Computational Algebra and Geometric Modeling

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Abstract

The aim of this workshop was to present new results, algorithms, developments and applications of effective algebraic geometry (computational algebra) in geometric modeling. On the one hand, Algebraic Geometry has developed an impressive theory targeting the understanding of geometric objects defined algebraically. On the other hand, geometric modeling makes frequent use of virtual shapes based on algebraic models in practical and difficult problems. As recent and interesting developments have shown, these two domains can benefit from each other. This workshop contributed to reinforce the natural bridge which exists between these two areas, allowing a better analysis of key problems and of related approaches.

1 Overview of the Field

Both effective algebraic geometry (i.e. computational algebra) and geometric modeling deal with algebraic manifolds, typically curves and surfaces, but from distinct perspectives. Algebraic geometry traditionally studies their theoretical properties, while geometric modeling uses these curves and surfaces to represent objects on a computer for industrial design, manufacture, architecture, and entertainment. The subjects histories are also disjoint, with geometric modeling residing in the realm of computer science and using mathematical tools from numerical analysis and approximation theory while algebraic geometry is firmly anchored within classical mathematics. However, this characterization overlooks an important overlap. Geometric modelers need to prove properties of their modeling algorithms and characterize their solutions, while algebraic geometry is turning more to applications, with computation and visualization becoming important tools.

Starting in the 1980's geometric modeling began to use tools developed in algebraic geometry. By the turn of the century, this interaction deepened with methods arising in geometric modeling having an impact on computational algebraic geometry, as well as a significantly deeper use of notions from algebraic geometry by geometric modeling. Those developments led to a pivotal conference in 2002 on Algebraic Geometry and Geometric Modeling (AGGM) held in Vilnius and organized by Ron Goldman and Rimas Krasauskas, who spearheaded these deepening interactions. This workshop was the first conference where the two communities collaborated. The value of further cultivating these interactions was immediately recognized, and since 2002 there have been further meetings in Nice (2004), Barcelona (2006), Lijiang, China (2009) and BIRS, Banff, Canada (2013). More recently, these interactions have become part of a broader trend for the use of algebraic geometry in applications, which has led to the creation of a SIAM activity group on algebraic geometry and a SIAM journal partially devoted to this burgeoning field.

The past twenty years have seen a trend of deepening collaborations between Algebraic Geometry and Geometric Modeling with ideas and methods flowing in both directions. There remains a significant opportunity for further development, both deepening these interactions, and broadening them both within the subjects and by bringing in people who use these same objects in different fields. The purpose of this international conference on ”Computational Algebra and Geometric Modeling” was to continue and to deepen these growing interactions with the additional goal of focussing on effectiveness by using techniques and algorithms from computational algebra (effective algebraic geometry).

2 Recent Developments and Open Problems

The interactions between geometric modeling and effective algebraic geometry are driven by natural developments in each field. In effective algebraic geometry it is the study of computational methods (i.e. computational algebra) while in geometric modeling, it is the drive to master and use more theoretical tools in the study of their basic objects. We provide here a list (non exhaustive) of some particular topics of strong collaboration in relation to this workshop.

2.1 Syzygies of rational parameterizations

A central problem in geometric modeling is to find the implicit equation for a rational curve or surface starting from a parameterization. Indeed, this change of representation has a lot of applications, in particular it allows one to drastically simplify intersection problems between parameterized curves and surfaces. In the most common practical situations, the source of parameterizations is either the projective line (rational Bézier curve), the projective plane (rational triangular Bézier surface) or a product of two projective lines (rational tensor product surface).

The implicization problem amounts to eliminating variables in a system of polynomial equations. Therefore, standard tools such as Gröbner bases or resultants can be used to solve the implicitization problem. However, these two methods are not efficient enough in practice because of the presence of base points, which are the source points where the parameterization is not well defined. Instead, the method of moving curves and surfaces is used because of its flexibility in the presence of base points. This method was introduced by Sederberg and Chen [45] for the case of parameterized curves, and then was regularly extended and improved during these last two decades for the case of surfaces (e.g. [20, 19, 39, 10]). The basic idea is to build a square matrix whose entries are linear forms in the implicit variables and whose determinant yields the implicit equation. As expected, the base points are here again the main difficulty. Syzygies of the coordinates of parameterizations play a key role in building such implicitization matrices ([17]) and these syzygies attracted a lot of attention from the commutative algebra community. Several extensions of the original method of moving curves and surfaces have then been introduced, built on the use of sophisticated tools from computational algebra (blow-up algebras, approximation complexes, Castelnuovo-Mumford regularity) [13, 3, 14, 25]. These approaches also lead to implicit matrix representations of a parameterization defined by a singular (i.e. non-square) matrix whose entries are linear forms in the implicit variables and whose rank drops exactly at the points in the image of the parameterization [12, 6, 9, 5].

Besides the computation of implicit representations of parameterized curves or surfaces, syzygies of parameterizations have been studied and analyzed because syzygies reveal many geometric properties, such as singularities. The importance of the first syzygy module of the coordinates of a parameterization goes back to the paper [22] where the concept of a μ-basis is introduced. In the context of rational plane and space curves, μ-bases are deeply linked to their singularities [49, 11, 52, 33, 32]; in the case of surfaces, there also exist some results in the same direction [4, 42] but a lot of problems, including the extraction of singularities, are still open in the case of surfaces.

As a natural extension, syzygies of a parameterization can be interpreted as the equations of the symmetric algebra of the ideal generated by the coordinates of the parameterization, and hence the equations of the Rees algebra of this ideal. This algebra has a strong geometric content since this algebra is the algebraic definition of the blow-up of the parameterization along the ideal (which is the defining ideal of the base points, up to saturation). This link between μ-bases, syzygies and the equations of the Rees algebra was first noticed in [18]. Soon this link became an active area of research, especially in the commutative algebra community [8, 16, 15, 28, 38, 21].

All the above developments lead to a ”geometry of syzygies” paradigm for parameterizations, that is to say the extraction of geometric information, such as the image, the degree, or the singularities, from the syzygies of the coordinates of parameterizations. Recent developments, such as those presented during this workshop, show that this approach is not limited to the parameterization of curves or surfaces, but can also be useful for dense volume parameterizations, for instance to compute a distance function to a rational curve or surface.

2.2 Semi-algebraic functions on general domains

NURBS (Non Uniform Rational B-Spline) models are nowadays the standard representation of complex geometry in CAGD. This accurate description of shapes is semi-algebraic, meaning that NURBS can be generated by gluing together high order rational Bézier (algebraic) patches. NURBS curves and surfaces are built from the B-spline functions that encode a piecewise algebraic function with a prescribed regularity at the seams. Despite many advantages, these representations also have important drawbacks. For example the difficulty in locally refining a NURBS surface and its topological restrictions may require many such patches with trims for designing complex geometry. To overcome these difficulties, an active area of research is to look for new blending functions in order to define new representations of shapes. Important classes of such functions are the T-spline [44, 46], LR-spline blending functions [24], or hierarchical splines [23, 27], that have been recently devised in order to efficiently perform local refinement.

More generally, an important problem is to analyze spline spaces associated to general subdivisions. So far, algebraic methods based on homological techniques have been extensively developed in order to provide new results, in particular regarding the dimension of these new spline spaces; see [41] and the references therein.

2.3 Algebraic methods for solving and designing in geometric modeling

Many problems in geometric modeling can be translated into an algebraic problem, either to design a new class of curves and/or surfaces, or to solve a geometric problem. Indeed, as the basic ingredients in geometric modeling are described (piecewise) algebraically, it is very often possible to turn a geometric problem into a system of polynomial equations. These equations are usually very particular, having strong structures (invariance under a certain group action, shape of the polynomial equations, determinantal formulation, base locus, etc.) and with relatively low degree and number of variables. Therefore, algebraic techniques can be used to take into account these specificities to design efficient and reliable algorithms for solving geometric problems. The class of ruled surfaces is a typical example where some specific algebraic methods, here computations with Grassmann coordinates ("geometry of lines"; [40]) have been used and developed in order to tackle difficult problems such as intersection [51, 30], fitting, collision detection [31, 34], or design under constraints. Another example are cyclides. Dupin and Darboux cylcides can be used both to model and to blend between existing models. Syzygies and μ-bases have been used successfully to analyze Dupin cyclides [29, 35], and Clifford algebras have been invoked to investigate Darboux cyclides [37, 36].

Some current researchers use this approach on other particular classes of surfaces, such as surfaces that are invariant a certain group action (e.g. under certain rotations or shears[1, 2]). There are also several works applying algebraic methods in order to solve a specific problem in degenerate situations, such as the extraordinary points of a Cattmull-Clark subdivision surface [50] where the complete treatment is still open for non-uniform subdivision schemes. Another useful application of algebraic techniques is to extract simple geometric primitives (cylinders, cones or tori) from the least possible number of points in a 3D point cloud [43]; these interpolation problems are still waiting for a complete and efficient answer. Finally, there is also the use of algebraic approaches to define new classes of curves and surfaces for design purposes. Typical examples here are the definition of PH B-spline curves, as an extension of the classical PH Bézier curves, quantum Bézier and B-spline curves and surfaces, as extensions of classical Bezier and B-spline curves and surfaces [7, 47, 48], and the definition of new toric Bézier patches [26], as well as methods for changing their representations (e.g. implicitization).

3 Scientific Progress Reported During the Workshop

This workshop was attended by 38 participants with strong expertise and interest in the fields of effective algebraic geometry and geometric modeling. This meeting was the occasion to have fruitful discussions that will hopefully lead to new collaborations. During the workshop, we had 27 talks of 45 minutes each in which new advances where reported. Below, we briefly list these talks under the themes mentioned in the previous section.

Syzygies of rational parameterizations

- Nicolas Botbol: Shapes of the simplest minimal free resolutions in $P^1 \times P^1$
- Laurent Buse: Syzygies and distance functions to parameterized curves and surfaces
- Falai Chen: Implicitization using moving planes and moving quadrics
- Carlos D'Andrea: On minimal generators of the ideal of moving curves following a rational plane parametrization
- Eliana Duarte: Implicitization of tensor product surfaces in the presence of a generic set of base-points
- Xiaohong Jia: Moving planes and moving spheres following dupin cyclides
- Shen Liyong: Strong μ-bases for Rational Tensor Product Surfaces
- Haohao Wang: Quaternion Surfaces
- Xuhui Wang: Complex, Hyperbolic and Parabolic Rational Curves

Semi-algebraic functions on general domains

- Peter Alfeld: Software for the Analysis of Multivariate Splines
- Bert Juettler: Interpolation by Low Rank Spline Surfaces
- Bernard Mourrain: Space of spline functions on general domains
- Jorg Peters: Geometric continuity, analysis at irregularities, and subdividal enclosures
- Ragni Piene: Algebraic splines and generalized Stanley-Reisner rings
- Alexandra Seceleanu: A hands-on approach to tensor product surfaces of bidegree (2,1)
- Amos Ron: Slicing the simplex
- Nelly Villamizar: Geometrically continuous splines on surfaces of arbitrary topology

Algebraic methods for solving and designing in geometric modeling

- Chandrajit Bajaj: Algebra and Geometry of Reproducing Hilbert Space Kernels
- Andre Galligo: Extraction of cylinders, cones and tori, from minimal point sets
- Rida Farouki: Rational rotation-minimizing frames on polynomial space curves: recent advances and open problems
- Ron Goldman: Algebraic surfaces of revolution and algebraic surfaces invariant under scissor shears: similarities and differences
- Tom Sederberg: An Inverse Eigenvalue Problem for Subdivision Surfaces
- Josef Schicho: How to Count Euclidean Embeddings of Rigid Graphs
- Lucia Romani: Pythagorean-Hodograph B-Spline curves
- Wenping Wang: On Configurations Formed by Two Ellipsoids
- Rimvydas Krasauskas and Severinas Zube: Rational patches on Darboux and isotropic cyclides and their modeling applications

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