NEW TRENDS IN STOCHASTIC ANALYSIS WORKSHOP 23W5060

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The workshop was a success. Participants enjoyed the atmosphere of CMO and interacted in a productive way. We hope to be able to reorganize a similar workshop in the future.

1. Overview of the field

Stochastic analysis is an extremely wide line of research in probability, which relies at its core on some fundamental deep and basic principles. To be more specific, the term stochastic analysis encompasses both stochastic integration of Itô and Stratonovich type, as well as Malliavin calculus techniques. It is certainly one of the most active areas of probability theory, with ramifications in stochastic PDEs, Gaussian fields theory, limit theorems in probability, rough paths analysis, stochastic differential geometry on the one hand, and applications to theoretical physics, finance modeling, biophysics and machine learning on the other hand. Within this framework, our proposed workshop focused on both theoretical and applied aspects of stochastic analysis. One can single out two decisive moments in the History of this topic: the initial burst of creativity by Malliavin at the end of the 70's, and Hairer's theory of regularity structures in 2013. Building on those two pathbreaking contributions, stochastic analysis has evolved into a huge and solid theory, which has developed fertile connections with numerous fields within and outside the probability realm. Our workshop originated from the wish to give a broad overview and to gather active experts in the area, favoring interdisciplinary and intergenerational exchanges that could possibly lead to new developments in the field. The following items detail some of the topics we focused on.

- **Rough paths theory and regularity structures:** As mentioned above, the theory of regularity structures introduced by M. Hairer [3](Fields medal 2014, Breakthrough price 2021) can be seen a wide extension of the rough paths analysis, which encompasses the definition of rough paths indexed by $\mathbb{R}^n$, a richer rough paths structure indexed by trees, products of distributions, an additional group structure for renormalizations, and evaluation of singularities. This extremely deep and rich theory has paved the way to the study of numerous physical systems which were previously out of reach. Among the workshop participants, H. Shen and K. Matzeski reported about recent developments concerning relevant systems such as KPZ and other growth models, Φ^4_3-equations and Yang-Mills measures.
- **Linear or non-linear stochastic partial differential equations:** A number of significant recent contributions have also allowed a better understanding of more classical stochastic PDEs. In the workshop, the expert C. Mueller discussed most recent progress in ergodic properties and the definition of stochastic PDEs with power type coefficients. R. Balan gave a talk on her recent result of Gaussian fluctuation for the solution to stochastic wave equation with time-independent rough noise. Among the most exciting and challenging systems in this area is the so-called parabolic Anderson model. This model can be described by a simple linear PDE which can be written formally as follows:
$$\frac{\partial u}{\partial t} = \Delta u + u\,\dot{W}\,, \quad t \geq 0, \quad x \in \mathbb{R}^d\,, \tag{1}$$
where $\dot{W}$ is a noisy term, usually described as a Gaussian field. In spite of its simple form, equation (1) exhibits all sorts of non classical behaviors in terms of moment estimates, growth rate in time and space, or energy landscape. It is also related to polymer measures, whose behavior in a rough environment is the subject of recent very deep studies. X. Chen, J. Wang and D. Khoshnevisan devoted their talks to this challenging topic.
- **Stochastic processes and random fields on geometric structures :** Stochastic analysis is also tightly related to the theory of Dirichlet forms, which in turn provides a convenient and useful tool to construct diffusion processes and random fields in geometric structures like graphs, fractals, or Gromov-Hausdorff limits of Riemannian manifolds. While diffusion processes in such spaces have extensively been studied in the last 20 years, the investigation of random fields (fractional noises) in this setting is still fairly new and it opens the door to the study of stochastic PDEs in those environments. The connection with the study of equations like (1) gave way to fruitful interactions during the workshop. Among the senior participants, K.T. Sturm presented some fundamentals of the theory; also C. Lacaux and L. Chen reported some of the most recent developments in the construction of random fields on fractals.
- **Limit theorems:** The discovery by Nourdin and Peccati of the connection between Malliavin calculus and Stein's method has led to a burst of new research in stochastic analysis. Subjects like dimension free estimates for the rate of convergence in limit theorems, Sudakov-Fernique theorem or

Slepian inequality can now be generalized to Wiener chaos by methods of Malliavin calculus. They represent important tools when investigating a wide number of demanding contexts, such as central and non-central limit theorems, stochastic PDEs, spin glasses or polymer measures. In another direction, estimates of Malliavin-Stein type can be successfully applied to geometric problems, involving in particular random graphs defined on the points of a homogeneous Poisson process. A number of new applications include Poisson-Voronoi approximation and Boolean models. Among our participants, F. Viens presented recent results and applications.

- **Rough paths and machine learning:** Another most exciting new trend in rough paths analysis concerns applications of so-called signature method to data analysis. It has been successfully advocated that signatures of paths are excellent features, which can feed a neural network and produce outstanding results for classification or prediction problems. Relevant applications include character recognition, classification of patients and evaluation of trading policies. The participants of our workshop had the opportunity to learn from the expert T. Cass and X. Geng some of the most recent advances in this direction.

2. Talks

The workshop counted with a total of 27 talks, given by a mixture of senior and junior participants. Thursday afternoon was devoted to talks by graduate students and an open problem session.

(1) W.-T. Fan. *Longtime behavior of stochastic waves on 1-dimensional metric spaces and their genealogies.*
Stochastic reaction-diffusion equations are important models in mathematics and in applied sciences such as spatial population genetics and ecology. However, for many reaction terms and noises, the solution notion of these equations is still missing in dimension two or above, hindering the study of spatial effect on stochastic dynamics. In this talk, I will discuss a new approach, namely, to study these equations on general 1-dimensional structures (including metric graphs and fractals) that flexibly parametrizes space. This enables us to assess in great detail the impact of spatial effect on the co-existence and the genealogies of interacting populations. We will focus on recent results on extinction/survival probability, quasi-stationary distribution, asymptotic speed and other long-time behaviors for stochastic reaction-diffusion equations of Fisher-KPP type. Based on joint work with Rick Durrett, Wenqing Hu, Greg Terlov, and ongoing work with Zhenyao Sun, Oliver Tough and Yifan (Johnny) Yang.

(2) M. Salins. *Stochastic partial differential equations with superlinear forcing.*
Many standard existence and uniqueness results for stochastic partial differential equations assume that the deterministic and stochastic forcing terms are Lipschitz continuous with at most linear growth. Another common set of assumptions requires the deterministic forcing to be dissipative with at most polynomial growth. I present some recent results about much more general sufficient conditions on superlinear deterministic and stochastic forcing terms that guarantee that mild solutions to stochastic partial differential equations exist, are unique, and never explode. Results

compare various settings including bounded and unbounded spatial domains and the cases of dissipative and accretive superlinear deterministic forces.

(3) P. Fatheddin *Asymptotic Behavior of Stochastic Navier-Stokes and Schrodinger Equations.*
We will consider the stochastic Navier-Stokes and stochastic Schrodinger equations and discuss their asymptotic limits such as large and moderate deviations, central limit theorem and the law of the iterated logarithm. To achieve the large deviation principle, we apply both techniques available in the literature: Azencott method and the weak convergence approach and compare the two methods. The Azencott method is then used to derive the law of the iterated logarithm. Also I will discuss my recently published book for graduate students: "Teaching and Research in Mathematics: A Guide with applications to Industry".

(4) J. Leon. *Forward integral of bounded variation coefficients with respect to HÃűlder continuous processes.*
In this talk, we study the existence and an integration by parts formula for forward stochastic integrals of the form $\int_0^T f(Y_s)dY_s^-$. Here, the integral is interpreted in the Russo and Vallois sense, Y is a Hölder continuous process with exponent bigger than 1/2 and f is a deterministic function of bounded variation.
As a consequence of the integration by parts formula, we can obtain a representation for the solutions to some equations of the form

$$X_t = x + \int_0^t \sigma(X_s)dY_s^-, \quad t \in [0,T],$$

where σ is a discontinuous coefficient with bounded variation.
Finally, we interpret the local time of fractional Brownian motion as the trace term of a forward integral.

(5) C. Mueller. *The radius of star polymers in low dimensions and for small time.*
Studying the end-to-end distance of a self-avoiding or weakly self-avoiding random walk in two dimensions is a well known hard problem in probability and statistical physics. The conjecture is that the average end-to-end distance up to time n should be about $n^{3/4}$.
It would seem that studying more complicated models would be even harder, but we are able to make progress in one such model. A star polymer is a collection of N weakly mutually-avoiding Brownian motions taking values in $\mathbf{R}^d$ and starting at the origin. We study the two and three dimensional cases, and our sharpest results are for $d = 2$. Instead of the end-to-end distance, we define a radius R_T which measures the spread of the entire configuration up to time T. There are two phases: a crowded phase for small values for T, and a sparser phase for large T where paths do not interfere much. Our main result states for $T < N$, that R_T is approximately proportional to $T^{3/4}$ and also to $N^{1/4}$.

(6) R. Balan *Hyperbolic Anderson model with time-independent rough noise: Gaussian fluctuations.*
In this talk, we introduce the hyperbolic Anderson model in dimension 1, driven by a time-independent rough noise, i.e. the noise associated with the fractional Brownian motion of Hurst index $H \in (1/4, 1/2)$. The goal of the talk will be to show that, with appropriate normalization and centering, the spatial integral of the solution

converges in distribution to the standard normal distribution, and to estimate the speed of this convergence in the total variation distance. For this, we use some recent developments related to the Stein-Malliavin method. More precisely, we use a version of the second-order Gaussian Poincaré inequality developed by Nualart, Xia, Zheng (2022) for the similar problem for the parabolic Anderson model with rough noise in space (colored in time). To apply this method, we need to derive first some moment estimates for the increments of the first and second Malliavin derivatives of the solution. These are obtained using a connection with the wave equation with delta initial velocity, a method which is different than the one used in the parabolic case. This talk is based on joint work with Wangjun Yuan (University of Luxembourg).

(7) J. Wang. *The stochastic heat equation on Heisenberg groups.*
In this talk we will study stochastic heat equation $\partial_t u = \frac{1}{2}\Delta u + u\dot{W}_\alpha$ defined on the Heisenberg group where Δ is the hypoelliptic Laplacian and $\{\dot{W}_\alpha; \alpha > 0\}$ is a family of Gaussian space-time noises which are white in time and have a covariance structure generated by $(-\Delta)^{-\alpha}$ in space. We will give a proper description of the noise W_α, and prove that the stochastic heat equation can be solved in the Itô sense when $\alpha > \frac{n}{2}$. We also give some basic moment estimates for the solution $u(t,x)$. This is a joint work with F. Baudoin, C.Ouyang, and S. Tindel.

(8) K. Matzeski. *The dynamical Ising-Kac model converges to Φ^4 in three dimensions.*
The Glauber dynamics of the Ising-Kac model describes the evolution of spins on a lattice, with the flipping rate of each spin depending on an average field in a large neighborhood. Giacomin, Lebowitz, and Presutti conjectured in the 90s that the random fluctuations of the process near the critical temperature coincide with the solution of the dynamical Φ^4 model. This conjecture was proved in one dimension by Bertini, Presutti, Ruediger, and Saada in 1993 and the two-dimensional case was proved by Mourrat and Weber in 2014. Our result settles the conjecture in the three-dimensional case.
The dynamical Φ^4 model is given by a non-linear stochastic partial differential equation which is driven by an additive space-time white noise and which requires renormalization of the non-linearity in dimensions two and three. The renormalization has a physical meaning and corresponds to a small shift of the inverse temperature of the discrete system away from its critical value.
This is joint work with Hendrik Weber and Paolo Grazieschi.

(9) X. Geng. *Finiteness of the radius of convergence for the logarithmic signature.*
It was conjectured by T. Lyons and N. Sidorova in 2006 that the logarithmic signature of a BV path always has finite radius of convergence unless the path is a line segment. Such a property is closely related to the study of rough differential equations from the Lie-algebraic perspective. In their original paper, the conjecture was confirmed for two special classes of paths: piecewise linear paths and paths that are monotone in one direction. In this talk, We discuss some recent progress on this problem in both probabilistic and deterministic settings. The heart of the main strategy lies in understanding the dynamics of path developments onto suitably chosen Lie groups. This is based on joint work with S. Wang as well as ongoing joint work with H. Boedihardjo and S. Wang.

(10) E. Nualart. *Everywhere and instantaneous blowup of parabolic SPDEs.*
We consider the non-linear stochastic heat equation defined on the whole line driven by a spaceÂŋ time white noise. We will first recall some known results on the almost sure blow up for this type of equations. We then give sufficient conditions for the solution to blow up everywhere and instantaneously almost surely. The main ingredient of the proof is the study of the spatial growth of stochastic convolutions using techniques from Malliavin calculus and Poincare inequalities .
This is a joint work with Davar Khoshnevisan and Mohammud Foondun.

(11) L. Luo. *Logarithmic Sobolev inequalities on non-isotropic Heisenberg groups.*
We study logarithmic Sobolev inequalities with respect to a heat kernel measure on finite-dimensional and infinite-dimensional Heisenberg groups. Such a group is the simplest non-trivial example of a sub-Riemannian manifold. First we consider logarithmic Sobolev inequalities on non-isotropic Heisenberg groups. These inequalities are considered with respect to the hypoelliptic heat kernel measure, and we show that the logarithmic Sobolev constants can be chosen to be independent of the dimension of the underlying space. In this setting, a natural Laplacian is not an elliptic but a hypoelliptic operator. Furthermore, these results can be applied in an infinite-dimensional setting to prove a logarithmic Sobolev inequality on an infinite-dimensional Heisenberg group modelled on an abstract Wiener space.

(12) H. Bessaih. *Synchronization of stochastic models with applications.*
In the first part of the talk, we consider a system of two coupled stochastic lattice equations driven by additive white noise processes. Our objective is to investigate its longtime behavior. A system synchronizes if all elements eventually exhibit the same behavior. We show a synchronization for this system. To describe this phenomenon, we prove the upper semi continuity of the family of attractors with respect to the attractor of a specific limiting stochastic system. In the second part of the talk, we investigate a complex network consisting of finitely many nodes that are coupled in a deterministic and stochastic way. The behavior of each node is described by an evolution equation that includes a reaction diffusion equation driven by a multiplicative noise. We prove that the system synchronizes towards a deterministic equation. Furthermore, we describe various concepts of synchronization

(13) W. Vickery. *Intrinsic Noise on the Torus and Parabolic Anderson Model.*
We present an intrinsic definition for a family of noise that is colored in space and white in time on the d-dimensional torus. We also show existence and uniqueness results for the parabolic Anderson model (linear stochastic heat equation) driven by this noise. We demonstrate intermittency by producing exponential lower bounds on the second moments using the Feynman-Kac formula for the solution.

(14) W. Xu. *Periodic homogenisation for dynamical ϕ^4_2.*
We consider the periodic homogenisation problem for dynamical ϕ^4_2, a toy model that combines both renormalisation in singular SPDEs and homogenisation in a single problem. We show that in this situation, the two limiting procedures commute. Joint work with Yilin Chen (Peking University).

(15) D. Khoshnevisan. *On the valleys of the stochastic heat equation.*
We consider a generalization of the parabolic Anderson model driven by space-time white noise, also called the stochastic heat equation, on the real line. High peaks of solutions have been extensively studied under the name of intermittency, but

less is known about spatial regions between peaks, which may loosely refer to as valleys. We present two results about the valleys of the solution. Our first theorem provides information about the size of valleys and the supremum of the solution over a valley. More precisely, we show that the supremum of the solution over a valley vanishes as the time variable tends to infiniry, and we establish an upper bound of $\exp(-const.t^{1/3})$ for the rate of decay. We demonstrate also that the length of a valley grows at least as $\exp(+const.t^{1/3})$ as t gets large. Our second theorem asserts that the length of the valleys are eventually infinite when the initial data has subgaussian tails.
This is based on joint work with Kunwoo Kim (POSTECH) and Carl Mueller (Rochester).

(16) C. Pacheco. *Processes in random environment and its applications.*
We describe the connection between the Sinai random walk and the Brox diffusion, both processes with a random environment, the first one in discrete time and the second in continuous time. In doing so, we mention how one uses information of the environment to infer the behavior of the processes, which leads to predictions. An important tool we use is the so-called excursion theory of stochastic processes. In addition we present an application in financial modelling and pricing.

(17) T. Cass. *Topologies, functions and measures on unparameterised path space.*
The signature is a non-commutative exponential that appeared in the foundational work of K-T Chen in the 1950s. It is also a fundamental object in the theory of rough paths (Lyons, 1998). More recently, it has been proposed, and used, as part of a practical methodology to give a way of summarising multimodal, possibly irregularly sampled, time-ordered data in a way that is insensitive to its parameterisation. A key property underpinning this approach is the ability of linear functionals of the signature to approximate arbitrarily closely (in the uniform topology) any compactly supported and continuous function on (unparameterised) path space. We use this conext to present some new results on the properties of a selection of topologies on the space of unparameterised paths. We discuss various related consequences and applications. This is based on joint work with Willliam Turner and, if time permits, some results in a joint paper with Terry Lyons and Xingcheng Xu.

(18) H. Shen. *2D Yang-Mills.*
In an earlier work with Chandra, Chevyrev and Hairer [1], we constructed the local solution to the stochastic Yang-Mills equation on 2D torus, which was shown to have gauge equivariance property and thus induces a Markov process on a singular space of gauge equivalent classes. In this talk, we discuss a more recent work with Chevyrev [2], where we consider the Langevin dynamics of a large class of lattice gauge theories on 2D torus, and prove that these discrete dynamics all converge to the same limiting dynamic constructed in [1]. Using this universality result for the dynamics, we show that the Yang-Mills measure on 2D torus is the universal limit for these lattice gauge theories. We also prove that the Yang-Mills measure is invariant under the dynamic. Our argument relies on a combination of regularity structures, lattice gauge-fixing, and Bourgain's method for invariant measures.

(19) L. Chen. *Dirichlet fractional Gaussian fields on the Sierpinski gasket.*

In this talk, we study Dirichlet fractional Gaussian fields on the Sierpinski gasket. Heuristically, such fields are defined as distributions $X_s = (-\Delta)^{-s} W$, where W is a Gaussian white noise and Δ is the Laplacian with Dirichlet boundary condition. The construction is based on heat kernel analysis and spectral expansion. We also discuss regularity properties and discrete graph approximations of those fields. This is joint work with Fabrice Baudoin.

(20) E. Camrud. *Hypocoercive exponential decay in entropy for Hamiltonian Monte Carlo.*
We consider a family of Markov chains, encompassing unadjusted Hamiltonian Monte Carlo schemes and splitting schemes of the kinetic Langevin diffusion, commonly used in Markov chain Monte Carlo algorithms. Assuming that the target distribution satisfies a logarithmic Sobolev inequality, we adapt to this discrete-time framework the modified entropy method used to establish the hypocoercive exponential decay in entropy for the continuous-time kinetic Langevin process. This yields non-asymptotic quantitative bounds in relative entropy (hence in Wasserstein and total variation distances) for the law of the chain at a given time with respect to its target. Since the schemes are second-order in the stepsize, we get in this general settings a complexity of order $d/\varepsilon^{1/4}$ (when the second, third and fourth derivatives of the potential are bounded independently from d) with ε the error tolerance for the relative entropy, and of order $(d/\varepsilon)^{1/4}$ in the weakly interacting mean field case.

(21) H. Chen. *Multiplicative Stochastic Heat Equation on Compact Riemannian Manifolds of Nonpositive Curvature.*
Using the method introduced by Le Chen and Jingyu Huang, I will study the well-posedness of the Multiplicative Stochastic Heat Equation with colored noise on Compact Riemannian Manifolds. Unlike previous work on this equation, the Fourier transform is not employed. The main tools are Gaussian type bounds for the Heat kernel. Existence of the second moment of the solution in large time for all compact Riemannian manifolds will be shown. In small time, the difficulty induced by nonuniqueness of geodesics will be presented, along with how to overcome it on manifolds of nonpositive curvature.

3. Organization of the workshop

Our workshop was held in a hybrid format and included researchers at many different career stages. We tried our best to foster interactions and provide a comfortable environment for all participants.

3.1. **Interaction between participants.**

To facilitate the interaction with virtual participants, we tried to schedule virtual speakers at the beginning of a section. In this way we could check the technology and also have conversations with them prior to their talks.
We made an effort to always include early-career researchers, and in particular graduate students, in conversations during the coffee breaks. In this way they were rapidly introduced to more senior participants, which facilitated their interaction throughout the workshop.
After the open problem session we encouraged both junior and senior participants

to have conversations regarding career paths and other academic advice such as job applications, letters of recommendation, and workshop organization.

3.2. **Open problem session.** On Thursday afternoon we gathered all participants in an informal session where we gave the opportunity to anyone to present or elaborate on a research question, conjecture or open problem. Since most of the presenters had given their talks by that time, the explanation and introduction to the problem was easier for both presenters and participants. This presentations were mostly made on the blackboard and we took pictures of them for those who were interested in recording the discussion. Among various topics discussed in this session, C. Lacaux presented some further questions on the study of Gaussian and stable random fields on fractals. A junior participant, William Salkeld, also discussed some open problems related to his research.

3.3. **Takeaways from hybrid delivery.** Video and sound quality of the online talks was most of the times good, the size of the screen fitted well the in-person participants. It was still difficult to accommodate board talks or explanations from in-person participants for the virtual participants.

3.4. **Suggestions for improvement.** Gather the slides of all talks of the workshop *prior* to the beginning of the workshop to avoid momentary issues with email servers.

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