calabi Yau n-folds. In "Complex multiplication, Griffiths-Yukawa couplings, and rigidity for families of hypersurfaces" we have shown, that there exist polarized complex variation of Hodge structures $\mathbf{V}$ and $\mathbf{V}'$ on S and S', respectively, and a Hodge isometry $R^n g_* \mathbf{C}_Z \cong pr_1^* \mathbf{V} \otimes pr_2^* \mathbf{V}'$. Does such a decomposition exist for $R^n g_* K_Z$ where K is a number field, or perhaps even over $K = \mathbf{Q}$? And, does the existence of such a decomposition has any geometric interpretation?

Problem 4. (James D. Lewis) Let X be an algebraic $K3$ surface, and consider the transcendental regulator

$$\Phi : \mathrm{CH}^2(X, 1) \to \frac{H^{2,0}(X)^\vee}{H_2(X, \mathbf{Z})},$$

where $\mathrm{CH}^2(X, 1)$ is Bloch's higher Chow group. Consider the subgroup V of classes in $\mathrm{CH}^2(X, 1)$ obtained in the following way. Let $C \subset X$ be an irreducible nodal rational curve, $\sigma : \tilde{C} \xrightarrow{\approx} C$ its normalization, with points $P,\ Q \in \tilde{C}$, for which $\sigma(P) = \sigma(Q)$ is a nodal point on C. On $\tilde{C}$ there is a rational function $\tilde{f}$ for which $\mathrm{div}_{\tilde{C}}(\tilde{f}) = P - Q$. Since $\mathbf{C}(\tilde{C}) = \mathbf{C}(C)$, $\tilde{f}$ corresponds to a rational function f on C for which $\mathrm{div}_C(f) = 0$. Then $\xi := \{(f, C)\} \in \mathrm{CH}^2(X, 1)$ defines a higher Chow cycle. If $\omega \in H^{2,0}(X)$, then the value $\Phi(\xi)(\omega)$ is described as follows. Consider $f : C \to \mathbf{P}^1$. Then one can argue that $f^{-1}[0, \infty]$ is a closed 1-cycle that bounds a real 2-dimensional membrane Γ. Then

$$\Phi(\xi)(\omega) = \int_\Gamma \omega \quad (modulo\, periods\ H_2(X, \mathbf{Z})).$$

What can one say about the image $\Phi(V)$?

Problem 5. (Rolf Schimmrigk) Determine the relation between the conductor of modular Calabi-Yau variety and the level of the corresponding modular form.

Background: Weil's important contribution to the Shimura–Taniyama conjecture was his observation of the relation between the conductor of the modular elliptic curve and the level of the corresponding modular form. Recently it has been shown that many Calabi–Yau varieties are modular in the sense that the Mellin transform of the Hasse–Weil L-functions of the variety are modular forms of weight four at some level N. In order to understand this modularity of Calabi–Yau varieties more systematically, it would be useful to have an generalization of Weil's conductor observation to higher dimensional varieties, even if only conjecturally.

Problem 6. (Rolf Schimmrigk) Establish the relation between Kuga–Sato varieties and modular Calabi–Yau manifolds.

Background: Many Calabi–Yau threefolds have recently been shown to be modular. It has also been known for some time that cusp Hecke eigenforms forms of weight k admit a motivic interpretation in terms of the Kuga–Sato variety (Deligne, Scholl). We therefore encounter a situation which is rather different from the one encountered in the context of elliptic curves and abelian varieties, where it is known from Faltings' proof of the Tate conjecture that varieties over $\mathbf{Q}$ with the same L-function are isogeneous. In the context of Calabi–Yau varieties we have two different varieties with the same modular form. This raises the question what precisely the relation is between such modular Calabi-Yau varieties and the corresponding Kuga–Sato variety, associated to its modular form.

Problem 7. (Klaus Hulek) In [HV] we discuss two rigid (birational) Calabi-Yau varieties called X_1 and X_9 and prove that they are both modular with the same associated modular form f_6 which is the unique normalized newform of weight 4 and level 6. By a conjecture of Tate there should be a correspondence between X_1 and X_9 which explains this fact. We can actually prove that X_1 and X_9 are not birational. It would be interesting to exhibit a correspondence as predicted by Tate's conjecture.

Comment: The Calabi-Yau variety X_1 is also birational to the Barth-Nieto quintic and to Verrill's variety associated to the root lattice A_3 (cf [HSvGvS] and [SY]). The Barth-Nieto quintic has (birationally) a double cover $\tilde{X}_1$ which is again a Calabi-Yau variety with the same modular form. Here too X_1 and $\tilde{X}_1$ are not birational (they have different Euler numbers). But in this case the double cover gives the correspondence required by Tate's conjecture.

The following problem seems to have been posed by several people, and is posted here by Klaus Hulek.

Problem 8. (Hulek, et al.) For which Hecke eigenforms f of weight 4 does there exist a rigid Calabi-Yau variety X such that the L-series of X equals the Mellin transform of f (up to finitely many primes)?

Comment: Hulek became aware of this problem when B. Mazur asked him this question in Oslo in June 2003. He was meanwhile informed by D. van Straten that Straten himself had asked this question to B. van Geemen at an earlier occasion.

References for Problems 7 and 8

[HK] K. Hulek and H. Verrill, *On modularity of rigid and non-rigid Calabi-Yau varieties associated to the root lattics* A_4. arXiv:math.AG/0304169.

[HSvGvS] K. Hulek, J. Spandaw, B. van Geemen and D. van Straten, *The modularity of the Barth-Nieto quintic and its relatives* . Adv. Geometry **1** (2001), 263–289.

[SY] M.-H. Saito and N. Yui, *The modularity conjecture for rigid Calabi-Yau threefolds over* $\mathbf{Q}$. J. Math. Kyoto Univ. **41** (2001), 403–419.

Problems of Andrey Todorov

Introduction. Noriko Yui suggested to pose some problems about CY manifolds. The problems that I suggested are mainly connected to the construction of CY manifolds whose moduli space is a locally symmetric space. Up to now only one example is known, namely Borcea-Voisin manifold. The problems suggested in this note are related to the conjecture of Oort–Andre–Mazur which states that the set of manifolds of CM types are dense in the moduli space if and only if the moduli space is a locally symmetric manifold. This conjecture is related to some questions concerning rational conformal field theories.

A. Locally Symmetric Spaces as Moduli of Polarized CY Manifolds

Problem 9. Characterize all CY threefolds whose moduli spaces are locally symmetric spaces.

B. Gross in [gr] classified all the symmetric domains that are also tube domains and over them one can construct a variations of Hodge structure of weight three with dim$H^{3,0} = 1$.

Problem 10. This is an open problem posed by B. Gross. Can one find a geometric realization of the variations of the Hodge structure described in [gr]?

Problem 11. Show that the moduli space of CY threefolds that are double cover of $\mathbf{CP}^3$ ramified over eight planes in a general position is a locally symmetric space associated with $\mathbf{SU}(3,3)/\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$. If the moduli space of CY manifolds that are double covers of $\mathbf{CP}^n$ ramified over $2n+2$ planes is a locally symmetric space then it should be $\mathbf{SU}(n,n)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$.

One can show that the moduli space of a CY threefold is a locally symmetric space of rank greater than or equal to two if and only if the Yukawa coupling has no quantum corrections. It is a well known fact that any moduli space of CY manifolds that is one dimensional is a locally symmetric space and the famous example of Candelas and coauthors shows that there exists a CY manifolds whose moduli space is one dimensional locally symmetric space and there are quantum corrections. In the Candelas example the action of the mapping class group on the upper half plane is not arithmetic. It will be interesting to construct CY manifolds whose moduli space is one dimensional and action of the mapping class group is arithmetic on the upper half plane. I do not know how arithmetically it is related to the existence of quantum corrections to Yukawa coupling.

Problem 12. This problem was also discussed in [Bo]. It was stated that I. Dolgachev conjectured that the moduli space of CY manifolds that are double covers of $\mathbf{CP}^n$ ramified over $2n+2$ planes is the tube domain $S_n(\mathbf{C}) + \sqrt{-1}S_n^+(\mathbf{C})$, where $S_n(\mathbf{C})$ is the space of $n \times n$ Hermitian matrices and $S_n^+(\mathbf{C})$ is the space of positive Hermitian matrices.

The basis of proposing Problem 12 is the following lemma:

Lemma: Let $\mathbf{C}^{2n}$ be equipped with a Hermitian metric $\langle u, u\rangle$ with signature (n, n). Let $\mathbf{C}^{2n} = V \oplus \overline{V}$, where $\langle u, u\rangle$ when restricted to V is positive and on $\overline{V}$ is negative. Then $\wedge^n(V \oplus \overline{V})$ is a variation of Hodge Structures of weight n with $\dim_{\mathbf{C}} H^{n,0} = 1$. This variation of Hodge structures is parameterized by $\mathbf{SU}(n,n)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$.

B. CY Manifolds, Conic Singularities and M. Reid Conjecture

It is natural to ask if the generic point of the discriminant locus of an odd dimensional CY manifold corresponds to a manifold with conic singularity. This is not true. Let us take double covers of $\mathbf{CP}^3$ ramified over eight planes in a general position. After the resolution of the singularities we will get a CY threefold. The discriminant locus corresponds to a double covering which is ramified over eight planes three of them meeting in one point. So the generic point of the discriminant locus does not correspond to a threefolds with conic singularities since the monodromy group around these points is finite. The monodromy group of a conic singularity is infinite.

Remark. The homological mirror conjecture predicts that CY threefolds should have always conic singularities. The example of double cover of $\mathbf{CP}^3$ ramified over eight planes in a general position shows that one needs to modify this part of the homological mirror conjecture.

Problem 13. Suppose that M is a CY manifold whose moduli space is not a locally symmetric space. Is it true in this case that the generic point of the discriminant locus corresponds to a CY manifold with a conic singularity?

This problem is closely related to Miles Ried's conjecture that the moduli spaces of all CY threefolds are connected. So one can ask the following question:

Problem 14. Is it true that a CY threefold such that its moduli space is a locally symmetric space then the moduli space is contained in the discriminant locus of the moduli space a CY manifold and the generic point of the discriminant locus corresponds to a manifold with a conic singularity?

C. The Analogue of the Dedekind Eta Function

In [To03] we proved the following Theorem:

Theorem. Let $\mathfrak{M}_L$(M) be the moduli space of the polarized CY manifold. Let $\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ be the relative dualizing sheaf on $\mathfrak{M}_L$(M). Let det$(\Delta_{\tau,1})$ be the regularized determinant of the Laplacian of the CY metric acting on $(0,1)$ forms. Then locally we have: det$(\Delta_{\tau,1}) = \langle\omega_\tau,\omega_\tau\rangle\,|\eta|^2$, where ω_τ is a family of holomorphic n forms and η is a holomorphic function.

As a Corollary we get:

Corollary. Let Γ_L is the subgroup in the mapping class group which preserve the polarization class. According to Sullivan and Kazdhan for CY manifolds the group $\Gamma_L/[\Gamma_L,\Gamma_L]$ is finite. (See [Bour].) Let $N = \#\Gamma_L/[\Gamma_L,\Gamma_L]$. Then there exist exists a section η^N of the line bundle $\left(\omega^*_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}\right)^{\otimes N}$ such that det$(\Delta_{\tau,1}) = \langle\omega_\tau,\omega_\tau\rangle\,|\eta|^2$.

Let $\tau\in\mathfrak{M}_L$(M). Then we know that τ corresponds to a CY threefold M$_\tau$. Let us denote by ω_τ a non-zero holomorphic threeform on M$_\tau$. Let $\beta\in H_3$(M,**Z**), then we will denote by

$$\langle\tau,\beta\rangle := \int\limits_\beta \omega_\tau.$$

Problem 15. Can one find a product formula for the analogue of the Dedekind eta function of CY threefolds

$$\eta^N = \exp\left(2\pi\sqrt{-1}\,\langle\gamma,\tau\rangle\right)\times\prod_i\left(1-\exp 2\pi\sqrt{-1}\,\langle\tau,\beta_i\rangle\right),$$

around points of maximal degenerations, which would mean that around such points the monodromy operator has an index of unipotency $n+1$, β_i are the vanishing invariant cycles of the monodromy operators of infinite order and $\tau =(\tau^1,...,\tau^N)$ are the flat local coordinates? For a discussion of the product formulae for automorphic forms see [Bo1].

Problem 15 is closely related to paper [BCOV] and more precisely to the counting problem of elliptic curves on the CY threefold.

Problem 16. Prove that det$\Delta_{\tau,1}$ is bounded on the moduli space $\mathfrak{M}_L$(M) of any CY manifold M.

This problem will follow directly if one can prove that the coefficients a_k for $k=-n,...,1$ of the short term asymptotic expansion

$$Tr\left(\exp\left(-t\Delta_{\tau,1}\right)\right) = \sum_{k=-n}^{1}\frac{a_k}{t^k} + a_0 + ...$$

are constants. We prove that a_0 is a constant if M is a CY. The solution of Problem 16 will show that the analogue of the Dedekind eta function η^N vanishes on the discriminant locus. This will imply that the section $\overline{\eta}^N$ constructed in [To03] will be related to the algebraic discriminant as defined by Gelfand, Kapranov and Zelevinsky. The analogue of the Dedekind eta function for algebraic polarized K3 and Enriques surfaces was discussed in [Bo], [JT95], [JT96], [jt], [JT99] and [Y].

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There were unfortunately several last minutes cancellations. S.-T. Yau had to cancel his participation as the Chinese Premier was visiting the United States in the same period as the workshop, and he had to go to Washington DC to meet the Premier. A. Klemm had to cancel his participation as he was not able to find a replacement for his teaching (he has just moved to University of Wisconsin from Berlin). N. Shepherd-Barron had to cancel his participation due to a family reason.

Chapter 41

p-adic variation of motives (03w5104)

December 13–18, 2003

Organizer(s): Kevin Buzzard (Imperial College), Robert Coleman (UC Berkeley), Matthew Emerton (Northwestern University), Eyal Goren (McGill University)

The background

Langlands' conjectures, made in the 1970s, predict an extraordinary link between automorphic forms (essentially analytic objects) and representations of Galois groups (much more algebraic objects). In fact, strictly speaking, the link conjecturally relates automorphic forms to representations of even bigger groups, whose existence is yet to be established and about which we shall say very little. Langlands also made local conjectures and conjectured that the local and global conjectures should be compatible with one another. The link had already been established for automorphic forms on GL_1 when Langlands made his conjectures—indeed the link in this case was essentially equivalent to the main theorems of local and global class field theory. For automorphic forms on other groups, a lot is known about the local case ([12] for example) and the function field case ([19]), but the conjectures are still wide open in the number field case. One should also add that serious breakthroughs in the mod p version of the local conjectures have been made in [26]. The existence of the link (if it could be proved) has many consequences, for example it would give new ways of building automorphic forms via base change and automorphic induction. In a few cases these constructions have already been made, as a consequence of a lot of work ([1], for example). The local and global compatibility of Langlands' conjectures states basically that the local component at a place v of a global automorphic representation should contain essentially the same data as the restriction of the associated global ℓ-adic Galois representation to the corresponding local Galois group, at least for v and ℓ coprime. In the few cases where the global conjectures are known, these local and global compatibilities have frequently also been established (see [12], for example).

The beginnings of the theory of p-adic modular forms also emerged in the 1970s, thanks to work of Serre ([25]) and Katz ([17]). This theme lay dormant for a while afterwards, until it was taken up again by Hida in an important series of papers in the 1980s ([14],[15] and others). Hida developed a theory of p-adic families of modular forms, and one consequence of his work was that one could attach Galois representations to certain p-adic modular forms.

These two stories, Langlands' conjectures and p-adic families, have enjoyed a healthy amount of interplay over the years, but it is only apparently recently that people have started to make deeper observations about a hitherto missing piece of the puzzle—the local and global theory of so-called p-adic automorphic forms. Let us make some attempt of describing the missing pieces in some degree of concreteness, at least for the group GL_2. On the automorphic side, if one considers continuous representations of the group $\mathrm{GL}_2(\mathbf{Q}_p)$ acting on a p-adic vector space, there are many many more such representations than the usual "admissible" representations which arise in the classical theory. On the Galois side, if one considers 2-dimensional p-

adic representations of the absolute group of $\mathbf{Q}_p$ then there are far more continuous representations than potentially semi-stable ones—and specialisations of Hida families at forms of negative weight typically give rise to representations which are not potentially semi-stable. The link between automorphic forms and Galois representations is frequently made through arithmetic geometry, and there are far more p-adic modular forms than classical modular forms. These three observations are all well-known, but it has taken a long time before people have begun to understand precisely which Galois representations, which p-adic modular forms, and which representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ are the "correct" ones, and what the links should be. Indeed, our understanding is still very limited, and for more general reductive groups it is still in its infancy. Our Banff workshop was organised to bring together researchers in these areas, to report on recent progress in our understanding of "the big picture" and to raise precise open problems in this area.

Let us conclude these general remarks with some observations about the current state of the art as of the end of 2003, for GL_2 and the general case. Amongst the p-adic modular forms, it seems that the overconvergent ones are the ones of the most interest. Indeed, in the important paper [8], Coleman shows that finite slope overconvergent forms lie in families just as Hida's ordinary forms did. It is also known that ordinary forms are overconvergent. Unfortunately Coleman's work does not generalise too easily to other groups, and we are still searching for a good definition of an overconvergent automorphic form. Good definitions seem to be known for tori ([6]), groups which are compact at infinity ([7]), and certain unitary groups ([16]).

Schneider and Teitelbaum have begun a systematic study of so-called "locally analytic representation theory": the representation theory of p-adic lie groups on p-adic vector spaces ([21], [22], [23] [24]). In particular, they have introduced several important notions of "admissibility", which cut out various Abelian categories of representations. Emerton has introduced a notion of Jacquet module functors in the context of this theory ([9], [10]); passing to the Jacquet module of a representation is the analogue in representation theory of passing to the finite slope part of the space of overconvergent modular forms, in the theory of p-adic modular forms.

In the paper [11] Emerton has applied the techniques of locally analytic representation theory to construct many new examples of eigenvarieties, which however do not parameterise families of p-adic automorphic forms *per se*, merely p-adic analytic families of Hecke eigenvalues. This represents an important breakthrough, although it raises questions as well as answering them—for example, do the eigenvarieties that Emerton constructs coincide with those that are constructed by other, more classical, means? These are important questions that are only just being formulated.

Finally, amongst the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, Fontaine has singled out the Hodge-Tate and de Rham representations. Here the situation is more delicate. The representations associated to classical modular forms are known to be de Rham. However the Dieudonné module associated to a p-adic modular form is only 1-dimensional in general, by recent work of Kisin ([18]). On the other hand, Hodge-Tate representations are probably too general to be of interest. Breuil ([2], [3]) has made some important observations in this area, formulating very precise links between the irreducible admissible locally algebraic p-adic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ (as defined in [20] and [9]) and 2-dimensional de Rham representations, and he has also made some insightful conjectures concerning a mod p version of the theory.

Although the above picture is of a theory that is clearly only in its infancy, the theory has already had some non-trivial applications. Building on work of Wiles, Taylor and his coworkers used Hida families to verify many non-solvable cases of Artin's conjecture for 2-dimensional representations in [5]. Kisin has verified the Fontaine-Mazur conjecture for the Galois representations coming from overconvergent p-adic modular forms in [18]. Chenevier's construction of families of automorphic forms in his thesis were used by him and Bellaïche to prove new cases of the Bloch-Kato conjecture. Coleman's theory of the eigencurve explains computational observations of Gouvêa and Mazur, although more precise computations of Buzzard and Gouvêa have thrown up much more precise conjectures that still remain unproven (although see [4] and [13]).

The talks

Coleman and Mazur developed the theory of "eigencurves", geometric objects parameterising overconvergent modular eigenforms, and in their paper they raise many questions about the geometry of these objects. Several talks at the conference were about these questions, and the generalisation of the construction to other situations.

Work on explicitly computing regions of eigencurves corresponding to small slope forms has been done by Emerton and Coleman-Stevens-Teitelbaum. The first attempts to work at higher slopes were ideas due to Smithline, and the first concrete results were due to Kilford in his thesis, where he computed the fibre of the 2-adic eigencurve above some explicit points in weight space. Kilford talked about these results at the conference. This work was recently extended by Buzzard and Kilford, who manage to compute the pre-image in the 2-adic eigencurve of an annulus at the boundary of weight space. Buzzard and Calegari, in joint work, have recently managed to deduce from these results that the 2-adic eigencurve is proper over weight space, answering one of the questions raised by Coleman and Mazur in this particular case (Much of the work on this result was in fact done on the plane home after the conference finished.)

Kassaei in his talk illustrated how the Coleman-Mazur ideas could be extended to give p-adic families of automorphic forms associated to certain unitary groups, and as a consequence showed how one could deduce Gouvêa-Mazur-like results about the automorphic forms on these unitary groups. Kassaei also explained an important new construction of analytic continuation of overconvergent eigenforms in this case, which enabled him to prove that overconvergent eigenforms of small slope were classical, a generalisation of an old result of Coleman. Kassaei's new proof seems to use much less machinery than Coleman's, involving an "explicit" gluing process, and should certainly have applications in other areas. This is work in progress of Kassaei.

Stevens in his talk explained the status of his work with Ash on generalising the theory of eigencurves to cohomological eigencurves for GL_n. Here a new phenomenon comes into play, that of torsion in cohomology, which makes the construction much more delicate. As a consequence it is still not yet quite a theorem that overconvergent eigenforms for GL_n lie in analytic families of the expected dimension, and it indeed seems to be the case that this is probably not true in general. On the other hand, he has enough of a theory to, for example, construct the symmetric square of a family of modular forms, by interpolating the symmetric squares of the classical forms in the family.

The representation associated to a p-adic modular form was initially constructed via the theory of pseudorepresentations, and hence very little could be said about the local behaviour of such a representation at p. Iovita's talk (joint work with Stevens) was on a direct geometric construction of the representations. This construction of course sheds new light on known facts, for example on Kisin's work on the Dieudonné module associated to a p-adic modular form, and will no doubt have other applications.

Gouvêa, Buzzard and Stein have all done extensive computations of slopes of classical forms, in an attempt to better understand the general phenomenon of local constancy of slopes of modular forms. Buzzard has made some very precise conjectures about slopes, essentially saying that, for fixed tame level and prime, in many cases one can predict all the slopes of all classical modular forms of all weights. These conjectures were very ad-hoc and based mostly on computer calculations. Herrick has made some much more conceptual observations about these formulae and in his talk he gave some very detailed conjectures which have a lot more structure to them.

The so-called $\mathcal{L}$-invariant has played an important role in the local theory of modular forms and elliptic curves. There have been several different definitions of the $\mathcal{L}$-invariant, due to Coleman, Teitelbaum, Kato-Kurihara-Tsuji, and Fontaine. All definitions are now known to coincide and one crucial piece of work in this area was the theorem of Greenberg and Stevens, who used Hida families to relate the $\mathcal{L}$-invariant to L-functions. Hida's talk was about the current state of play in this area.

Moving away from GL_2, Tilouine in his talk gave an update of his progress for the group GSp_4. Much progress has been made on this group in the last decade—work of Weissauer has attached ℓ-adic representations to eigenforms, and Tilouine and his co-authors,in a series of papers, have developed enough of the theory to generalise work of Wiles and Taylor-Wiles to this situation. Tilouine talked about the ordinary Λ-adic version of this theorem, which is joint work with Genestier.

Urban in his talk reminded us about the known approaches to proving conjectures of Bloch-Kato type, by interpreting the cohomology groups which arise in the conjectures as groups of extension classes of representations, and relating these representations to automorphic forms. Classically, Mazur and Wiles used classical automorphic forms on GL_2 to deduce the main conjecture for GL_1, and Urban talked about higher-dimensional versions of this construction. Here there are immense technicalities to be overcome but serious progress has been made.

Greenberg in his talk gave us the state of the art about pseudo-null submodules. It is sometimes of great help in Iwasawa theory to know that certain modules which arise naturally in the theory have no non-zero pseudo-null submodules, and Greenberg gave an overview of many cases where this is now known.

Kisin in his talk announced a new and very strong modularity theorem of the form "ρ modular mod p implies ρ modular". Kisin's new insight is how to deal with local deformation conditions in cases which had hitherto been thought intractable. His arguments rely on Breuil's theory of linear algebra associated to finite flat group schemes over DVRs over which p is highly ramified. Kisin's main new idea in this area is a beautiful way of avoiding the technical troubles that Breuil, Conrad, Diamond and Taylor had with their local deformation problems at p and will certainly have other applications to modularity.

Niziol talk about her approach to Fontaine's conjectures via K-theory, a method which is now giving totally new proofs of the conjectures.

Finally, Coleman gave a short talk where he mentioned some history and raised some questions about Eisenstein series, series which have played a prominent role in several aspects of the theory, but whose overconvergence properties are still only just becoming known. Some explicit results about overconvergence of 2-adic Eisenstein series are now known, thanks to work of Buzzard and Kilford, but Coleman emphasized that it is important to understand the general case.

One of the most exciting things about the workshop was that many of the talks were on unpublished work. In particular Kassaei's talk on overconvergent forms of small slope being classical and Kisin's talk on deformation rings were all on work that was not even in preprint form at the time.

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Chapter 42

Coordinate Methods in Nonselfadjoint Operator Algebras (03w5110)

December 13–18, 2003

Organizer(s): Allan Donsig (University of Nebraska), Michael Lamoureux (University of Calgary)

An operator algebra is an algebra of operators on a Hilbert space, which is closed in a suitable topology. The most studied operator algebras are the selfadjoint ones: the norm-closed C*-algebras, which can be thought of as the noncommutative generalization of topological spaces, and the weakly-closed von Neumann algebras, the noncommutative generalization of measure spaces. Dropping the requirement that the operator algebra is closed under the adjoint operation gives a wide range of algebras with connections to various branches of mathematics. Examples include the algebras generated by a single operator or by a finite family of operators, and algebras constructed from dynamical systems. Even the algebras generated by finitely many (non-commuting) isometries have an interesting theory quite different from that of the corresponding C*-algebras.

Historically, some of the most challenging problems in the study of operators on Hilbert space showed the necessity of nonselfadjoint techniques. The invariant subspace problem, that every linear operator has a non-trivial invariant subspace, is a classic hard problem in operator theory. Positive results for specific families of operators, such as Lomonosov's result for operators which commute with a compact, or Brown, Chevreau and Pearcy's result for contractions whose spectrum contains the unit circle, rely on algebraic ideas. See, for example, the presentation of Lomonosov's result in Radjavi and Rosenthal's *Invariant Subspaces*. Reed and Enflo showed (independently) that there are operators on Banach spaces without invariant subspaces; the invariant subspace problem for an operator on a (separable) Hilbert space remains open. Also in the 1950's, Sz. Nagy's work on dilations of contractions demonstrated the power of nonselfadjoint techniques, and dilation results continue to play an important role in both operator theory and operator algebras.

A more formal study of nonselfadjoint operator algebras began with Kadison and Singer's 1960 paper on triangular operator algebras, the work of Ringrose on nest algebras in the early sixties, and Gohberg and Krein's book *An introduction of the theory of linear nonselfadjoint operators* from 1965. Also in this decade, connections between these algebras and dynamic systems were made, such as in Arveson and Josephson's work which demonstrated that the nonselfadjoint version of the operator algebra crossed product contained more information about the dynamics than the full selfadjoint crossed product. Understanding the structure of these algebras, in particular describing the ideals (closed or otherwise), became a focus of several important works, including those of Erdos and Power, McAsey, Muhly and Saito, and others. Davidson's text on nest algebras provides a comprehensive reference on the subject, although there have been a number of important development's since then, such as Orr's Interpolation Theorem for continuous nests and Annousis and Katsoulis's Russo-Dye Theorem for nest algebras.

In the last fifteen years, a new front has been opened, studying subalgebras of almost finite (AF) C*-

algebras. These algebras, obtained as inductive limits of subalgebras of matrices, provide a tractable class of algebras which nevertheless contain many of the surprising properties of the wider family of all nonselfadjoint algebras. In an analytic sense, the triangular AF algebras are small, and somewhat discrete, which means many of the functional analytic tools required to study them are not overly complicated. Remarkably, the K-theory for these algebras is typically rather simple, contracting to the K-theory of the self-adjoint diagonal. Describing the larger algebra in term of the smaller diagonal is a useful strategy.

Coordinate methods allow for such a description, and form an important general tool in the study of nonselfadjoint operator algebras. They allow these abstract algebras to be realized concretely as function algebras using a suitable convolution multiplication. This conceptually simple description masks serious technical issues, such as building a general framework for coordinates and describing the operator algebra norm in terms of the functions on the coordinates. Nonetheless, coordinates have been fruitfully applied to a wide variety of nonselfadjoint operator algebras, leading to progress on classification, structural results, representation theory, and ideal structure, as well as yielding connections with dynamical systems and crossed product constructions.

The goal of this workshop is to bring together researchers in nonselfadjoint operator algebras and related areas, to unify and broaden the technical machinery of coordinate methods to a wider class of nonselfadjoint operator algebras. Key problems include classifications of these algebras and precise descriptions of the ideal structure, the family of homomorphisms between algebras, and related properties. Applications to the study of single operators, commuting and noncommuting family of operators, dilations, semigroups, free actions, and dynamical systems, are also of interest.

The key topics are discussed more fully below.

Single operators and their algebras

The earliest results in nonselfadjoint operator theory were theorems that describe basic properties of a single linear operator in terms of the algebra it generates: for instance, questions of invariant subspaces, dilations, nilpotency, and other properties are best revealed by studying a full algebra and not just the single operator. Marcoux, in joint work with Farenick and Forrest, described the connection of amenability of a single operator in terms of the amenability of the Banach space of operators it generates. Such results on amenability are key theorems in harmonic analysis and C*-theory (e.g. equivalent to nuclearity), and these results are fundamental steps for nonselfadjoint algebras.

Families of isometries and graph algebras

From single operators, one next considers algebras generated by families of isometries; these include the free semigroup algebras as well as the graph algebras, wherein the structure of the isometries is encoded in a specified, directed graph. These include a large family of key examples of nonselfadjoint algebras, and one goal of this program is to develop techniques that can analyze this class of operators, as well as extend the results in C^*-theory for these type of structures. Davidson, Hopenwasser, Katsoulis, Kribs, Larocque, Peters all spoke on various aspects of these graph algebras.

Dynamical systems and crossed products

The evolution of a physical system over time is the basic model for a dynamical system. More generally, one considers the action of a group on a space, or on another algebra; the crossed product construction encodes both information about the space and the action in one algebra of operators. It is well-known that the nonselfadjoint variant of the crossed product contains more information than the large selfadjoint algebra; techniques to study these algebras effectively are still being developed. One particular challenge is that many of the existing methods involve some assumption of discreteness - an action of a discrete group, an r-discrete groupoid, and so on. An important research direction is to develop methods for the continuous case. Haataja, Lamoureux, Peters spoke on these topics.

Invariants

Invariants are a key tool in developing a classification theory for mathematical algebras. In nonselfadjoint algebras, many of the standard invariants (K-theory, homology, etc) of noncommutative geometry do not apply, or contract to a simple description of the algebra's diagonal, and thus more innovative invariants are needed. Pitts spoke on some joint work with Donsig on isomorphism invariants for triangular subalgebras, including the notion of the twist.

Spectral theorems

A spectral theorem describes an algebra or module as a space of functions on some fundamental object called the spectrum; for nonselfadjoint algebras, typically the spectrum is given by some maximal Abelian selfadjoint algebras (masa), and operators are analyzed, and synthesized through a reference to this masa. Bimodules and ideals are often effectively characterized by these techniques. Katavolos spoke on masa bimodules and Todorov on ternary masa-bimodules

Subalgebras of W^*-algebras

There is an important distinction between algebras that are closed in the norm topology, in the weak operator topology, and in the W^* topology; the analytical techniques used to work with these various closures are quite different, as one sees in the difference between norm-closed C^*-algebras and W^*-closed von Neumann algebras. Erlijman and Solel both gave talks on such variants, one of subfactors and invariants for von Neumann algebras, the other on W^*-correspondences. Davidson presented a Kaplansky-like density theory that it relevant to all three topologies for free semigroup algebras.

Open problems

One afternoon was devoted to presenting open problems, many of which had been hinted at in the expositions above. Listing relevant and important new challenges is a valuable contribution to the discipline. Kribs pointed out a number of basic questions concerning the free semigroupoid algebra L_G arising from a directed graph that remain to be resolved; for instance, when can $n \times n$ matrices of functionals be represented by collections of n-vectors (property A_n). Also with graph algebras, Peters suggested there should be some way to introduce a shift on standard Bratelli diagrams, to obtain a groupoid and a corresponding nonselfadjoint algebra; how would this compare to the AF C^*-algebra described by the Bratelli diagram? Solel noted here there is a connection with the fixed point algebra, which may be a useful direction to pursue. Is the a classification for co-cycles on Cuntz-Krieger algebras, in analogy with Solel's classification of co-cycles on AF-groupoids by extended asymptotic ranges? Given an isomorphism of a Cuntz-Krieger groupoid, does the k-th level set map onto itself (or it's negative)? It's not even known if the zero-th set maps to itself. An old problem of Larry Brown was brought up by Davidson: is there an isomorphism of the Calkin algebra which reverses Fredholm index? Also for free semigroups algebras, when are there wandering vectors? Can one find a description of all completely contractive Schur idempotents; patterns in the matrix representation are important, but it is not easy to see how to build them all. One could ask the same question in the case of continuous nest algebras. Pitts considers some problems in AF algebras: for instance, given an operator with zero spectrum, can one find a triangular limit subalgebra containing this operator? Marcoux illustrated a more specific problem in UHF algebras: given an operator with zero trace, can it be expressed as a commutator? The answer is yes in finite dimensions, in other examples it can be expressed as a sum of two commutators, and there seems to be no obstruction to doing it in general with just one, but the answer is not known. Grossman considered some nonselfadjoint problems that arise in real problems with seismic imaging, including characterizing minimum phase operators which represent physical attenuation of seismic waves.

Titles and abstracts:

Ken Davidson: A Kaplansky theorem for free semigroup algebras

Abstract: A free semigroup algebra is the unital weak operator topology closed algebra generated by n isometries with pairwise orthogonal ranges. We show that the unit ball of the norm closed algebra is weakly dense in the whole ball if and only if the weak-$*$ closure agrees with the weak operator closure. This fails only when the weak closure is a von Neumann algebra but the weak-$*$ closure is not — and no examples of this phenomenon are known to exist.

Juliana Erlijman: On braid type subfactors and generalisations

Abstract: I will discuss a few aspects related certain construction of families of subfactors from braid group representations and of their extension to subfactors from braided tensor categories, as well as some techniques for computing important invariants for some of the examples.

Jeff Grossman: Minimum-phase preserving filters

Abstract: This talk is intended as part of the "open problems" session for the workshop. I'll begin by introducing causality and minimum phase conditions that come up in wave propagation and seismic imaging

problems. In signal processing and imaging, we typically think of linear operators acting in L^2 as filters. The class of stationary (translation-invariant) linear filters corresponds to the convolution operators; and it is known that among these stationary filters, the ones which preserve minimum phase are precisely those described as convolution by a minimum phase function. So we ask the question: which nonstationary filters, if any, preserve minimum phase?

Steve Haataja: Inverse semigroups and crossed products

Alan Hopenwasser: Subalgebras of graph C*-algebras

Abstract: After a review of the groupoid associated with a graph C*-algebra, I will discuss the spectral theorem for bimodules. This says that a bimodule over a natural masa is determined by its spectrum iff it is generated by its Cuntz-Krieger partial isometries iff it is invariant under the gauge automorphisms. This contrasts notably with the situation for principal groupoids. If the edges of the graph are suitably ordered then (for finite graphs), there is a natural nest and a natural nest subalgebra associated with the order. I will describe the Cuntz-Krieger partial isometries which are in the nest algebra and the spectrum of the nest algebra.

Aristides Katavolos: Some results and problems on masa bimodules

Elias Katsoulis: Isomorphisms of algebras associated with directed graphs

Abstract: Given countable directed graphs G and G', we show that the associated quiver algebras A_G, $A_{G'}$ are isomorphic as Banach algebras if and only if the graphs G are G' are isomorphic. For quiver algebras associated with graphs having no sinks or no sources, the graph forms an invariant for algebraic isomorphisms. We prove that the quiver algebra A_G, associated with a graph G with no sources, is isometrically isomorphic to the disc algebra $alg(G)$ of the universal Cuntz-Krieger graph C*-algebra $C^*(G)$. This allows us to extend our classification scheme to subalgebras of graph C*-algebras of Cuntz-Krieger type. We also show that given countable directed graphs G, G', the free semigroupoid algebras L_G and $L_{G'}$ are isomorphic as dual algebras if and only if the graphs G and G' are isomorphic. In particular, similar free semigroupoid algebras are unitarily equivalent. For free semigroupoid algebras associated with locally finite directed graphs with no sinks, the graph forms an invariant for algebraic isomorphisms as well. (Joint work with D. Kribs.)

David Kribs: Directed graph operator algebras

Abstract: Every directed graph generates a family of operator algebras. They go by such names as Cuntz-Krieger or C-K-Toeplitz algebras, free semigroupoid algebras, quiver algebras, etc. Initial motivations came from dynamical systems, but now the study of these algebras has taken on a life of its own. Work on the nonselfadjoint subclass has been fruitful because it has been possible to link deep properties of the algebras with simple properties of the underlying directed graph in ways not possible for the C*-algebra case, and at the same time many new interesting examples have been discovered. I shall begin with a general discussion then touch on some specific results from joint works with Jury, Katsoulis and Power.

Michael Lamoureux: Continuous coordinate methods in nsa algebras

Abstract: Many of the tractable examples of nonselfadjoint operator algebras involve some assumption of discreteness: r-discrete groupoids, discrete group actions, graph algebras, and atomic nests, to name a few. To deal with more general nsa algebras that arise from dynamical systems and continuous group actions, we need more powerful tools to analyze the structure of the these algebras. We examine the continuous analogues for useful discrete coordinate methods.

Philippe Larocque: A spatial model for m λ-commuting isometries

Abstract: In this talk, we will describe a model for m isometries satisfying $V_iV_j = \lambda_{i,j}V_jV_i$ (in a Hilbert space). Basically, to (almost) every such m-tuple, a subset of Z^m can be chosen in such a way that m isometries can be defined on it and these isometries are approximately unitarily equivalent to the original m isometries.

Laurent Marcoux: On amenable operators

Abstract: A Banach algebra A is said to be amenable if all (continuous) derivations of A into dual Banach A-bimodules M are inner. In this talk, we shall discuss the amenability of norm closed, singly generated algebras of operators on a Hilbert space. (Joint work with D.R. Farenick [Regina] and B.E. Forrest [Waterloo].)

Justin Peters: Cocycles on Cuntz-Krieger groupoids

Abstract: We examine $Z^1(G; R)$ where G is a Cuntz-Krieger groupoid. We begin with a representation theorem for cocycles. This theorem yields a connection between the dynamics of the shift map on path space, and properties of cocycles. In AF groupoids, the bounded cocycles and the integer-valued Cocycles play important roles. We look at these classes in the Cuntz-Krieger context.

David Pitts: Isomorphism invariants for subdiagonal triangular subalgebras of regular C*-inclusions

Abstract: A pair of unital C*-algebras (C, D) is a regular C*-inclusion if D is a MASA in C whose normalizers span C and is such that every pure state on D has a unique extension to a state on C. When this occurs, there exists a faithful conditional expectation $E : C \to D$. Following Arveson, a norm-closed subalgebra A with $D \subset A \subset C$ is triangular and subdiagonal if $A \cap (A)^* = D$ and $E|_A$ is a homomorphism.

For C*-diagonals, Kumjian introduced an isometric isomorphism invariant, which he called the twist. I will describe a class $E(A)$ of linear functionals on A which plays an role in the context of triangular subdiagonal algebras similar to that of the twist and which gives an invariant under bounded isomorphism. I will also discuss several questions about when this invariant is a complete invariant. (Joint work with Allan Donsig.)

Baruch Solel: Hardy algebras associated with W*-correspondences

Abstract: I shall discuss the construction of the Hardy algebras (which are the weak closures of the tensor algebras), their representations, canonical models for the representations and Schur-class operator functions.

Ivan Todorov: Normalisers, ternary rings of operators and reflexivity

Abstract: A ternary ring of operators is a subspace of $B(H, K)$ closed under the triple product $(T, S, R) \to TS^*R$. A ternary masa-bimodule is a ternary ring of operators which is also a bimodule for two maximal Abelian selfadjoint algebras. In this talk a relation between ternary masa-bimodules and normalisers of some classes of operator algebras will be exhibited. The role ternary masa-bimodules play in operator synthesis will be described.

List of Participants

Davidson, Kenneth (Fields Institute)
Donsig, Allan (University of Nebraska)
Duncan, Benton (University of Nebraska)
Erlijman, Juliana (University of Regina)
Forrest, Brian (University of Waterloo)
Grossman, Jeff (University of Calgary)
Haataja, Steve (University of Nebraska)
Hopenwasser, Alan (University of Alabama)
Katavolos, Aristides (University of Athens)
Katsoulis, Elias (East Carolina University)
Kribs, David (University of Guelph)
Lamoureux, Michael (University of Calgary)
Larocque, Philippe (University of Waterloo)
Marcoux, Laurent (University of Waterloo)
Peters, Justin (Iowa State University)
Pitts, David (University of Nebraska)
Solel, Baruch (Israel Institute of Technology)
Todorov, Ivan (Queen's University Belfast)

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Two-day Workshop Reports

Chapter 43

Northwest Functional Analysis Workshop (03w2310)

March 27–29, 2003

Organizer(s): Michael Lamoureux (University of Calgary), Anthony Lau (University of Alberta), Ian Putnam (University of Victoria), Nicole Tomczak-Jaegermann (University of Alberta)

The subject of Functional Analysis is now a well-established one and a strength of the Canadian Mathematical community. Western Canada, in particular, has many researches working in the field and several well-established groups.

Roughly speaking, the community can be divided into three distinct groups: C^*-algebras and non-commutative geometry, Banach algebras and amenability and geometric functional analysis.

We were very pleased that there was an excellent level of participation, especially from graduate students and post-doctoral fellows. We were also happy that many senior visitors were able to attend. The meeting went a long way in strengthening the ties between the different groups present. We hope to continue to have this meeting on a regular basis and there seems to be a great deal of support for this idea.

Geometric Functional Analysis

Geometric Functional Analysis deals with the geometry of finite and infinite dimensional spaces.

ALEXANDER LITVAK gave a talk on "ASYMPTOTIC BEHAVIOUR OF HIGH-DIMENSIONAL CONVEX BODIES". The main subject of Asymptotic Geometric Analysis is, in most general terms, the study of various geometric parameters of high-dimensional convex bodies (such as, for example, volumes) and of their projections and sections, and of the asymptotic behaviour of these parameters as the dimensions tends to infinity. Several basic examples of this approach were given, such as the concentration of measure phenomenon and Dvoretzky's theorem on almost Euclidean sections, as well as a series of new results of a similar flavor.

VLADIMIR TROITSKY talked on "MINIMAL VECTORS IN BANACH SPACES". The method of minimal vectors was introduced in 1998 by Ansari and Enflo in order to prove the existence of invariant subspaces for certain classes of operators on a Hilbert space. It turns out that the method works in general Banach spaces. A variant of the method also shows that a large class of quasinilpotent operators on arbitrary Banach spaces have hyperinvariant subspaces. As it also applies in the spaces where there are known examples of operators without invariant subspaces; this dashes out the hopes that the method of minimal vectors alone could solve the invariant subspace problem.

RAZVAN ANISCA talked on "UNCONDITIONAL DECOMPOSITIONS IN SUBSPACES OF $l_2(X)$". This work continues and generalizes the series of constructions of Banach spaces without an unconditional basis done in the framework of arbitrary Banach spaces by Komorowski and Tomczak-Jaegermann in the 1990's. The talk presents characterizations of a Hilbert space in terms of "good" structure of all subspaces of $l_2(X)$. An additional interest stems from the connection with a long standing problem in Banach spaces on a structure of complemented subspaces of Banach spaces with unconditional basis.

BUNYAMIN SARI gave a talk "ON THE STRUCTURE OF THE SET OF SPREADING MODELS OF ORLICZ SPACES". The notion of a spreading model is an important tool in the Banach space theory. Roughly speaking, a spreading model of a Banach space X is another Banach space whose norm is obtained by stabilizing at infinity the norm on a non-degenerate sequence of vectors in X. Such a stabilization is achieved as a neat (and immediate) application of Ramsey Theorem. What type of spreading models may exist on a given Banach space X was a central problem which have been widely investigated since the spreading models were first introduced in 70's.

Abstract Harmonic Analysis

Abstract Harmonic analysis relates to the studies of Banach algebras of spaces of measures or functions associated to (unitary representations of) a locally compact group. Two locally compact groups are isomorphic if and only if certain associated Banach algebras (i.e. Fourier algebras or the group algebras) are isometrically isomorphic. Consequently, the study of various Banach algebras and their geometric properties reveal deep structural properties of the underlying group. For example, the classical result of B.E. Johnson asserts that the group algebra is amenable if and only if the underlying group is amenable. An analogous result for the Fourier algebra has been proved only very recently by Z.J. Ruan and it involves so called operator amenability. A characterization of amenability in terms of a deep combinatorial property of Følner led to a strong relationship to the recent study of amenable unitary representation of locally compact groups by M.E. Bekka.

During the BIRS workshop, reports related to the above were made by Garth Dales on homological properties of modules over locally compact groups, by Volker Runde on non-amenability of the Fourier and Fourier-Stieltjes algebra, by Monica Ilie on characterization of completely bounded homeomorphisms on the Fourier algebra viewed as an operator space and their ranges, and by Ross Stokke on quasi-approximate units on the group algebra related to Følner conditions.

C^*-algebras and Non-commutative Geometry

C^*-algebras arose as mathematical models for quantum mechanical systems. In classical mechanics, one studies a geometric object, phase space, and an algebraic one, the observables of the system. The passage from the former to the latter is simply by taking the continuous functions on the space. Thus, the algebra of observables is commutative. The passage to quantum mechanics means that one considers algebras of operators on Hilbert space which are, in general, non-commutative. This means that the phase space is lost. The program of non-commutative geometry, as proposed by Alain Connes, is to develop the tools of conventional geometry in the setting of non-commutative operator algebras. This program has had many connections with other areas of mathematics: index theory for manifolds, topology, dynamical systems and number theory.

Marcelo Laca (Victoria) gave a talk on the structure of Hecke C^*-algebras. These were first constructed by Bost and Connes and arise naturally from number theoretic information. Igor Nikolaev (Calgary) gave a talk on the C^*-algebras which arise from dynamical systems which are closely linked with Riemann surfaces. In particular, the K-theory of these algebras provide information regarding the dynamics. Inhyeop Yi (Victoria) gave a talk about the general structure and K-theory of C^*-algebras arising from hyperbolic dynamical systems and shifts of finite type.

List of Participants

Anisca, Razvan (University of Alberta)
Bami, Mahmood Lashkarizadeh (University of Alberta)
Binding, Paul (University of Calgary)
Brenken, Berndt (University of Calgary)
Dales, Garth (Leeds University)
Erlijman, Juliana (University of Regina)
Farenick, Doug (University of Regina)
Gibson, Peter (University of Calgary)
Goncalves, Daniel (University of Victoria)
Goncalves, Maria Inez Cardoso (University of Victoria)
Gordon, Yehoram (Israel Institute of Technology)
Graham, Colin (University of British Columbia)
Ilie, Monica (University of Alberta)
Laca, Marcelo (University of Victoria)
Lamoureux, Michael (University of Calgary)
Lau, Tony (University of Alberta)
Litvak, Alexander (University of Alberta)
Mohanty, Parasar (University of Alberta)
Namioka, Isaac (University of Washington)
Nikolaev, Igor (University of Calgary)
Phillips, John (University of Victoria)
Putnam, Ian (University of Victoria)
Reznikoff, Sarah (University of Victoria)
Runde, Volker (University of Alberta)
Rychtar, Jan (University of Alberta)
Sari, Bunyamin (University of Alberta)
Sourour, Ahmed Ramzi (University of Victoria)
Stokke, Ross (University of Alberta)
Tandra, Haryono (University of Alberta)
Tcaciuc, Adi (University of Alberta)
Tomczak-Jaegermann, Nicole (University of Alberta)
Troitsky, Vladimir (University of Alberta)
Yi, Inhyeop (George Washington University)
Zizler, Peter (Mount Royal College)

Chapter 44

BIRS Math Fair Workshop (03w2311)

April 10–12, 2003

Organizer(s): Ted Lewis (University of Alberta), Andy Liu (University of Alberta)

The BIRS Math Fair workshop was unusual for BIRS in that its focus was Education rather than research. The participants were teachers from elementary schools, junior high schools, colleges and universities, and also people from other institutions and organizations that have a deep interest in Mathematics Education.

The purpose of the workshop was to help teachers learn how to run a successful math fair, to exchange information about math fairs, and to put the members of this diverse group in contact with each other. The deeper purpose is to change the mathematical culture in the classroom, and after five years of experience we believe that this is beginning to happen.

It must be stressed that the sort of math fair that we are talking about is radically different from a typical science fair. Without going into too much detail, the four main tenets are that the math fair be non-competitive (no prizes), that it be all-inclusive (not just for the elite students), that it be interactive (not a poster session) and that it be based on problem-solving.

The problem that we have now is to disseminate the news about the success of math fairs. Workshops are one way of helping teachers learn about math fairs, helping them sustain their efforts, and letting them share experiences with co-workers. As well, workshops build trust between teachers and other educators.

Teachers were invited to the workshop on the condition that they subsequently hold a math fair in their own schools. All participants received a booklet that contains the underlying principles for the math fair.

The workshop dealt with what constitutes a good problem for a math fair, included several examples, and described several different types of math fairs that are based on the guidelines. Many of the participants had already organized math fairs at their schools, and although there was great variation in the details all followed the guidelines set out in our booklet.

How does a teacher find problems that are suitable for the math fair? Do you begin with a curriculum topic and design an appropriate puzzle, or do you start with a challenging puzzle and try to fit it to the curriculum? The workshop advocated the latter approach, and spent some time having the participants find ways to adapt a good puzzle to the curriculum. A few days ago we visited a math fair organized by one of the workshop participants, and saw that this adaptation was taking place.

One of the most valuable and spontaneous aspects of the workshop occurred when the teachers who had already conducted math fairs began sharing information about their experiences. The ones who had not yet had a math fair asked many questions and picked up the enthusiasm from those that did. There were some common fears experienced by teachers who had done the math fair for the first time: They want their students to succeed and have a tendency to intervene when students are presented with an unfamiliar task. The math fair works best when, as one teacher put it, you let the students take ownership of their problems. This is a difficult thing for teachers to do, especially when they know that the result is going to be on public display.

Because of the uncertainty of a new venture, many teachers will limit either exposure or participation on their “first-time” math fair. Discussions about this indicated that subsequent math fairs would be greatly expanded, and that the math fair would become a regular part of the students’ math activities.

Bill Ritchie, one of the participants in mentioned below, told the teachers that they should be immensely proud of what they are doing, that they are part of a group that could revolutionize how mathematics is being learned in North American school systems. (Bill Ritchie is the CEO of Binary Arts. He has maintained close contact with many schools throughout North America, and is very familiar with what is going on in many school districts.)

List of Participants

Bessette, Patricia (St. James School)
Borges-Couture, Paula (Good Shepherd School)
Darroch, Judy (Lendrum Elementary School)
Dumanski, Micheal (St. Gerard School)
Estabrooks, Manny (Red Deer College)
Ewasiuk, Lindy (St. Clement School)
Friesen, Sharon (Galileo Education Network Association)
Gluwchynski, Jennifer (St. Michael School)
Hartmangatti, Suzanna (St. Philip School)
Hodak, Laura (Lendrum Elementary School)
Hohn, Tiina (Grant McEwan Community College)
Holloway, Tom (University of Alberta)
Kowalchuk, Auriana (Consulting Services, Edmonton Public Schools)
Lagu, Indy (Mt. Royal College)
Lewis, Ted (University of Alberta)
Liu, Andy (University of Alberta)
Lovallo, Patti (Killarney Junior High School)
McCulloch, Chalaine (St. Michael School)
McKie Grenier, Kelly (Galileo Education Network Association)
Melnyk, Linda (St. Francis of Assisi)
Mitchell, Shirley (Pacific Institute for the Mathematical Sciences)
Pawliuk, Heather (St. Dominic School)
Porter, Kyle (St. Patrick School)
Poulin, Tracy (Lorelei School)
Prefontaine, Suzanne (Holyrood School)
Raymaakers, Chris (John Ware Elementary School)
Ritchie, Bill (Binary Arts)
Rozycki, Angela (Edmonton Catholic schools)
Skinner, Emma (University of Alberta and Malcolm Tweddle Elementary School)
Slen, Gail (John Ware Elementary School)
Springer, Jean (Mt. Royal College)
Sun, Wen-Hsien (Chiu Chang Publishers)
Thiell, Jane (Louis St. Laurent Jr High/High School)

Chapter 45

The Regression Discontinuity Method in Economics: Theory and Applications (03w2312)

May 15–17, 2003

Organizer(s): Thomas Lemieux (University of British Columbia), David Card (University of California, Berkeley)

Objectives

The objective of the workshop is to bring together a group of applied economists interested in specific applications of the RD method and a group of econometricians interested in the estimation of non-linear and discontinuous regression models. We will organize the workshop in a way sense of potential applications of state-of-the art econometric techniques.

List of Participants

Abadie, Alberto (John F. Kennedy School of Government)
Angrist, Joshua (Massachusetts Institute of Technology)
Auld, Christopher (University of Calgary)
Battistin, Erich (Institute for Fiscal Studies)
Card, David (University of California, Berkeley)
Chen, Susan (University of North Carolina at Chapel Hill)
DiNardo, John (University of Michigan)
Fortin, Nicole (University of British Columbia)
Gyimah-Brempong, Kwabena (National Science Foundation)
Hirano, Keisuke (University of Miami)
Imbens, Guido (University of California, Berkeley)
Kane, Thomas (University of California, Los Angeles)
Lee, David (University of California, Berkeley)
Lemieux, Thomas (University of British Columbia)
Matsudaira, Jordan (University of Michigan)
McCrary, Justin (University of Michigan, Ann Arbor)
Porter, Jack (Harvard University)

Rettore, Enrico (University of Padova)
Riddell, Craig (University of British Columbia)
Ridder, Geert (University of Southern California)
Van der Klaauw, Wilbert (University of North Carolina at Chapel Hill)

Chapter 46

Theoretical Physics Institute (University of Alberta) Annual Symposium 2003 (03w2314)

August 28–30, 2003

Organizer(s): Helmy Sherif (University of Alberta), Frank Marsiglio (University of Alberta)

The Theoretical Physics Institute (TPI) at the University of Alberta consists of members from three different departments (Physics, Mathematical and Statistical Sciences, and Chemistry). The idea of this Symposium was to bring together members of the Institute, their students and postdoctoral fellows, as well as colleagues from other universities in the West, for two days designed to promote exchange of ideas and collaboration. In keeping with this intent the workshop centred around two main activities:

- Plenary review talks by speakers from institutes in Western Canada and the Western United States
- Short presentations by participants; this included presentations from faculty members, visitors, RAs/PDFs, and graduate students. These talks covered topics ranging from theoretical biophysics to advanced loop calculations in particle physics.

Professor Werner Israel gave a very lucid talk about the status of research regarding the question of the cosmological constant. The issue is that the expansion of the universe appears to be accelerating and observations give a small effective cosmological constant. Some of the old puzzles concerning vacuum fluctuations and their role in the evolution of the universe were discussed. Various theoretical approaches and proposed mathematical models to answer these questions were reviewed.

Another topic reviewed was the subject of Lattice Gauge Calculations in particle physics. A review of the historical development of this technique was given. The method of Monte Carlo simulations was described as most appropriate for these calculations. Results obtained for heavy flavour physics (i.e. systems made up of massive quarks) were discussed. Similarities and differences with lattice simulations in condensed matter physics were noted; in particular the problems posed by the 'chiral extrapolation' and the 'quenched fermion' approximation were noted.

The appearance of order in many body systems such as atoms and nuclei was the subject of a talk entitled "Order from randomness in many body systems". Rotational and vibrational excited states in nuclei are examples of such orderly behaviour. The talk discussed novel techniques that allow physicists to carry out elaborate calculations of the structure of atomic nuclei without resort to the calculational short cuts and truncations used in the past. Calculations such as "the no-centre shell model calculations" are now able to describe many of the regularities in light atomic nuclei.

In plasma physics and condensed matter, some overviews were given; a couple of talks focused on exact cluster methods, including Quantum Monte Carlo and a path integral approach. An entire session was devoted to superconductivity, mainly concerning the high temperature cuprate materials. However, no unifying principles emerged. It remains the case that the many body problem is a very difficult one, and the emphasis is on improved algorithms for existing theoretical methods. It may also be the case that complicated materials simply cannot be described by rather simple models.

Finally, in mathematics it is clear that quantum field theories in physics remain an inspiration for novel mathematical insights.

Terry Gannon explained what monstrous moonshine is: a mysterious relation in pure math between number theory (e.g. modular functions) and algebra (e.g. the Monster finite group). The present explanation of this uses perturbative string theory, or what is essentially the same thing, conformal field theory (i.e. conformally invariant quantum field theory in 1+1 dimensional space-time). The idea is that the algebra describes the symmetry of the conformal field theory, and the number theory concerns its partition function and 1-point 1-loop functions. Borcherds noticed that the conformal field theory can be replaced by something which is mathematically simpler: a purely algebraic structure called a vertex operator algebra. For example, the familiar notion of a "quantum field" changes from being an operator-valued distribution on space-time, to being a formal power series in a complex variable z whose coefficients are operators on state-space.

This resulting algebraic structure is still quite complicated though, and many people have argued that we still don't have our finger on the fundamental underlying principle which connects number theory with algebra. Gannon concluded his talk by speculating on what that underlying principle could be. We should ask ourselves what is so remarkable about quantum field theories in 2-dimensions. The answer, he suggested, is the possibility of braid group statistics (i.e. anyons). Thus, a natural guess for the underlying principle of moonshine is that it involves the braid group. His talk was concluded by briefly explaining how the braid group can be used to explain a baby example of moonshine: the modularity of lattice theta functions.

A workshop of this nature necessarily covers a wide variety of topics. Nonetheless, it seems that many areas in the sciences share common ground insofar as the techniques to describe them share the same mathematical basis. Many of these potential connections remain to be exploited.

List of Participants

Blinov, Nicholas (University of Alberta)
Blokland, Ian (University of Alberta)
Boninsegni, Massimo (University of Alberta)
Campbell, Bruce (University of Alberta)
Covaci, Lucian (University of Alberta)
Das, Saurya (University of Lethbridge)
Dixon, John (University of Alberta)
Dogan, Fatih (University of Alberta)
Gannon, Terry (University of Alberta)
Gortel, Zbigniew (University of Alberta)
HedayatiPoor, Mohammad (University of Alberta)
Israel, Werner (University of Victoria)
Johnson, Calvin (San Diego State University)
Kim, Wonkee (University of Alberta)
Knigavko, Anton (McMaster University)
Kovalyov, Mikhail (University of Alberta)
Kryukov, Sergei (University of Lethbridge)
Kunzle, Hans-Peter (University of Alberta)
Legare, Martin (University of Alberta)
Lovallo, Chris (University of Alberta)
Luchko, Tyler (University of Alberta)
Marsiglio, Frank (University of Alberta)
Moody, Robert (University of Alberta)
Moroni, Saverio (University of Alberta)
Page, Don (University of Alberta)
Rezania, Vahid (University of Alberta)

Roy, Pierre-Nicholas (University of Alberta)
Samson, John (University of Alberta)
Sengupta, Supratim (University of Alberta)
Sherif, Helmy (University of Alberta)
Tanaka, Kaori (University of Saskatchewan)
Tran, Chuong (University of Alberta)
Vardarajan, Suneeta (University of Alberta)
Vos, Ken (University of Lethbridge)
Woloshyn, Richard (TRIUMF)
Woolgar, Eric (University of Alberta)
Ziegler, Tom (University of Calgary)

Chapter 47

MITACS-PIMS Health Canada Meeting on SARS (03w2315)

September 4–6, 2003

Organizer(s): Jianhong Wu (York University)

Objectives

The main purpose of this workshop is to bring together international leaders and active researchers working in the areas related to the modeling, simulations and analysis of the transmission dynamics of SARS and other infectious diseases, to further the fruitful interplay among mathematical, statistical, epidemiological sciences and operations research, in order to speed up the process of finding effective tests and prevention and control measures.

List of Participants

Becker, Niels (Australian National University)
Boer, Rob (RAND Corporation)
Brauer, Fred (University of British Columbia)
Cuff, Wilfred (Health Canada)
Curtis, Lori (Health Canada)
Day, Troy (Queen's University)
Earn, David (McMaster University)
Ekeland, Ivar (PIMS & University of British Columbia)
Feng, Zhilan (Purdue University)
Glasser, John (Centers for Disease Control and Prevention)
Gumel, Abba (University of Manitoba)
Gupta, Arvind (Mathematics of Information Technology and Complex Systems)
Hethcote, Herb (University of Iowa)
Hsieh, Ying-Hen (National Chung Hsing University)
Jacobson, Zack (Health Canada)
Jolly, Ann (Health Canada)
Liu, Rongshen (York University)
Ma, Junling (McMaster University)
Ma, Renjun (University of New Brunswick)
Mykitiuk, Roxanne (York University)

Radoeva, Detelina (York Centre for Health Studies)
Riley, Steven (Imperial College of Science, Technology & Medicine)
Sahai, Beni (Cadham Provincial Laboratory)
Watmough, James (University of New Brunswick)
Wu, Jianhong (York University)
Yan, Ping (Health Canada)
Zeng, Qingling (York University)
Zhang , Shenghai (Health Canada)
Zhu, Huaiping (York University)
van den Driessche, Pauline (University of Victoria)

Chapter 48

Canadian Mathematics Chairs Meeting (03w2313)

September 18–20, 2003

Organizer(s): Ted Bisztriczky (University of Calgary), Bob Erdahl (Queens), Yvan Saint-Aubin (Montreal)

Objectives

This is the fifth annual meeting of chairs of Canadian mathematics departments, held at BIRS. Heads will get to know each other; we will find out how other math departments work, and bring this information back to our own department. There will be a Special Session on the Math Department Industry Interface.

The math institutes have taken the lead in promoting closer ties between universities and industry, and it is important that math departments become aware of the dynamics of these important initiatives. We have been able to attract Rob Calderbank, who had a particularly good vantage point to follow the dynamics on the interface between mathematics departments and industry. Rob was formerly Vice President, Research, of A T & T; he will report on the difficulties Bell Labs recently came face-to-face with, and how these calamitous events will result in out-sourcing of industrial research. MITACS is the other main contributor for the Saturday Special Session the contributors will describe how MITACS is shaping the university-industry interface.

List of Participants

Ahmed, S. Ejaz (University of Windsor)
Allen, O. Brian (University of Guelph)
Alvo, Mayer (University of Ottawa)
Anderson, Robert V. (Universite du Quebec a Montreal)
Archibald, Tom (University of Acadia)
Bisztriczky, Tibor T. (University of Calgary)
Bland, John S. (University of Toronto)
Calderbank, Rob (AT & T)
Ekeland, Ivar (Pacific Institute for the Mathematical Sciences)
Erdahl, Bob (Queens University)
Garner, Cyril W. L. (Carleton University)
Gilligan, Bruce C. (University of Regina)
Gowrisankaran, Kohur (McGill University)

Gupta, Arvind (Mathematics of Information Technology and Complex Systems)
Hailes, Jarett (Mathematics of Information Technology and Complex Systems)
Holzmann, Wolfgang H. (University of Lethbridge)
Keast, Patrick (Dalhousie University)
Krause, Guenter (University of Manitoba)
Lau, Anthony T. (University of Alberta)
Leger, Christien (Centre de Recherches Mathématiques)
Madras, Neal (York University)
Marcus, Brian (University of British Columbia)
Moody, Bob (University of Alberta)
Reilly, Norman (Simon Fraser University)
Saint-Aubin, Yvan (Universite de Montreal)
Teare, Betty (University of Calgary)
Trummer, Manfred (Pacific Institute for the Mathematical Sciences)
Tuszynski, Jack (Mathematics of Information Technology and Complex Systems)
Valeriote, Matthew A. (McMaster University)
Vaughan, David C. (Wilfrid Laurier)
Wright, Graham (Canadian Math Society)
Zorozito, Frank (University of Waterloo)

Chapter 49

West Coast Operator Algebra (03w2307)

October 16–18, 2003

Organizer(s): Berndt Brenken (University of Calgary), Bruce Blackadar (University of Nevada, Reno)

List of Participants

Arveson, William (University of California, Berkeley)
Blackadar, Bruce (University of Nevada Reno)
Boersema, Jeffrey (Seattle University)
Brenken, Berndt (University of Calgary)
Burns, Michael (University of Victoria)
Deaconu, Valentin (University of Nevada, Reno)
Elliott, George (University of Toronto)
Floricel, Remus (University of Ottawa)
Greene, Devin (University of California, Irvine)
Hirshberg, Ilan (University of California, Berkeley)
Itza-Ortiz, Benjamin (University of Ottawa)
Kahng, Byung-Jay (University of Nevada)
Kaliszewski, Steve (Arizona State University)
Katsura, Takeshi (University of Oregon)
Kemp, Todd (Cornell University)
Laca, Marcelo (University of Victoria)
Lamoureux, Michael (University of Calgary)
Latremoliere, Frederic (Univerisity of California, Berkeley)
Lau, Antony To-Ming (University of Alberta)
Lin, Huaxin (University of Oregon)
MATUI, Hiroki (Chiba University)
Morrison, Scott (University of California, Berkeley)
Nikolaev, Igor (University of Calgary)
Ozawa , Narutaka (University of California, Berkeley)
Packer, Judith (University of Colorado)
Phillips, John (University of Victoria)
Phillips, N. Christopher (University of Oregon)
Putnam, Ian (University of Victoria)
Quigg, John (Arizona State University)
Ramsay, Arlan (University of Colorado Boulder)

Rangipour, Bahram (University of Victoria)
Reznikoff, Sarah (University of Victoria)
Ruiz, Efren (University of Oregon)
Runde, Volker (University of Alberta)
Shlyakhtenko, Dimitri (University of California, Los Angeles)
Sourour, Ahmed Ramzi (University of Victoria)
Spielberg, Jack (Arizona State Univerity)
Takesaki, Masamichi (University of California, Los Angeles)
Ventura, Belisario (California State University, San Bernardino)
Webster, Corran (University of Nevada, Las Vegas)

Chapter 50

The World Bank Thailand SEQI Project (03w2308)

October 16–18, 2003

Organizer(s): Andy Liu (University of Alberta)

Over the period October 1 to 18, 2003, a two-day workshop was conducted at BIRS at the invitation of PIMS. It is titled "Secondary Education Quality Improvement", and is part of a much larger project sponsored by the World Bank through the Thailand Government at the University of Alberta. The participants in this workshop are nine mathematics educators from various Teaching Training Institutes in Thailand, plus workshop organizer Prof. Andy Liu from the University of Alberta.

The group arrived around 1 pm. on Thursday October 1. Since the rooms were not ready, we proceeded directly to our first session right after a late lunch. The discussion is centred around the disappearance of Euclidean Geometry from the North American school curriculum and what we are doing to try to remedy the situation. A textbook prepared at the University of Alberta was distributed.

The second session was held Friday morning, examining a once-experimental course in Discrete Mathematics at the University which uses a mathematical novel as the innovative text. It has had a phenomenally successful ten years and is now a fixture in the department's course offering. A notebook which is prepared at the University of Alberta and serves as a companion volume was distributed.

The third session was in Friday evening. Here we discussed a course at the University of Alberta to perspective teachers in the elementary classroom, but which has impact on secondary education. A very successful aspect of this course is the concept of a Math Fair. A Math Fair booklet was distributed along with the textbook, both prepared at the University of Alberta.

Due to a local road race, it was decided to cancel the last session on Saturday morning in order to avoid the roadblock by departing earlier than originally planned.

We have found the workshop enormously beneficial. The Thai group was particularly impressed with the facilities at BIRS and the warm welcome from manager Andrea Lundquist. They had a chance to wander into town on Thursday night, plus taking part in a side trip to the Sulphur Mountain on Friday afternoon. We are most grateful to PIMS for its generosity in sponsoring this workshop.

List of Participants

Akejariyawong, Mana (Thepsatri)
Hematulin, Apichai (Nakonratchasima)
Jinagool, Kannika (Surin)
Kruehong, Chaisongkram (Suratthani)
Liu, Andy (University of Alberta)
Monmongkol, Prasit (Rajanagarindra)

Nitipreecha, Songsak (Nakonratchasima)
Promthai, Poungthong (Chiangmai Rajabhat Institute)
Ratanaudomchok, Somchit (Sakonnakorn)
Sungamullig, Gobgul (Petburiwitayalongkorn)

Focused Research Group Reports

Chapter 51

Regularity for Hypergraphs (03frg004)

May 10–24 2003

Organizer(s): P.E. Haxell (University of Waterloo), V. R¨odl (Emory University), J. Skokan (University of Illinois at Urbana-Champaign)

Introduction

The Regularity Method for hypergraphs is a newly emerging technique that grew out of the famous Regularity Lemma of Szemerédi for graphs [53]. The purpose of this Focused Research Group was to bring together the experts who developed the Regularity Method for hypergraphs with some other leading researchers in extremal hypergraph theory, so that all participants could learn the technical details of the new method, and so that new applications of the method to important extremal problems in hypergraphs could be found. The workshop was structured in a way that allowed a lot of informal discussion. Each session was led by a workshop participant, who usually spent some time giving something like a formal lecture on the session topic, but also strongly encouraged the other participants to contribute ideas, ask questions and make suggestions. No time limits were imposed on sessions and each typically lasted several hours.

This report outlines the topics of the discussion sessions, and closes with a short section highlighting the main accomplishments of the workshop. The topics are loosely arranged into three categories. In Section 51 the basics of the Regularity Method are described, Section 51 details some applications of the method, and in Section 51 other extremal hypergraph problems are discussed, in particular two Ramsey theoretic questions that were solved at BIRS by the participants working as a group. Each subsection heading notes the name of the participant who led the session on that topic. Finally in Section 51 the specific results and accomplishments of the workshop are noted.

The Regularity Method

The regularity lemma for graphs and hypergraphs

The Regularity Lemma for graphs (V. Rödl)

Because it was the inspiration for the new regularity method for hypergraphs, part of a session was devoted to the regularity lemma for graphs. While proving his famous Density Theorem [52], E. Szemerédi invented an auxiliary lemma which later proved to be a powerful tool in extremal graph theory. This lemma and its improved version named the Regularity Lemma [53], assert that an arbitrary large graph can be approximated by "random-like" graphs.

More precisely, let $G = (V, E)$ be a graph and $A, B \subset V$ be a pair of disjoint sets of vertices of G. Denote by $E(A, B)$ the set of all edges of G between A and B, i.e., $E(A, B) = \big\{\{a, b\} \in E\colon\ a \in A,\ b \in B\big\}$, and let $e(A, B) = |E(A, B)|$. The *density* of the pair (A, B) is defined by $d(A, B) = e(A, B)/|A||B|$. The pair (A, B) is called ε*-regular* if for any $A' \subset A$, $B' \subset B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have $|d(A, B) - d(A', B')| < \varepsilon$.

Theorem 1 (Szemerédi's Regularity Lemma [53]) *For every $\varepsilon > 0$ and $l > 0$, there exist integers L and n_0 such that any graph $G = (V, E)$ with $n \geq n_0$ vertices admits a partition $V = V_1 \cup \cdots \cup V_t$, where $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$, $l \leq t \leq L$, and all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j) are ε-regular.*

The lemma below and its generalizations are crucial for most applications of the Regularity Lemma. We refer to the combined use of the Regularity Lemma followed by the Counting Lemma as *The Regularity Method*.

Lemma 2 (Counting lemma) *For any ν, $d > 0$, there exists $\varepsilon > 0$ such that the following holds. Let F be a graph with vertex set $\{w_1, \ldots, w_k\}$ and let $G = (V, E)$ be a graph and $V_1, \ldots, V_k$ be disjoint subsets of V, all of size n. If for every edge $\{w_i, w_j\} \in E(F)$ the pair (V_i, V_j) is ε-regular with density d, then there are $(1 \pm \nu)d^{|E(F)|}n^k$ copies of F in G with w_i mapped onto a vertex of V_i for all $1 \leq i \leq k$.*

To understand this lemma it is best to look at the case in which F is the complete graph K_k. It is easy to see that if G' is a k-partite graph, where each partite set has size n, in which edges between partite sets are generated with probability d, then G' contains $(1 + o(1))d^{\binom{k}{2}}n^k$ copies of F. Lemma 2 says that if G is such that all pairs (V_i, V_j) are ε-regular with density d, then the number of copies of F in G is about the same as the number of copies in G'. In other words, regularity of G guarantees "random-like" behaviour in this sense.

The regularity lemma for hypergraphs (V. Rödl and J. Skokan)

Recall that in Szemerédi's Regularity Lemma the main structure which undergoes regularization is the edge set of a graph, and the auxiliary structure is a partition of the vertex set. Briefly, the 2-tuples (edges) are regularized with respect to the 1-tuples (vertices).

Unlike the graph case, there are several natural ways to define "regularity" for k-uniform hypergraphs. For example,

- for $k \geq 3$, if we naturally extend the concept of a regular pair and just regularize the k-tuples versus 1-tuples, as e.g. in [11, 23, 44], we obtain a **weak $\boldsymbol{\delta}$-regularity**. Then, one can easily prove the Regularity Lemma, but the natural analogue to the counting lemma fails to be true.
- A more refined approach is to consider an auxiliary partition of the l-tuples for each $l < k$ (concept of $\boldsymbol{(\delta, 1)}$**-regularity**). This was done in [8, 21]. However, there was no attempt to prove a companion counting statement and it is an open question whether it is even possible.

A breakthrough came when Frankl and Rödl [22] modified hypergraph $(\delta, 1)$-regularity for $k = 3$ and developed a concept of $\boldsymbol{(\delta, r)}$**-regularity**. They succeeded in proving both the regularity lemma and the counting lemma for the case $F = K_4^{(3)}$, where $K_4^{(3)}$ is the complete 3-uniform hypergraph on 4 vertices. The general 3-uniform hypergraph counting lemma corresponding to Lemma 2 for graphs was proved later by Nagle and Ródl [42].

Subsequently, the regularity lemma from [22] was extended by Rödl and Skokan [48] to k-uniform hypergraphs for arbitrary $k \geq 3$.

This workshop session discussed in detail all three above concepts of regularity, and the reasons why the rather complicated concept of (δ, r)-regularity is needed (including counterexamples to the counting statement for the weak δ-regularity). The speakers also described the statement of the hypergraph regularity lemma.

The counting lemma (J. Skokan)

As noted above, many applications of Szemerédi's Regularity Lemma for graphs are based on Lemma 2. To generalize this result for k-uniform hypergraphs, we consider the following random environment:

i) a vertex partition $\mathcal{H}^{(1)} = V_1 \cup \ldots \cup V_{k+1}$, $|V_1| = \ldots = |V_{k+1}| = n$,

ii) a random $(k+1)$-partite graph $\mathcal{H}^{(2)}$, the edges of which are generated with probability d_2,

iii) a random $(k+1)$-partite 3-uniform hypergraph $\mathcal{H}^{(3)}$, whose edges are chosen from triangles of $\mathcal{H}^{(2)}$ independently with probability d_3, and

iv) for $i = 4, \ldots, k$, a random $(k+1)$-partite i-uniform hypergraph $\mathcal{H}^{(i)}$, whose edges are chosen from copies of $K_i^{(i-1)}$ in $\mathcal{H}^{(i-1)}$ independently with probability d_i.

It is easy to show that under the above assumptions, the number of copies of $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$ is

$$(1+o(1))\prod_{j=2}^{k} d_j^{\binom{k+1}{j}} n^{k+1}, \tag{51.1}$$

where $o(1) \to o$ as $n \to \infty$. Any counting lemma should show that (51.1) is also true in the setup produced by the corresponding regularity lemma for k-uniform hypergraphs.

Guided by the hypergraph regularity lemma of Frankl and Rödl [22], Nagle and Rödl [41] proved a corresponding counting lemma for 3-uniform hypergraphs

Lemma 3 (Counting lemma for 3-uniform hypergraphs [41]) *Let $s \geq 3$ be an integer. For every $\mu > 0$ and $d_3 \in (0,1]$ there exists $\delta > 0$ such that for every $d_2 \in (0,1]$ there exist $\varepsilon > 0$ and integers r and m_0 such that the following assertion holds.*

If $\mathcal{G}$ is an s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$, where $|V_i| = m > m_0$ for $1 \leq i \leq s$, and $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition such that

(1) $\mathcal{G}$ is (ε, d_2)-regular, and

(2) $\mathcal{G}$ underlies $\mathcal{H}$, and $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$,

then $\mathcal{H}$ contains $(1 \pm \mu) d_2^{\binom{s}{2}} d_3^{\binom{s}{3}} m^s$ copies of $K_s^{(3)}$.

There is hope that the concept of (δ, r)-regularity will allow one to prove a generalization of Lemma 2 and Lemma 3, at least for $F = K_{k+1}^{(k)}$. This is stated as Conjecture 4 below.

Conjecture 4 (see Conjecture 1.16, p. 6 in [49]) *For any $\nu > 0$ and any $k \in \mathbb{N}$, the following is true: $\forall d_k \ \exists \delta_k \ \forall d_{k-1} \ \exists \delta_{k-1} \ \ldots \ \forall d_2 \ \exists \delta_2 \ \exists r \in \mathbb{N}$ such that if $\mathcal{H}^{(k)}$ is a k-uniform hypergraph and $\{\mathcal{G}^{(l)}\}_{l=1}^{k}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(k+1,k)$-complex, where $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$, with $\mathcal{G}^{(k)} = \mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G}^{(k-1)})$ and $\mathcal{G}^{(1)} = W_1 \cup \cdots \cup W_{k+1}$, where $|W_i| = n$ for all i, then $\mathcal{H}^{(k)}$ contains at least*

$$(1-\nu)\prod_{l=2}^{k} d_l^{\binom{k+1}{l}} \times n^{k+1}$$

copies of $K_{k+1}^{(k)}$.

For $k = 4$, Conjecture 4 was proved by Rödl and Skokan (see [49]). The techniques of [42] seem to be extendible to the counting of arbitrary 4-uniform hypergraphs F. The presentation in this session provided a detailed proof of Lemma 3 (focusing on $s = 4$) and outlined the major differences between this proof and the proof of Conjecture 4 for $k = 4$.

Alternative proof of the counting lemma (Y. Peng)

The proof of the counting lemma (Lemma 3) is rather technical, mostly due to the fact that the 'quasi-random' hypergraph arising after applying the regularity lemma of Frankl-Rödl is sparse and consequently is difficult to handle. Recently, Kohayakawa, Rödl and Skokan [36] found a simpler proof of the counting lemma in the easier dense case. Their result applies to k-uniform hypergraphs for arbitrary k.

Lemma 5 *[36] Let $s \geq 3$ be an integer. For every $\mu > 0$ and every $d \in (0,1]$, there exist $\delta_0 > 0$ and $m_0 > 0$ such that the following holds. If*

(1) $\mathcal{G}$ is a complete s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$, where $\left|V_i\right| = m \geq m_0$ for $1 \leq i \leq s$, and

(2) $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition $V = \bigcup_{i=1}^{s} V_i$ and $\mathcal{H}$ is $(\delta, d, 1)$-regular with respect to $\mathcal{G}$, where $\delta \leq \delta_0$,

then $\mathcal{H}$ contains $(1 \pm \mu) d^{\binom{s}{3}} m^s$ copies of $K_s^{(3)}$.

The subject of this session was the paper "Counting small cliques in 3-uniform hypergraphs" by Peng, Rödl and Skokan [43], where for $k = 3$, the harder, sparse case is reduced to the dense case. In particular, it is shown that a 'dense substructure' randomly chosen from the 'sparse δ-regular structure' is δ-regular as well. This makes it possible to count the number of cliques (and other subhypergraphs) using the Kohayakawa-Rödl-Skokan result and provides an alternative proof of the counting lemma in the sparse case. Since the counting lemma in the dense case applies to k-uniform hypergraphs for arbitrary k, there is a possibility that the approach of this paper can be adapted to the general case as well.

Characterizing Hypergraph Quasi-randomness (B. Nagle)

An important development regarding Szemerédi's Lemma showed the equivalence between the property of ε-regularity of a bipartite graph G and an easily verifiable property concerning the neighbourhoods of its vertices [1] (cf. [15]). This characterization of ε-regularity led to an algorithmic version of Szemerédi's Lemma [1].

Similar problems were also considered for hypergraphs. In [10], [27] and [36], various descriptions of quasi-randomness of k-uniform hypergraphs were given. These notions of hypergraph quasi-randomness coincided with a special case of the quasi-randomness provided by the Frankl-Rödl Regularity Lemma.

The hypergraph regularity of [22] renders quasi-random "blocks of hyperedges" (i.e. (δ, r)-regular triads) which are very sparse. This situation leads to technical difficulties in its application. Moreover, as was shown in [13], some easily verifiable conditions analogous to those considered in [10] and [36] fail to be true in the setting of [22]. However, in [13] and [12], some necessary and sufficient conditions for the Frankl-Rödl notion of hypergraph quasi-randomness were established. These conditions enabled the design in [12] of an algorithmic version of a hypergraph regularity lemma in [22].

To understand the above notions it is best to look at the graph analogues. In what follows, we consider a fixed bipartite graph Γ with bipartition $X \cup Y$. For fixed positive constants α and ε, we assume $d(X, Y) \sim_{\varepsilon} \alpha$, where by $a \sim_{\gamma} b$, we mean $(1+\gamma)^{-1} \leq a/b \leq 1+\gamma$. We denote by $\deg_{\Gamma}(x)$ the number of vertices that are neighbours of x in the graph Γ, and by $\deg_{\Gamma}(x_1, x_2)$ the number of vertices that are neighbours of both x_1 and x_2 in Γ.

The property of ε-regularity of Γ is a "global" property in the sense that it asserts a fact about every pair of reasonably large subsets of its vertex classes X and Y. An important development regarding Szemerédi's Lemma showed the equivalence between this global regularity property of Γ and a fairly simple "local" property concerning the neighbourhoods of the vertices in X. Given positive reals α, ε and ε', consider the following two properties:

$$\mathbf{G_1} = \mathbf{G_1}(\varepsilon) \qquad \Gamma \text{ is } \varepsilon\text{-regular with density } d(X,Y) \sim_{\varepsilon} \alpha.$$

$$\mathbf{G_2} = \mathbf{G_2}(\varepsilon') \qquad (i)\ \deg_{\Gamma}(x) \sim_{\varepsilon'} \alpha|Y| \text{ for all but } \varepsilon'|X| \text{ vertices } x \in X,$$

$$(ii)\ \deg_{\Gamma}(x_1, x_2) \sim_{\varepsilon'} \alpha^2|Y| \text{ for all but } \varepsilon'|X|^2 \text{ pairs } x_1, x_2 \in X.$$

It was shown in [1] (cf. [15]) that properties $\mathbf{G_1}$ and $\mathbf{G_2}$ are equivalent in the following sense.

Theorem 6 (Alon, Duke, Lefmann, Rödl, Yuster, [1]) *For any $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that*

$$\mathbf{G_1}(\varepsilon') \Rightarrow \mathbf{G_2}(\varepsilon).$$

Similarly, for any $\varepsilon' > 0$, there exists $\varepsilon > 0$ such that

$$\mathbf{G_2}(\varepsilon) \Rightarrow \mathbf{G_1}(\varepsilon').$$

The equivalence of Properties $\mathbf{G_1}$ and $\mathbf{G_2}$ tells us that the notion of ε-regularity is equivalent to a condition concerning uniformity of degrees and codegrees. Since degrees and codegrees concern only vertices and pairs of vertices, and not large subsets as in the definition of ε-regularity, Property $\mathbf{G_2}$ is a "local" criterion for the regularity of graphs.

As mentioned earlier, the equivalence of Properties $\mathbf{G_1}$ and $\mathbf{G_2}$ played the crucial role in the algorithmic version of Szemerédi's Regularity Lemma in [1].

Theorem 7 (Constructive Regularity Lemma, [1]) *For every $\varepsilon > 0$ and every positive integer k, there exists an integer $Q = Q(\varepsilon, k)$ such that every graph G with $n > Q$ vertices admits an ε-regular partition into $t+1$ classes for some $k < t < Q$ and such a partition can be found in $O(M(n))$ sequential time, where $M(n)$ denotes the time needed for the multiplication of two $(0,1)$ matrices of size n.*

In the hypergraph setting, there is also a natural candidate for the "local" property $\mathbf{H_2}$ that should correspond to the global property $\mathbf{H_1}$ of regularity in the Frankl-Rödl sense. However, as shown in Dementieva, Haxell, Nagle and Rödl [12], unfortunately these two properties are not fully equivalent. Nevertheless, there is an equivalence in a special case, namely when the regularity parameter $r = 1$. This result was discussed in detail during the session.

Despite the inequivalence of properties $\mathbf{H_1}$ and $\mathbf{H_2}$, in [12], an algorithmic version of a special case of Frankl and Rödl's Hypergraph Regularity Lemma was obtained, namely, when $r = 1$.

Theorem 8 ([12]) *Let δ and γ with $0 < \gamma \leq 2\delta^4$, integers t_0 and ℓ_0 and function $\varepsilon(\ell) > 0$ be given. Let T_0, L_0, and N_0 be those constants guaranteed by the Frankl-Rödl Regularity Lemma. There exists a constant k so that, given any 3-uniform hypergraph $\mathcal{H} \subseteq [N]^3$, $N \geq N_0$, one may in time $O(N^k)$ find a $(\delta, 1)$-regular, $(\ell, t, \gamma, \varepsilon(\ell))$-partition of $\mathcal{H}$, for some t and ℓ satisfying $t_0 \leq t \leq T_0$ and $\ell_0 \leq \ell \leq L_0$.*

This restricted version of the lemma is still sufficient for some applications, for example for Theorem 17 described below. Therefore, for such applications, corresponding efficient algorithms exist.

The following problem attempts to make the Frankl-Rödl regularity lemma fully algorithmic.

Problem 9 *Let δ and γ with $0 < \gamma \leq 2\delta^4$, integers t_0 and ℓ_0 and functions $\varepsilon(\ell) > 0$, and $r(t, \ell)$ (integer valued) be given. Let T_0, L_0, and N_0 be those constants guaranteed by the Frankl-Rödl regularity lemma. Does there exist a constant k so that, given any 3-uniform hypergraph $\mathcal{H} \subseteq [N]^3$, $N \geq N_0$, one may in time $O(N^k)$ find a $(\delta, r(t, \ell))$-regular, $(\ell, t, \gamma, \varepsilon(\ell))$-partition of $\mathcal{H}$, for some t and ℓ satisfying $t_0 \leq t \leq T_0$ and $\ell_0 \leq \ell \leq L_0$?*

The combination of an algorithmic version of the Hypergraph Regularity Lemma and the Counting Lemma would be very helpful in solving many constructive hypergraph problems.

The current algorithmic version of the Hypergraph Regularity Lemma seen in Theorem 8 delivers only the special case $r = 1$. As one sees in the hypothesis of the Counting Lemma, however, one requires $r > 1$ to apply counting. As such, there is not a direct link between the current Theorem 8 and the Counting Lemma. A positive solution to Problem 9 would allow one to combine an algorithmic version of the Hypergraph Regularity Lemma with the Counting Lemma.

Very recently, exciting but partial progress on Problem 9 was obtained. It was shown by Dementieva, Haxell, Nagle and Rödl that one can indeed combine the current algorithmic version of the Hypergraph Regularity Lemma seen in Theorem 8 with a form of the Counting Lemma, despite the fact that Theorem 8 only delivers $r = 1$. It is hoped that further constructive applications may now ensue.

Applications of the Regularity Method

Thresholds for Ramsey properties of hypergraphs (M. Schacht)

We denote by $\mathbb{G}^{(k)}(n, p)$ the binomial random k-uniform hypergraph with n vertices and edges occurring independently with probability $p = p(n)$. It is well known that for many interesting properties $\mathcal{P}$ of hypergraphs there exists a critical function $\widetilde{p} = \widetilde{p}(n)$ around which the behaviour of $\mathbb{G}^{(k)}(n, p)$ suddenly changes with respect to $\mathcal{P}$. More precisely, we say $\widetilde{p} = \widetilde{p}(n)$ is a *threshold for the property* $\mathcal{P}$ if $\mathbb{G}^{(k)}(n,p)$ *asymptotically almost surely* (with probability tending to 1 as $n \to \infty$) satisfies $\mathcal{P}$ if $p \gg \widetilde{p}$ (i.e., $p(n)/\widetilde{p}(n) \to \infty$ as $n \to \infty$), while $\mathbb{G}^{(k)}(n,p)$ asymptotically almost surely fails to satisfy $\mathcal{P}$ for $p \ll \widetilde{p}$.

For two k-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$ we use the arrow notation $\mathcal{G} \to (\mathcal{H})^e_r$ to abbreviate the following Ramsey type statement: *For every r-colouring of the edges of $\mathcal{G}$ there exists a monochromatic copy of $\mathcal{H}$.* Obviously, the property $\mathcal{G} \to (\mathcal{H})^e_r$ is increasing and hence as a consequence of [5] it has a threshold. The study of the threshold function for the Ramsey property $\mathbb{G}^{(2)}(n,p) \to (\mathcal{H})^e_r$ for fixed graphs $\mathcal{H}$ and fixed integers r was initiated by Łuczak, Ruciński, and Voigt in [38]. In [45, 46], Rödl and Ruciński solved the problem completely for graphs ($k = 2$). They proved the following for graphs $\mathcal{H}$ different than forests. For a fixed k-uniform hypergraph $\mathcal{H}$ with at least one edge we define the k-density $m_k(\mathcal{H})$ as follows

$$m_k(\mathcal{H}) = \max\{d_k(\mathcal{H}')\colon\ \mathcal{H}' \subseteq \mathcal{H} \text{ and } e_{\mathcal{H}'} \geq 1\}\,,$$

where $d_k(\mathcal{H}')$ is defined as

$$d_k(\mathcal{H}') = \begin{cases} \frac{e_{\mathcal{H}'}-1}{v_{\mathcal{H}'}-k} & \text{if} \quad e_{\mathcal{H}'} > 1 \\ \frac{1}{k-1} & \text{if} \quad e_{\mathcal{H}'} = 1 \end{cases}$$

and $e_{\mathcal{H}}$ ($v_{\mathcal{H}}$) denotes the number of edges (vertices) of $\mathcal{H}$.

Theorem 10 (Rödl–Ruciński) *For all graphs $\mathcal{H}$ with at least one cycle and for all integers $r \geq 2$, there are constants $0 < c < C$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(2)}(n,p) \to (\mathcal{H})^e_r\right) = \begin{cases} 0 & \textit{if} \quad p \leq cn^{-1/m_2(\mathcal{H})} \\ 1 & \textit{if} \quad p \geq Cn^{-1/m_2(\mathcal{H})} \end{cases}$$

The proof of Theorem 10 utilises the Szemerédi Regularity Lemma [53], despite the fact that the result deals with sparse random graphs.

The general problem for k-uniform hypergraphs $k > 2$ is still wide open. It is conjectured that Theorem 10 extends naturally to "most" hypergraphs $\mathcal{H}$ with 2 replaced by k. (There might be some class of exceptional hypergraphs similar to forests in the graph case.)

Conjecture 11 *For "most" k-uniform hypergraphs $\mathcal{H}$ and for all integers $r \geq 2$, there are constants $0 < c < C$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \to (\mathcal{H})^e_r\right) = \begin{cases} 0 & \textit{if} \quad p \leq cn^{-1/m_k(\mathcal{H})} \\ 1 & \textit{if} \quad p \geq Cn^{-1/m_k(\mathcal{H})} \end{cases}$$

In [47] Rödl and Ruciński verified Conjecture 11 for $r = 2$ and $\mathcal{H} = K_4^{(3)}$, the complete 3-uniform hypergraph on four vertices. The proof of that case involves the hypergraph regularity lemma of Frankl and Rödl [22] for 3-uniform hypergraphs and a corresponding counting lemma which estimates the number of copies of $K_4^{(3)}$ in a regular 4-partite 3-uniform hypergraph (see Section 51). We believe that the counting tools described in Sections 51 and 51 ([41] and [43]) combined with the techniques of [46] can be applied to establish Conjecture 11 for 3-uniform hypergraphs $\mathcal{H}$ different from $K_4^{(3)}$. Moreover, the recent progress in developing the regularity lemma for k-uniform hypergraphs and the accompanying counting lemmas (see [48, 49]), hopefully, shed light in the study of thresholds for other Ramsey properties of random k-uniform hypergraphs.

Rödl, Ruciński, and Schacht previously worked on that problem. As a first step they were able to verify Conjecture 11 for arbitrary $k \geq 2$ and $r \geq 2$ in the case when $\mathcal{H}$ is a k-partite k-uniform hypergraph.

Theorem 12 (Rödl–Ruciński–Schacht) *For all integers $k \geq 2$ and $r \geq 2$ and every k-uniform k-partite hypergraphs $\mathcal{H}$ with at least one edge, there exists a constant $C > 0$ such that for every $p = p(n) \geq Cn^{-1/m_k(\mathcal{H})}$*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \to (\mathcal{H})^e_r\right) = 1\,.$$

We discussed their approach in this session of the workshop. In particular, due to a discussion initiated by Łuczak, we were interested whether this approach could be used to extend Theorem 12 to the corresponding Turán problem.

The Turán problem is a density version of the Ramsey type question. Here we are interested in the minimum size of a colour class to ensure a monochromatic copy of $\mathcal{H}$. Turán [54] first solved that problem for complete graphs $\mathcal{H}$ in the deterministic setting. Erdős, Stone, and Simonovits [16, 18] then generalised Turán's Theorem to arbitrary graphs $\mathcal{H}$.

Theorem 13 (Erdős–Stone–Simonovits) *For every graph $\mathcal{H}$ and $\eta > 0$, there exists an $n_0 > 1$ such that if an n-vertex graph $\mathcal{F}$ with $n \geq n_0$ contains more than*

$$\left(1 - \frac{1}{(\chi(\mathcal{H}) - 1)} + \eta\right)\binom{n}{2}$$

edges, where $\chi(\mathcal{H})$ is the chromatic number of $\mathcal{H}$, then $\mathcal{F}$ contains at least one copy of $\mathcal{H}$.

Theorem 13 can be viewed as a result for subgraphs $\mathcal{F}$ of the random graphs $\mathbb{G}^{(2)}(n,p)$ with $p \equiv 1$. Hence, naturally the question arises for which $p = p(n)$ Theorem 13 remains asymptotically almost surely (a.a.s.) true for subgraphs $\mathcal{F}$ of $\mathbb{G}^{(2)}(n,p)$ with $\binom{n}{2}$ replaced by $p\binom{n}{2}$ (the expected number of edges in the random graph $\mathbb{G}^{(2)}(n,p)$). It was shown that there exists a threshold for the Turán property discussed above (see, e.g., [28, Chapter 8]), even though it is not a monotone property. If $p = p(n)$ is such that the expected number of copies of subgraphs $\mathcal{H}'$ of $\mathcal{H}$ in $\mathbb{G}^{(2)}(n,p)$ is much smaller than the expected number of edges of $\mathbb{G}^{(2)}(n,p)$, then it is not hard to show that $\mathbb{G}^{(2)}(n,p)$ a.a.s. fails to satisfy the Turán property for $\mathcal{H}$. Conjecture 14 below, first formulated by Kohayakawa, Łuczak, and Rödl in [32], demonstrates the belief that this is the only obstacle.

Conjecture 14 (Kohayakawa–Łuczak–Rödl) *For every graph $\mathcal{H}$ containing at least one edge and $\eta > 0$, there exists a constant $C > 0$ such that if $p = p(n) \geq Cn^{-1/m_2(\mathcal{H})}$, then $\mathbb{G}^{(2)}(n,p)$ a.a.s. satisfies the following Turán type property. If $\mathcal{F}$ is a subgraph of $\mathbb{G}^{(2)}(n,p)$ with more than*

$$\left(1 - \frac{1}{(\chi(\mathcal{H}) - 1)} + \eta\right)p\binom{n}{2}$$

edges, then $\mathcal{F}$ contains at least one copy of $\mathcal{H}$.

So far, there are a few results in support of Conjecture 14. Any result concerning the tree-universality of expanding graphs, or any simple application of Szemerédi's regularity lemma for sparse graphs, gives Conjecture 14 for $\mathcal{H}$ a forest. The cases in which $\mathcal{H} = K_3$ and $\mathcal{H} = C_4$ are essentially proved in Frankl and Rödl [20] and Füredi [24], respectively, in connection with problems concerning the existence of some graphs with certain extremal properties. The case for $\mathcal{H} = K_4$ was proved by Kohayakawa, Łuczak, and Rödl [32] and the case in which $\mathcal{H}$ is a general cycle was settled by Haxell, Kohayakawa, and Łuczak [25, 26] (see also Kohayakawa, Kreuter, and Steger [31]). Very recently Gerke et al. settled the case $\mathcal{H} = K_5$. In [35] and [51] some weaker versions are obtained for arbitrary l and $\mathcal{H} = K_l$.

Due to the fruitful discussion during the workshop, mentioned earlier, some further progress towards Conjecture 14 was achieved. Rödl, Ruciński, and Schacht may extend their proof of Theorem 12 to the corresponding Turán problem. This, e.g., would verify Conjecture 14 for arbitrary bipartite graphs $\mathcal{H}$.

Dirac's theorem for hypergraphs (A. Ruciński)

A substantial amount of research in graph theory continues to concentrate on the existence of Hamiltonian cycles. The following classical theorem of Dirac from 1952 [19] is one of the best known results in graph theory.

Theorem 15 (Dirac) *Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ contains a Hamiltonian cycle. Moreover, there is an example showing that this is best possible.*

The study of Hamiltonian cycles in hypergraphs was initiated by Bermond where, however, a different definition than the one considered here was introduced. Here, by a *Hamiltonian cycle* in a 3-uniform hypergraph with n vertices we mean a spanning subhypergraph with n edges that admits an ordering $v_1, \ldots, v_n$ of the vertices so that all n triples $\{v_i, v_{i+1}, v_{i+2}\}$ (indices modulo n) are edges of the subhypergraph. Katona and Kierstead [29] proved that for a Hamilton cycle to exist, it is sufficient that all pairs belong to at least $5n/6$ edges. They also suggested that the following conjecture might be true.

Conjecture 16 *Every* 3*-uniform hypergraph with* $n \geq 4$ *vertices in which every pair of vertices belongs to at least* $n/2$ *edges contains a Hamilton cycle.*

The support for this conjecture stems from a construction of an edge-maximal, 3-uniform hypergraph with each pair degree at least $\lfloor n/2 \rfloor - 1$, not containing a Hamiltonian cycle (see [29, Theorem 3]).

Rödl, Ruciński and Szemerédi have proved an asymptotic version of this conjecture. We say that a 3-uniform hypergraph H is an (n, γ)-graph if H has n vertices and every pair of vertices belongs to at least $(1/2 + \gamma)n$ edges.

Theorem 17 *Let* $\gamma > 0$*. Then, for sufficiently large* n*, every* (n, γ)*-graph contains a Hamiltonian cycle.*

The proof of this Theorem is based on the hypergraph regularity lemma and its accompanying counting lemma for $k = 3$ (see Section 51). This proof was the main topic of the session.

During the workshop the three participants continued to work on the stronger version, where γ is totally eliminated, and are now about to complete the proof of the following extension of Theorem 17. A 3-uniform hypergraph on n vertices with every pair belonging to at least $n/2$ edges will be called a *Dirac* 3-graph.

Theorem 18 *For sufficiently large* n*, every Dirac 3-graph contains a Hamiltonian cycle.*

Extremal Hypergraph Problems

Stability and structure in Turán problems (D. Mubayi)

In this session, several extremal problems concerning Turán questions in hypergraphs were discussed.

Ramsey-Turán problems for hypergraphs.

For an l-graph $\mathcal{G}$, the Turán number $\text{ex}(n, \mathcal{G})$ is the maximum number of edges in an n-vertex l-graph $\mathcal{H}$ containing no copy of $\mathcal{G}$. The limit $\pi(\mathcal{G}) = \lim_{n\to\infty} \text{ ex } (n, \mathcal{G})/\binom{n}{l}$ is known to exist. The Ramsey-Turán density $\rho(\mathcal{G})$ is defined similarly to $\pi(\mathcal{G})$ except that we restrict to only those $\mathcal{H}$ with independence number $o(n)$. This definition is motivated by the fact that the densest graphs without a fixed graph usually have large independence sets; so what happens if we do not allow large independent sets? A result of Erdős and Sós [17] states that $\pi(\mathcal{G}) = \rho(\mathcal{G})$ as long as $\mathcal{G}$ is in some sense locally dense. Therefore a natural (first) question is whether there exist $\mathcal{G}$ for which $\rho(\mathcal{G}) < \pi(\mathcal{G})$.

Another variant $\tilde{\rho}(\mathcal{G})$ proposed in [17] requires the stronger condition that every set of vertices of $\mathcal{H}$ of size at least εn ($0 < \varepsilon < 1$) has density bounded below by some threshold (we omit the precise formulation). By definition, $\tilde{\rho}(\mathcal{G}) \leq \rho(\mathcal{G}) \leq \pi(\mathcal{G})$ for every $\mathcal{G}$. However, even $\tilde{\rho}(\mathcal{G}) < \pi(\mathcal{G})$ is not known for very many l-graphs $\mathcal{G}$ when $l > 2$.

Let $\alpha \in (0, 1), l \geq 2$ and let $\mathcal{H}_n$ be an l-graph on n vertices. $\mathcal{H}_n$ is (α, ξ)-uniform if every ξn vertices of $\mathcal{H}_n$ span $(\alpha \pm \xi)\binom{\xi n}{l}$ edges. A recent result of Mubayi and Rödl implies the following:

Theorem 19 (Mubayi-Rödl) *For all* $\tilde{\delta}$*, there exist* δ, r, n_0 *such that, if* $n > n_0$ *and* $\mathcal{H}_n$ *is* (α, δ)*-uniform, then all but* $\tilde{\delta}\binom{n}{r}$ r*-sets of vertices of* $\mathcal{H}_n$ *induce a subsystem that is* $(\alpha, \tilde{\delta})$*-uniform.*

Theorem 19 has important consequences for Ramsey-Turán problems for hypergraphs. In particular, it allows one to prove a phenomenon similar to supersaturation for Turán problems for hypergraphs. This could perhaps be the first step of a "Ramsey-Turán" analogue (to hypergraphs) of the celebrated Erdős-Stone-Simonovits theorem of extremal graph theory.

A slightly weaker version of Theorem 19 was proved independently by Alon, de la Vega, Kannan, and Karpinski [3]. Their motivation was to obtain an efficient sampling method for approximating r-dimensional Maximum Constrained Satisfaction Problems. Another application of Theorem 19 is

Theorem 20 *Let* $\mathcal{F}$ *be a fixed* l*-graph, and* $c > 0$*. Then there is an* n_0 *and* r' *such that: If* $\mathcal{H}$ *is an* n *vertex* l*-graph (*$n > n_0$*) such that the deletion of any* cn^l *edges of* $\mathcal{H}$ *leaves an* l*-graph that admits no homomorphism into* $\mathcal{F}$*, then there exists* $\mathcal{H}' \subset \mathcal{H}$ *on* r' *vertices, that also admits no homomorphism into* $\mathcal{F}$*.*

The special case of Theorem 20 when $\mathcal{F}$ is a complete graph was also recently proved by Alon and Shapira [2]. We hope that Theorem 19 applies more generally to show that global properties of an l-graph imply some local structure. The following problem was posed during the session.

Problem 21 *Find other applications of Theorem 19.*

The structure of extremal hypergraphs.

The Turán problem for hypergraphs is one of the oldest unsolved problems in combinatorics. In all known examples, there exists a 3-graph containing no copy of $\mathcal{F}$ with close to ex$(n, \mathcal{F})$ edges with a reasonable structure. Our goal is to make this statement precise.

Definition 22 *A directed hypergraph of rank three is a hypergraph whose edges consist of one element sets, ordered two element sets, and three element sets.*

Let $\mathcal{H}$ be a directed hypergraph of rank three with vertex set $\{v_1, \ldots, v_h\}$. Then $\mathcal{G}$ is a *recursive blow up* of $\mathcal{H}$ if the vertices of $\mathcal{G}$ can be partitioned into $V_1 \cup \ldots \cup V_h$ and

- if $x \in V_i, y \in V_j, z \in V_k$, then $xyz \in \mathcal{G}$ if and only if $v_i v_j v_k \in \mathcal{H}$
- if $x, y \in V_i, z \in V_j$, then $xyz \in \mathcal{G}$ if and only if $(v_i, v_j) \in \mathcal{G}$
- the construction giving directed edges and triples as above is repeated in V_i if and only if $v_i \in \mathcal{H}$.

As an example, the standard construction of a 3-graph with density $5/9$ and no copy of $\mathcal{K}_4^{(3)}$ is a recursive blow up of $\mathcal{H} = \{(x, y), (y, z), (z, x), \{x, y, z\}\}$. This and all other known examples are motivation for the following

Conjecture 23 *Let $\mathcal{F}$ be fixed a 3-graph. Then there exists a directed hypergraph of rank three $\mathcal{H} = \mathcal{H}(\mathcal{F})$ and an n_0 such that for all $\varepsilon > 0$: there is a 3-graph $\mathcal{G}$ on $n > n_0$ vertices*

- *with density at least $\pi(\mathcal{F}) - \varepsilon$,*
- *containing no copy of $\mathcal{F}$, and*
- *$\mathcal{G}$ is a recursive blow up of $\mathcal{H}$.*

A weaker version of Conjecture 23, obtained by interchanging the quantifiers $\exists \mathcal{H}, \forall \varepsilon$, can be proved by the Hypergraph Regularity Lemma. This was observed during the workshop by Rödl, and Simonovits.

Cycles in hypergraphs.

The Turán problem for cycles in graphs is notoriously hard. It seems natural to ask the same question for hypergraphs, but we need a meaningful definition of cycle. There are several possibilities (see, e.g., Duke [14]), one of which is the following: A t-cycle $\mathcal{C}_t$ in a hypergraph is a sequence of t distinct edges $A_1, \ldots, A_t$, with $A_i \cap A_j \neq \emptyset$ if and only if $j = i + 1$ (modulo t). For r-graphs, it is now a natural question to ask for ex$(n, \mathcal{C}_t)$.

This definition was also initially motivated by the following question of Erdős: Determine $f_r(n)$, the maximum size of a family of r-sets of an n element set such that whenever $A \cap B = C \cap D = \emptyset$, we have $A \cup B \neq C \cup D$. When $r = 2$, this is just ex(n, C_4), and probably inspired Erdős' question. For 3-graphs the forbidden configuration in Erdős' problem is a $\mathcal{C}_4$.

Theorem 24 (Mubayi-Verstraëte [39]) *Let $\mathcal{C}_t$ be an r-uniform t-cycle. Then*

$$\left\lfloor \frac{t-1}{2} \right\rfloor \binom{n-1}{r-1} \leq \text{ex}(n, \mathcal{C}_t) \leq 3 \left\lfloor \frac{t-1}{2} \right\rfloor \binom{n-1}{r-1}.$$

Conjecture 25 (Mubayi-Verstraëte) *Let $\mathcal{C}_t$ be an r-uniform t-cycle. Then, as $n \to \infty$, ex$(n, \mathcal{C}_t) = (1 + o(1)) \left\lfloor \frac{t-1}{2} \right\rfloor \binom{n-1}{r-1}$.*

Sharp Turán results for 3-uniform hypergraphs (M. Simonovits)

The aim of this session was to discuss two new results in extremal hypergraph theory. In general, the Turán problem in hypergraphs is very difficult, and so any particular instance that can be solved is of significant interest.

The following theorem proves a conjecture of Mubayi and Rödl. Here the 3-uniform hypergraph $F_{3,2}$ consists of the vertices $\{a, b, c, d, e\}$ and the triples abc, abd, abe and cde.

Theorem 26 (Füredi, Pikhurko, Simonovits) *Let $\mathcal{H}$ be a 3-uniform hypergraph with n vertices that does not contain a copy of $F_{3,2}$. Then $\mathcal{H}$ has at most $(4/9 + o(1))\binom{n}{3}$ triples.*

This theorem is best possible, because there exists a hypergraph with $(4/9)\binom{n}{3}$ triples that does not contain a copy of $F_{3,2}$. A notable feature of the proof is that it first establishes a stability result for the problem. In other words, it is first shown that any $F_{3,2}$-free hypergraph that has close to $(4/9)\binom{n}{3}$ triples must have a structure that is very close to the extremal example mentioned above.

A second significant fact about this problem is that the extremal hypergraph has two classes of vertices, canonically joined to each other by triples, but with a high asymmetry: one of the classes is twice as large as the other. Such asymmetry in non-degenerate hypergraph extremal configurations is very rare. Surprisingly, similar asymmetric configurations were found for the hypergraph Ramsey problems that were studied at the workshop (Section 51), in several different contexts. It would be very interesting to understand the reasons for asymmetry in problems where the original conditions are symmetric.

The second theorem discussed in this session was the analogous result for the Fano plane, proved by Füredi and Simonovits. In this case the extremal configuration has $(3/4)\binom{n}{3}$ triples.

Hypergraph Ramsey numbers for paths (T. Łuczak)

This session focused on Ramsey numbers for paths in graphs and hypergraphs. As defined in Section 51, the Ramsey number $R_k(H)$ of a graph H is defined to be the smallest integer m such that every colouring of the edges of the complete graph K_m with k colours contains a monochromatic copy of H, that is, a copy whose edges are all the same colour. For a 3-uniform hypergraph H the Ramsey number $R_k(H)$ is defined analogously, for colourings of the complete 3-uniform hypergraph $K_m^{(3)}$. It was proved by Bondy and Erdős [4] that $R_2(P_n) \leq 2n - 1$ for the path P_n with n vertices, and they conjectured that $R_3(P_n) \leq 4n - 3$. A few years ago Łuczak [37] proved an asymptotic version of this conjecture, using Szemerédi's regularity lemma for graphs [53]. The aim of this session was to investigate whether the new regularity method for hypergraphs could be used to extend the classical Bondy-Erdős result to the much more complex problem of finding $R_2(P_n^3)$ asymptotically, where P_n^3 is a hypergraph path with n vertices. There are two natural definitions for a 3-uniform hypergraph path with vertices $v_1, \ldots, v_n$: the *loose path* LP_n^3 with triples $v_1v_2v_3, v_3v_4v_5, v_5v_6v_7, \ldots, v_{n-2}v_{n-1}v_n$, and the *tight path* TP_n^3 with triples $v_1v_2v_3, v_2v_3v_4, v_3v_4v_5, \ldots, v_{n-2}v_{n-1}v_n$.

This proposal became the subject for a series of sessions involving all seven of the workshop participants who were present for the first week, who with a great deal of work and discussion succeeded in solving both of these problems. The value of $R_2(LP_n^3)$ is asymptotically $5n/4$, while $R(TP_n^3)$ is asymptotically $4n/3$. The arguments turn out to be quite different in the loose and tight cases, and require different approaches. The loose path argument is simpler and can be solved with a weaker form of hypergraph regularity, whereas the tight case requires the full force of the new method. Some of the intermediate results from the proof of Theorem 17 and insights from this work were key in the solution to the tight path problem. These two results will be joint papers and are currently being written up by J. Skokan and P. Haxell.

Conclusions

Here we highlight some specific accomplishments of the Focused Research Group, with references to the sections where they are noted in more detail.

- Two new results on hypergraph Ramsey numbers for paths (Section 51) were proved by the seven participants present during the first week of the workshop.

- The progress of Rödl, Ruciński, and Szemerédi on Theorem 18 (Section 51) partly took place during the workshop.
- The progress of Rödl, Ruciński, and Schacht on Conjecture 14 (Section 51) was prompted by suggestions of Łuczak at the workshop.
- progress of Luczak and Simonovits on their joint project on the structure of graphs of large minimum degree not containing subgraphs from a given family.
- A question posed by Łuczak at the workshop asks for a characterization for the notion of (γ, δ, r)-graph regularity, which is an essential concept in the definition of the "local" hypergraph property $\mathbf{H_2}$ (see Section 51). During the workshop, Nagle and Rödl found an efficiently verifiable characterization of (γ, δ, r)-regular graphs.
- The progress on Problem 9 of Dementieva, Haxell, Nagle and Rödl noted in Section 51 partly took place during the workshop.
- Some progress on Conjecture 4 was made (see Section 51), with contributions from several workshop participants. Proving this conjecture now seems quite feasible.

List of Participants

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Chapter 52

Topology and Analysis: Complementary Approaches to the Baum-Connes and Novikov Conjectures (03ss002)

May 24–June 7, 2003

Organizer(s): Nigel Higson (Pennsylvania State University), Jerry Kaminker (Indiana University Purdue University Indianapolis), Shmuel Weinberger (University of Chicago)

Objectives

The proposers are presently the recipients of a Focused Research Group grant from NSF on this topic. The summer of 2003 would be the final one funded by the grant and it would be particularly appropriate to have a summer school on the topic to survey the work done over the three years of the grant and to formulate the next set of problems which will be worked on. There have been a series of summer schools and training session in Europe over the past few years on this topic and while senior researchers from Canada and US have been able to participate, there has been less opportunity for younger mathematicians and graduate students from North (or South) America to take part. A program as we are requesting would contribute to changing that situation.

The specific objectives of the workshop would be to merge more closely the the geometric and analytic approaches to the study of discrete groups from the point of view of the Novikov conjecture. The analytic approach is based on studying the set of irreducible representations of the fundamental group of a manifold as a dual object for the group. The geometric approach is based on Gromov's notion of asymptotic properties of groups. While some connections have been made, there is much remaining to be developed. Currently under the Focused Research Grant work is proceeding well, but it has proved fruitful to continue to bring together workers in these different aspects. The dynamics of the action of the group on various spaces has also started to play an increasingly important role. A meeting like the one being proposed would bring together people who already are acquainted, but the informality of the activities and the extended period would be likely to solidify collaborations and continue to establish a subject in its own rite.

It should be emphasized that the lectures given would be expository and aimed at educating both graduate students and young researchers, (as well as senior scientists) in less familiar aspects of the subjects. In particular, we would have a lecture series on Group C*-algebras and Connections with Dynamics,and one on Geometric Group Theory and connections with Noncommutative Geometry.

List of Participants

Davis, Jim (Indiana University)
Guentner, Erik (University of Hawaii)
Higson, Nigel (Pennsyvania State University)
Julg, Pierre (Universite d'Orleans)
Kaminker, Jerry (Indiana University Purdue University Indianapolis)
Khalkhali, Masoud (University of Western Ontario)
Nest, Ryszard (University of Copenhagen)
Valette, Alain (Universite de Neuchatel)
Yu, Guoliang (Vanderbilt University)

Chapter 53

Quantum Algorithms and Complexity Theory (03frg052)

June 7–21, 2003

Organizer(s): Richard Cleve (University of Calgary), Umesh Vazirani (University of California, Berkeley), John Watrous (University of Calgary)

Objectives

The objective is to further develop the theory of quantum algorithms and complexity theory, as well as quantum information theory and cryptography. BIRS offers a unique opportunity for a small number of outstanding researchers in quantum computing to interact intensively for a period of a couple of weeks.

List of Participants

Aaronson, Scott (University of California, Berkeley)
Aharonov, Dorit (Hebrew University)
Ambainis, Andris (University of Latvia)
Buhrman, Harry (Centrum voor Wiskunde en Informatica)
Cleve, Richard (University of Calgary)
Farhi, Edward (Massachusetts Institute of Technology)
Hoyer, Peter (University of Calgary)
Klauck, Hartmut (Princeton University)
Mosca, Michele (University of Waterloo)
Regev, Oded (Institute for Advanced Study)
Santha, Miklos (Université Paris Sud)
Tamon, Christino (Clarkson University)
Tapp, Alain (Universite de Montreal)
Vazirani, Umesh (University of California, Berkeley)
Watrous, John (University of Calgary)
Yao, Andrew (Princeton University)
de Wolf, Ronald (Centrum voor Wiskunde en Informatica)

Chapter 54

Problems in Discrete Probability (03frg003)

July 12–26, 2003

Organizer(s): Robin Pemantle (Ohio State University), Yuval Peres (University of California, Berkeley), Peter Winkler (Bell Labs, Lucent Technologies)

Objectives

The purpose of the Institute for Elementary Studies series of workshops is to bring together talent from different fields (esp. combinatorics, probability, computer science and statistical physics) to solve problems and pursue new ideas in discrete probability. Since its creation in 1992, the group has opened up new fields (e.g. dynamic percolation) and produced numerous results in Markov chains, physical models, and randomized algorithms. One focus of the planned meeting will involve Markov fields on general graphs, which have many current algorithmic applications. In these models a large network of nodes is given, where each node can be in a small number of possible states. Every node interacts only with its immediate neighbours, yet from this local interaction, global structure can emerge when the interactions are strong enough. Such models have been studied for decades by mathematical physicists who focused on the case where the underlying network was an Euclidean lattice. Some of the concepts from the physics literature (phase transitions, critical exponents) retain their importance far beyond the domain where they were created, but other concepts from the geometry of graphs (isoperimetric inequalities, flows) are important as well.

List of Participants

Achlioptas, Dimitris (Microsoft Research)
Galvin, David (Microsoft Research)
Holroyd, Alexander (University of British Columbia)
Kenyon, Claire (Ecole Polytechnique)
Lyons, Russell (Indiana University)
Pemantle, Robin (Ohio State University)
Peres, Yuval (University of California, Berkeley)
Pippenger, Nicholas (University of British Columbia)
Propp, Jim (University of Wisconsin)
Randall, Dana (Georgia Institute of Technology)
Virag, Balint (Massachusetts Institute of Technology)

Winkler, Peter (Bell Labs, Lucent Technologies)

Chapter 55

Arithmetic of Fundamental Groups (03rit007)

September 6–20, 2003

Organizer(s): David Harbater (University of Pennsylvania), Florian Pop (University of Bonn)

Objectives

This two-week Focused Research Group is intended as an opportunity for mathematical collaboration and communication among mathematicians who have been working on related open problems concerning Galois groups and fundamental groups, from complementary perspectives. It is intended to provide a forum for researchers in this area to exchange information on the latest advances in the field, and to promote new research in the area. There are several topics we intend to consider:

1. Open aspects of the an Abelian conjecture, particularly concerning affine curves over algebraically closed fields of characteristic p. Given such a field, can the curve be determined from its fundamental group? Currently this is known only in genus 0, over the algebraic closure of F_p (a result of Tamagawa). In the other direction, given a curve (e.g. the affine line), can the field be determined by the fundamental group of the curve? Currently, this is known for countable fields only if one of the fields is the algebraic closure of F_p (also a result of Tamagawa).

2. The study of embedding problems for varieties, as a way of understanding their fundamental groups and the realization of Galois groups. This includes not only the case of curves over a rather general field, but also the situation of higher dimensional varieties (thus relating to the higher dimensional Abhyankar conjecture). The associated birational problem is also of real interest. Some group-theoretic obstructions to solving split embedding problems are known, but are presumably not sufficient to guarantee a proper solution, particularly in the higher dimensional case and over fields that are not large. Determining which embedding problems can be solved essentially comes down to understanding the tower of branched covers of the given variety.

3. The Galois action on the Teichmuller tower of covers. In particular, we would like to study the arithmetic of the moduli spaces M(g,r) via special loci in moduli. In a related vein, there is the problem of refining GT via the full Teichmuller tower, to better approximate G_Q (i.e. to obtain the best possible approximation to G_Q that arises from this approach). Such a refinement would act on the full tower; it should be the full group of "nice" outer automorphisms of the tower; and it should fit into an inverse system that stabilizes (as one goes up in the tower). In addition, we would want to relate this refined

GT to versions that have appeared in the literature via another approach, viz. adding cycle conditions to the definition of GT (beyond the usual three).

4. Other topics relating to the arithmetic of fundamental groups, including methods of constructive Galois theory, such as rigidity; the lifting and reduction of covers, with consideration of the resulting arithmetic and geometry; the ramification behaviour of covers, especially in characteristic p; Galois deformations, from the point of view both of moduli spaces and universal deformation spaces; Galois representations and Galois modules.

List of Participants

Abhyankar, Shreeram (Purdue University)
Bouw, Irene (University of Essen)
Chinburg, Ted (University of Pennsylvania)
Guralnick, Robert (University of Southern California)
Harbater, David (University of Pennsylvania)
Kani, Ernst (Queen’s University)
Koenigsmann, Jochen (University of Freiburg)
Lehr, Claus (Université de Bordeaux I)
Matignon, Michel (Université de Bordeaux I)
Minac, Jan (University of Western Ontario)
Nakamura, Hiroaki (Max-Planck-Institut fur Mathematik)
Pop, Florian (University of Bonn)
Pries, Rachel (Columbia University)
Saidi, Mohamed (Max-Planck-Institut fur Mathematik)
Schneps, Leila (Universite de Paris 6)
Stevenson, Katherine (California State University, Northridge)
Stix, Jakob (University of Bonn)
Szamuely, Tamas (Hungarian Academy of Sciences)
Tamagawa, Akio (Kyoto University)
Wewers, Stefan (University of Bonn)

Chapter 56

Mathematical Models for Plant Dispersal (03frg304)

September 21–October 2, 2003

Organizer(s): Mark A. Lewis (University of Alberta), James Bullock (NERC Centre for Ecology and Hydrology)

Scientific Background

The ability of plants to move into new environments and adapt to global change depends crucially upon the dispersal of the plant seeds [2]. The probability density function describing the spatial redistribution of seeds about a parent plant ('dispersal kernel') has been the subject of intensive mathematical and biological study. Classical mathematical theory of travelling waves, nonlinear PDEs and related integral models as well as detailed biological studies have shown that it is this dispersal kernel that determines the rate at which plants can spread spatially when introduced into new environments [26], or when responding to changing environmental conditions [5].

The importance of dispersal applies equally to invasive pest plants (many of which are extremely costly to agriculture), to persistence of threatened plants and species, and to the movement of indigenous plants, such as hemlock and spruce, in response to climate change. Thus plant dispersal plays a key role in today's most pressing ecological concerns: invasive species and adaption of vegetation to global climate change and conservation biology.

While invasive species in North America extract an immense ecological and economic toll (with estimated costs exceeding $130 billion US per year), the impact of costs and changes incurred by vegetation response to climate change is unclear. However, one thing is certain: in northern Canada and Fennoscandinavia the best estimates to date indicate vegetation will have to move at rates exceeding 1000 meters per year to keep up with changing temperature isoclines [17].

Mathematicians and quantitative biologists have addressed the problems of plant spread using a variety of different models, including reaction-diffusion, integrodifference, random-walk and simulation models. This has lead to a very rich and broad theory for calculating rates of spread. The theory goes back to the work of Fisher [7], Kolmogorov [11] and others in the 1930's using traveling wave analysis of parabolic PDE models, and extends to modern day with recent results on spread rates in populations with long-distance (non-diffusive) dispersal [12, 6], stage structure [19], spread in fluctuating environments [23, 18], stochastic spread [16, 5], and spread in the presence of secondary ecological interactions [14].

However, the mathematical models underlying the theory assume that the dispersal kernel, describing the possible dispersal distances, is known with arbitrary accuracy. The primary stumbling block in applying the theory to real plant spread has turned out to be uncertainty in the shape of the dispersal kernel, particularly over long distances. It can be shown that small changes in the "tails" of the dispersal kernel can result in order-of-magnitude changes in predicted spread rates—the rare, long-distance dispersal events described by the tails of the kernel are the dominant factor in the determining rate of spread [13, 10].

While plant seeds can be redistributed by wide variety of mechanisms (animals, birds, water, ballistic dispersal and so forth), a primary mechanism for plant seed dispersal is movement by wind flow. The quest to accurately determine the tail of dispersal kernels for wind-dispersed seeds has proceeded in at least three different ways: (1) Theoretically derive kernels, based on underlying assumptions about the dispersal process. These analytically derived kernels arise as solutions, or approximate solutions to mechanistic PDE submodels [23, 10]. (2) Computationally derive kernels, based on models for turbulent wind flow, ranging from Large Eddy Simulation of Navier Stokes wind flow to simulation of the related Fokker-Planck approximation for 3D turbulent and wind-based dispersal in the the atmospheric boundary layer [24, 25, 23]. (3) Empirically derive kernels, using accurate measurement of highly detailed seed trap data over long distances [2, 1, 9, 21].

Each of the above approaches has strengths and weaknesses. Theoretically derived kernels are of great use in models of plant spread because they are relatively tractable by analysis. The weakness of this approach is that its relative simplicity means it may not adequately describe important long-distance dispersal events. Computational models, based on approximations for Navier-Stokes flow have a solid mechanistic basis for the long-distance dispersal, but are computationally intensive, and difficult to simplify. Empirical models have the benefit that they are based on real dispersal data but it is often logistically infeasible to measure extremely long-distance dispersal of seed.

Focused Research Group Results

The meeting brought together mathematicians and quantitative biologists. This cross-disciplinary research environment led to specific advances in the modelling of plant dispersal. The focused research group comprised of James Bullock (NERC, Dorset), David Greene (Concordia), Steve Higgins (UFZ, Leipzig), Mark Lewis (Alberta), Annemarie Pielaat (Alberta), Tom Robbins (Utah), Merel Soons (Utrecht), Oliver Tackenberg (Regensburg). Prior to the FRG, each group member had made significant contributions to the study of long-distance dispersal and biological invasions (see References). The composition was divided evenly between senior (Bullock, Greene, Higgins and Lewis) and junior (Pielaat, Robbins, Soons, Tackenberg) researchers.

The group tackled three major problems: (1) How to accurately estimate population spread rates using empirical dispersal data, fitted to to nonlinear integrodifference models [15], (2) The formulation of a generalized dispersal function that precisely predicts long-distance wind-mediated seed dispersal, based on physics of the atmospheric boundary layer [22] (3) Comparison of computational models for wind-mediated seed dispersal [3]. Each of these group efforts is being written up in a paper (given above).

Accurate prediction of spread rates. The mathematical description of population growth and spread we considered is the integrodifference equation. Here, a discrete-time model is coupled with the dispersal and non-overlapping generations are assumed. This model is written as

$$n_t(x) = \int_\Omega f(n_t(y))\, K(x,y)\, dy, \tag{56.1}$$

where Ω is the region over which the population is spreading, $n_t(x)$ is the population density at point x and time $t = 0, 1, \ldots$, f describes nonlinear population dynamics, and $K(x, y)$ is the dispersal kernel describing movement from y to x. More complex versions of the model include x in higher space dimensions, and n_t a vector of interacting species or stages within a population. Population spread rate describes the asymptotic speed at which a locally introduced population $n_0(x)$ asymptotically travels in space and time. Calculation of the spread rate in this model was pioneered by Weinberger [26]. As described in the "Scientific Background" section, such spread rates are crucial in the context of invading populations and environmental change.

Despite the Weinberger's elegant theory, the process of using fitted dispersal kernels to estimate population spread rates has been problematic. Parametric kernels with different shaped 'tails' produce drastically different estimates for spread rate, despite providing similar fits to the measured dispersal.

We proposed and tested a new method for weighting the fit of parametric dispersal kernels to data on dispersal distance so as to reduce bias in the predicted spread rate. This is a brand new approach, that came out of a "break-out" discussion subgroup. The weighting is based on a calculation that involves the steepness of the leading edge of the invading population $n_t(x)$, which can be calculated using a new "empirical estimator" method [4]. Our tests showed that it can produce accurate and reliable estimates for population spread rates where none were possible previously. The paper, outlining the new method, its application and its testing, has been 70 % written up during the meeting [15] and will be soon submitted for publication.

A generalized dispersal kernel To date there is no widely accepted dispersal function for describing wind-mediated seed dispersal in terms of atmospheric boundary layer parameters (such as wind speed, turbulent mixing parameters and so forth). A group member, Robbins [23], recently derived the existing "Okubo-Levin" [20] dispersal kernel from first principles. This involved deriving a stochastic differential equation (SDE) model for individual individual seeds, and analyzing it using a Fokker-Planck expansion to yield an Eulerian description of particle movement in the form of a partial differential equation (PDE). Subsequent analysis of the PDE model using similarity methods yields an explicit formula for the Okubo-Levin kernel for the deposition rate of particles on the ground [20] in the limit where the seeds have no inertia.

While the Okubo-Levin model is analytically tractable, it misses some important details. We determined three major factors that are left out: (1) variation in the mean wind speed (2) unstable boundary layer dynamics at low wind speeds and (3) temporal autocorrelation in the turbulent wind dynamics. These factors can be included in the computational simulations (below), and it was the critical comparison the computational models and data (below) that made it clear the importance of these three features. Our goal was to incorporate each of these factors into the explicit formulation of the kernel, and to compare predictions, based on the atmospheric measurements against dispersal data in the Bullock and Clarke [1] and Tackenberg [25] data sets.

We found analytical methods to incorporate the wind speed variation and unstable boundary layer dynamics into the explicit formula, yielding a "modified Okubo-Levin" model. We have not yet found a way of incorporating the autocorrelation, but have devised a general approach to do this which we believe will be successful after further work. We tested the model predictions against data sets when possible. Our goal is to write up this work as a paper [22].

Critical comparison of computational models for deriving kernels. Three approaches for computational derivation of dispersal kernels have been proposed recently. They are all individual-based and are linked in their fundamental form, but differ in complexity.

A SDE model was developed by Robbins [23]. This assumes that vertical wind fluctuations are uncorrelated and the wind profile is neutrally stable, but allows for inertial forces acting on the seed. The "STG" model of Soons [24] is a trajectory model that takes into account autocorrelation in wind fluctuations and simulates a realistic wind flow pattern. However, it does not allow for the effect of boundary layer instability (thermals). "PAPPUS", developed by Tackenberg [25] is a highly realistic model which uses measured data on horizontal and vertical wind profiles, and thus will represent the effects of boundary layer instability on long distance seed dispersal.

Our approach was to compare these models by using them to derive kernels in relation to particular measured seed dispersal data. Targeted data sets were from Bullock and Clarke [1] and Tackenberg [25], which represented a range of meteorological and ecological conditions. Models were parameterized using relevant meteorological measures from these field sites. We included a fourth model in this analysis; the "modified Okubo and Levin" model, referred to above, to represent a much simplified, but theoretically attractive analytical solution to kernel estimation.

Model accuracy was assessed simply by determining the relationship with dispersal data, but models were also compared in terms of differences in spread rate estimates (determined using the Weinberger [26] moment generating function method). Long distance dispersal (i.e. longer tails to the dispersal kernel) was facilitated by the incorporation of autocorrelated wind fluctuations and thermals. Both lift seeds into higher wind profiles, which have stronger horizontal wind velocities. However, simpler models are accurate in some cases, and the group identified possible extensions to the simpler models to incorporate varying conditions. Extensions to the modified Okubo and Levin model are detailed above. The SDE model of Robbins [23] was also developed to represent autocorrelation accurately. A paper is in preparation [3].

List of Participants

Bullock, James (NERC Centre for Ecology and Hydrology)
Greene, David (Concordia University)
Higgins, Steven (UFZ - Centre for Environmental Research)
Lewis, Mark (University of Alberta)
Pielaat, Annemarie (University of Alberta)
Robbins, Tom (University of Utah)
Soons, Merel (Universiteit Utrecht)
Tackenberg, Oliver (Universität Regensburg)

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[4] J.S. Clark, L. Horvath, M.A. Lewis: On the estimation of spread rate for a biological population, *Statistics and Probability Letters* **51**:225–234 (2001).

[5] J.S. Clark, M.A. Lewis, L. Horvath: Invasion by extremes: Population spread with variation in dispersal and reproduction, *Am. Nat.* **157**:537–554 (2001).

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Research in Teams Reports

Chapter 57

Restricting Syzygies of Algebraic Varieties (03rit551)

April 3–6, 2003

Organizer(s): David Eisenbud (Mathematical Science Research Institute), Sorin Popescu (Stony Brook University), Mark Green (University of California, Los Angeles), Klaus Hulek (Fachbereich MathematikUniversitat Hannover)

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Chapter 58

Asymptotic Dynamics of Dispersive Equations with Solitons (03rit401)

April 18–26, 2003

Organizer(s): Stephen Gustafson (University of British Columbia), Kenji Nakanishi (Nagoya University, Princeton University), Tai-Peng Tsai (University of British Columbia)

Our collaboration during our stay in the BIRS has resulted in very fruitful outcomes. We have obtained results sufficient for about one and a half papers. In the following we describe the background, what we have obtained, and what we plan to do next.

Background

Solutions of dispersive partial differential equations (with repulsive nonlinearities) tend to spread out in space, although they often have conserved L^2 mass. There has been extensive study in this subject, usually referred to as scattering theory. These equations include Schrödinger equations, wave equations and KdV equations. When the nonlinearity is attractive, however, these equations possess solitary wave solutions (solitons) which have localized spatial profiles that are constant in time. To understand the asymptotic dynamics of general solutions, it is essential to study the interaction between the solitary waves and dispersive waves. The matter becomes more involved when the linearized operator around the solitary wave possesses multiple eigenvalues which correspond to excited states. The interaction with eigenvectors is very delicate and very few results are known.

For nonlinear Schrödinger equations with solitons, there are two types of results:

1. Control of the solutions in a finite time interval and construction of all-time solutions with specified asymptotic behaviours (scattering solutions). The first kind of results does not allow sufficient time for the excited states interaction to make a difference. In contrast, for scattering solutions the excited state interaction is effectively eliminated and scattering solutions may indeed be very rare.

2. Asymptotic stability of solitons, assuming the spectrum of the linearized operator enjoys certain properties (for example has only one eigenvalue or has multiple "well-placed"). The data are often assumed to be localized so that the dispersive wave has fast local decay. Currently, only perturbation problems can be treated for large solitons, while more general results can be obtained for small solitons.

What we have done

During our stay in BIRS, we first studied small solutions of the equation

$$i\partial_t\psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \qquad \psi(0,\cdot) = \psi_0 \in H^1(\mathbf{R}^3). \tag{58.1}$$

We assume ψ_0 is small in H^1, but we do not assume ψ_0 is in $L^1(\mathbf{R}^3)$, as is usually assumed. The equation possesses small solitary wave solutions which do not move in space, and is hence a good first-step model problem.

The importance of H^1 results (i.e., with non-localized data) is in that it is intimately related to the Hamiltonian or conservative structure, and more shortly, persistence global in time, in contrast against weighted L^2, whose smallness persists only for short time due to dispersion, and L^1, which may be instantaneously lost and therefore does not seem to have physical relevance. A related motivation is, as more eigenvalues are present, the dispersive component tends to decay very slowly. It is thus essential to be able to remove the localization assumption on the data.

Assume that $-\Delta + V$ supports only one eigenvalue $e_0 < 0$. There is a family of small nonlinear bound states Q_E satisfying

$$(-\Delta + V)Q + \lambda Q^3 = EQ, \qquad E \sim e_0.$$

They give exact solutions $Q_E(x)e^{-iEt}$ to (58.1). Let $\mathcal{L}_E$ denote the corresponding linearized operator. For any ϕ sufficiently small in H^1, it can be decomposed as

$$\phi = (Q_E + \xi)e^{-i\omega}$$

for a unique set of $E, \omega \in \mathbf{R}$ and $\xi \in H_c(\mathcal{L}_E)$. Since $\psi(t)$ is uniformly small in H^1, there is a well-defined set of functions $E(t), \omega(t) \in \mathbf{R}$ and $\xi(t) \in H_c(\mathcal{L}_{E(t)})$ such that

$$\psi(t,x) = (Q_{E(t)}(x) + \xi(t,x))e^{-i\omega(t)}. \tag{58.2}$$

$Q_{E(t)}$ and $\xi(t,x)$ are the solitary and dispersive wave components, respectively. We want to study the asymptotic stability of the solitary wave component and the asymptotic completeness of the dispersive wave component. We have obtained the following results, to be collected in the paper "Asymptotic Stability and Completeness in Energy Space for Nonlinear Schrödinger Equations with Small Solitons."

1. Asymptotic stability and completeness. When ψ_0 is sufficiently small in $H^1(\mathbf{R}^3)$, $\psi(t)$ can be uniquely decomposed as in (58.2), with differentiable $E(t)$, $\omega(t)$ and $\xi(t) \in H_c(\mathcal{L}_{E(t)})$. We have $E(t) \in (E_{\min}, E_{\max})$ and

$$\|\xi\|_{L_t^2 W_x^{1,6} \cap L_t^\infty H_x^1} \le C \|\psi_0\|_{H^1}.$$

Moreover, there exist $E_\infty \sim e_0$ and $\xi_+ \in H_c(\mathcal{L}_{E_\infty}) \cap H^1$ such that $E(t) \to E_\infty$ and

$$\left\| \psi(t) - Q_{E_\infty} e^{-i\omega(t)} - e^{-iE_\infty t} e^{t\mathcal{L}_\infty} \xi_+ \right\|_{H^1} \to 0, \qquad \text{as } t \to \infty. \tag{58.3}$$

2. Wave operator. For any set of $E_\infty \in (E_{\min}, E_{\max})$ and $\xi_+ \in H_c(\mathcal{L}_{E_\infty}) \cap H^1$ with $\|Q_{E_\infty}\| + \|\xi_+\|_{H^1}$ sufficiently small, there is a solution $\psi(t)$ of (58.1) such that (58.3) holds for some $\omega(t) \in \mathbf{R}$.

3. Examples of slow decay. For any non-increasing function $f(t)$ which goes to zero as $t \to \infty$, there exists a solution $\psi(t)$ of (58.1), decomposed as in (58.2), and a sequence t_j, $j = 1, 2, 3, \ldots$ with $t_j \to \infty$ as $j \to \infty$, such that

$$\|\xi(t_j)\|_{L^2_{\mathrm{loc}}} \ge f(t_j). \tag{58.4}$$

Besides the above results for Eq. (58.1), we have also estimated explicitly all small eigenvalues of the linearized operator for

$$i\partial_t \psi = -\Delta\psi - |\psi|^{p-1}\psi, \qquad x \in \mathbf{R}^d, \qquad \psi(0,\cdot) = \psi_0, \tag{58.5}$$

when p is closed to the critical exponent p_c for stability and blow-up, $p_c = 1 + 4/d$. This confirms a picture conjectured by M.I. Weinstein (he made some unpublished computations for the 1D case), with greater details.

Next project

We plan to extend the known results for (58.5) to the Hartree equations

$$i\partial_t\psi = -\Delta\psi - (\frac{1}{|x|^\alpha} * |\psi|^2)\psi, \qquad \psi(0,\cdot) = \psi_0, \qquad 0 < \alpha < d. \tag{58.6}$$

The equation is similar to (58.5) but is subtler due to the nonlocalness of the nonlinearity. The case $\alpha = 2$ corresponds to the critical case $p = p_c$ for (58.5). The cases $\alpha < 2$ are subcritical. The case $\alpha = 1$ corresponds to the case $p = 1 + 2/d$ for (58.5) and is the borderline for long-range potential. It is equivalent to the Schrödinger-Poisson system and is of fundamental importance. The stability of vacuum and asymptotic completeness for small solutions in $H^1(\mathbf{R}^3)$ are well known for both repulsive and attractive nonlinearities. We expect similar results about solitons for (58.5) to hold for (58.6). We plan to divide the investigation into the following steps.

Linear analysis:

1. No hidden symmetry: this corresponds to the nonexistence of nontrivial solutions for a certain linear equation associated to the linearized operator around the soliton.

2. Wave operator estimates for $\alpha > 1$: Extend the result of K. Yajima – S. Cuccagna on the $L^p - L^p$ estimates for the wave operator for linearized operator to Hartree equations. The essential difficulty lies on the nonlocalness of the convolution. We restrict $\alpha > 1$ to ensure that the potential terms are of short range as in Yajima-Cuccagna.

3. Modified wave operator estimates for $\alpha = 1$: In this case the potential terms are of long range and the wave operator needs to be modified. We hope we may still prove certain decay and Strichartz estimates for the linear evolution.

Nonlinear analysis:

4. As a preliminary step for Step 5, (and independent of Steps 1–3), we wish to study (58.1) in the case V supports one eigenvector but is of long range, i.e., $V(x) \sim |x|^{-1}$ as $|x| \to \infty$. We use this step to investigate the effect of the long range potential on the (small) soliton.

5. Nonlinear dynamics of Hartree equation (58.6) with $\alpha = 1$. Since the linearized operator is expected to possess many eigenvalues by physical arguments, one cannot hope to study the initial value problem using the known machinery. One may, however, study the nonlinear wave operator and construct scattering solutions. For this step we will only consider localized data in weighted space, $\int_{\mathbf{R}^3} |\psi_0(x)|^2(1+x)^{1+\varepsilon}dx < \infty$, as in the small solution case.

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Chapter 59

Topological Orbit Equivalence for Cantor Minimal Systems (03rit002)

April 26–May 10, 2003

Organizer(s): Thierry Giordano (University of Ottawa), Ian Putnam (University of Victoria), Christian Skau (Norwegian University of Science and Technology)

The project is an ongoing one; the first paper appeared in 1995. The program is to classify, up to orbit equivalence, group actions and, more generally, étale equivalence relations on Cantor spaces. During the RIT period at BIRS, we worked on the case of minimal, free $\mathbb{Z}^2$ actions on Cantor sets.

The following set of notes was written by Ian Putnam as a survey of the current state of the program. The RIT result is covered in the last section.

Introduction

We will be considering dynamical systems, usually minimal, on a Cantor set. By a Cantor set, we mean a compact, totally disconnected metric space with no isolated points. For dynamical systems, we include free actions of a countable group by homeomorphisms. However, our definition, which follows below, will include more general systems. The main problem is to understand the orbit structure of such systems. Specifically, if we are given two such systems, is there a homeomorphisms between the underlying spaces which carries the orbits of one system to the orbits of the other?

This is the natural extension to the topological case of the program in ergodic theory initiated by Henry Dye [Dy], who considered invertible measure preserving transformations of a Lebesgue space. This was continued by many others, most notably Krieger [Kr1] and Connes, Feldman and Weiss [CFW]. In another direction is the Borel case.

Group actions

Suppose that X is a compact metric space and that G is a countable, Abelian group. (In fact, Abelian is not really required here.) Suppose that φ is an action of G on X by homeomorphisms. That is, for every a in G, there is a homeomorphism $\varphi^a : X \to X$ which satisfy $\varphi^{a+b} = \varphi^a \circ \varphi^b$ and $\varphi^0(x) = x$, for all a, b in G and x in X.

We assume that the action is *free*; that is, if $\varphi^a(x) = x$ for some x in X and a in G, then $a = 0$. We say that the action is *minimal* if, for every point x in X, its orbit under the action, $\{\varphi^a(x) \mid a \in G\}$, is dense in X.

Étale equivalence relations

The group actions which we described above are our main objects of interest. However, we will expand the class of dynamical systems which we are considering. This extended class will be equivalence relations endowed with some extra structure. It includes the case of a group action by considering the relation in which the equivalence classes are the orbits. This extension is not merely done for the sake of maximal generality; we will use some of the others (AF-relations) in an essential way.

We begin with some notation and basic ideas. For the first part, we will allow more general topological spaces than just Cantor sets. We let X be a compact metric space and we consider an equivalence relation R on X. Shortly, we will restrict to the case that R has countable equivalence classes.

We let r and s (for range and source) denote the two canonical projections from R to X; $s(x, y) = x, r(x, y) = y$. We say that R is *minimal* if every equivalence class is dense in X.

Definition 1 *Let X be a compact metrizable space, R be an equivalence relation on X and $\mathcal{T}$ be a topology on R. We say that $R, \mathcal{T}$ is étale if*

1. *$\mathcal{T}$ is Hausdorff, second countable and σ-compact,*

2. *the diagonal $\{(x, x) \mid x \in X\}$ is open in R,*

3. *the maps $r, s : R \to X$ are local homeomorphisms; that is, for every (x, y) in R, we may find an open set U in $\mathcal{T}$ such that $r(U)$ and $s(U)$ are open in X and $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms,*

4. *if U and V are open sets as above, then the set*
$$UV = \{(x, z) \mid (x, y) \in U, (y, z) \in V, \text{ for some } y\}$$
is also open and

5. *if U as above is open, then so is $U^{-1} = \{(x, y) \mid (y, x) \in U\}$.*

When $\mathcal{T}$ is understood, we simply say that R is étale.

(We make some remarks on the terminology, which comes from the theory of groupoids. An equivalence relation is also a principal groupoid. The term 'étale' is relatively recent; in the past these have also been known as 'r-discrete groupoids with counting measure as Haar system'. See [Ren, Pat, GPS2].)

This may seem an unusual definition, so we spend some time elaborating on it. The idea is that $\mathcal{T}$ provides R with the structure of a dynamical system. The key point is item 3. Consider (x, y) in R and let U be as in part 3 of the definition. Consider the restriction of s to U, $s|U$. The map $\gamma = r \circ (s|U)^{-1}$ is a homeomorphism from $s(U)$ to $r(U)$, both open subsets of X. The graph of γ is simply U, which is contained in R. So we may think of R as being made up of the graphs of local homeomorphisms of X. This collection is closed under composition (part 4) and under taking inverses (part 5). Part 2 of the definition is the analogue of freeness of an action.

Let R be an étale equivalence relation on X. If U is an open subset of R as in part 3 above, then we refer to U as a *graph*.

It is probably worthwhile to give a very simple example of a relation which does not admit such a topology. Let R be the relation on the unit interval $[0, 1]$ whose classes are all singletons, except for $\{0, 1\}$. It is clear that there is no local homeomorphism from a neighbourhood of 0 to a neighbourhood of 1, taking 0 to 1 and whose graph is contained in the relation.

Here are some basic facts to keep in mind. We state them without proof.

1. There are relations, even minimal ones, which admit no such topology.

2. If R is étale, then its equivalence classes are countable. (One shows that, for any x in X, $r^{-1}\{x\} \subset R$ is discrete and closed and then uses σ-compactness.)

3. If R is étale and compact in the topology, then the topology must be the relative topology from $R \subset X \times X$. Moreover, in this case, the equivalence classes are finite. In fact, there is a uniform upper bound on the number of elements in an equivalence class.

4. If R is étale and has an infinite equivalence class, then $\mathcal{T}$ is *not* the relative topology of $X \times X$.

5. If R has an étale topology, it may not be unique. (This may seem a little surprising. Just to make things clear, the topology on X is fixed.)

We note one easy consequence of the definitions.

Theorem 2 *Suppose that R is an étale equivalence relation on X and that R' is an open sub-equivalence relation of R. Then R' is also étale, in the relative topology from R.*

Finally, we note that there is a natural notion of invariant measure for étale relations. A measure μ is R-invariant [Ren] if

$$\mu(r(U)) = \mu(s(U)),$$

for all graphs $U \subset R$. We let $M(X, R)$ denote the set of all R-invariant probability measures on X. There is a notion of amenability for relations. Describing this would take us too far afield, but we note that amenable relations always possess such measures [Ren]. We will have nothing to say about the case that there are no finite R-invariant measures.

Group actions (revisited)

We recall the situation of the earlier section, where X is a compact metric space, G is a countable Abelian group and φ is a free action of G on X. Our relation of interest in this case is

$$R_\varphi = \{(x, \varphi^a(x)) \mid x \in X, a \in G\}.$$

To topologize R_φ, we use the fact that the map sending (x, a) in $X \times G$ to $(x, \varphi^a(x))$ is bijective, since the action is free. We give G the discrete topology and $X \times G$ the product topology and use the map to transfer this to R_φ. In other words, a sequence, $(x_n, \varphi^{a_n}(x_n))$ converges to $(x, \varphi^a(x))$ in this topology if and only if x_n converges to x in X and a_n converge to a in the discrete topology. (See [Ren].)

This means that the graph of each homeomorphism φ^a is a compact open set in R_φ. Letting U be this graph, the associated map γ from our earlier discussion is just φ^a. One can think of this as the special case where our maps γ are actually given by global rather than just local homeomorphisms of X.

Theorem 2 takes on a new significance in this context: if one considers $R = R_\varphi$ arising from a group action, it is possible that open subequivalence relations need not be themselves group actions. More than just possible, this will be a critical step for us later.

AF-relations

In this section, we introduce one of the most important classes of étale relations called AF-relations [Ren, GPS2]. The terminology (which actually comes from C^*-algebra theory) represents 'approximately finite'. In these examples, the underlying space is totally disconnected.

Definition 3 *An étale relation R on X is an* AF-relation *if X is compact, metrizable and totally disconnected and if there are*

$$R_1 \subset R_2 \subset \cdots$$

such that $\cup_n R_n = R$ and $R_n \subset R$ is a compact open subequivalence relation, for each n.

We will have much more to say about these examples in a later section. For the moment, we want to point out that by including these relations, we are expanding significantly from group actions by noting the following.

Theorem 4 ([GPS2]) *Let φ be a free action of a countable group G on a compact, totally disconnected metric space, X. The relation R_φ is an AF-relation if and only if the group G is locally finite; that is, there is an increasing sequence of finite subgroups of G, $G_1 \subset G_2 \subset \cdots$ whose union is G.*

We give a sketch of the proof. We first suppose that G is locally finite and choose a sequence of subgroups as in the theorem. For each $n \geq 1$, we let

$$R_n = \{(x, \varphi^a(x)) \mid x \in X, a \in G_n\}.$$

It is easy to see that each R_n is a compact open subrelation of R_φ and their union is R_φ.

To prove the converse statement, we suppose that R_φ may be written as an increasing union of compact open subrelations $R_n, n \geq 1$. Select a finite subset F of G. We will argue that the subgroup of G generated by F, denoted $< F >$ will be finite. It is fairly easy to see that this then implies that G is locally finite. Consider U, the union of the graphs of the elements of F, which is compact in R_φ. Since the R_n form an increasing open cover, U is contained in R_n, for some n. Since R_n is a subrelation, it is fairly easy to check that the graph of any element $< F >$ is also in R_n. This means that the orbits of any point in X under $< F >$ is finite. By the freeness of the action, this implies that $< F >$ is finite.

C^*-algebras (briefly)

The notion of an étale equivalence relation comes from C^*-algebra theory. We will not use any C^*-algebra theory in the remainder of these notes or even in the complete proofs. However, we take a few moments to give some idea of the connections. More information may be found in [Ren, Pat].

A C^*-algebra, A, (briefly) is a *-algebra over the complex numbers equipped with a norm $\| \cdot \|$. That is, we have addition, scalar multiplication and a ring multiplication, which is, in general, not commutative. There is also a conjugate linear involution $a \to a^*$ satisfying $(ab)^* = b^* a^*$. The ring multiplication need not, in general, have a unit, but the algebras we construct here will be unital. Of course, the algebraic operations should be continuous in the topology coming from the norm. Moreover, regarded as a metric space with $d(a, b) = \| a - b \|$, A should be complete. Finally, the norm should satisfy the C^*-condition, $\| a^* a \| = \| a \|^2$, for all a in A. This condition may seem obscure to the non-expert, but is really quite powerful.

The first example is the complex numbers, with $*$ being complex conjugation and the usual norm. The second example is $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. The $*$ operation is conjugate transpose and the norm is

$$\| a \| = \sup\{\| a\xi \|_2 | \ \xi \in \mathbb{C}^n, \| \xi \|_2 = 1\},$$

for all a in $M_n(\mathbb{C})$, where $\| \cdot \|_2$ denotes the l^2-norm on $\mathbb{C}^n$. This example can be easily generalized to the algebra of bounded linear transformation on a complex Hilbert space, by replacing $\mathbb{C}^n$ by the Hilbert space.

If R is an étale equivalence relation on a space X (not necessarily Cantor), we may construct a C^*-algebra as follows. Let $C_c(R)$ denote the set of continuous, compactly supported complex-valued functions on R. It is a linear space in an obvious way. The product and involution are defined by the formulae

$$\begin{aligned} (f \cdot g)(x, y) &= \sum_{(x,z) \in R} f(x, z) g(z, y) \\ f^*(x, y) &= \overline{f(y, x)}, \end{aligned}$$

for all f, g in $C_c(R)$ and (x, y) in R. It is a subtle point here that the product $f \cdot g$ is again in $C_c(R)$. The proof uses the étale property of R.

The formula above for the product should remind one of matrix multiplication. Indeed, if $X = \{1, 2, \ldots, n\}$ and $R = X \times X$, then this algebra is just $M_n(\mathbb{C})$.

The issue of a norm is more subtle. For each point x_0 in X, one can consider the Hilbert space of l^2 sequences on its equivalence class, $[x_0]$. Each f in $C_c(R)$ defines a linear transformation on this Hilbert space by

$$f\xi(x) = \sum_{(x,y) \in R} f(x, y) \xi(y),$$

for x in $[x_0]$. This transformation is bounded and we define

$$\| f \| = \sup\{\| f \|_{x_0} | \; x_0 \in X\},$$

where $\| \cdot \|_{x_0}$ denotes the operator norm for the Hilbert space $l^2[x_0]$. Again in the finite case above, this gives the same norm on $M_n(\mathbb{C})$. Of course, it is necessary to prove that this supremum is finite. Finally, the algebra $C_c(R)$ is usually not complete in this norm. We complete it to obtain a C^*-algebra which is denoted by $C^*_r(R)$ called the reduced C^*-algebra of R. The reason for the subscript r and the term 'reduced' is that there are other choices for the norm. The one above is arguably the most interesting. For amenable equivalence relations, all (reasonable) norms are the same.

Isomorphism and Orbit equivalence for étale relations

There are two natural notions of equivalence between two étale relations which we describe now. Classifying systems up to orbit equivalence is our main objective.

Definition 5 ([GPS2]) *Let (X, R) and (X', R') be two étale relations.*

1. *We say that (X, R) and (X', R') are* orbit equivalent *and write $(X, R) \sim (X', R')$ if there is a homeomorphism $h : X \to X'$ such that*
$$h \times h(R) = R'.$$
That is, the map h carries R-equivalence classes exactly to R'-equivalence classes.

2. *We say that (X, R) and (X', R') are* isomorphic *and write $(X, R) \cong (X', R')$ if there is a homeomorphism $h : X \to X'$ such that*
$$h \times h(R) = R'$$
and such that $h \times h : R \to R'$ is a homeomorphism.

The first notion is probably the most natural one for dynamics. However, much of what we are doing really uses the topology on R and this makes the second important. It is worth pointing out here that what is really going on is that the topologies which are given to our relations are not usually unique. Given an orbit equivalence h from R to another relation, we may use $(h \times h)^{-1}$ to transfer the other topology back to R, but it may not agree with the original from R.

Let us also take a moment here to explain why we concentrate on totally disconnected spaces. Just to be specific, suppose that φ is a free action of the group G on the compact, connected space X and ψ is a free action of the group H on the compact, connected space Y. Also suppose that $h : X \to Y$ is an orbit equivalence. This means that, for x in X and a in G, we may find b in H such that $h(\varphi^a(x)) = \psi^b(h(x))$. Fix a for the moment and for each b in H, let B_b be the set of x where the equation above holds. The sets $B_b, b \in H$ form a countable partition of the space X. It is fairly easy to check that each of these sets is closed. By a result of Sierpinski, as X is connected, one of these sets must be all of X and the rest are empty. We will not pursue this, but it allows a rather precise (and very restrictive) description of the map h.

We have hinted at the importance of AF-relations. It leads us to the following definition.

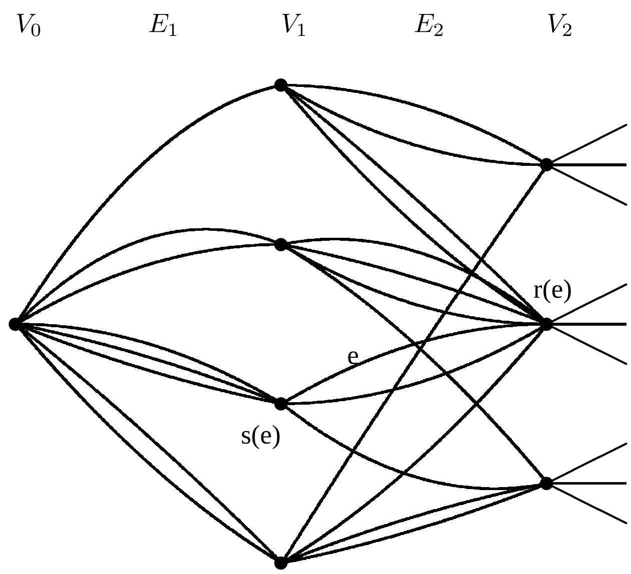
Definition 6 ([GPS2]) *Let (X, R) be an étale relation with X totally disconnected. We say that (X, R) is* affable *if (X, R) is orbit equivalent to an AF-relation (X', R').*

The reason for the terminology follows from the last comment of the previous paragraph. If (X, R) is orbit equivalent to an AF-relation, we may use the orbit map to transfer the topology to R. That is, R may be given a new topology in which it is AF. So that R is 'AF-able' or affable.

The construction of AF-relations

One thing which was missing from our definition of AF-relations earlier was a general method for producing such systems. We present this now.

We begin with a Bratteli diagram: a locally finite, infinite directed graph as shown below. (See [HPS, Ef].)



It consists of a vertex set V which is partitioned into a sequence of non-empty finite sets, $V_n, n \geq 0$, and an edge set, E, which is also partitioned into a sequence of non-empty finite sets, $E_n, n \geq 1$. Each edge e in E_n has a source, $s(e)$ in V_{n-1}, and a range, $r(e)$ in V_n. (This is a different use of the terms range and source than earlier, but it should not cause any confusion.) For simplicity, we assume that V_0 consists of a single vertex and for every other vertex v, $r^{-1}\{v\}$ and $s^{-1}\{v\}$ are non-empty. (There are no sources, other than in V_0, or sinks.)

The space X is the set of infinite paths in the diagram. That is,

$$X = \{(e_1, e_2, \ldots) \mid e_n \in E_n, r(e_n) = s(e_{n+1}), n \geq 1\}.$$

It is given the relative topology from the product space $\Pi_n E_n$ in which it is compact, metrizable and totally disconnected. For each $N \geq 0$, we define

$$R_N = \{(e, f) \mid e, f \in X, e_n = f_n, \text{ for all } n > N\}.$$

This set is given the relative topology of the product $X \times X$. It is easy to check that R_N is a compact étale equivalence relation (and hence each equivalence class is finite), and that R_N is an open subset of R_{N+1}, for all N. We define

$$R = \cup_{N=0}^{\infty} R_N$$

and it is given the inductive limit topology.

It is not difficult to check that R is an étale equivalence relation on X. It is worth considering for a moment the local homeomorphisms we described earlier. Suppose that $(e_1, e_2, \ldots, e_N)$ and $(f_1, f_2, \ldots, f_N)$ are two finite paths in the diagram, ending at the same vertex $v = r(e_N) = r(f_N)$. We let U be the clopen set in R_N

$$U = \{(e', f') \mid e'_n = e_n, f'_n = f_n, n \leq N, e'_n = f'_n, n > N\}.$$

The map γ is

$$\gamma(e_1, e_2, \ldots, e_N, e'_{N+1}, e'_{N+2}, \ldots) = (f_1, f_2, \ldots, f_N, e'_{N+1}, e'_{N+2}, \ldots)$$

for all sequences of the form $(e_1, e_2, \ldots, e_N, e'_{N+1}, e'_{N+2}, \ldots)$ in X.

We denote X by $X(V, E)$ and $R = R(V, E)$.

Theorem 7 ([GPS2]) *Let R be an AF-relation on a totally disconnected compact metrizable space X. Then (X, R) is isomorphic to $(X(V, E), R(V, E))$, for some Bratteli diagram V, E.*

Invariants for Cantor étale relations

We will introduce two invariants for Cantor minimal systems. These will be ordered Abelian groups [Go, Ef]. By an order on an Abelian group, G, we mean a subset G^+ such that $G^+ \cap (-G^+) = \{0\}$, $G^+ + G^+ \subset G^+$ and $G^+ - G^+ = G$. The set G^+ is usually referred to as a positive cone. An order in the usual sense is obtained by setting $a \geq b$ if and only if $a - b$ is in G^+. Also, our ordered groups will have a distinguished positive element. Such an element, u, is called an order unit if for every a in G^+, we have $nu - a$ is in G^+, for some $n \geq 1$.

We let $C(X, \mathbb{Z})$ denote the continuous $\mathbb{Z}$-valued functions on X. It is an Abelian group with the operation of pointwise addition. If E is a clopen subset of X, we let χ_E denote its characteristic function, which is in $C(X, \mathbb{Z})$.

We define $B(X, \varphi)$ to be the subgroup of $C(X, \mathbb{Z})$ generated by the functions $\chi_{r(U)} - \chi_{s(U)}$, where U is a compact graph in R.

We define $B_m(X, R)$ to be the subgroup of $C(X, \mathbb{Z})$ generated by the functions f such that $\int_X f d\mu(x) = 0$, for all μ in $M(X, R)$. (In the case that there are no invariant probability measures, we have $B_m(X, R) = C(X, \mathbb{Z})$.) It is clear that $B(X, R) \subset B_m(X, R)$.

Definition 8 *Let R be an étale relation on the Cantor set X. We define*

$$D(X, R) = C(X, \mathbb{Z})/B(X, R),$$

with positive cone

$$D(X, R)^+ = \{[f] \mid f \in C(X, \mathbb{Z}), f \geq 0\},$$

and order unit $u = [1]$.

We also define

$$D_m(X, R) = C(X, \mathbb{Z})/B_m(X, R),$$

with positive cone

$$D_m(X, R)^+ = \{[f] \mid f \in C(X, \mathbb{Z}), f \geq 0\},$$

and order unit $u = [1]$.

Although the notation and context are slightly different, a version may be found in [HPS]. Notice that $D_m(X, R)$ is a quotient of $D(X, R)$.

Earlier, we had two notions, isomorphism and orbit equivalence, between Cantor étale relations. We now spell out the precise sense in which our 'invariants' are invariant.

Theorem 9 *Let (X, R) and (X', R') be two étale relations on Cantor sets. If $h : X \to X'$ is a homeomorphism which implements an isomorphism between the relations, then the map $h^*[f] = [f \circ h]$, $f \in C(X', R')$, is an isomorphism from $D(X', R')$ to $D(X, R)$ mapping $D(X', R')^+$ onto $D(X, R)^+$ and preserving the order units. If $h : X \to X'$ is a homeomorphism which implements an orbit equivalence between the relations, then the map $h^*[f] = [f \circ h]$, $f \in C(X', R')$, is an isomorphism from $D_m(X', R')$ to $D_m(X, R)$ mapping $D_m(X', R')^+$ onto $D_m(X, R)^+$ and preserving the order units.*

The first statement is quite easy. For the second, it is fairly easy to check that an orbit equivalence will induce a bijection between the sets of invariant measures of the two systems.

The invariants of an AF-relation

The structure of the invariants introduced in the last section are quite well-understood for AF-relations.

First, we want to describe briefly how, if one is given a Bratteli diagram V, E, the invariant $D(X(V,E), R(X,R))$ can be computed. For each $n \geq 0$, let $\mathbb{Z}^{V_n} = \{f : V_n \to \mathbb{Z}\}$. This group is given the simplicial or standard order, $f \in \mathbb{Z}^{V_n+}$ if and only if $f(v) \geq 0$ for all v in V_n. The edge set E_n gives a group homomorphism $\alpha_n : \mathbb{Z}^{V_{n-1}} \to \mathbb{Z}^{V_n}$ as follows. Either one can consider E_n as providing a rectangular adjacency matrix and the homomorphism is simply multiplication by this matrix, or equivalently, we have, for f in $\mathbb{Z}^{V_{n-1}}$,

$$\alpha_n(f)(v) = \sum_{r(e)=v} f(s(e)), v \in V_n.$$

This provides an inductive system of ordered Abelian groups.

Theorem 10 (See [HPS]) *For an AF-relation, $(X(V,E), R(V,E))$, the group $D(X(V,E), R(V,E))$ is the inductive limit of the system $(\mathbb{Z}^{V_n}, \alpha_n)$, in the category of ordered Abelian groups.*

Such a group is called a *dimension group*. A fundamental result in the subject is the following result.

Theorem 11 (Effros-Handelman-Shen [Ef, Go]) *A countable, ordered Abelian group G, G^+ is a dimension group if and only if*

1. *it is unperforated: if g is in G and ng is in G^+ for some positive integer n, then g is in G^+, and*
2. *it satisfies the Riesz interpolation property: if g_1, g_2, h_1, h_2 are in G such that $g_i \leq h_j$ for all $1 \leq i, j \leq 2$, then there exists g in G such that $g_i \leq g \leq h_j$ for all $1 \leq i, j \leq 2$.*

The point is that it is relatively easy to find groups which satisfy the two conditions of the theorem. To any such group, we may find an AF-relation, (X, R), having $D(X, R)$ isomorphic to that group.

An *order ideal* in an ordered Abelian group G, G^+ is a subgroup H such that $H \cap G^+$ generates H as a group and whenever g in G^+ and h in $H \cap G^+$ satisfy $g \leq h$, then g is in H. A dimension group is *simple* if the only order ideals are 0 and G. (See [Go].)

Theorem 12 (See [HPS]) *An AF-relation (X, R) is minimal if and only if the associated dimension group $D(X, R)$ is simple.*

The strategy for orbit equivalence results

We now take a few lines to set out our strategy, without being too precise about all the terms, for proving orbit equivalence results. There are three steps:

1. Classify minimal AF-relations.
2. Let R be a minimal AF-relation on a Cantor set X. Suppose that $Y_1, Y_2 \subset X$ are 'small' closed subsets and $\alpha : Y_1 \to Y_2$ is a homeomorphism. Show that the relation

   $$R \vee Graph(\alpha)$$

   is orbit equivalent to R. Here, $\vee$ denotes the equivalence relation generated by the two sets.
3. For a free, minimal action φ of the group G on the Cantor set X, find a sequence $R_1 \subset R_2 \subset \cdots$ of compact open subequivalence relations of R_φ, whose union, denoted by R, is minimal, and Y_1, Y_2, α as above such that

   $$R_\varphi = R \vee Graph(\alpha).$$

With these steps, the problem of orbit equivalence for actions of the group G is reduced to that of AF-relations. Of course, the third step above depends on the group G. We will see that we have a complete answer for the case $G = \mathbb{Z}$ and a partial one for the group $G = \mathbb{Z}^2$.

This is the same strategy used by Dye in the ergodic measure preserving case. For the first step, there is only AF-relation in this case, up to orbit equivalence. For the second step, the meaning of 'small' in the definition of the sets Y_1 and Y_2 is measure zero. Then the result needed for that step is trivial. The third step is done by using Rohlin partitions for the appropriate group. It is the classic Rohlin lemma for $G = \mathbb{Z}$. This can be extended to include amenable groups.

Classification of AF-relations

One of the most important features of AF-relations is that they may be classified up to isomorphism and also (at least in the minimal case) up to orbit equivalence by the invariants we have discussed.

Theorem 13 (Elliott-Krieger [Kr2]) *For AF-relations (X, R), the triple $(D(X, R), D(X, R)^+, [1])$ is a complete invariant for isomorphism.*

Building on this, one may also obtain the following result, but the hypothesis of minimality is also needed.

Theorem 14 (Giordano-Putnam-Skau [GPS1]) *For minimal AF-relations (X, R), the triple $(D_m(X, R), D_m(X, R)^+, [1])$ is a complete invariant for orbit equivalence.*

We will not discuss the proofs of these. The second result appears in [GPS1] as a consequence of the classification for $\mathbb{Z}$-actions. In hindsight, this seems to be putting the proverbial cart before the horse. A direct proof can be given, and it now seems much more logical to proceed with this result first.

The absorption theorem

We now turn our attention to the second step of our strategy. That is, showing that a minimal AF-relation may be enlarged slightly and remain orbit equivalent to the result. Here, the topological case is much more subtle than, say, the measurable. The precise result, which we refer to as the absorption theorem, follows below.

Theorem 15 ([GPS2]) *Let X, R be a minimal AF-relation. Suppose that Y_1 and Y_2 are closed subsets of X and $\alpha : Y_1 \to Y_2$ is a homeomorphism such that the following hold.*

1. $$R \cap (Y_1 \times Y_2) = \emptyset,$$
2. $\mu(Y_1) = \mu(Y_2) = 0$, *for all μ in $M(X, R)$,*
3. $R \cap (Y_i \times Y_i)$ *is an étale relation on Y_i, for $i = 1, 2$,*
4. α *is an isomorphism from $(Y_1, R \cap (Y_1 \times Y_1))$ to $(Y_2, R \cap (Y_2 \times Y_2))$.*

Then the relation

$$R \vee Graph(\alpha)$$

is orbit equivalent to R.

Minimal $\mathbb{Z}$-actions

Our main objective is the classification up to orbit equivalence. However, this seems a good time to note the following result regarding isomorphism for the relations coming from $\mathbb{Z}$-actions.

Theorem 16 (Boyle, see [GPS1]) *Let φ and ψ be two minimal $\mathbb{Z}$-actions on Cantor sets. The relations R_φ and R_ψ are isomorphic if and only if φ is conjugate to ψ or to ψ^{-1}.*

Now we return to the problem of orbit equivalence, concentrating on the group $G = \mathbb{Z}$.

Theorem 17 (Giordano-Putnam-Skau [GPS1]) *Let φ be a minimal action of $\mathbb{Z}$ on the Cantor set X. Then the relation R_φ is orbit equivalent to an AF-relation. (That is, R_φ is affable.)*

The following is an immediate consequence of this result and Theorem 14.

Corollary 18 *The triple $(D_m(X,R), D_m(X,R)^+, [1])$ is a complete invariant for orbit equivalence for the class of Cantor systems consisting of minimal AF-relations and minimal $\mathbb{Z}$-actions.*

We will sketch a proof of Theorem 15, showing how the absorption theorem of the last section is used.

Begin by selecting a sequence of clopen sets $U_1 \supset U_2 \supset \cdots$, whose intersection is a single point y. For each $n \geq 1$, let R_n denote the relation generated by $\{(x, \varphi^1(x)) \mid x \in X - U_n\}$. Notice that if any U_n were empty, R_n would be R_φ. As it is, since U_n is open and φ is minimal, any point in X will enter U_n after a finite number of iterations of φ or φ^{-1}. From this, it follows that the R_n-equivalence class of the point in X is finite. A slightly more careful analysis involving the continuity of the return times of φ on U_n shows that R_n is compact and open. It is clear that $R_n \subset R_{n+1}$, for all $n \geq 1$. We let $R = \cup_n R_n$, which is an AF-relation. It is easy to check that R is minimal. In fact, every R-class is also a φ-orbit, except for the orbit of the point y. We let $Y_1 = \{y\}$, $Y_2 = \{\varphi^1(y)\}$ and $\alpha = \varphi^1$. We are then in a position to apply the absorption Theorem 15. (It is surprisingly easy here to check the hypotheses.) Moreover, we have

$$\begin{aligned} R_\varphi &= (\cup_n R_n) \vee \{(y, \varphi^1(y))\} \\ &= R \vee Graph(\alpha) \\ &\sim R \end{aligned}$$

and we are done.

Minimal $\mathbb{Z}^2$-actions

The results of this section are quite recent and are in preparation [GPS3]. We consider a minimal, free action, φ, of the group $\mathbb{Z}^2$ on the Cantor set X. Before we can state our main result, we need some basic notions about cocycles. The first definition is a standard one, although our interest is only in integer-valued cocycles. The next two are new, as far as we know.

Definition 19 *Let R be an étale equivalence relation on X. A* cocycle *or more accurately a* 1-cocycle *for R is a continuous function*

$$\theta : R \to \mathbb{Z}$$

such that

$$\theta(x,z) = \theta(x,y) + \theta(y,z),$$

for all $(x,y),(x,z)$ in R.

Definition 20 *Let (X,φ) be a minimal free $\mathbb{Z}^2$ Cantor system and let C be a subset of $\mathbb{Z}^2$. A cocycle θ for R_φ is* positive *with respect to C if $\theta(x, \varphi^n(x)) \geq 0$ for all $x \in X$ and $n \in C$.*

We say that θ is strictly positive if it is positive and θ is a proper as a map from $\{(x, \varphi^n(x)) \mid x \in X, n \in C\}$ to $\mathbb{Z}$.

Definition 21 *Let (X,φ) be a minimal free $\mathbb{Z}^2$ Cantor system and let θ be a cocycle for R_φ. For any positive integer M, we write $\theta \leq M^{-1}$ if $|\theta(x,\varphi^n(x))| \leq 1$, for all x in X and n in $\mathbb{Z}^2$ with $|n| \leq M$, where $|n| = |(n_1,n_2)| = \max\{|n_1|,|n_2|\}$ denotes the L^∞ norm on $\mathbb{Z}^2$.*

For any a, b in $\mathbb{Z}^2$ which generate it as a group, we define

$$C(a,b) = \{ia + jb \in \mathbb{Z}^2 \mid i,j \geq 0\}.$$

Theorem 22 *Let (X, φ) be a free, minimal $\mathbb{Z}^2$ Cantor system. Suppose that for every pair of generators, a, b, of $\mathbb{Z}^2$ and every positive integer M, there is a cocycle θ such that θ is strictly positive on $C(a, b)$ and $\theta \leq M^{-1}$. Then R_φ is orbit equivalent to an AF-relation. (That is, R_φ is affable.)*

Corollary 23 *The triple $(D_m(X, R), D_m(X, R)^+, [1])$ is a complete invariant for orbit equivalence for the class of Cantor systems consisting of minimal AF-relations, minimal $\mathbb{Z}$-actions and free, minimal $\mathbb{Z}^2$-actions satisfying the hypotheses of Theorem 22.*

The actual theorem has a slightly weaker version of the hypothesis. In any case, the condition is a little strange and we have very little insight at this point whether or not it is reasonable. We know of no free, minimal $\mathbb{Z}^2$action which does not satisfy the condition. We know of two classes of examples which do satisfy the hypothesis which we describe now.

Let p be prime (although the result is surely true for any integer greater than 1). Let X be the p-adic integers. That is, $X = \Pi_{n \geq 0}\{0, 1, \ldots, p-1\}$. It is a group with addition done coordinate-wise modulo p and with carry over to the right. Suppose that α and β are two elements such that $i\alpha + j\beta = 0$ only if $i = j = 0$ and such that the subgroup they generate is dense. (It is not difficult to find such pairs.) We define a $\mathbb{Z}^2$-action by rotation by α and β; that is, $\varphi^{(i,j)}(x) = x + i\alpha + j\beta$, for all x in X and (i, j) in $\mathbb{Z}^2$. This system satisfies the hypotheses of the theorem.

For the second example, we let S^1 be the circle, which we write as $\mathbb{R}/\mathbb{Z}$. Suppose that α, β are real numbers such that $1, \alpha, \beta$ are linearly independent over the rationals. We begin with the $\mathbb{Z}^2$-action on S^1 obtained by rotating by α and β - see the formula in the last example. It is possible to 'cut' the circle along an orbit of this action (or even countably many orbits). Take a single point and replace it by two points separated by a gap. Repeat this process for each point in its orbit under the $\mathbb{Z}^2$-action, using smaller and smaller gaps. The result is a Cantor set which we denote by X. The action extends to X in an obvious way and this is a free minimal $\mathbb{Z}^2$-system. It also satisfies the hypotheses of the theorem.

We conclude with a few general remarks about the hypothesis of the theorem. Let (X, R) be a minimal Cantor étale equivalence relation. If f is in $C(X, \mathbb{Z})$, then $bf(x, y) = f(y) - f(x)$ is called a *coboundary*. The set of all cocycles form an Abelian group under addition. The coboundaries form a subgroup and we let $H^1(X, R)$ be the quotient group. If we now restrict to the case of a free, minimal $\mathbb{Z}^2$-action, φ, we may find a canonical copy of the group $\mathbb{Z}^2$ in $H^1(X, R_\varphi)$. Specifically, for a in $\mathbb{Z}^2$, let $\theta_a(x, \varphi^b(x)) =< a, b >$, where $<, >$ denotes the usual inner product. If every cocycle is equal to one of these (up to coboundaries), then it is not hard to see that the hypothesis of the theorem fails. (It is impossible to make small cocycles from the θ_a.) We note that it does not imply that the conclusion fails.

It is interesting to note that in our first example, we have a short exact sequence

$$0 \to \mathbb{Z} \to H^1(X, R_\varphi) \to \mathbb{Z}[1/p] \to 0.$$

In particular, the group H^1 is rank two, but slightly larger than $\mathbb{Z}^2$. In the second example, $H^1(X, R_\varphi) \cong \mathbb{Z}^3$. So in both cases, the cohomology is slightly larger than just $\mathbb{Z}^2$, but it is sufficiently large to provide enough cocycles for application of the theorem.

List of Particpants

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Chapter 60

Field Theory and Cohomology of Groups (03rit305)

April 26–May 10, 2003

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Most of our time at the BIRS was spent investigating a conjecture which suggests that the Galois group of the maximal quadratic extension of a field has a strong regularity property.

Specifically, the conjecture states that if $1 \to R \to S \to G \to 1$ is a minimal presentation of $G = \mathrm{Gal}(F_q/F)$, where F_q is the maximal quadratic extension of F, then the lower-2-central series of S has a regular intersection with R, in the following sense.

Let $R^{(1,S)} = R$, and $R^{(n+1,S)}$ be the subgroup of $R^{(n,S)}$ generated by $[R^{(n,S)}]^2$ and $[S, R^{(n,S)}]$. Let $\langle S^{(n)} | n \geq 1\rangle$ be the lower-2-central series of S. Then we have

Conjecture 1 $R \cap S^{(n+1)} = R^{(n,S)}$.

It should be noted that this is a very special property, which is not true for just any group. For example, if G is described as the quotient $\mathbb{Z}/4\mathbb{Z}$, with $S = R = \mathbb{Z}$, then $R \cap S^{(3)} \neq R^{(2,S)}$.

This conjecture was motivated by attempts to find concrete interpretations in field theory for the Milnor conjecture [8] (now a theorem of V. Voevodsky [14]). This celebrated result gives (in the context described here) an isomorphism between the mod 2 Milnor K-theory $K_*^M(F,2)$ and the mod 2 cohomology of G, $H^*(G)$. Many consequences of this isomorphism have since been derived, for example, enormous advances were made in the (Quillen) K-theory of rings of algebraic integers. However, as of this writing, no simple field-theoretic interpretation of the result is known.

The conjecture stated above would provide such an interpretation. Indeed, for small values of n, it has already been shown that the conjecture is equivalent to the isomorphism given by the Milnor conjecture in low degrees. Specifically, the equality $R \cap S^{(3)} = R^{(2,S)}$ is equivalent [9] to surjectivity in Merkurjev's theorem [5, 6] (the Milnor conjecture for $n = 2$). Also, (assuming the equalities for $n < 3$) the equality $R \cap S^{(4)} = R^{(3,S)}$ is equivalent to injectivity in the Milnor conjecture for $n = 3$. (This result was originally proved by Merkurjev, Suslin, and Rost [7, 11, 12].) Thus, there is strong evidence that the conjecture is in fact a group-theoretic interpretation of the Milnor conjecture.

We were able to test the conjecture in small degrees for many Galois groups $G = \mathrm{Gal}(F_q/F)$ using the computer algebra system MAGMA [1]. Needless to say, no counterexamples were found.

In addition, we were able to prove the conjecture in a variety of special cases, including Demuskin groups, the maximal pro-2-quotient of the absolute Galois group of $\mathbb{Q}$, and the pro-2-completions of fundamental groups of Riemann surfaces. The proofs for these cases rely on techniques developed by John Labute [3, 4]. Of course, the fundamental group of a Riemann surface need not be a Galois group of the type we consider.

But the conjecture is a purely group-theoretic statement, and the fundamental group of a Riemann surface is sufficiently similar to the groups we consider to be an interesting case.

We also investigated the connection between our conjecture and various strengthenings of the Milnor conjecture. We considered the conjecture of Positselski and Vishik [10] and the conjectures of Carlsson [2]. Although we were unable to establish a concrete connection, these papers do suggest techniques which might be helpful in proving our conjecture.

In the short term, we plan to write a paper presenting the conjecture and giving two partial proofs: one for all our Galois groups in small degrees, and another in all degrees for some class of examples. Already these results provide some interesting consequences of Voevodsky's theorem and give strong restrictions on possible absolute Galois groups. In the long term, we plan to continue our investigation of the conjecture and its connections to other results in field theory.

It is a pleasure to thank the sponsors of the BIRS for the opportunity to advance our work in such a pleasant setting. In addition, we thank the staff of the BIRS (Andrea Lundquist, Brent Kearney, and Robert Moody), for their hospitality and unfailing good humour. We all look forward to returning to the BIRS in the near future.

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Chapter 61

Representation Theory of Linearly Compact Lie Superalgebras and the Standard Model (03rit004)

July 26–August 16, 2003

Organizer(s): Alberto De Sole (Harvard University), Victor Kac (Massachusetts Institute of Technology), Aleksey Rudakov (University of Tronheim), Minoru Wakimoto (Kyushu University)

A linearly compact Lie algebra is a topological Lie algebra whose underlying space is a topological space isomorphic to the space of formal power series over $\mathcal{C}$ in finite number of variables with formal topology. Examples include the Lie algebra of formal vector fields W_n on an n-dimensional manifold M and its closed infinite-dimensional subalgebras. Cartan's list of simple linearly compact Lie algebras consists of four series: W_n and its subalgebras of divergence 0 vector fields, Hamiltonian vector fields and contact vector fields.

In the "super" case, i.e., when M is a supermanifold, the answer is much more interesting: there are ten series and also five exceptional Lie superalgebra of vector fields, denoted by $E(1,6)$, $E(4,4)$, $E(3,6)$, $E(3,8)$ and $E(5,10)$ [1].

Here comes a possible connection to the Standard Model: it turns out that the maximal compact subalgebras of $E(3,6)$ and $E(5,10)$ are $K = su_3 \times su_2 \times u_1$ and su_5, respectively, whereas the corresponding compact Lie groups are the groups of symmetries of the Standard and the Grand Unified Model respectively. Of course, K uniquely embeds in su_5, and it turned out that this embedding extends to the embedding of $E(3,6$ in $E(5,10)$. Moreover, the "negative part" of $E(5,10)$ as a su_5 module decomposes with respect to K precisely into the multiplets of leptons and quarks as described by the Standard Model.

In [2] representation theory of $E(3,6)$ was developed, and some further observations were made on its connections to the Standard Model. In [3] some initial progress was made on representation theory of $E(5,10)$.

The program consisted of mathematics and physics parts:

I. Mathematics part.

   First we reviewed the known results on representation theory of $E(3,6)$ and $E(5,10)$ and connections between them. Next, we found new singular vectors for $E(5,10)$ as compared to [3] and made some progress in proving that there are no other singular vectors. We are hopeful that the methods we developed in BIRS will lead to a complete representation theory of $E(5,10)$. We also hope that a complete representation theory of $E(3,6)$ and $E(5,10)$ and connections between them will shed a new light both on the Standard Model and the Grand Unified Model.

II. Physics part.

We had a general review on quantum field theories and the Standard Model [4]. The topics covered in the review sessions are:

1. Lagrangian and propagator in free field theories: free boson, free fermion and free vector field.
2. Gauge invariance in QED: local $U(1)$-invariance, Ward identities, Faddeev–Popov ansatz.
3. Non Abelian gauge theories: Yang–Mills Lagrangian, Faddeev–Popov ansatz and ghost fields.
4. Spontaneous symmetry breakdown: Higgs mechanism.
5. Grand unified theories.
6. Possible interpretation of the exceptional infinite dimensional Lie superalgebras $E(3,6)$ and $E(5,10)$ as "hidden" symmetries of a quantum field theory.

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Chapter 62

Variance of Quasi-Coherent Torsion Cousin Complexes (03rit005)

August 2–16, 2003

Organizer(s): Joseph Lipman (Purdue University), Suresh Nayak (Chennai Mathematical Institute), Pramathanath Sastry (University of Toronto)

Grothendieck Duality is a subject having numerous applications in Algebraic Geometry, as well as its own intrinsic attractiveness. The basic ideas are well known, but because of the underlying complexity in the details, the situation with respect to full expositions is not yet entirely satisfactory.

The participants in this program have been working on some foundational matters in the area, with the intention of publishing a small book, now nearing completion, containing three separate papers. This volume constitutes a reworking of the main parts of Chapters VI and VII in Hartshorne's "Residues and Duality" [7], in greater generality, and by a local, rather than global, approach.

"Greater generality" signifies that we work throughout with arbitrary (quasi-coherent, torsion) Cousin complexes on (noetherian) formal schemes, not just with residual complexes on ordinary schemes. And what emerges at the end is a duality pseudofunctor on the category of composites of compactifiable maps between those formal schemes which admit dualizing complexes.[1]

"Local approach" signifies that the compatibilities between certain pseudofunctors associated to smooth maps on the one hand and to closed immersions on the other (base-change and residue isomorphisms…), compatibilities which underly the basic process of pasting together these two pseudofunctors, are treated by means of explicitly-defined—through formulae involving generalized fractions—maps between local cohomology modules over commutative rings. This way of dealing with compatibilities seems to us to have advantages over the classical one, not least of which is that the connection between local and global behaviours is made transparent, the latter being defined entirely in terms of the former. In regard to relative complexity, one might for instance compare Chapter 6 of [8], where the compatibilities we need are taken care of, with [2, Chap. 2, §7], where the compatibilities needed in the global approach of [7, Chap. VI, §2] are discussed. (To follow the global approach, one would have to redo everything for formal schemes, with the added complication introduced by the necessary presence of the derived torsion functor.)

The papers in this volume continue efforts, begun in [1], to generalize all of Grothendieck duality theory to noetherian formal schemes. Why formal schemes (aside from their just being there)? For one thing, the category F of formal schemes contains the category of ordinary schemes, that is, formal schemes whose structure sheaf has the discrete topology. Also, F contains the opposite category of the category of local homomorphisms of complete noetherian local rings. Thus the category of formal schemes offers, potentially, a framework for treating local and global duality results as aspects of a single theory.

[1] Nagata showed that every separated finite-type map of (noetherian) schemes is compactifiable; this is not known to be so for formal schemes, and seems likely to be false.

In [1], the fundamental duality and flat base change theorems are proved for *pseudo-proper* formal-scheme maps. The most notable obstruction to dealing with more general separated pseudo-finite-type maps is that we know of no theorem to the effect that such a map is *compactifiable,* that is, factors as an open immersion followed by a pseudo-proper map. Nevertheless, we can still work with those pseudo-finite separated formal-scheme maps which can be built up from pseudo-proper maps and open immersions, i.e., consider the subcategory F^0 of F having the same objects, but only those maps which are compositions of compactifiable ones. The category F^0 includes all separated finite-type maps of ordinary noetherian schemes, since, by the above-mentioned theorem of Nagata, they are compactifiable. And indeed, we are able to extend the main theorem in [7] to F^0, as follows.

A basic problem is to paste together, in a natural way, the above pseudofunctor, denoted $(-)^!$, for pseudo-proper maps and the inverse image pseudofunctor $(-)^*$ on the category of open immersions into a pseudo-functor, still denoted $(-)^!$, on all of F^0. One would like to have a natural abstract pasting procedure in the spirit of Prop. 3.3.4 in [4, p. 318], a Proposition which, as indicated before, applies to ordinary schemes, but which cannot be applied to formal schemes because we don't know that the composition of two compactifiable maps is still compactifiable.

Nayak's paper "Pasting pseudofunctors and Grothendieck duality," provides *an applicable such procedure.*

Sastry's paper "Duality for Cousin complexes" provides, in many situations (see below), a *concrete, canonical realization of the pseudofunctor* $(-)^!$.

The approach taken overlaps—and was inspired by—that in [7, Chap. 7], but it is both more concrete and more general. It begins with the canonical pseudofunctor $(-)^\sharp$ to whose construction the joint paper "Pseudofunctorial behaviour of Cousin complexes on formal schemes" of Lipman, Nayak and Sastry is devoted. Roughly speaking, $(-)^\sharp$ is defined over a suitable category $\mathbb{F}_{\mathrm{c}}$ of formal schemes X with codimension functions Δ, assigning to each object (X,Δ) the category $\mathrm{Coz}_\Delta(X)$ of quasi-coherent torsion Δ-Cousin $\mathcal{O}_X$-complexes.

Briefly, having in mind that $(-)^\sharp$ is meant to be a concrete approximation to $(-)^!$, one first describes the functor $f^\sharp$ for f a closed immersion or a smooth map, by "Cousinifying" the usual concrete realizations (extended to formal schemes) in these cases. Then, noting that every $\mathbb{F}_{\mathrm{c}}$-map factors locally as (smooth)$\circ$(closed immersion), one defines $(-)^\sharp$ for such factorizable maps by pasting. All this is done canonically, so finally it is possible to define $(-)^\sharp$ globally by gluing the local definitions. Carrying this all out involves careful attention to a great many details, a good portion of which have already been dealt with by Huang in [8], where he constructed, in essence, the restriction of $(-)^\sharp$ to Cousin complexes with vanishing differentials.

In [7, Chap. 6, §3], Hartshorne describes the construction of a pseudofunctor $(-)^\Delta$ on residual complexes over noetherian schemes (i.e., those Cousin complexes which are "pointwise dualizing"). See also [2, §3.2]. Our pseudofunctor $(-)^\sharp$ is more general, because it operates on a larger class of Cousin complexes, and over formal schemes, but each $f^\sharp$ does take residual complexes to residual complexes. It should be said, however, that the basic elements of the strategy for constructing $(-)^\sharp$, as outlined in the preceding paragraph, can all be found in [7].

Let us return to Sastry's paper. The proof of the Duality Theorem in [7, Chapter 7] begins with a trace map $f_*f^\Delta K \to K$, of graded modules, defined when $f\colon X \to Y$ is a finite-type map of noetherian schemes and K is a residual $\mathcal{O}_Y$-complex. What is called there the Residue Theorem states that when the map f is proper, trace is a map of complexes. Using local residues, Sastry defines, for every $\mathbb{F}_{\mathrm{c}}$-map $f\colon (X,\Delta_1) \to (Y,\Delta)$ and every Δ-Cousin $\mathcal{O}_Y$-complex F, a functorial trace

$$\mathrm{Tr}_f(F)\colon f_*f^\sharp F \to F\,;$$

and proves: *for pseudo-proper* f, $\mathrm{Tr}_f(F\,)$ *is a map of complexes* (Trace Theorem).

Via the basic properties of the functor $f^!$ constructed by Nayak (see above) for any composition $f\colon X \to Y$ of compactifiable maps, the Trace Theorem enables the construction of a canonical pseudo-functorial derived-category map

$$\gamma^!_f(F\,)\colon f^\sharp F \to f^! F \qquad \big(F \in \mathrm{Coz}_\Delta(Y)\big).$$

Applying the usual Cousin functor E makes this an *isomorphism* $f^\sharp F \cong E(f^!F)$. Moreover, $\gamma^!_f$ itself is an isomorphism whenever f is flat or F is an injective complex. One finds then, with Q the canonical functor from the category of complexes to the derived category, that if one restricts to flat maps and Cohen-Macaulay complexes (the derived-category complexes isomorphic to $Q(C)$ for some Cousin complex C), or to Gorenstein complexes (the derived-category complexes isomorphic to $Q(C)$ for some *injective* Cousin complex C), then, $Qf^\sharp E$ *is a pseudofunctor satisfying the expected conditions for a duality pseudofunctor.* Using $\gamma^!_f$, Sastry also proves a *canonical Duality Theorem* for pseudo-proper maps $f\colon (X,\Delta') \to (Y,\Delta)$ and Δ-Cousin $\mathcal{O}_Y$-complexes F: *the pair* $\big(f^\sharp F, \mathrm{Tr}_f(F\,)\big)$ *represents the functor* $\mathrm{Hom}_Y(f_*C,\ F\,)$ *of* Δ'*-Cousin* $\mathcal{O}_X$*-complexes* C.

In summary, $f^\sharp$ *is a canonical concrete approximation to the duality functor* $f^!$.

Finally, the canonicity of $\gamma^!_f$ and uniqueness properties of residual complexes enable one to draw closer to the holy grail of defining canonically a duality pseudofunctor $(-)^!$ for all pseudo-finite-type maps $f\colon X \to Y$, at least in the presence of bounded residual complexes (or equivalently, dualizing complexes), and under suitable coherence hypotheses. The idea, taken from [7], is to define $f^!$ as being dualization on Y with respect to a fixed residual complex $\mathcal{R}_Y$ (i.e., application of the functor $\mathcal{H}om^\bullet_Y(-,\mathcal{R})$), followed by $\mathbf{L}f^*$, followed by dualization on X with respect to the residual complex $f^\sharp\mathcal{R}$.

* * *

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Chapter 63

Invariant Manifolds for Stochastic Partial Differential Equations (03rit003)

August 16–30, 2003

Organizer(s): Tom´as Caraballo (Universidad de Sevilla), Jinqiao Duan (Illinois Institute of Technology), Kening Lu (Michigan State University), Bjorn Schmalfuss (University of Applied Sciences)

Randomness or uncertainty is ubiquitous in scientific and engineering systems. Stochastic effects are not just introduced to compensate for defects in deterministic models, but are often rather intrinsic phenomena. Taking stochastic effects into account is of central importance for the development of mathematical models of many phenomena in physics, mechanics, biology, economics and other disciplines. Macroscopic models in the form of partial differential equations for these systems contain such randomness as stochastic forcing, uncertain parameters, random sources or inputs, and random initial and boundary conditions. Stochastic partial differential equations are appropriate models for randomly influenced systems.

Although many useful techniques exist to investigate deterministic partial differential equations as nonlinear dynamical systems, fundamental issues about studying stochastic partial differential equations as random dynamical systems remain unsolved.

In order to investigate stochastic partial differential equations from a dynamical systems point of view, we need to establish a theory for invariant manifolds for stochastic partial differential equations. As in deterministic systems, we expect that invariant manifolds, especially stable and unstable manifolds, to be essential for describing and understanding dynamical behavior of nonlinear random systems.

Recently, Duan, Lu and Schmalfuss [4, 5] have proved the existence of stable and unstable invariant manifolds at deterministic stationary points for a special class of stochastic partial differential equations with a multiplicative or additive white noise. The approaches are based on a random graph transform with a random fixed point theorem and Lyapunov Perron's method. On the other hand, Caraballo, Langa and Robinson [3] proved the existence of a local unstable manifold at the origin for a stochastic Chafee-Infante reaction-diffusion equation by truncating the equation in a suitable way, and proving the existence of inertial manifolds for the truncated equation. To establish the general theory of invariant manifolds for general stochastic partial differential equations, new ideas are needed to be developed.

As a research team at Banff, we have investigated the existence of invariant manifolds at a stationary process for general stochastic partial differential equations under the framework of exponential dichotomy which has a quantitative interpretation as a gap condition. The graph of this manifold is a fixed point of a certain transformation. Under the assumption that the nonlinearity is differentiable, we obtain manifolds with a sufficiently smooth graph. We have made much progress and expect to complete a paper by the end of this year. In the meantime, we have initiated two more new projects related to invariant manifolds.

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Chapter 64

Local Uniformization and Resolution of Singularities (03rit006)

August 16–September 6, 2003

Organizer(s): Steven Dale Cutkosky (University of Missouri), Franz-Viktor Kuhlmann (University of Saskatchewan)

This research in teams project was devoted to a seemingly impossible classical problem, for which there has recently been encouraging progress: local uniformization (a local form of resolution of singularities) in positive characteristic. In recent years, the participants have made independent progress on this problem, and the main purpose of this meeting was to share the insights and methods that have been developed. One of our participants, Shreeram Abhyankar has proven local uniformization up through dimension 3 in positive characteristic [1]. Although this theorem was proven in the mid 1960s, this is still the strongest general result.

During our discussion we made progress on several central problems which arise in local uniformization and valuation theory, and identified important areas of focus. Below is a short list of some of the focus problems.

1.) Kuhlmann has shown in earlier versions of the papers [9] and [10] that local uniformization is always possible after taking a finite extension, and that no extension is needed in the case of Abhyankar places (places satisfying equality in the Abhyankar inequality). The first result can also be deduced from de Jong's work on resolution after a generically finite extension [7]. However, Kuhlmann's proof uses only valuation theoretic methods, so that it is entirely different from de Jong's approach. We discussed Kuhlmann's proof, and possible removal of the necessity of taking a finite extension in the case of non-Abhyankar places. For this one would need, among other things, to find a transcendence basis of a given valued algebraic function field such that the rational function field generated by it has the same value group as the function field itself. This is closely related with the description of all valuations on rational function fields [11]. We discussed whether it can be done in an "easy" way in characteristic 0; the ideas developed in this discussion shall be worked out in a subsequent research project.

Another ingredient of local uniformization is the elimination of (tame and wild) ramification. Abhyankar's Lemma is an instance of elimination of tame ramification. A short discussion of its proof inspired a generalization to a broader ramification theoretical context [14]. A theorem proved by Epp [8] in 1972 and a theorem proved in Kuhlmann's thesis and applied in [10] are instances of elimination of wild ramification. It would now be desirable to investigate elimination of wild ramification more systematically. This should play a key role in the search of a solution to the problem of avoiding extensions of the function field.

2.) Teissier [15] has recently made outstanding progress on the problem of local uniformization by deformation to the associated graded ring of the valuation. This ring is in general not finitely generated, and requires completion, so a number of very interesting technical problems arise in commutative algebra and valuation theory. We discussed some of these problems. For example, his approach needs an infinite-dimensional form of the Implicit Function Theorem or of Hensel's Lemma. Such a theorem could possibly be proved by general ultrametric techniques, building on work done by Kuhlmann [12]. The results of this paper shall be extended to the infinite-dimensional case.

3.) We discussed and analyzed Zariski's original proof of local uniformization in characteristic zero, and its obstruction to generalization in positive characteristic. We discussed its relationship with the problem of the existence of defect in finite extensions of valued fields, the problem of local monomialization [6] in positive characteristic, and the relationship of this problem with the difficulties arising in 1.) and 2.). We discussed Kuhlmann's work [13] on the classification of Artin-Schreier-extensions with defect and what it could tell us about a crucial example given in [6]. As an improvement of the present form of the paper [13], it was suggested to work out in detail the deformation of Artin-Schreier-extensions with defect into purely inseparable extensions with defect, and to draw the connection with deformations in the sense of algebraic geometry.

4.) Given an analytically unramified local ring R, dominated by a valuation ν, find a prime ideal P in the completion $\hat{R}$ such that ν extends naturally to a valuation of the same rank as ν which dominates $\hat{R}/P$. This problem has arisen independently in the work of Cutkosky [3], [5], [4] and Teissier [15]. P can be defined naturally if ν has rank 1. In fact, P can be chosen to be nonsingular, and have other nice properties reflected in the value group if R has equicharacteristic zero (as shown in [3], [5], [4]). We identified examples showing that if the rank is larger than one then P is not uniquely determined, although it may be possible to impose conditions on P so that it is canonical.

5.) Suppose that R is an analytically unramified local ring, and ν is a valuation which dominates R. In dimension 2 the structure of the semigroup of R and the structure of a generating sequence are well understood. It would be desirable to understand completely the structure of these semigroups, and of a generating sequence in higher dimensions. The connection of these questions with the key polynomials of MacLane, as generalized by Vaquié, and with methods involving pseudo convergent sequences of Kuhlmann [11] were discussed. It appears that key polynomials can be deduced in an easy way from the approach used in [11]; this remains to be worked out in detail.

6.) Knaf has extended Kuhlmann's local uniformization for Abhyankar places to the arithmetic case, working over discrete valuation rings instead of ground fields [9]. In this result certain types of ramification have to be excluded. At BIRS, Knaf and Kuhlmann worked on the paper [10], achieving the same generalization for non-Abhyankar places, and discussed possible approaches to remove the restrictions concerning ramification.

A discussion not directly related to our project produced another little paper [2]. The valuations of interest here are of infinite rank. While such valuations do not appear in the context of local uniformization, they do appear in the context of the model theory of valued fields, which has turned out to be tightly connected to local uniformization through the phenomenon of the defect.

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http://math.usask.ca/fvk/Valth.html

Chapter 65

Modular invariants and NIM-reps (03rit552)

October 3–18, 2003

Organizer(s): Matthias R Gaberdiel (Eidgenössische Technische Hochschule Zürich), Terry Gannon (University of Alberta)

Most of our time during this program was directed at extending the known results on D-brane charges of WZW models in string theory. Let us begin by sketching the context.

String theory contains two sorts of strings: open strings (*i.e.* strings with two end-points), and closed strings (*i.e.* strings that have the topology of a circle). These strings propagate in some background space (such as for example Minkowski space). The end-points of open strings lie on in general multi-dimensional hyperspaces that are called D-branes. D-branes are dynamical structures in their own right, and much of their behaviour is captured by their charges. At least for certain examples, the charges $q_a \in Z$ associated to a brane labelled by a, obey an identity of the form

$$D_a\, q_b \equiv \sum N_{ab}^c\, q_c \qquad (\mathrm{mod}\, M)\,, \tag{1}$$

where D_a and M are integers, and the coefficients N_{ab}^c are the so-called fusion coefficients (or more generally, NIM-reps). Understanding D-brane charges is a natural and fundamental problem in string theory.

One of the best-understood string theories are the Wess-Zumino-Witten (WZW) models, for which the background space is a Lie group manifold. The algebraic structure governing the WZW models are affine Kac-Moody algebras, and so these models typically involve very pretty mathematics. Our working hypothesis has been that any natural question asked of a WZW model, should have a Lie theoretic answer. For example, the possible D-branes preserving the full affine algebra symmetry are parametrised by the integrable highest weights of a given level.

What has been worked out already [1, 2, 3] are the charges of D-branes living on a simply connected Lie group manifold $\widetilde{G}$, and preserving the full affine symmetry. They found that the charges q_λ equalled the Weyl dimension of the associated finite-dimensional representation $\overline{\lambda}$, and that M was a certain factor of the level k plus the dual Coxeter number $h^\vee$.

One of the two main questions we addressed, was to extend this to non-simply connected Lie groups G. Each such G corresponds to some subgroup Z_0 of the centre Z of the simply connected cover $\widehat{G}$. This particular question has not been seriously addressed in the literature, partly because it is difficult to find good expressions for the corresponding NIM-rep coefficients N_{ab}^c. We obtained the charges q_a whenever the order of the subgroup Z_0 is a prime. We also showed with concrete examples that the situation for composite $|Z_0|$ is similar though technically more difficult. For primes $p \neq 2$, the answer is reminiscent of that for simply connected groups; however when $|Z_0| = 2$ (*e.g.* $G = SO(3)$) there is an obstruction which unexpectedly trivialises the possible charges. We are currently in the process of typing up this paper [4] and expect to submit it to the preprint server hep-th and the journal JHEP in the very near future.

There are also D-branes which preserve the affine algebra only up to some twist, *i.e.* up to a symmetry of the corresponding Dynkin diagrams. We computed the relevant NIM-rep coefficients in a previous paper [5]. At BIRS [6] we worked out the associated charges, and found them to be dimensions of representations of the corresponding invariant algebra. We also found that the twisted M equalled the untwisted M of [1]. The twisted M had been previously calculated by Braun [7] using completely independent means (K-theory) but the charges were not known.

Fitting this into a broader perspective, the question of D-brane charges can be asked for every possible toroidal partition function. We have a good understanding now of these partition functions for the WZW models: apart from certain small exceptional levels, they correspond to a choice of (not necessarily simply connected) Lie group, and a choice of Dynkin diagram automorphism. The naturality of our new papers [4] and [6] becomes obvious from that perspective. The only generic WZW D-brane charge question which has yet to be addressed, then, corresponds to the twists of the models on non-simply connected groups.

Our two weeks at BIRS were two of our most productive ever. We found the environment beautifully conducive to research. We also found Andrea Lundquist very helpful. We certainly look forward to our next visit!

List of Participants

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[3] P. Bouwknegt, P. Dawson and A. Ridout, D-branes on group manifolds and fusion rings, *JHEP* **0212** (2002), 065; `hep-th/0210302`.

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[5] M.R. Gaberdiel and T. Gannon, Boundary states for WZW models, *Nucl. Phys.* **B639** (2002), 471; `hep-th/0202067`.

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[7] V. Braun, Twisted K-theory of Lie groups, `hep-th/0305178`.

Summer School Reports

Chapter 66

PIMS-IMA Graduate Industrial Mathematics Modelling Camp (GIMMC)

May 17–22, 2003

Organizer(s): Chris Bose (University of Victoria), Ian Frigaard (University of British Columbia), Huaxiong Huang (York University), Rachel Kuske (University of British Columbia), Jack Macki (University of Alberta), Fadil Santosa (Institute for Mathematics and its Applications)

Overview

The 6th Annual PIMS-IMA Graduate Mathematics Modelling Camp (GIMMC) took place at BIRS this year. Thirty-four graduate students from North America participated in the programme. It was cosponsored by the Institute for Mathematics and its Applications (IMA).

GIMMC is designed to give graduate students in the Mathematical Sciences an opportunity to learn techniques of mathematical modelling under the supervision and guidance of experts in the field.

GIMMC is the first leg of the PIMS-IMA Industrial Mathematics Forum which also includes the PIMS-IMA Industrial Problem Solving Workshop (IPSW).

In a first session, the mentors presented the problems, and for the remainder of the week, they guided a group of graduate students through to a resolution, this culminated in a group presentation and a written document at the end of the week.

Industrial Mentors and Problems

Emily Stone (Utah State University): *Modelling PCR Devices for Fun and Profit*
The group developed models based on differential equations for Polymerase Chain Reaction, used to amplify sequences of DNA. The goal was to identify the initial concentration of DNA, using only observations of the process over time.

Richard Braun (University of Delaware): *Thin Fluid Film Drainage Mathematical Models of a Boundary of a Thin Fluid*
A mathematical model for the evolution of a thin film boundary was developed and studied analytically and numerically to get a description of the thin film boundary decay rate. This quantity is important in quality control for the production of surfactants.

Sonja Glavaski (Honeywell): *Stability of Hybrid Systems using Sum of Squares (SoS) Programming Approach: VCCR System Example*
Hybrid systems are human controlled dynamic processes, such as air-conditioning systems and car transmissions, which have different states. Stability and control of these systems was studied with Lyapunov functions and Sum of Squares methods.

David Misemer (3M): *Modelling Polymer Purification by Counter-current Extraction*
The purification process in the production of adhesives is necessary for avoiding side-effects, as in medical patches, or malfunctions, as in electrical devices. The group built two models of purification via counter-current exchange, analysed both models and compared their results with experimental data.

Fadil Santosa (IMA & University of Minnesota): *Solar Car Racing Strategy*
The team developed models for power consumption in a car powered by solar energy. Using optimal control methods, they developed optimal racing strategies for a variety of weather, road, and racing conditions.

Robert Piché (Tampere University of Technology, Finland): *Converting Machine Tool Measurements into a CAD Model*
Manufacturers of machine tools often rebuild and modify existing machine tools, incorporating new technology to meet customer requirements at significantly lower cost. A geometrical algorithm based on level set methods was developed to give a mathematical description of the measurements, which can then be input into a CAD package.

For more information please see **www.pims.math.ca/industrial/2003/gimmc/**.

List of Participants

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Chapter 67

Preparatory Workshop for the 2003 AMS/MSRI von Neumann Symposium (03msri257)

June 22–26, 2003

Organizer(s): Robert Bryant (Duke University)

This small workshop for about fifteen participants is designed to allow a more leisurely introduction to the background material of the von Neumann symposium for the benefit of graduate students and postdoctoral mathematicians who are interested in attending the symposium (at MSRI in Berkeley, August 11–20, 2003).

Participants should have some familiarity with the basics of calculus on manifolds (differential forms, Stokes' Theorem, de Rham cohomology) and have taken at least an introduction to Riemannian geometry. Topics to be covered include an introduction to minimal submanifolds and calibrations, holonomy, examples, fiber bundles and connections, and symplectic geometry. In addition, reading material will be recommended for further preparation for the Von Neumann Symposium.

Local lodging and possibly some additional funding will be available to those selected to participate. Applications will be considered on a space-available basis.

Chapter 68

IMO TRAINING CAMP (03ss001)

June 24–July 10, 2003

Organizer(s): Bill Sands (University of Calgary)

The 2003 IMO Training Camp started on Tuesday June 24 with the arrival in Calgary of the six student Team members and two of the three adult Team members. Also arriving were three Edmonton students chosen, along with three students from the Calgary area, to participate in the Calgary portion of the Camp.

The participants were:

Team members: Robert Barrington Leigh, Olena Bormashenko, David Han, Oleg Ivrii, János Kramár, Jacob Tsimerman;

Adult trainers: Andy Liu (Leader), Richard Hoshino (Deputy Leader), Robert Morewood (Leader Observer), Elena Braverman;

"Local" (Alberta) students: Radoslav Marinov, David Rhee, and Brian Yu from Edmonton; Sarah Sun from Okotoks; and Dennis Cheung and Peter Zhang from Calgary.

Everyone was housed in a dormitory-style Residence on campus, two students to a room, with each Team member paired up with a "local" student. All participants were issued Meal Cards which could be used at various food outlets on campus.

Training began in the morning of June 25, with lectures, problem sets, and not-too-serious contests. Two other Calgary students, Boris Braverman and Hongyi Li, were invited to take part in this training during the day.

Graham Wright arrived in Calgary in the morning of Thursday June 26, and that afternoon he and I made a trip to a nearby shopping mall to scout out the pants that needed to be purchased as part of the Team uniforms. Later we took the students back to the mall to get fitted.

Later that evening, Richard Hoshino arrived, having been delayed by his involvement in the National Camp. In his absence, University of Calgary faculty member Elena Braverman had helped out with the training.

On Friday June 27 was the "Media Day". The next morning, all Team members and local students checked out of Residence, and were taken on an excursion to the Calgary zoo. This was the last event of the Camp for the "local" students, and in the afternoon the Team members were driven to the BIRS facility in Banff for the remainder of the Training Camp.

BIRS was a wonderful setting for the Camp, drawing rave comments from all participants for its accommodations, its food, and its elk! Also its mountains, with a couple of the Team members in particular making regular trips up Tunnel Mountain (including after dark on occasion, I hear), and even getting most of the way up Mount Rundle one evening, accompanied by Robert Morewood.

Because of the SARS epidemic that was in the news at the time, to be allowed into Japan our Team members all had to obtain a medical certificate stating that they were free of SARS symptoms and had not been at a SARS-infected site for two weeks before leaving Canada. This was arranged by BIRS, and on July 3 all Team members were examined by the medical staff on hand at the Banff Centre.

Besides the concentrated training that took place at BIRS, the Team was taken on two excursions: on July 3, a cold rainy day, we drove to the Columbia Icefields (where it was snowing) with a stop at Lake Louise on the way back; and on July 7, a beautiful day, we all went on a lengthy hike through Corey Pass and Edith Pass, arriving back at BIRS too late for supper in fact, which required a trip to a local pizza restaurant instead.

On July 5 Robert Morewood left the Camp to spend that night with his family in Vancouver. Andy Liu left the Camp the next morning to fly to Vancouver, and Andy and Robert met up in Vancouver Airport late that morning for the long flight to Tokyo (the site of the IMO) to help prepare the contest. They were out of contact with the rest of the Team until the competition was over. The Team stayed at Banff to continue training under the supervision of Richard Hoshino, and with the help of Elena Braverman, who had joined the Camp in Banff on June 30, and Terry Gannon of the University of Alberta, who was invited to assist with the training starting July 5 as a replacement for Robert and Andy.

The Team left for Japan on the morning of July 10.

Many thanks to:

- The staff and management at BIRS, especially **Andrea Lundquist** and **Robert Moody**, who made our stay there so memorable; also, **Gary Margrave**, the Calgary representative of PIMS, and **Robert Moody**, the BIRS head, were both very supportive of the idea that the IMO Camp should be at BIRS.
- **Elena Braverman** of the Department of Mathematics and Statistics of the University of Calgary, and **Terry Gannon** of the Department of Mathematics of the University of Alberta, who were both Trainers during the IMO Camp.
- **Betty Teare**, Budgets and Administration Manager of the Department of Mathematics and Statistics of the University of Calgary, who helped to arrange the site of the Media Launch, booked the food, and took the pictures at the Media Launch.
- **Greg Harris**, of Media Relations at the University of Calgary, for assistance in setting up and running our successful and enjoyable Media Launch in the Learning Commons on campus.
- **Sylvia Kokts-Poreitis**, who helped meet the IMO Team at Calgary airport on June 24 and drove a couple of them to the University. She also drove Robert Barrington Leigh to a nearby high school so he could write an exam during the Calgary portion of the IMO Camp. Also **Anthony Fink**, Sylvia's husband, who drove to the airport later in the evening of June 24 to pick up Robert Morewood.
- Former IMO Team member (and now University of Calgary student) **Alex Fink**, who helped out during the Calgary part of the IMO Camp.

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