WOMEN IN COMMUTATIVE ALGEBRA III

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1 Goals of the Workshop

Commutative algebra stands as a dynamic domain in mathematical research, interconnected with various disciplines. Its classical roots lie in algebraic geometry and number theory, but more recently it has forged compelling links with combinatorics, statistics, topology, and information theory.

One of the influential founders of commutative algebra is the female mathematician Emmy Noether. However, as is the case in many mathematical subfields, conscious effort is essential to ensure the visibility of women's contributions. Noteworthy progress has been made, evidenced by events such as the special session on *Women in Commutative Algebra –100 years of Idealtheorie in Ringbereichen* at a 2020 AMS meeting, special sessions in commutative algebra at the 2017, 2019, 2021 and 2023 *AWM Research Symposia*, as well as a AWM special session on *Women in Commutative Algebra* at the 2023 *Joint Mathematics Meetings*. These were well-attended sessions consisting solely of female speakers.

Rather than focusing on presentation and dissemination of results, the *Women in Commutative Algebra* (WICA) workshop series focuses on research collaboration in small groups. The algebraists in attendance, ranging from early-career mathematicians to leaders in the field, conduct research on cutting edge topics in commutative algebra. This five-day workshop *Women in Commutative Algebra III* is the third conference in the WICA series. The first two installments took place in Banff, Canada in 2019 and Trento, Italy in 2023. We were motivated to plan a third conference because of the success of the former conferences, the demand for participation which could not be met by the available space in the first two conferences, and also because we wish to connect the networks of women in mathematics in North America and Europe to their counterparts in Latin America.

The major goals of the WICA network are the following:

1. Make advances in commutative algebra through state-of-the-art research.
2. Promote the research of women in commutative algebra.
3. Develop a research network among women commutative algebraists by facilitating collaborative interaction between junior and senior mathematicians.
4. Bolster leadership among women researchers.

5. Provide mentoring for early career mathematicians.
6. Advance the academic careers of women algebraists including those from historically underrepresented groups, from universities with a teaching focus and small colleges, and those isolated geographically from potential collaborators.

2 Overview of the Field and Focus Areas

In the dynamic landscape of commutative algebra, the past few years have witnessed remarkable breakthroughs, with several major conjectures being successfully resolved. These achievements have not only provided definitive answers to long-standing questions but have also opened up new and intriguing avenues for exploration within the field. McCullough and Peeva's construction of counterexamples to the Eisenbud-Goto Conjecture, Hochster and Ananyan, followed by Erman–Sam–Snowden, settling Stillman's question on bounds for projective dimension, Walker's resolution of the total rank conjecture for modules of finite length, André's completion of the proof for the longstanding Direct Summand conjecture in mixed characteristic, and Briggs's proof of the Vasconcelos Conjecture showcase the most impressive recent advances. These breakthroughs resonate with yet-unresolved conjectures, such as the Buchsbaum-Eisenbud-Horrocks conjecture.

These transformative developments collectively render commutative algebra a richer and more nuanced research area, inspiring further exploration and pushing the boundaries of mathematical understanding. To showcase the importance of the field, two focus programs in this area are taking place: a spring 2023 semester program on Commutative Algebra at SLMath and a spring 2025 thematic program in Commutative Algebra and Applications at the Fields Institute.

In the wake of recent advancements, such as numerous long-standing conjectures being resolved in recent years, commutative algebra continues to offer many avenues for exploration. The workshop facilitated collaborative efforts in targeted areas. The topics of investigation in the workshop are described below.

- **Combinatorial commutative algebra** is concerned with the study of algebraic and combinatorial properties of (monomial) ideals associated to combinatorial objects such as graphs, posets and simplicial complexes. The aim is to understand how algebraic invariants and properties are influenced by combinatorial properties and vice versa. This branch of mathematics merges techniques from combinatorics and commutative algebra. It seeks to understand and exploit the inherent combinatorial properties of algebraic objects, leading to insights into both algebra and combinatorics. It finds applications in various areas, including coding theory, algebraic statistics, and algebraic geometry, contributing to a deeper understanding of the connections between algebraic and combinatorial structures. A group led by Jennifer Biermann and Emanuela de Negri focused on advances in combinatorial commutative algebra.
- **Arrangements of hyperplanes** Hyperplane arrangement theory is a branch of mathematics that investigates the geometric and algebraic properties arising from the arrangement of hyperplanes in affine or projective spaces. A hyperplane arrangement is a finite collection of hyperplanes. The theory explores the combinatorial structure of these arrangements, focusing on the regions they partition the space into, known as chambers, and the intersection properties of the hyperplanes. Central to hyperplane arrangement theory is the study of the associated combinatorial objects, such as the poset of regions and the characteristic polynomial of the arrangement. Applications of hyperplane arrangement theory extend to various fields, including algebraic geometry, topology, and combinatorics, making it a versatile and fundamental area of research in mathematics. The group leaders Szpond and Guardo utilizes the combinatorial structure of hyperplane arrangements to study questions related to projections of point configurations.
- **Semigroup rings** Semigroup rings are algebraic structures formed by associating a commutative ring with a semigroup; specifically, the elements of the semigroup act as exponents for monomial generators of the ring. The study of these algebraic structures involves understanding the interplay between algebraic properties of the ring and the combinatorial structure of the semigroup. Semigroup rings provide a versatile framework for investigating questions about algebraic varieties and rings at large. For

instance, semigroup rings are central to the study of toric varieties. Scholars are actively investigating questions related to their structure, associated primes, and homological properties such as the Cohen-Macaulay property and regularity, which contribute to a deeper understanding of their algebraic and geometric behavior. Famous open problems abound in this research area, most notably Wilf's conjecture establishes an inequality that relates three fundamental invariants of a numerical semigroup: the minimal number of generators (or the embedding dimension), the Frobenius number, and the number of gaps. The group led by Kriti Goel and Hema Srinivasan is preparing a paper on this topic.

- **Toric ideals, degenerations of Grassmannians and matroid varieties** Toric ideals, rooted in the theory of algebraic toric varieties, provide a powerful framework for understanding geometric objects parametrized by lattice points in convex polytopes. They have applications in diverse fields, including coding theory and optimization. Degenerations of Grassmannians involve studying families of subspaces in vector spaces that parameterize degenerations of more general varieties. This area bridges algebraic geometry and representation theory, offering insights into the geometric structure of moduli spaces. Matroid varieties, inspired by matroid theory, capture combinatorial structures associated with linearly independent sets and extend the concept of Grassmannians to more general combinatorial objects. Research in these fields often explores connections between toric geometry, degenerations of Grassmannians, and matroid varieties, illuminating the intricate relationships between algebraic structures and combinatorial objects in the pursuit of a deeper understanding of geometric phenomena. The group lead by Fatemeh Mohammadi and Laura Bossinger focuses on topics relevant to this circle of ideas.

- **Reduction ideals of determinantal ideals** Reductions of determinantal varieties form an area of study at the intersection of algebraic geometry and commutative algebra. Investigating families of varieties defined by the vanishing of minors of a matrix is a fundamental concept in linear algebra. The coordinate ring of the blow-up of the affine space along a determinantal subvariety is an example of a Rees algebra. Reduction ideals aid in the study of these Rees algebras. Besides its connections to resolution of singularities, the study of Rees algebras plays an important role in many other active areas of research including multiplicity theory, equisingularity theory, asymptotic properties of ideals, and integral dependence. This is a current area of interest within commutative algebra and presents many unresolved challenges. In particular, the Rees algebras of determinantal ideals are an object of active study. Understanding reductions of determinantal varieties not only contributes to the theoretical foundation of algebraic geometry but also has practical implications in fields such as geometric modeling and computer-aided design. A group led by Mostafazardeh and Lisa Seccia studied determinantal ideals of Hankel matrices.

- **Invariant theory** Invariant theory is a branch of mathematics that focuses on the study of algebraic objects and their properties under transformations. Central to this field is the investigation of invariants, which are quantities that remain unchanged under the action of a particular group of transformations. Historically, invariant theory emerged in the 19th century through the pioneering work of mathematicians like David Hilbert and Emmy Noether. The primary goal is to identify and understand algebraic structures and geometric objects that remain invariant when subjected to symmetries or transformations, often described by groups such as permutation groups or linear algebraic groups. Invariant theory has applications across various mathematical disciplines, including algebraic geometry and representation theory, and finds practical use in physics and computer science. Modern developments in invariant theory involve sophisticated algebraic and geometric techniques, contributing to a deeper understanding of symmetry-related phenomena and providing tools for solving problems in diverse areas of mathematics and its applications. In our workshop, a group led by Emilie Dufresne and Nelly Villamizar studies invariant theory with applications to graph-theoretic problems.

3 Conference Highlights

There were 29 participants in the meeting coming from each of the following countries: Argentina, Belgium, Brazil, India, Italy, Kazakhstan, Germany, Mexico, Poland, Spain, Switzerland, United Kingdom, United

States, Turkey. Geographic diversity was one of the main goals and highlights of the conference as it allowed mathematicians from different part of the world to meet, who would have not otherwise had a chance to do so. In particular, a goal of the workshop was to connect researchers from Latin America to those from North America, Europe, and beyond.

Building on the model of the previous conferences in this series, we divided the WICA III participants into six working groups. Each group was directed by leaders in the field, either senior or showing exceptional promise, who were given considerable freedom to direct their research group. The rest of the group members were be selected based on applications and the recommendation of the team leaders. The organizers worked with the leaders in selecting group members with the appropriate training, expertise, and interest for each project. In advance, we asked the leaders to put forth a specific problem (or circle of problems) from their area of interest into a short project description. These project descriptions were distributed to the team members several months prior to the workshop in order to aid them in acquiring the necessary background to hit the ground running.

Much of the time of the workshop was be dedicated to actively working on the proposed problems, with some time allocated to progress reports and to short presentations of problems of interest to participants which may foster interactions across working groups. In particular, during the first day of the workshop there were project introductions given by each of the group leaders. The fourth day of the workshop ended with informal progress reports. The progress reports are expanded upon in section 4 of this document. We found this to be a very efficient model during the previous two WICA workshop leading to high productivity for the working groups. Informal discussion groups were formed around various various aspects of the profession such as choice of publishing venues, grant writing, and creating a network of collaborators.

3.1 Poster presentations

There was also a poster session for the most junior participants to showcase the research they have done outside of the workshop. The following posters were presented:

- Aslı Musapaşaoğlu (Sabanci University, Turkey) – ”The edge ideals of t-spread d-partite hypergraphs”
- Sudeshna Roy (Tata Institute of Fundamental Research, India) – “Computing epsilon multiplicities in graded algebras”
- Kumari Saloni (Indian Institute of Technology Patna, India) – ”Ratliff-Rush filtration, Hilbert coefficients and the reduction number of integrally closed ideals”
- Dayane Santos de Lira (Universidade Federal Rural do Semi-Árido, Brasil) – ”Gorenstein ideals and Macaulay inverse system”.

4 Scientific Progress Made

4.1 Working group on Combinatorial Commutative Algebra

The members of the group were:

- **Jennifer Biermann** – Hobart and William Smith Colleges, USA (leader),
- **Emanuela De Negri** – Università di Genova, Italy, (leader),
- **Oleksandra Gasanova** – Universität Duisburg–Essen, Germany,
- **Aslı Musapaşaoğlu** – Sabanci University, Turkey,
- **Sudeshna Roy** – Tata Institute of Fundamental Research, India.

The object of their investigation is the class of double determinantal ideals, introduced by Li. Fix m, n, r to be integers bigger than 1, and let $X_q = (x_{ij}^q)$, with $q = 1, ..., r$, be $m \times n$ matrices of distinct indeterminates.

Let K be a field and let $R = K[x_{ij}^q \,|\, 1 \leq i \leq m,\ 1 \leq j \leq n, 1 \leq q \leq r]$. Consider

$$H = (X_1, \cdots, X_r) \text{ and } V = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}$$

the horizontal and the vertical concatenation of the matrices $X_1, ..., X_r$, and let $I = I_t(H) + I_s(V)$ be the ideal generated by the t-minors of H and by the s-minors of V. The ideal I is a **double determinantal ideal**.

The starting point of their work is the following result of Fieldsteel and Klein:

Theorem 4.1 ([**?**]). *: The minors generating I are a Gröbner basis with respect to any diagonal order.*

In the case when $s = t = 2$ the initial ideal $\mathrm{in}(I)$ is generated by the 2-diagonals of the matrices H and V, in particular it is square-free. Let $\Delta^r_{m,n}$ be the simplicial complex associated to the double determinantal ideal

$$I^r_{m,n} = I_2(H) + I_2(V)$$

generated by the 2-minors of the horizontal and the vertical concatenation of the $m \times n$ matrices $X_1, ..., X_r$.

In [**?**], the authors study the case when $r = 2$. The facets of $\Delta^2_{m,n}$ are described, and formulas for the multiplicity and the regularity are given. Moreover Gorensteiness is characterized.

During the week in Oaxaca the group studied the complex $\Delta^r_{m,n}$, with $r \geq 2$. As a first result two equivalent descriptions of the facets of the complex were obtained: the first one generalizing the description given in [**?**], the second one giving a very different point of view. The second description gives a very nice formula for the multiplicity of the initial ideal $\mathrm{in}(I^r_{m,n})$, which is the cardinality of the set of the facets of $\Delta^r_{m,n}$. The group were able to prove:

Theorem 4.2. *The multiplicity of $I^r_{m,n}$ is given by the trinomial coefficient*

$$\binom{m+n+r-3}{m-1, r-1, n-1}.$$

In their continuing work the group members hope to prove the following conjecture:

Conjecture 4.3. *Assume $m \leq n$. Then $\mathrm{reg}(R/I^r_{m,n}) = m + \min\{n, r\} - 2$ and $R/I^r_{m,n}$ is Gorenstein if and only if $m = n = r$.*

4.2 Working group on Semigroup Rings

The group consisted of :

- **Saipriya Dubey** – Indian Institute of Technology Dharwad, India,
- **Kriti Goel** – Università of Bilbao, Spain, (leader),
- **Nil Sahin** – Bilkent University, Turkey,
- **Srishti Singh** – University of Missouri, USA,
- **Hema Srinivasan** – University of Missouri, USA (leader).

A few weeks before the meeting, the leaders shared with everyone in the group a file with problems and some reading materials. The group then discussed several questions on numerical semigroups and some recent works.

Semigroup rings are toric rings of the form $k[x_1, \ldots, x_m]/I_A$ where I_A is the binomial prime ideal associated to an $n \times m$ matrix $A = \{a_{ij}\}$ whose columns minimally generate a subsemigroup of $\mathbb{N}^n$. Without loss of generality, we may assume that the gcd of $(a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m) = 1$. We denote the semigroup by $\mathcal{A}$ and the semigroup ring by $k[A]$ where k is a field. The embedding dimension of $k[A]$ is e, and the dimension of $k[A]$ is the rank of A.

When $n = 1$, $\mathcal{A} = \langle a_1, \ldots, a_m \rangle$ is called a **numerical semigroup**, and the corresponding semigroup ring is called the numerical semigroup ring of embedding dimension m. Thus, numerical semigroup rings are one-dimensional Cohen-Macaulay domains and are also the coordinate ring of the affine monomial curve in A^m parametrized by $t^{a_1}, \ldots t^{a_m}$. $\mu(I_A)$ denotes the minimal number of generators of an ideal I_A which in this case is also the number of equations defining the associated monomial curve.

In this project, the group considered a few problems on semigroup rings.

Problem 4.4. *Let $\mu(n) = \max\{\mu(I_A) \mid edim(A) = n\}$, where A is a symmetric numerical semigroup minimally generated by n elements. Is there an upper bound for $\mu(n)$ for a fixed n?*

This is a hard problem and has been open for a long time. There are bounds for $\mu(a_1, n)$ where a_1 is the smallest positive integer in the semigroup A.

The group soon converged on a discussion of some numerical semigroups which Judith Sally had introduced on a different context. They enlarged that class to a class of semigroups which they call Monomial Curves of Sally Type. For any monomial curve of multiplicity e, the embedding dimension is $\leq e$. The semigroups with embedding dimension equaling the multiplicity have been thoroughly studied and described by Kunz. We consider the case where the embedding dimension is one less. To be precise, we a numerical semigroup belongs to S_e multiplicity e and embedding dimension and width are both $e - 1$. These have not been studied before and the group undertook to classify these monomial curves by the number of equations needed to define them as well as other invariants such as Betti Numbers.

The researchers are also considering further generalizations to monomial curves of width $e - 1$ but lower embedding dimension or those with embedding dimension equalling width in general with a fixed multiplicity. In addition they have a longer term project to continue to collaborate over Zoom and Overleaf and write the free resolutions of these monomial curves explicitly.

4.3 Working group on Fermat-type Configurations of Points and their Projections

The research group in Oaxaca consisted of:

- **Elena Guardo** – Università di Catania, Italy (leader),
- **Giovanna Ilardi** – Naples University, Italy,
- **Zhibek Kadyrsizova** – Nazarbayev University, Kazakhstan,
- **Justyna Szpond** – University of the National Education Commission, Poland (leader).

The group studied Fermat-type configurations of points and their projections.

A **Fermat-type configuration of points**, denoted $\mathcal{F}_N^n$ is the finite set of points of locally (in the complex topology) maximal multiplicity in the arrangement defined by linear factors of the polynomial

$$\prod_{0 \leq i < j \leq N} (x_i^n - x_j^n)$$

for some positive integers n, N.

The group investigated initial examples of such configurations in $\mathbb{P}^3$, so $N = 3$ and in particular they studied their projections to $\mathbb{P}^2$. This is motivated by recent developments of the theory of **geproci sets**, see [**?**]. They were particularly interested in the geometry of the projected sets of points and the Weddle loci associated to the original sets. These are, roughly speaking, loci of points such that projecting from them one obtains an image enjoying additional, sometimes unexpected algebraic and/or geometric properties.

Along these lines the group obtained the following two main results

Theorem 4.5. *Let Z be subset of eight points with coordinates $(\pm 1 : \pm 1 : \pm 1 : 1)$ in $\mathbb{P}^3$ and let $P = (a_0 : a_1 : a_2 : a_3)$ be a general point in $\mathbb{P}^3$. Then the image of Z under the projection π_P from P imposes independent conditions on cubics in $\mathbb{P}^2$. The ninth intersection point of the pencil of cubics passing through $\pi_P(Z)$ has coordinates expressed by degree 10 homogeneous polynomials in the variables a_0, a_1, a_2, a_3. The zero set of these polynomials is a union of 18 lines in $\mathbb{P}^3$, which intersect exactly in points of $\mathcal{F}_3^2$.*

Theorem 4.6. *The $2-$Weddle locus of Z is the union of six lines, the edges of the coordinate tetrahedron in $\mathbb{P}^3$. The $3-$Weddle locus of Z is the union of the aforementioned six lines and 28 lines determined by pairs of points in Z.*

The group continues investigations of other Fermat-type configurations of points. It is expected that a manuscript resulting from these investigations will be submitted to a research journal by the end of the year.

4.4 Working group on Gröbner Degenerations of Grassmanians

This group consisted of:

- **Lara Bossinger** – Mathematics Institute of the UNAM, Mexico (leader),
- **Anna Brosowsky** – University of Michigan, USA,
- **Fatemeh Mohammadi** – KU Leuven, Belgium (leader),
- **Janet Page** – North Dakota State University,
- **Alexandra Seceleanu** – University of Nebraska–Lincoln.

A *Gröbner degeneration* is a technique in algebraic geometry used to systematically deform a given ideal or variety into more manageable ideals, such as monomial or toric ideals. This process produces a parametric family of ideals that preserves several geometric and algebraic properties of the original ideal. Gröbner degenerations are a powerful tool for analyzing the invariants of ideals and varieties under perturbations or deformations.

In this project, the researchers focus on Gröbner degenerations of determinantal ideals. Specifically, they compute the monomial Gröbner degenerations of these ideals and seek an ordering of their monomials such that the resulting ordering yields linear quotients. The goal is to demonstrate that these ideals possess cellular monomial free resolutions. They examine a family of weight vectors derived from matching fields and show that the corresponding monomials exhibit the desired properties. Additionally, they aim to explicitly describe their minimal free resolution supported on a CW-complex, or equivalently, their cellular resolution.

The group's objective is to identify a family of weight vectors and monomial initial ideals such that all their powers exhibit the aforementioned properties. Recent work by Mohammadi and Clarke [**?**] has established these results for a specific family of weight vectors in the case of the Grassmanian Gr$(3, n)$. The group plans to generalize these results to other weight vectors and extend their results to arbitrary Grassmanians Gr(k, n).

The weight vectors they consider arise from permutations of the set $\{1, \ldots, n\}$. The following theorem identifies a family of such weight vectors that lead to toric degenerations of Gr(k, n). It also shows that the associated polytopes are related to Gelfand-Tsetlin polytopes by mutations.

Theorem 4.7 (Clarke-Mohammadi-Zaffalon 2024 [**?**])**.** *Fix $n \in \mathbb{Z}$ with $n > 0$ and a subset $K \subseteq [n]$. If $\sigma \in S_n$ is a permutation that avoids the patterns* 4123*,* 3124*,* 1423*, and* 1324*. Then the corresponding weight vector gives rise to a toric degeneration of Gr$(K; n)$. In particular, the associated polytopes of these toric varieties are all combinatorially mutation-equivalent to the Gelfand-Tsetlin polytope.*

A second goal is to show that, for these permutations, the corresponding algebra satisfies specific properties. Let $X = (x_{ij})$ be a $k \times n$ matrix of variables. Consider the polynomial ring $R = K[x_{ij}]$ and $M_{\mathbf{a}} = \langle m_1, \ldots, m_k \rangle$ be the associated monomial ideal (obtained as initial ideal of determinantal ideal w.r.t. the corresponding weight vector) . Define the polynomial ring $S = K[u_1, \ldots, u_k]$, along with the following two maps:

$$\varphi : S \to R(M) \quad \text{with} \quad \varphi(x_{ij}) = x_{ij} \quad \text{and} \quad \varphi(u_j) = m_j t$$

and

$$\psi : K[u_1, \ldots, u_k] \to K[m_1, \ldots, m_k] \quad \text{with} \quad \psi(u_j) = m_j.$$

The researchers will first show that the ideal $\ker(\varphi)$ is of fiber type. In other words, the degrees of the generators of $\ker(\varphi)$ are either $(*, 1)$ or $(0, *)$.

4.4.1 Rees ring

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring and $M = \langle m_1, \dots, m_k \rangle \subset R$ be a monomial ideal whose generators are all of same degree. Then the Rees ring

$$R(M) = \oplus_{\ell \geq 0} M^\ell t^\ell = R[m_1 t, \dots, m_k t] \subset R[t]$$

is naturally bigraded with $\deg(x_i) = (1, 0)$ and $\deg(m_j t) = (0, 1)$ for all $i = 1, \dots, n$ and $j = 1, \dots, k$. Now, we define a polynomial ring $S = K[u_1, \dots, u_k]$ whose variables are corresponding to the generators of M. Similarly, we define a bigrading on S such that $\deg(x_i) = (1, 0)$ and $\deg(u_j) = (0, 1)$ for all i, j.

Note that the map φ is a natural surjective homomorphism of bigraded algebras, and the map ψ has a kernel which defines the toric ideal corresponding to M. The following conjecture will be studied.

Conjecture 4.8. *The toric ideal* $\ker(\psi)$ *is quadratically generated.*

The ultimate goal is to compute the Rees algebra of these ideals, describe their generators, and demonstrate that they are all quadratically generated.

During the meeting in Oaxaca the group members formulated the conjecture and computed many examples. Since then they have been holding regular meetings and continue working online.

4.5 Working group on Reduction Ideals of Determinantal Ideals

The members of the group are

- **Katie Ansaldi** – Wabash College, USA,
- **Maral Mostafazardeh** – Federal University of Rio de Janeiro, Brazil (leader),
- **Dayane Santos de Lira** – Universidade Federal Rural do Semi-Árido, Brasil,
- **Lisa Seccia** – University of Neuchâtel (leader),
- **Yevgeniya "Jonah" Tarasova** – University of Michigan, USA.

Let H be a generic square **Hankel matrix** of size n. By a coordinate section degeneration of H we mean

$$H[1] = \begin{pmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_2 & x_3 & \dots & x_n & x_{n+1} \\ \vdots & \vdots & \dots & \vdots & \\ x_{n-1} & x_n & \dots & x_{2n-3} & x_{2n-2} \\ x_n & x_{n+1} & \dots & x_{2n-2} & 0 \end{pmatrix}, \dots, H[n-2] = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \dots & x_n & x_{n+1} \\ x_3 & x_4 & x_5 & \dots & x_{n+1} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ x_{n-1} & x_n & x_{n+1} & \dots & 0 & 0 \\ x_n & x_{n+1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

Note that in each such r-degeneration $H[r]$, the base ring is the polynomial ring $R_r := k[x_1, \dots, x_{2n-r-1}]$. The goal of the group's work is as follows.

Problem 4.9. *Study the special fiber and the reduction ideals of the ideal* $I := I_{n-1}(H[r])$.

When $r = 0$, the variety defined by the ideal of sub-maximal minors of $H[0]$ is well-understood. In particular, it is known that a minimal reduction of $I_{n-1}(H[0])$ is given by the gradient ideal of the determinant of $H[0]$, say the polynomial f. Therefore, $\langle \nabla \det H[0] \rangle$ is also a reduction of $I := I_{n-1}(H[r])$ in the smaller ring R_r. However it is not minimal, since the number of generators is greater than the dimension of the ring (= the analytic spread). As a first step, the group focused on the two extremal cases: the less degenerate case $r = 1$, and the most degenerate case $r = n - 2$.

The in-person discussions facilitated rapid progress in our project. The researchers have already achieved some results in specific cases and developed conjectures, supported by computer experiments, that will be the object of further investigation in the upcoming months. A a short summary of their achievements follows:

- **Case** $r = 1$**:** The special fiber $\mathcal{F}_R(I)$ is Gorenstein.
- **Case** $r = n - 2$**:** The equations of $\mathcal{F}_R(I)$ are given by Plucker relations + one "Laplacian"equation (inspired by Ramkumar-Sammartano).

  Moreover, they conjecture that:

  Conjecture 4.10. *Let* $\nabla(\det H[0]) = (f_1, \cdots, f_{2n-1})$ *where* f_i *is the partial derivative of* f *with respect to* x_i *in* R_r *. A minimal reduction of* $I_{n-1}(H[r])$ *is*

  $$J = (f_1 + f_2, f_3 + f_4, \cdots, f_{2n-5} + f_{2n-4}, f_{2n-3}, f_{2n-2}, f_{2n-1}).$$

  Note that these two extremal cases coincide when $n = 3$. Therefore, using both the results above, it has been proven that
- **Case** $n = 3$**:** The special fiber is F-rational, equivalently strongly F-regular.

The above partial results will be the group's starting point to address more general questions:

1. Find the equations of the $\mathcal{F}_R(I)$ for $r = 1$.
2. Is the special fiber always F-rational?
3. Find a minimal reduction whose generators have a nice expression in terms of the gradient.
4. When is $\mathcal{F}_R(I)$ Koszul?

Given the different time zones and schedules of the group members, the members of this group decided to split into two smaller subgroups, each focusing on a specific part of the project. Each subgroup meets online bi-weekly, and once a month, the two subgroups meet to discuss progress and exchange ideas. Additionally, they are planning to apply for funding through programs like "Research in Pairs" to facilitate another in-person meeting in the near future to continue their work on the project.

4.6 Working group on Invariant Theory

The members of the group are

- **Emilie Dufresne** – University of York, United Kingdom (leader),
- **Gabriela Jerónimo** – Universidad de Buenos Aires & CONICET, Argentina,
- **Jenny Kenkel** – Grinnell College, USA,
- **Haydee Lindo** – Harvey Mudd College, USA,
- **Nelly Villamizar** – Swansea University, United Kingdom (leader).

The **graph reconstruction conjecture** is a long-standing problem in Graph Theory. We start with a simple undirected graph $G = (V(G), E(G))$, where $V(G)$ is the set of vertices $\{x_1, \ldots, x_n\}$ (indexed by $[n] := \{1, \ldots, n\}$), and $E(G)$ is the set of edges of G, so $E(G) \subseteq \{\{i, j\} \mid i, j \in [n],\ i \neq j\}$.

A *card* of G is the isomorphism class of the subgraph G_k obtained from G by removing the vertex x_k and all the edges adjacent to it, that is, $V(G_k) := V(G) \setminus \{x_k\}$ and $E(G_k) := E(G) \setminus \{\{i, j\} \mid i = k, \text{ or } j = k\}$. The *deck* of G, written $\mathcal{D}(G)$ is then the multiset of the n possible cards (there can be repetitions). The assumption is that the original label of the vertices in each card as well as the deleted vertex have been forgotten. A graph is called *reconstructible* if and only if having $\mathcal{D}(G) = \mathcal{D}(H)$ implies that G and H are isomorphic as graphs.

Conjecture 4.11 (Kelly (1957)[**?**] and Ulam (1960) [**?**])**.** *If* $n \geq 3$*, then* G *is reconstructible.*

There are all sorts of versions of this conjecture, and a variety of partial results. In order to reword the reconstruction conjecture as an invariant theory problem, we consider $\mathbb{k}$-*weighted graphs*, simple graphs where each edge is given a non-zero weight. The invariant theoretic approach is not new, for example it is described in [**?**, Chapter 5] and [**?**]. Considering $\mathbb{k}$-weighted graphs, where $\mathbb{k}$ is any field includes the case $\mathbb{k} = \mathbb{F}_2$, which naturally corresponds to simple graphs (see for example [**?**]). The set V_n of $\mathbb{k}$-weighted graphs on n vertices form a vector space over $\mathbb{k}$ with basis vectors corresponding to each potential edge between any two elements of $[n]$ (that is, each 2-element subset of $[n]$). In other words, $\mathbb{k}$-weighted graphs correspond to $\mathbb{k}$-weighted adjacency matrices. Two $\mathbb{k}$-weighted graphs are *isomorphic* if relabelling the vertices of one turns it into the other. This corresponds to a linear action of the symmetric group S_n on V_n, and the ring of polynomial invariants $\mathbb{k}[V_n]^{S_n}$ can be used to decide whether two $\mathbb{k}$-weighted graphs are isomorphic , and in fact what one really needs is a separating set (see [**?**, Chapter 2]).

The study of separating invariants was initiated by Derksen and Kemper [**?**, **?**] just over 20 years ago. It returns to the roots of Invariant Theory: using invariants to distinguish between the orbits of a group action on some geometric or algebraic space. Roughly speaking, a separating set is a subset of the ring of invariants whose elements can be used to distinguish between any two orbits that can be distinguished using invariants. A separating set need not generate the ring of invariants and separating sets can be better behaved than the ring of invariants.

During and since the workshop the researchers in this group have constructed the space of multisets of n isomorphism classes of $\mathbb{k}$-weighted graphs on $n-1$ vertices using invariant theory. The space of isomorphism classes of $\mathbb{k}$-weighted graphs on n vertices then embeds into this new space, by mapping an isomorphism class of $\mathbb{k}$-weighted graphs to the corresponding deck. This allows them to formulate the graph reconstruction conjecture in terms of invariants. This is new even for the original conjecture (over the field with 2 elements), since previous invariant theoretic approaches only provided statements that would imply the graph reconstruction conjecture. Additionally, their construction provides a theoretical solution to the legitimate deck problem (i.e., determining whether a multiset of n isomorphism classes of $\mathbb{k}$-weighted graphs on $n-1$ vertices is the deck of some $\mathbb{k}$-weighted graph). Their approach relies on the notion of separating sets, which as in [**?**] one is required to consider in more generality than described above. Using polarisation (see [**?**]), they are then able to formulate a computational approach to verifying the graph reconstruction conjecture for $\mathbb{k}$-weighted graphs. The researchers have already completed the case $n = 4$ and are working towards the case $n = 5$, with an eye to a strategy for extending to the general case.

The group has been meeting regularly online since the workshop. Theor future plans include writing a survey article on the invariant-theoretic approach to the graph reconstruction conjecture, and arranging to meet again in person next Spring.

5 Outcome of the Meeting

The first WICA conference of 2019 resulted in the proceedings volume [**?**]. We have submitted a proposal for a similar volume to Springer for their AWM proceedings series. This anticipated proceedings volume will feature papers arising from working groups participating in the workshops WICA II (Trento, 2023) and the workshop WICA III (Oaxaca, 2024). The proceedings volume will be edited by Sara Faridi (Dalhousie University, Canada), Elisa Gorla (Universite de Neuchâtel, Switzerlans), Elisa Postinghel (Università di Trento, Italy) and Alexandra Seceleanu (University of Nebraska–Lincoln, USA). The papers in this volume will be peer-reviewed. Beyond such proceedings, each working group is expected to produce and disseminate at least one (and likely several) research articles detailing the results of their work.

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