Algebraic, Tropical, and Nonarchimedean Analytic Geometry of Moduli Spaces

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1 Overview of the Field and recent developments

Tropical geometry is a rapidly developing field, with close connections to algebraic and nonarchimedean analytic geometry as well as to piecewise-linear combinatorics. From the algebro-geometric perspective, *tropicalization* can be regarded as a modern degeneration technique in which varieties are replaced by polyhedral combinatorial objects. At the same time, one of the main recent developments of tropical geometry, encompassing the work of many authors, is its rigorous connection to nonarchimedean analytic geometry. We will discuss some recent notable applications of these connections to problems in algebraic geometry below.

There are a handful of basic setups for *tropicalization*, which we briefly review here. Some relationships between, and generalizations of these setups, will be touched on below. Explicitly or implicitly, they all involve *skeletons*, which one can temporarily understand as canonically defined polyhedral subcomplexes of Berkovich's nonarchimedean analytic spaces [7].

Tropicalization of toric varieties, and subvarieties thereof: Let K be a field, equipped with a nonarchimedean valuation $v\colon K \to \mathbb{R} \cup \{\infty\}$. Let $\mathbb{G}_m^n$ be an algebraic torus over K with cocharacter lattice N, and let $X \subset \mathbb{G}_m^n$ be a subvariety. The *tropicalization* of X is

$$\{(v_L(x_1), \ldots, v_L(x_n)) : (x_1, \ldots, x_n) \in X(L) : (L, v_L) \text{ a valued field extension of } (K, v)\} \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n.$$

The tropicalization of X is a subset of $\mathbb{R}^n$ that can be equipped with a polyhedral structure. This definition can be extended to subvarieties of normal toric varieties $X(\Sigma)$ by Kajiwara-Payne extended tropicalization [39, 51]. The outputs of extended tropicalization are polyhedral complexes inside $N_{\mathbb{R}}(\Sigma)$, a space stratified by affine spaces $\mathbb{R}^{n_i}$ in a way that is compatible with the fan Σ, and containing $N_{\mathbb{R}} \cong \mathbb{R}^n$ as a dense open.

Tropicalization of toroidal schemes: By work of Thuillier [58], the tropicalization of a toroidal embedding $U \subset X$ of schemes over a trivially valued field k may be defined as the *skeleton* of the beth space $X^{\beth}$ (see [58]). This can be identified with the generalized cone complex canonically associated to the boundary complex of $U \subset X$. This construction generalizes to toroidal Deligne-Mumford stacks by [3].

Tropicalization of semistable models [6, 7, 33]: Let K now be a complete and algebraically closed nonarchimedean field with valuation ring R. If X is a variety over K, $\mathcal{X}$ a strictly semistable model for X over R, and $H \subset \mathcal{X}$ is a horizontal divisor satisfying appropriate hypotheses, then there is a nonarchimedean skeleton $S(\mathcal{X}, H)$ associated to $\mathcal{X}$ and H, the *tropicalization* of X with respect to these data. For example,

if X is a curve over K, then a tropicalization of X is a metric graph, and if H is nonempty, the metric graph has unbounded rays.

A second, closely related family of results can be regarded roughly as the development of a stand-alone theory of tropical geometry. Many valuable contributions have developed and studied tropical objects (e.g. tropical curves, varieties, boundary complexes) in their own right. The Riemann-Roch theorem for graphs is a notable example [5] – logically independent from the Riemann-Roch theorem for curves, it inspired a great deal of work in specialization lemmas and tropical Brill-Noether theory as discussed later. Another setup is the rigorous foundational work on tropical schemes in development by J. Giansiracusa, N. Giansiracusa, O. Lorscheid, D. Maclagan, F. Rincon, and others (represented at the conference by the talk of J. Giansiracusa) [30, 42, 43, 31, 32, 35, 37, 38, 41, 44]. Such developments often presage rigorous connections to algebraic and nonarchimedean geometry down the road. Not all "synthetic" tropical objects have lives as tropicalizations, so in addition to studying their intrinsic combinatorial properties, it is interesting to study which ones arise from tropicalization, and in which ways. The talk of Kristin Shaw gave a successful instance of this study, motivated by the result of Vigeland [61] that a tropical cubic surface can contain more than 27 tropical lines [54]. Another approach is to study an alternate setting of *Artin fans* in which there are no obstructions to tropical objects being true tropicalizations, as in the work of Dhruv Ranganathan, building on work of Martin Ulirsch and previous work by several authors (see [1] and the references there, and see Section 2).

These recent developments in tropical and nonarchimedean geometry have permitted the application of tropical and nonarchimedean techniques to problems in algebraic geometry. One such application concerns enumerative geometry and Gromov-Witten invariants. This includes Mikhalkin's Correspondence theorem [46] and further refinements and generalizations by [29, 11, 47], some of which will be discussed below. Another broad theme has been the development of specialization theorems relating ranks of algebraic and tropical linear series [2, 4, 14], which led to new tropical and nonarchimedean proofs of the Brill-Noether and Gieseker-Petri Theorems [24, 36]. Other significant developments include the identification of analytic spaces with limits of tropicalizations [26, 51], and lifting theorems relating algebraic, tropical, and nonarchimedean analytic intersection theories [15, 49, 50].

Moduli spaces. The results and techniques of tropical geometry have deep applications to the study of moduli spaces, which were a broad theme of the conference. For example, it has been known for over 25 years, since the work of Culler and Vogtmann on Outer Space [25], that moduli spaces of metric graphs exhibit many properties similar to moduli spaces of Riemann surfaces. This analogy, in which Outer Space corresponds to Teichmüller space and the outer automorphism groups of free groups correspond to mapping class groups, has been enormously fruitful, leading to important developments in topology and geometric group theory [8, 25, 28, 62].

Such analogies between metric graphs, algebraic curves, and their moduli spaces were rediscovered by several groups of tropical geometers, who noticed that moduli spaces of tropical curves of genus g with n marked points has many properties analogous to those of the Deligne-Mumford moduli stack of stable curves of genus g with n marked points [10, 13]. Abramovich, Caporaso, and Payne formulated this analogy rigorously in a canonical and functorial framework, reinterpreting the moduli space of stable tropical curves as the skeleton of the nonarchimedean analytification of $\overline{M}_{g,n}$ over a trivially valued field [3], and showing that the tautological forgetful, clutching, and gluing maps on moduli spaces induce the analogous tropical maps via nonarchimedean analytification and retraction to the skeleton.

The identification of the moduli space of stable tropical curves with the skeleton of $\overline{M}_{g,n}$ is not only a satisfying explanation of an old analogy, it also has significant applications. For instance, the reduced rational homology of skeletons encode (up to a degree shift) the top-weight cohomology of open varieties, and the tropical modular interpretation of the skeleton of $\overline{M}_{g,n}$ has been used to find new nontrivial cohomology classes on $M_{1,n}$ and $M_{2,n}$, outside the stable range [21, 22]. Furthermore, the techniques used to prove this identification, which build on deep and powerful constructions due to Berkovich [7] and Thuillier [58], apply much more generally to arbitrary toroidal Deligne-Mumford stacks, and have already been used to study a wide range of moduli spaces with toroidal compactifications.

These developments have coincided with a significant number of similar results, using tropical geometry as a rigorous framework for understanding analogies between moduli spaces and combinatorial objects, and establishing new results in algebraic geometry. Examples include: the Deligne-Mumford moduli space of

stable curves, whose skeleton is the moduli space of stable tropical curves [3, 10]; the moduli space of admissible covers, whose skeleton is the tropical Hurwitz space [17, 19]; the moduli space of principally polarized abelian varieties [10, 20]; the Hassett moduli spaces of stable, weighted marked curves [16, 56]; relative stable maps [64]; Severi varieties parameterizing embedded nodal curves [40, 63]; and a generalization of Mikhalkin's celebrated Correspondence Theorem [46] by constructing the tropicalization of the moduli space of labeled parametrized marked rational curves in a toric variety [29] (see also Section 2).

This workshop brought together leading researchers from three fields that rarely have overlapping conferences: tropical geometry, classical algebraic geometry, and nonarchimedean analytic geometry. Moduli spaces were a broad, but not exclusive, theme. The workshop was also enriched by the presence of several topologists, including R. Jiminez Rolland, S. Galatius, and J. Giansiracusa. The meeting provided an opportunity to bridge all of these communities and advance the state of the art on all sides. Another objective of the conference was to help advanced graduate students and younger PhDs working in these areas to connect with peers and experts and encourage collaboration between the various fields. To that end, younger participants were encouraged to give mini-talks on the first day of the conference. These were short, 5-10 minute slide presentations designed to help introduce participants to the audience's fresher faces. Conference participants included 5 graduate students and 8 postdocs.

2 Presentation Highlights

The talks reviewed techniques in the subject, introduced the audience to open problems, and announced new results.

The opening talk was given by **Hannah Markwig**, who described joint work with Cavalieri, Johnson, and Ranganathan relating Okounkov and Panharipande's Gromov-Witten/Hurwitz correspondence to tropical curve counting techniques [18]. These techniques realize the GW/H correspondence via tropical Hurwitz numbers, which are essentially combinatorial, graph-theoretic invariants.

Martin Ulirsch and **Dhruv Ranganathan** both spoke about logarithmic geometry and Artin-Olsson fans. Ulirsch's talk was a mostly expository account of his work on how tropicalization can be understood as analytification of an Artin fan [55]. Ranganathan used this perspective to study the realizability problem for tropical curves. This is the problem of whether a tropical curve is the tropicalization of an algebraic curve. A consequence of Ranganathan's formalism is a version of Murphy's law for the realizability question, showing that, in a rigorous sense, this problem can be arbitrarily difficult.

On Monday evening we had the mini-talks, discussed previously in Section 1, by **Martha Bernal Guillén, Morgan Brown, Daniel Corey, Ashwin Deopurkar, Cristhian Garay López, Rita Jiménez Rolland, Yoav Len, Tif Shen,** and **Mattia Talpo**. The goal of these short presentations was to introduce the younger participants who ordinarily would not have a chance to advertise their work. This format seemed to be very effective, as a number of senior participants asked questions and followed up later in private conversations with the graduate students and young PhDs.

Ravi Vakil discussed analogies between the topology and arithmetic of certain moduli spaces, specifically the discriminant loci in configuration spaces of points in affine space [60]. He explained that these analogies should reflect properties of the corresponding motivic zeta functions.

Kristin Shaw talked about work with Ren and Sturmfels on tropicalization of del Pezzo surfaces [57]. Focusing on cubic surfaces, she described a rich combinatorial structure of metric trees on the tropical surface corresponding to the arrangement of 27 lines on the classical cubic surface.

Andreas Gross presented results on tropicalization of cycles and cycles classes [29], and relations to intersection theory in algebraic geometry, building on earlier lifting theorems, such as those in [49], and giving generalizations in the context of tropicalizations of subvarieties of toroidal embeddings, as defined by Ulirsch [59]. As an application, Gross proves a correspondence theorem for genus 0 descendent Gromov-Witten invariants, identifying the tropical descendants of Markwig and Rau [45] with genus 0 logarithmic Gromov-Witten invariants of toric varieties.

Nathan Pflueger discussed recent work expanding on an earlier result of Cools, Draisma, Payne, and Robeva, using tropical techniques to give a new proof of the Brill-Noether theorem. In the earlier work, the authors classify divisors on a certain metric graph, the chain of loops, so long as the edge lengths are sufficiently general. Pflueger expands this result to classify divisors on the chain of loops with arbitrary edge

lengths [52]. As a consequence, he obtains upper bounds on the dimensions of Brill-Noether loci on general curves of fixed gonality [53].

Valery Alexeev spoke on joint work with Thompson, giving generalizations of the Losev-Manin moduli space, which corresponds to the type A root system, to other ADE and extended ADE types. Just as the Losev-Manin space has several natural interpretations, including as a moduli space of weighted pointed rational curves, and as the toric variety corresponding to the type A Weyl fan, each of these generalized Losev-Manin spaces has several natural interpretations as well.

A central problem in tropical geometry is that the absence of subtraction causes many standard results from linear algebra to fail in the tropical setting. **Jeff Giansiracusa** continued his work with his brother Noah Giansiracusa establishing the foundations for tropical mathematics by constructing an analogue of the exterior algebra for idempotent semirings [32]. This construction should be a key step toward applying techniques from linear algebra, or deeper algebraic structures such as free resolutions, to problems in tropical geometry.

Soren Galatius reviewed recent joint work with Chan and Payne on the homotopy of moduli spaces of tropical curves [22]. As discussed in Section 1, the topology of these moduli spaces of combinatorial objects has deep applications to automorphism groups of free groups and the cohomology of $\overline{M}_{g,n}$.

In another take on the topology of moduli spaces, **Yuji Odaka** described a metric that one can put on the moduli space of curves such that, when the moduli space is completed with respect to this metric, one obtains the moduli space of graphs at the boundary [48].

While much of the work being done in tropical geometry concerns rank-one valuations, **Tyler Foster** explained how, in joint work with Hully and Ranganathan, many important results hold over higher-rank valuations as well [27]. Higher rank valuation rings occur naturally when one considers multi-stage degenerations in algebraic geometry, or equivalently, the local ring in a neighborhood of a flag of subvarieties. Now that the standard results have been established in this setting, it seems likely that the higher rank theory will find a number of applications.

Johannes Nicaise spoke on joint work with Payne and Schroeter [47] relating tropical geometry to motivic invariants of semialgebraic sets, in the framework of Hrushovski and Kazhdan [34]. This framework is applied to tropical refined curve counting, giving a precise geometric conjecture for the tropical refined curve counting invariants of Block and Göttsche [9] in terms of χ_{-y}-specializations of motivic measures in tropically defined semialgebraic subsets of relative Hilbert schemes of points and relative compactified Jacobians. Explicit tropical computations prove this conjecture in some nontrivial cases, including when $g = 1$.

3 Outcome of the Meeting and Scientific Progress Made

As mentioned in Section 1, this conference brought together leading experts and young researchers from several communities that seldom have the opportunity to discuss new ideas and learn from each other. Having a mixture of Ph.D. students, several postdocs and researchers in tropical geometry, classical algebraic geometry, and nonarchimedean analytic geometry from Europe, Asia, North and South America provided an ideal environment for exchanging perspectives and starting new collaborations.

The lectures, as described in Section 2, were well received, and the younger participants appreciated the opportunity to present their work to a specialized audience in the Monday mini-talks. The ample time between the lectures provided an opportunity for smaller groups to collaborate and begin new projects. Intangible benefits, such as building a community, establishing mathematical as well as personal connections, and of course, disseminating knowledge, were widely mentioned. We also got feedback from several participants about very concrete results from the workshop. These range from new projects conceived at the conference to existing projects completed and papers made ready for publication. Many participants stated that this was one of the best conferences ever attended.

We end this report with some of the testimonies gathered after the meeting (in alphabetical order):

Renzo Cavalieri: "The workshop was a really wonderful occasion to get together with several collaborators, and listen to extremely relevant work in the field. The workshop was masterfully organized, both in terms of number of talks, choice of topics, and quality of the lectures. It is certainly going to impact my research. I had the chance to discuss rather new projects with two distinct sets of collaborators. "

Ethan Cotterill: "Being able to participate in the tropical and nonarchimedean conference in Oaxaca helped advance my research program in a number of ways, for which I am very grateful. Meaningful inter-

actions outside of the talks included:
—A discussion with Farbod Shokrieh about the use of the abelian sandpiles module for Sage to compute discrete Jacobians and ranks of divisors on graphs;
—A discussion with Matt Baker in which he clarified to me the algorithm of Luo and Manjunath for deciding when rank-one divisors on metrized complexes lift to smooth curves;
—A discussion with Maria Angelica Cueto in which she explained to me the orthant-shooting algorithm for calculating the Chow polytope of an embedded projective variety from its tropicalization;
—Several discussions with César Lozano Huerta related to calculations of Chow forms of embedded projective curves; and
—Discussions with Nathan Pflueger and Melody Chan related to counting secant planes to linear series of general curves via Eisenbud–Harris limit linear series and tropical methods."

Maria Angelica Cueto: "The meeting was deeply inspiring to me. In addition to the talks by well established researchers, I enjoyed listening to the next generation of people working in the areas of the workshop. I had the opportunity to discuss ongoing projects with two of my collaborators (Hannah Markwig and Daniel Corey) and made serious progress towards completion of both articles. But most importantly, a 5 minutes short presentation by Daniel Corey on our joint work triggered discussions with Vivek Shende and Martin Ulirsch on cluster algebras, and faithful tropicalization. They made me aware of several results and contradicting perspectives on these subjects that are available in the literature. These conversations for sure saved us several months of literature search and helped clarify the state of the art in the subject as well as pointing out possible solutions to questions that have been puzzling us for several weeks. Attending this meeting was very fruitful!"

Cristhian Garay López: "I had meaningful conversations with Oliver Lorscheid, and I am planning to collaborate with him in the near future. The subjects I discussed with Oliver include hyperfields and log structures on algebraic varieties."

Noah Giansiracusa: "It was a fantastic workshop, one of the best I've ever been to! My brother and I had multiple conversations at the workshop inspired by the talks that we are currently working on (one project being families/moduli spaces of tropical linear spaces), and he and I also spoke with Chris Manon after Ravi's talk about a possible Grothendieck ring of tropical varieties and we hope to pursue this in the months to come."

Oliver Lorscheid: " Here are some points where I profited from the workshop:
—Progress in my collaboration with Martin Ulirsch on scheme theoretic skeleta via blue schemes.
—Discussions with Matt Baker and Ravi Vakil made clear that early ideas for Cech cohomology for tropical schemes would not work out, but that it requires more sophisticated methods, like matroids over hyperfields, in the sense of Baker's paper of the same name."

Hannah Markwig: "Angie and I talked a lot about modifications of genus two curves and hyperelliptic covers. Renzo, Dhruv and I talked about the Fock space approach, correspondence theorems for stationary descendant invariants, and planar Hurwitz numbers"

Dhruv Ranganathan: "The BIRS-CMO workshop on Moduli Spaces was a unique opportunity to present my work to experts in various aspects of the field. Ravi Vakil and I had the opportunity to advance our ongoing research project, and Diane Maclagan had a number of helpful suggestions and interesting questions regarding the work that I presented on the first day. In particular, she made me aware of a number of helpful and relevant techniques in polyhedral geometry that I was unaware of. More generally, it was also encouraging, as a young participant, to know that senior mathematicians expressed interest in the work that I have been doing. Andreas Gross, a fellow graduate student, and I have been working on similar but distinct aspects of tropical enumerative geometry for the last several years, and have rarely had the opportunity to talk in person for an extended length of time. We had such an opportunity at the conference and I believe both our long-term research ideas benefited greatly from this. Hannah Markwig, Renzo Cavalieri, and I had the chance to talk about new project ideas that I believe will be quite fruitful, as well as spend some time clarifying the details of a paper that is in preparation. The environment at CMO was extremely conducive to productive conversations between junior and senior participants as well, and I consider it one of the best conferences I have had the opportunity to attend."

Martin Ulirsch: "The amount of scientific interactions was mind-blowing and I managed to start several collaborations with other participants as well as significantly advance already ongoing ones with some of my collaborators. For example, together with Tyler Foster and Dhruv Ranganathan we started two new projects

that seem to be two sides of the same coin: a notion of a moduli stack of metrized curve complexes as well as a Riemann-Roch formula for logarithmic curves. Moreover, this workshop also gave me the chance to meet with two of my collaborators, Renzo Cavalieri and Melody Chan, and make progress in an ongoing project, whose aim is to develop a notion of a moduli stack of tropical curves. All in all, this was certainly one of the best workshops I have ever been to and I am hoping for a sequel."

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