17w5030 Workshop on *Splitting Algorithms, Modern Operator Theory, and Applications*

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September 17–22, 2017

This workshop is dedicated to the memory of Jonathan M. Borwein

1 Summary

The objective of this workshop was to bring together researchers with a strong interest in optimization algorithms based on monotone operator theory splitting, Both from mathematics and from the applied sciences, in order to survey the state-of-the-art of theory and practice, to identify emerging problems driven by applications, and to discuss new approaches for solving these problems.

Many of the participants had not met before. Various connections between diverse researchers have been established and strengthened. We thus expect this workshop to be the springboard for new innovative research and collaborations by its unique mix of experts whose areas of applications are broad, ranging from variational analysis, numerical linear algebra, machine learning, computational physics and crystallography.

2 Overview of the Field and Relationship with the Workshop

Over the past decade, many variants of operator splitting methods have been (re)discovered, and some of these have found unexpected applications [39]. These methods have been applied in a plethora of different areas, including partial differential equations [45, 47]. A major open question concerns the quantification of convergence rates, the understanding of the behaviour in infeasible cases, and the lack of satisfying explanations of the behaviour especially in nonconvex settings. The theory of monotone operators [10, 70] is relevant to these questions as it is the principal tool in understanding and analyzing the algorithms. Consequently, a substantial portion of the workshop deals with theoretical advances in monotone operator theory, especially as they pertain to algorithms and concrete, implementable methods. Connections have been built between mathematics, industry and physics, where splitting methods have been very successfully employed; see, e.g., [15, 39].

The splitting algorithms that were the main topic of this workshop have found significant real-world applications ranging from e.g., wavefront sensing [55] to road design [15]. The open questions surrounding these algorithms are not only of pure mathematical interest, but their resolution promises a real impact to the industrial world. The importance of this workshop was the potential of new knowledge that will make existing algorithms more efficient and expand their areas of applications through newly formed research connections.

The usage of splitting methods and the corresponding research activities have increased significantly especially in the past years; see, e.g., [10], [28] and [21], the references therein, as well as the references listed in the report.

The workshop realized our aim to reach out and include younger researchers (graduate students, postdoctoral fellows, and assistant professors) as well as women. In order to make the workshop most productive to junior experts as well as non-specialists, we have asked some of speakers specifically to write survey articles/tutorials for the accompanying conference proceedings volume. The networking opportunities at this workshop were particularly important to younger researchers and researcher at smaller institutions in terms of career planning and the formation of collaborative research programs.

A notable aspect of the talks delivered was the role that experimental mathematics played in the development of theoretical intuition, especially through visualization. The use of experimental results on benchmark problems has long been standard practice in research on numerical algorithms; however, the use of mathematical software to test theoretical hypotheses is not part of the mathematical mainstream yet. See the books by Bailey, Borwein and collaborators [6, 5, 7] for further information. We have dedicated this workshop to the memory of Jonathan Michael Borwein, a mathematical giant and one of the first strong supporters of this workshop.

3 Presentation Highlights

In this section, we highlight some of the recent developments and problems discussed at the workshop. In particular, we focus on recent scientific progress as well as contributions of participants to the workshop. The topics are grouped into areas, but common themes that arose throughout the conference are (i) the potential of splitting methods for solving large-scale and/or nonconvex problems, and (ii) the need for a theoretical foundation to explain their success.

3.1 Douglas–Rachford / ADMM-type Algorithms

The Douglas–Rachford algorithm [33], which is a linear implicit iterative method, was originally developed in 1956 for solving partial differential equations. In 1979, Lions and Mercier [54] extended the Douglas–Rachford algorithm to an operator splitting method for finding a zero of the sum of two maximally monotone operators.

The Douglas–Rachford algorithm was discussed in several talks and from different viewpoints. When applied to normal cone operators of two nonempty closed convex sets U and V, with associated projectors P_U and P_V as well as reflectors $R_U = 2P_U - \mathrm{Id}$ and $R_V = 2P_V - \mathrm{Id}$, the governing iteration takes the form

$$x_0 \in X, \qquad (\forall n \in \mathbb{N})\ \ x_{n+1} = \frac{\mathrm{Id} + R_V R_U}{2} x_n, \tag{1}$$

where Id denotes the identity operator of the Hilbert space X. Under appropriate assumptions, the so-generated sequence $(x_n)_{n\in\mathbb{N}}$ has the remarkable property that $(P_U x_n)_{n\in\mathbb{N}}$ converges to a solution of the underlying feasibility problem, i.e., to a point in $U \cap V$. More generally, one may try to find a zero of the sum of two maximally monotone operators. The Douglas–Rachford algorithm proceeds analogously but the projectors P_U and P_V are then replaced by the resolvents J_A and J_B (which are the proximity operators in the case of minimization of two functions). The method was rediscovered by different people working in different disciplines. Noteworthy is the application of the Douglas–Rachford algorithm in phase retrieval with a support constraint (as opposed to support and nonnegativity), where it is known as the *hybrid input-output (HIO)* algorithm, pioneered by Fienup [41] in 1982. (See also [11] for a view from convex optimization.) A very interesting development originates with Elser [37], who has very successfully applied the Douglas–Rachford algorithm to various continuous and discrete, *nonconvex* problems [39, 48]. In the physics community, the algorithm is now known as the *difference map algorithm* and its product space version à la Pierra [65] as *divide and concur*. A method closely related to the Douglas–Rachford algorithm is the *Alternating Direction Method of Multipliers (ADMM)* [18, 22, 50, 66].

Jim Burke described methods for solving large-scale affine inclusion problems on the product/intersection of convex sets, reporting on his recent work [26]. Robert Csetnek surveyed his 2017 work with Radu Boţ on the ADMM for monotone operators, focussing on convergence analysis and rates [20].

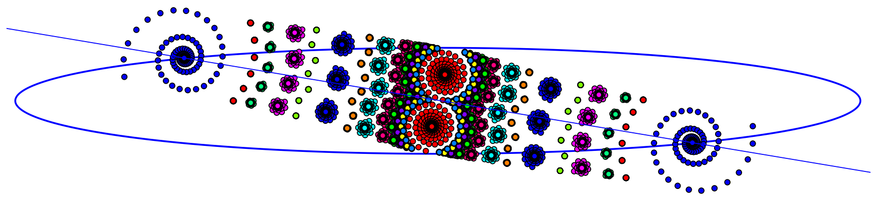

Figure 1: Basins of periodicity

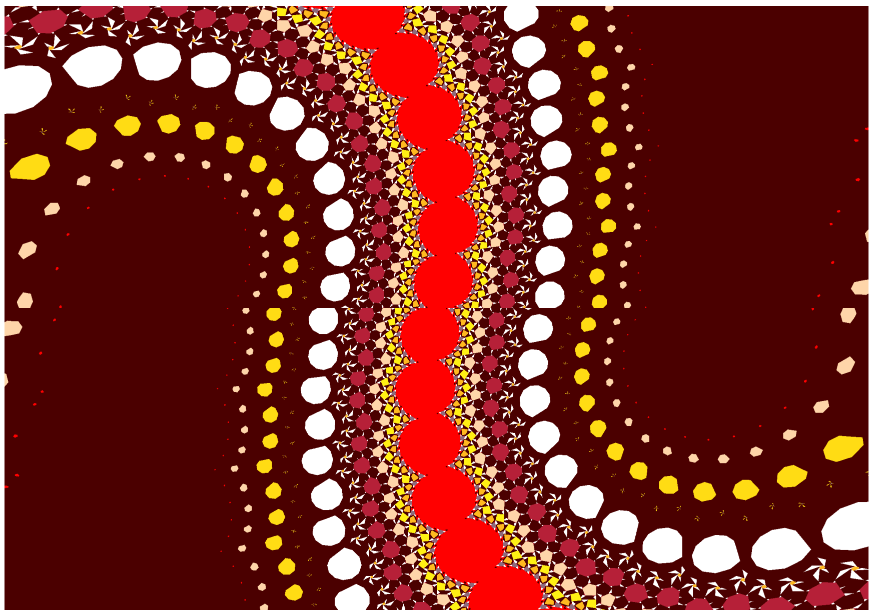

Figure 2: Dependence on the starting point

Scott Lindstrom reported on his recent work of applications of the Douglas–Rachford algorithm [19, 53]. It was very striking to observe that the behaviour in the Euclidean plane, for a line and a p-sphere, leads to breathtakingly beautiful pictures concerning the Douglas–Rachford algorithm applied to an ellipse and a line in the Euclidean plane.

In Figure 1, differently colored dots correspond to unique sequences of iterates with distinct starting points. The solutions for the feasibility problem are the two feasible points where the line intersects with the ellipse. Depending upon the starting location, sequences may converge to these solutions as the two blue sequences do on the far left and far right. However, if the algorithm starts elsewhere, sequences may be pulled into attractive instances of what we are calling basins of periodicity which prevent them from converging to the solution. Pictures like this are extremely valuable for studying how small changes to a problem (such as stretching a sphere into an ellipse) can cause drastic changes to the behaviour of a simple algorithm. They also illustrate what kinds of things can go "wrong." Zooming in, we find lovely swirls and stars for subsequences converging to periodic points. This particular image was created using Cinderella, and it appears on the poster for the Australian Mathematical Society's special interest group Mathematics of Computation and Optimization (MoCaO).

Next, the Douglas–Rachford algorithm is run starting from each individual pixel (Figure 2). We compute the first one thousand iterates before coloring the starting points according to which periodic point (or feasible point) their one thousandth iterate was nearest to. The apparent basins which emerge in this image appear more polyhedral than one might expect from such a problem. Works like this highlight the importance of parallel computing. Again, this picture is the main poster image for Australian Mathematical Society special interest group Mathematics of Computation and Optimization (MoCaO), and the colors were inspired by Australian aboriginal art.

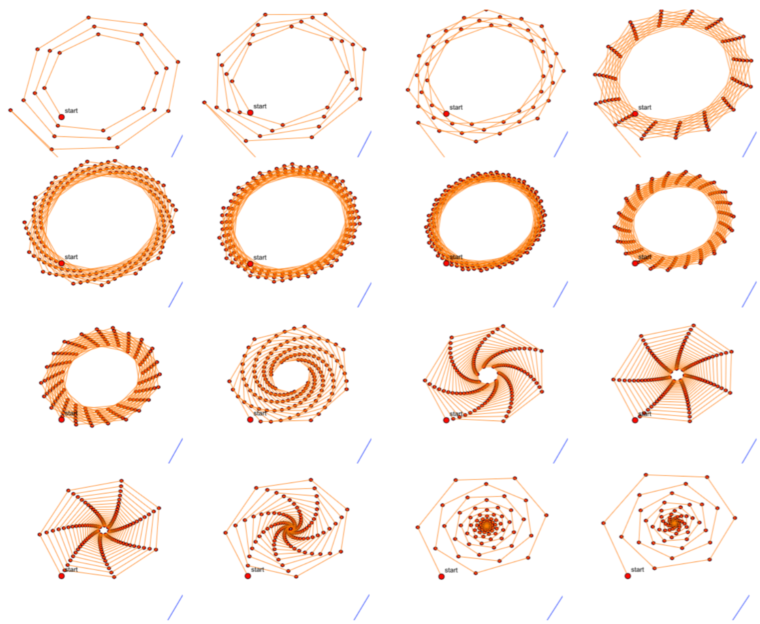

Figure 3: Rotating the line

Starting in the repelling basin for a pair of period-2 points and plotting every second iterate for the Douglas–Rachford method (see Figure 3), we make small changes to one of the sets. The set in question is a line, part of which is visible in the Bottom right corner of each frame. As we rotate the line, we see the "speed" at which iterates escape from the source basin decreases until eventually the source basin turns into a sink basin.

Panos Patrinos presented a new global convergence theory based on the *Douglas–Rachford envelope* as well as faster variants [64, 74].

Walaa Moursi surveyed her recent results on the Douglas–Rachford algorithm in the possibly inconsistent case [62]. In the classical feasibility setting (corresponding to minimizing a sum of indicator functions), the behaviour is now well understood: the shadow sequence approaches a generalized solution realizing the "gap" between the sets while the governing sequences escapes to infinity (see Figures 4 and 5 for the consistent and inconsistent case, respectively). This special case is useful to obtain least squares solutions when working in a product space [13]; see Figure 6. However, even in the classical convex-function setting, there are various open problems concerning the behaviour of the shadow sequences pertaining to boundedness and convergence to generalized solutions. (See [16] for additional information.)

Minh Dao reported on cases when the Douglas–Rachford algorithm converges in finitely many steps [14, 12]. Moreover, he also surveyed recent joint work with Hung Phan (on linear convergence [30]) and with Matt Tam (on a Lyapunov function approach [31]).

Shawn Wang discussed his new (unpublished) results on a regularized version of the Douglas–Rachford algorithm for finding minimum-norm solution for the sum of two maximally monotone operators.

Veit Elser shared his insights on a relaxed version of the Douglas–Rachford algorithm, termed "reflect-reflect-relax (RRR)" on hard combinatorial satisfiability problems including bit retrieval [38]. Interestingly, he viewed RRR as a sampling method. The choice of the optimal relaxation parameter is an open problem.

Pontus Giselsson reported on his very recent tight convergence rates on the Douglas–Rachford and related algorithms [40, 46].

A new concept, the "partial error bound condition" was the topic of Xiaoming Yuan's talk. He showed how this very general condition gives rise to many linear convergence results for ADMM. This is work in progress with Y. Liu, S. Zeng, and J. Zhang.

Fran Aragon Artacho reported on joint work with his student Ruben Campoy [2] on modification of the Douglas–Rachford algorithm to solve best approximation problems.

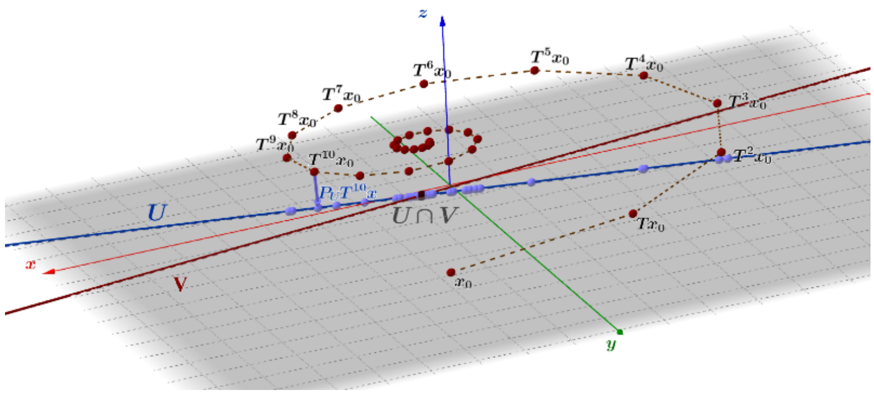

Figure 4: A GeoGebra snapshot that illustrates the behaviour of Douglas–Rachford method in the case of consistent feasibility problems. Two intersecting lines in $\mathbb{R}^3$, U the blue line and V the red line. The first few iterates of the governing sequence $(T^n x_0)_{n\in\mathbb{N}}$ (red points) and the shadow sequence $(P_U T^n x_0)_{n\in\mathbb{N}}$ (blue points) are also depicted.

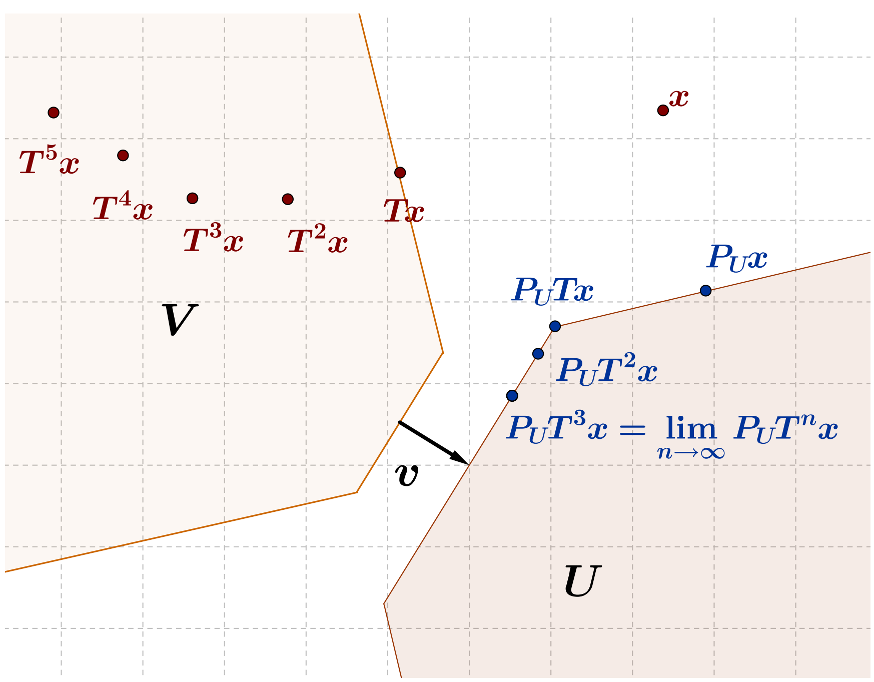

Figure 5: A GeoGebra snapshot that illustrates Douglas–Rachford method in the case of inconsistent feasibility problems. Two nonintersecting polyhedral sets in $\mathbb{R}^2$, U and V. The first few iterates of the governing sequence $(T^n x)_{n\in\mathbb{N}}$ (red points) and the shadow sequence $(P_U T^n x)_{n\in\mathbb{N}}$ (blue points) are also depicted. Shown is the minimal displacement vector v as well.

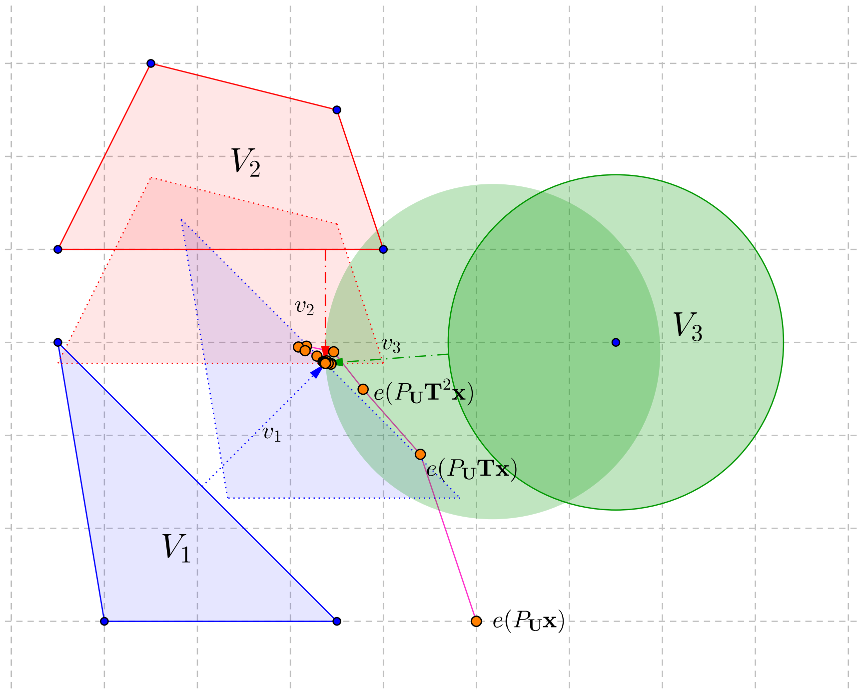

Figure 6: Obtaining least-squares solutions by employing Douglas–Rachford in a suitable product space.

Renata Sotirov discussed the quadratic shortest path problem, which is NP-hard, and solution strategies. One method was based on ADMM applied to a semidefinite programming relaxation. See [51] and the references therein.

3.2 Proximal gradient methods and their Accelerations

One of the highlights was the opening talk of the conference, held by Dr. Hedy Attouch, on the acceleration of first-order proximal gradient methods in the style of Nesterov [63]. This has been a subject of intense research over the past years. It is still not known whether the original FISTA method by Beck-Teboulle [17] has convergent iterates. Attouch drew the connection to continuous models and highlighted his many nice recent powerful results including [3]. It is also not known whether a fast version of Douglas–Rachford exists in the general case.

Sorin-Mihai Grad reported on joint work (in progress), with Radu Boţ, on an inertial forward-backward method for solving vector optimization problems. Specializing to the classical optimization case, one obtains inertial proximal point methods as studied by Alvarez and Attouch as well as Beck and Teboulle's ISTA.

Radu Boţ surveyed his recent work with Sebastian Banert on a novel algorithm [8] for solving difference-of-convex-functions optimization problems which were traditionally solved by Tao and An's algorithm [73].

3.3 Other Algorithms

Yura Malitsky considered non-stationary methods for solving variational inequalities based on the golden ratio.

Elena Resmerita reported on a new method for reconstructing positive solutions of inverse problems based on the Boltzmann–Shannon entropy [25].

Patrick Combettes and Jonathan Eckstein reported on very general new algorithmic framework [29] that allows for asynchronous computation. One interesting open problem is the choice of good parameters. Eckstein's talk focussed on a special case that is still very powerful [35].

Isao Yamada discussed hierarchical optimization problems and corresponding solution strategies by the hybrid steepest descent method. He also applied his algorithm for a certain statistical estimation problem, enhancing the popular LASSO technique. See [75, 76] for further information.

Evgeni Nurminski reported on current work (in progress) on solving monotone variational inequality problems using Fejer-type iterations.

Dominik Noll showed how, by combining local optimization methods tailored to lower-C^1 and upper C^1 functions with global optimization methods, one obtains robustness certificates for robust optimization problems arising in control engineering.

Reinier Diaz Millan discussed on-going joint work with Regina Burachik on algorithms for solving non-monotone variational inequalities [24].

Russell Luke surveyed a very general framework to obtain rate-of-convergence results for iterations of set-valued operators [56] while given a live demonstration of his `ProxToolbox` [67].

Max Goncalves reported on joint work with Jefferson Melo and Marques Alves on convergence results on a variable metric proximal ADMM [42, 43].

Jefferson Melo discussed a regularized variants of ADMM with an improved iteration complexity [43, 44, 58, 59].

A very general framework featuring quasi-nonexpansive operators was presented by Cegielski [27]. He demonstrated that it is closed under relaxations, convex combinations and compositions. In tandem with regularity properties, various rate-of-convergence results are obtained.

3.4 Convex Analysis, Variational Analysis, Control and Optimization, and Monotone Operator Theory

Aris Daniilidis reported on new (unpublished) work on the extension of Lipschitz functions related to the recent work [4].

Asen Dontchev surveyed the classical Hildebrand-Graves, Lyusternik-Graves, and Bartle-Graves theorems. Relating to [32], he formulated a conjecture concerning a nonsmooth Bartle-Graves theorem.

The importance of error bounds for analyzing convergence rates of first-order methods was demonstrated in Anthony So's talk which was based on [77]. So provided a new framework allowing for a unified treatment of various existing error bounds.

Genaro Lopez reported on joint work in progress (with A. Nicolae and U. Kohlenbach) on the moduli of regularity and uniqueness [52] as a tool for studying Fejer monotone

The classical Frank-Wolfe theorem states that a quadratic function that is bounded below on a convex polyhedron must attains its infimum. In his talk, Juan Enrique Martinez-Legaz discussed generalizations of this result to more general classes of convex sets (see [57] for the accompanying forthcoming paper).

Yao-Liang Yu reported on work (in progress) on conditions sufficient for guaranteeing that the proximal map of the sum of functions is a composition of the individual proximal maps. This work in progress has interesting applications since proximal operators are generally not easy to compute but are integral components in splitting algorithms.

Stephen Simons surveyed the beautiful framework of quasidense multifunctions which generalize monotone operators. Many results can be obtained with cleaner proofs, and the theory offers opportunities to deal with gradients of nonconvex functions. (See [69, 71, 72] for further information.)

Complementary to Yu's talk above, Samir Adly reported on recent work on finding the proximity operator for the sum of two functions. He offered a solution to this problem at the cost of suitably re-defining the proximity operator [1].

Yalcin Kaya reported on his work on a solving a nonsmooth optimal control problem asking to minimize the total variation of the control variables along a general function. An illustrative convex problem was fully analyzed with asymptotic results provided.

The importance of error bounds for convergence results of algorithms was highlighted in the talk by Adrian Lewis [34].

Rafal Goebel discussed necessary and sufficient conditions for pointwise asymptotic stability in terms of set-valued Lyapunov functions, its robustness under perturbations, and how it can be guaranteed in a control system by optimal control.

Regina Burachik presented recent joint work with Victoria Martin-Marquez analyzing (in)consistency of a convex feasibility problem via a dual support function formulation.

4 Outcome of the Meeting

The organizers will edit a Conference Proceedings volume entitled *Splitting Algorithms, Modern Operator Theory, and Applications*, published by Springer. A good number of the participants has indicated a strong interest to contribute to this volume; in addition, several researchers who were unable to attend the workshop have been invited to contribute as well, including: Stephen Boyd (Stanford), Amir Beck (Technion), Yair Censor (Haifa), Simeon Reich (Technion), Shoham Sabach (Technion), Claudia Sagastizabal (IMPA), Marc Teboulle (Tel Aviv), Michel Thera (Limoges), Lionel Thibault (Montpellier) Henry Wolkowicz (Waterloo). This volume will be dedicated to articles, surveys and tutorials related to the algorithm as outlined in Section 2.

Acknowledgment

The organizers thank Claudia Arias Cao Romero and Miguel Ricardo Altamirano Ibarra for their fantastic help in the preparation and realization of this workshop. They also thank Scott Lindstrom and Walaa Moursi for allowing us to use their striking illustrations.

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