Group Actions on Cantor Sets

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1 Overview of the Field

Compact, totally disconnected, metrizable spaces are omnipresent in mathematics. They arise, for instance, as infinite products of finite sets and as the boundary of infinite trees of finite degree. Every such space without isolated points is homeomorphic to Cantor's famous ternary set and is simply called *a Cantor set*. From the viewpoint of topology these spaces are completely understood.

It is a remarkable and persistent pattern in Group Theory that groups with exotic or surprising properties often admit a natural action on a Cantor set and that, moreover, this action provides an important tool to study their structure. Among these groups are R. Grigorchuk's famous groups of intermediate growth which act on a rooted binary tree and hence on the boundary of the tree. Grigorchuk's groups led to the very rich families of branch and self-similar groups, which contains numerous groups with unexpected properties. Among groups acting on Cantor sets we can also find the notorious groups constructed by R. Thompson. These groups gave rise to an enormous supply of generalizations, vaguely called *Thompson-like groups*, such as topological full groups and braided Thompson groups. In recent years these groups became the primary method for the construction of infinite simple groups with additional properties. Groups with actions on Cantor sets are also relevant in geometry, where they can be found as big mapping class groups of surfaces with a Cantor set removed. A Cantor set equipped with a suitable measure becomes a standard probability space and actions on these spaces provide the basis for measured group theory. Interactions between measured group theory and L^2-invariants led to a number of new results in group theory.

The central role of actions on Cantor sets is surprising given that the Cantor set itself has almost no useful structure. Additional structure can be used to improve the understanding and the research communities faced with this challenge work with different methods. In one direction, the Cantor set is considered as the boundary of some geometric object: a tree or a surface. On the other hand, the Cantor set can be equipped with a measure so that methods of measured group theory, L^2-methods and spectral theory can be applied. In yet another direction, there are interactions with geometric aspects of profinite groups.

2 Recent Developments and Open Problems

There is an ample literature regarding groups and profinite groups acting on rooted trees. A regular rooted tree T is a graph whose vertices are words in some alphabet X. The empty word is called the root of the tree. Two words are connected by an edge if one arises from the other by appending one letter. If $|X| = d$ one says that the tree T is d-regular. The boundary ∂T of the tree is the set of infinite non-backtracking paths starting from the root, i.e, $\partial T = \prod_{\ell=1}^{\infty} X$. If X is finite, then ∂T with the product topology is homeomorphic to a Cantor set. A regular rooted tree has a large automorphism group that contains interesting subgroups.

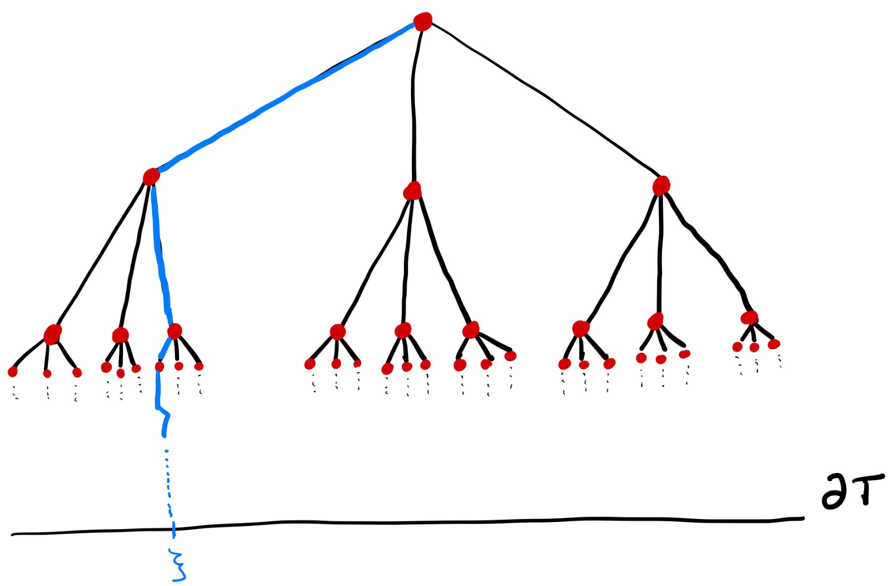


Figure 1: A 3-regular rooted tree and its boundary

Every subgroup of the automorphism group admits an actions on the boundary of the tree. Investigation of these subgroups led to the theory of branch groups and self-similar groups. Among these probably the most famous is Grigorchuk's group: the first group of intermediate word growth. Branch groups and self-similar groups have been used to give interesting examples regarding amenability [3] and dynamics [11]. Currently the subgroup structure of these groups attains a lot of attention. To name a particular problem: The existence (or non-existnce) of maximal subgroups of infinite index is currently only understood in specific families of branch groups. Moreover, the theory of self-similar groups of automorphisms of rooted trees is central in the recent construction [24] of a finitely generated simple group of intermediate word growth (with connections to topological full groups).

A rich family of groups can be constructed as homeomorphisms of Cantor sets with prescribed local behavior. We will refer to such groups as *Thompson-like groups*. In recent years Thompson-like groups became a central source for groups with surprising properties. The normal subgroups of Thompson-like groups can often be controlled which makes them the primary source in constructions of infinite simple groups.

- Topological full groups of minimal Cantor systems were shown to be amenable by Kate Juschenko and Nicolas Monod in [15] and give rise to the first examples of infinite amenable simple groups. Recently, Kionke-Schesler [17] constructed a new family of amenable simple groups using the theory of telescopes. The construction is inspired by the theory of branch groups and their groups provide the first 2-generated amenable simple groups.

- In his important paper [24], Nekrashevych puts special emphasis on groups acting on Cantor sets and exhibits a powerful constuction based on decompositions of actions of infinite dihedral groups on Cantor sets. As an application he obtains the first examples of finitely generated, simple, torsion-groups of intermediate growth. It would be very interesting to obtain more groups with these properties.

- Rachel Skipper, Stefan Witzel and Matthew Zaremsky [28] proved that commutator subgroups of suitable Röver-Nekrashevych groups give rise to simple groups with varying finiteness properties.

The study of topological groups appearing as automorphism groups of countable structures has a long tradition in the theory of topological groups. In this area there is a special interest in simplicity questions and extreme amenability. Recently, there has been some interesting geometric developments, as we outline below.

The definition of a coarse geometry type for such groups has been made possible due to seminal Work of Katryn Mann, Kasra Rafi, and Anschel Schaffer-Cohen [22], during 2019-2020, and first considerations of quasi-isometry of big mapping class groups acting on simplicial complexes associated to surfaces have been made in [6]. Actions of specific groups within this family, so-called big mapping class groups on hyperbolic simplicial complexes have been studied, among other sources in [2] in connection with the construction of non-trivial quasimorphisms, and in [5] in connection to Ivanov's rigidity type results.

In other direction, many researchers are applying geometric methods in the world of profinite groups. One classical, yet very relevant, point of view is the use of successful geometric approaches to study profinite groups. For instance, we have Bass-Serre Theory for pro-p and profinite groups. In [27], actions on *profinite graphs* are used to dissect a profinite group into amalgamated free products and HNN-extensions. In the same spirit, probabilistic finiteness properties of profinite groups have been defined in [19, 7] in an algebraic way. A natural problem is whether a "geometric" approach to probabilistic finiteness properties exists.

3 Workshop Structure

The workshop brought together researchers who deal with group actions on Cantor sets from various different perspectives. It included researchers from different fields, ranging from branch groups, Thompson-like groups over big mapping class groups to measured group theory and L^2-invariants related to group actions on Cantor sets. It was a unique opportunity, stimulating a transfer of methods and ideas.

The workshop was structured in a way to foster interactions between the participants. The schedule included only two research talks every day. In the remaining time, the participants discussed open problems and ideas in two working groups. We consider this approach to be very fruitful, as it stimulated new collaborations (to be discussed below).

3.1 Research Talks

Anja Randecker: Introduction to big mapping class groups.

This talk gave an introduction to big mapping class groups. Basic examples were discussed and the speaker explained the classification of big surfaces up to homeomorphism. The mapping class group and the pure mapping class group were defined. Coarsely bounded topological groups were defined and it was explained how quasi-isometry can be studied for coarsely bounded groups. The work of Mann-Rafi was sketched.

Volodymyr Nekrashevych: Simple groups of dynamical origins.

The speaker talked about groups of homeomorphisms of the Cantor set that are defined dynamically: e.g., as topological full groups of semigroups of local homeomorphisms. They discussed the relation between the properties of the dynamical systems and the associated groups. In particular, they focused on such properties as finite generation, amenability, subexponential growth, and torsion.

Rachel Skipper: Braiding groups of homeomorphisms of the Cantor set.

In this talk the speaker discussed some recent work on groups which connect self-similar and Higman-Thompson groups to big mapping class groups via "braiding". They explained some results on the topological finiteness properties of the resulting groups, which are topological generalizations of the algebraic properties of being finitely generated and finitely presented. The talk involved recent joint works with Xiaolei Wu (Fudan) and Matthew Zaremsky (Albany).

Roman Sauer: Cohomology of groups with unitary coefficients.

Roman Sauer discussed a higher Kazhdan property (T_n) based on a very inspiring joint paper with Uri Bader [1]. A group G has property (T_n) if $H^i(G, V)$ vanishes for every $0 \leq i \leq n$ and for every unitary representation V of G without non-zero G-invariant vectors. He explained that Kazhdan's property (T) is equivalent to (T_1) and discussed that (T) and property H_n^T (defined in the spirit of (T_n) using reduced homology) imply property (T_n). Bader-Sauer proved that the groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Z})$ have property (T_{n-2}). This result extends to other simple Lie groups and their lattices. This gives information of the stable range of the cohomology of arithmetic groups. It was also discussed that property H_n^T might also be important for the quasi-isometry classification of finitely generated nilpotent groups.

Matteo Vannacci: Iterated Wreath Products in Product Action, in search for new HJI groups.

A just infinite group is an infinite group without infinite proper quotients. A group is said to be hereditarily just infinite (HJI) if all of its finite index subgroups are just infinite. A classical classification theorem of Grigorchuk-Wilson states that a residually finite just infinite group is either: (a) a branch group or (b) an HJI group. Branch groups have been extensively studied (e.g. Grigorchuk group), but HJI groups remain a very mysterious class. The speaker reported on an on going project in the search for new HJI subgroups of iterated wreath products in product action.

Eduard Schlesler: From telescopes to frames and simple groups

The speaker explained the notion of a telescope introduced in joint work with Kionke [17]. He highlighted an example of a telescope of alternating groups that can be used to construct 2-generated infinite simple amenable groups. He illustrated the similarities to the classical construction of P. M. Neumann's branch groups. A number of additional applications were discussed.

Rostislav Grigorchuk: On subgroups of certain groups acting on a Cantor set.

In the beginning the speaker explained that branch groups act on the boundary of the tree, which is a Cantor set and then discussed maximal, weakly maximal and finitely generated subgroups of branch groups. The question which branch groups admit maximal subgroups of infinite index received a lot of attention in the literature. Grigorchuk mentioned that is would be good to have examples of finitely generated branch groups with uncountably many maximal subgroups of infinite index. In the second part he talked about maximal subgroups of TFG's (Topological Full Groups). Here the Cantor set is a cornerstone of the definition of TFG. The results presented in the first part are based on joint articles of the speaker with L.Bartholdi, D.Francoeur, P-H.Leemann and T.Nagnibeda. The second part is based on a joint work with Y.Vorobets.

Fabienne Chouraqui: An approach to the Herzog-Schönheim conjecture by covering graphs.

The speaker discussed the Herzog-Schönheim conjecture, an open question on coset partitions. Let G be a group and H_1,...,H_s be subgroups of G. If there exist $\alpha_i \in G$ such that $G = \bigcup\limits_{i=1}^{i=s} H_i\alpha_i$, and the sets $H_i\alpha_i$, $1 \leq i \leq s$, are pairwise disjoint, then $\{H_i\alpha_i\}_{i=1}^{i=s}$ is *a coset partition of* G. Let d_1,...,d_s denote the indices of H_1,...,H_s respectively. The coset partition $\{H_i\alpha_i\}_{i=1}^{i=s}$ has *multiplicity* if $d_i = d_j$ for some $i \neq j$. The Herzog-Schönheim conjecture is true for the group G, if any coset partition of G has multiplicity. In the 1980's, in a series of papers, M.A. Berger, A. Felzenbaum and A.S. Fraenkel studied the Herzog-Schönheim conjecture and they proved the conjecture is true for the pyramidal groups, a subclass of the finite solvable groups.

Fabienne Chouraqui studied the Herzog-Schönheim conjecture for free groups of finite rank and developed a new combinatorial approach, using covering spaces. Some sufficient conditions on the coset partition that ensure multiplicity were presented in the talk. Since every finitely generated group is a quotient of a free group of finite rank, these results extend to all finitely generated groups.

Tatiana Nagnibeda: Schreier graphs of self-similar actions and their spectra

This talk discussed self-similar groups and their actions on the boundary of an infinite regular tree. Schreier graphs of these actions constitute interesting families of graphs from the viewpoint of spectral graph theory providing examples of co-spectral graphs, graphs admitting exotic types of spectra and spectral measure, etc. Their self-similar nature provides keys to study them, from approximation by finite graphs to renormalization.

4 Scientific Progress Made

4.1 Working group 1: Dynamics of self-similar groups and related topics

The first working group discussed a number of problems, questions and recent developments related to groups, dynamics and in particular, to groups acting on rooted trees. Participants proposed problems and explained related results. The goal was to identify directions where the participants expertise could be combined to investigate previously unexplored directions. The following areas and ideas have been discussed.

1. Matteo Vannacci proposed to study the uniqueness of the branch action of a branch group. It is known from the work of Grigorchuk and Wilson [10] that if a branch group admits a *suitable* action on a rooted tree, then this action is essentially unique "up to deleting some layers" (see Theorem 1 in [10]). In particular, for just-infinite branch groups with a suitable action, the unique tree is the structure lattice of basal subgroups. A branch action is suitable, if the following two conditions hold

   ($*$) For each vertex u of the tree T the stabilizer of u in G acts as a transitive cyclic group of prime order on the descendants of u.

   ($**$) whenever u and u' are incomparable vertices and $v < u$, there is an element $g \in G$ that stabilizes u' and moves v.

   In addition, it is known that the action of a weakly branch group on the boundary of the tree is unique (see [25, 2.4.37]). This leads to two interesting questions

   Question 1.
   - *Can the conditions ($*$) and ($**$) be removed or weakened?*
   - *Given a branch group G: Describe the space of all trees on which G acts.*

   Several sessions were devoted to describing the problems and some background in detail. For the second question, the structure of the lattice of basal subgroups seems to be a central object. We discussed the lattice of basal subgroups and sketched an alternative proof of [25, 2.4.37] for branch groups. In regard to the first question we focused on [10, Thm. 1] to see if it is possible to remove at least condition ($*$). In the end we found an interesting example of a branch group that acts on an uncountable set of trees that cannot be obtained by deletion of layers from some fixed tree. For each of these trees either condition ($*$) or condition ($**$) is violated. In particular, this led us to consider the following question:

   Question 2. *Can condition ($*$) be generalized to: For each vertex u of the tree T the stabilizer of u in G acts primitively on the descendants of u.*

   In addition, it would be interesting to see if the result of Grigorchuk-Wilson can be generalized to weakly branch groups. Many more questions arose in discussion and they are the subject of an ongoing collaboration among some participants of the working group.

2. Eduard Schlesler explained the notion of telescopes (see [17]). Very roughly a telescope is a directed system of groups that contains various commuting images of some fixed group B. Telescopes are inspired from the theory of groups acting on rooted trees and were used to construct groups that answered a number of open problems. Eduard Schesler suggested to use telescopes to produce a new explicit example of an infinite simple group of non-uniform exponential word growth. This idea seems to have a lot of potential. In discussions the question arose whether the same could be done for constructing an infinite simple group of intermediate word growth. This is the current topic of work in progress among some participants of the working group.

3. Sebastián Barbieri gave an introduction to *self-simulable* groups. A group G is called self-simulable if every computable action of G on an effectively closed subset of the Cantor set is a topological factor of a subshift of finite type. Mainly, a few results that ensure that a nonamenable group has that property were mentioned and there was a discussion about finding interesting examples within the class of self-similar groups.

4. Fabienne Chouraqui explained the Herzog-Schönheim conjecture and discussed recent developments. The conjecture asserts that if a group G is partitioned into $k \geq 2$ cosets $a_1G_1, \dots, a_kG_k$ of subgroups of finite index, then at least one of the indicies $|G : G_i|$ occurs twice. The conjecture received a lot of attention, in particular in finite group theory. The famous Newman-Mirsky theorem proves the conjecture for $\mathbb{Z}$. In addition, the answer is positive for nilpotent groups. However, a general answer seems currently to be out of reach. The working group discussed a number of approaches and tried to identify new unexplored directions. We had the impression that this is a very challenging problem.

5. Santiago Radi Severo explained *arboreal Galois representations* and some of their applications (see [14] for an introduction). Let K be a global field and let K^{sep} denote the separable closure. Let $f \in K[X]$ be polynomial of degree $d \geq 2$. For every suitable element $\alpha \in K$ the set of inverse images $\bigcup_{n \in \mathbb{N}_0} \{x \in K^{\text{sep}} \mid f^n(x) = \alpha\}$ are the vertices of a d-regular rooted tree, where x is connected to y, if $f(x) = y$. The absolute Galois group of K acts on this tree, giving rise to an aboreal Galois representation. Going back to work of Ondoni in the 1980's, arboreal representations can be used to prove results about the density of the set of prime divisors for elements in a sequence $a_0, a_1 = f(a_0), a_2 = f(a_1), \dots$ with $f \in \mathbb{Z}[X]$.

   In these investigations a group theoretic invariant - the fixed point proportion - appears that seems to be of independent interest. Let $G \leq \mathrm{Aut}(T_d)$ be a closed, self-similar and level transitive subgroup. Let G_n be the finite group $G/\mathrm{St}_G(n)$. We define

$$\mathrm{FPP}(G) = \lim_{n \to +\infty} \frac{|\{g \in G_n \mid g \text{ fixes a vertex in level } n\}|}{|G_n|}.$$

   Note that the limit exists because the sequence is non-increasing. Moreover, the above invariant can be re-written in measure-theoretic terms, using the Haar-measure of G. Finally, it is known that if, for all $n \geq 1$ and for $X = \{0, \dots, d-1\}$,

$$\text{the natural action of } \mathrm{St}_G(n-1) \text{ on } vX \text{ is transitive for all } v \in T_d^{n-1}, \tag{1}$$

   and G is self-similar, then $\mathrm{FPP}(G) = 0$ (using martingales). During we workshop we discussed about branch groups with condition (1). It seems interesting to investigate whether these groups satisfy $\mathrm{FPP}(G) = 0$. On the other hand, we proposed to study the behaviour of $\mathrm{FPP}(G)$ for groups that do not satisfy condition (1).

6. In the talk by R. Grigorchuck the interesting question was raised whether there are finitely generated branch groups with uncountably many non-isomorphic maximal subgroups. Eduard Schesler proposed a strategy to construct such examples building on a paper of Kionke-Schesler [18]. This is the topic of current work in progress.

4.2 Working group 2: Big mapping class groups

The second working group focused their sessions on the study of Polish group properties of mapping class groups of infinite-type surfaces (i.e. big mapping class groups).

Participants explained well-known results and posed problems along with possible solutions. The following was discussed:

1. Oscar Molina (Yulu) gave an introductory talk about big mapping class groups and associated subgroups such as pure mapping class groups and compactly-supported mapping class groups.

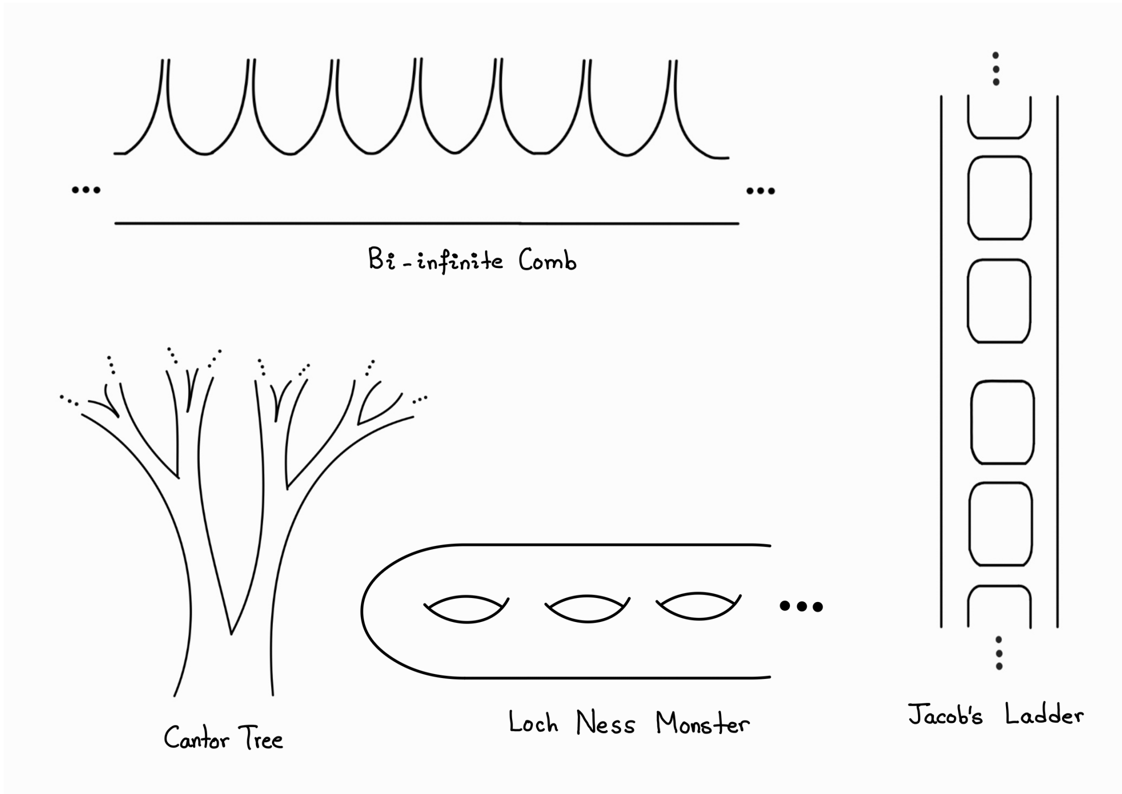


Figure 2: Some examples of infinite type surfaces, by Oscar Molina (Yulu)

2. Anja Randecker and Jess Hernndez Hernndez commented on the characterization of pure mapping class groups satisfying automatic continuity ([8]) and attempts to generalize such characterization to big mapping class groups. In this guise, Ulises Ariet Ramos-Garca proposed to study the *algebraic perfect rank* ([12, Definition 4.3]) of big mapping class groups in order to determine which big mapping class groups with automatic continuity also have the Steinhaus Property (it is expected that automatic continuity is equivalent to the Steinhaus property on the class of Polish groups).

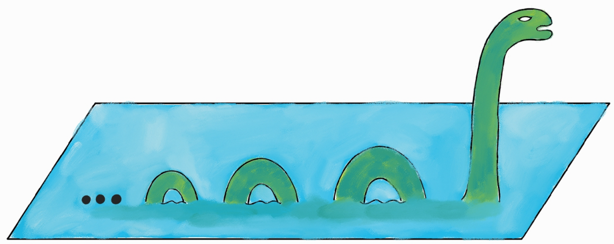
3. Carlos Prez Estrada talked about the non-extreme amenability of big mapping class groups, pure mapping class groups and compactly-supported mapping class groups following [20]. When discussing the case of infinite-type surfaces with finitely many ends (and thus with infinite genus), Jess Hernndez Hernndez remarked a flawed argument on Yusen's proof and how to fix it. It is worth mentioning that Jess Hernndez Hernndez and Carlos Prez Estrada communicated such discussion to Yusen, who later published a corrected version of his work ([21]).

4. Motivated by Yusen Long's result above, Jareb Navarro talked about the extreme amenability of automorphism groups of countable Fraïssé structures in terms of Structural Ramsey Theory following [16]. This motivated the use of Structural Ramsey Theory to exactly determinate which subgroups of big mapping class groups are extremely amenable. This resulted in an ongoing collaboration among participants of the working group.

One of the tools used in [21] to discard extreme amenability on subgroups of mapping class groups was to exhibit finite-order elements. The working group was able to generalize such idea and to reprove the main results of [21] in the following sense:

Theorem 1. *Let S be an infinite-type surface and let $\Gamma \leq Map(S)$. If there exist an essential curve α on S, a homeomorphism $g \in \mathrm{Homeo}^+(S)$ with mapping class in Γ and a natural number $n \in \mathbb{N}$ for which the set $\{\alpha, g(\alpha), \ldots, g^{n-1}(\alpha)\}$ bounds a finite-type subsurface of S, then Γ is not extremely amenable.*

Figure 3: Artistic Vision of the Loch Ness Surface by Oscar Molina (Yulu)



The main idea for proving Theorem 1 was to consider any closed subgroup $\Gamma \leq \mathrm{Map}(S)$ as the automorphism group of a Fraïssé limit as in [13] and then disproving the Ramsey property for the associated Fraïssé class of finite substructures embedable in the corresponding Fraïssé limit. Since such limit is obtained by enriching the structure of the curve graph $C(S)$ with the orbits of the canonical actions of Γ on $C(S)^n$ as n-ary relations, the working group got interested in the structural Ramsey theory of the *classical* curve graphs. Concretely, the following was posed:

Question 3. *Given a surface S (not necessarily of infinite-type), do the Ramsey property and the combinatorial Ramsey property hold for the class of finite subgraphs of $C(S)$?.*

By [4, Theorem 1.1] and [26] it was concluded that for surfaces of infinite genus, the combinatorial Ramsey property does hold but the Ramsey property does not.

It is expected that in one year the working group gets to fully answer Question 3 and to strengthen Theorem 1 by weakening its technical conditions.

5 Outcome of the Meeting

In total this was a very productive workshop. The structure of the workshop was helpful and enabled the participants to have long discussions and to give informal introductory talks on related topics. The small number of research talks gave the working groups a sufficient amount of time to work. The working groups collected a number of promising question and this inspired a number of ongoing scientific collaborations. We expect that joint publications of workshop participants will appear. We asked the participants to mention the CMO, if an article was initiated in this workshop.

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