Dirichlet-to-Neumann Maps: Spectral Theory, Inverse Problems and Applications (16w5083)

Michael Levitin (University of Reading)
Adrian Nachman (University of Toronto)
Lauri Oksanen (University College London)
Iosif Polterovich (Université de Montréal)

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1 Overview of the Field

The five day workshop "Dirichlet-to-Neumann Maps: Spectral Theory, Inverse Problems and Applications (16w5083)" that took place in Oaxaca brought together researchers exploring different aspects of the Dirichlet-to-Neumann maps. The two central subjects of the meeting were inverse problems and spectral geometry.

The study of inverse problems for the Dirichlet-to-Neumann operator goes back to the celebrated work of Calderón [11] and is a well established subject with many applications, notably to geophysics and medical imaging. The landmark results on Calderón's inverse problem include [49] where Sylvester and Uhlmann, both of whom participated in the workshop, solved the problem in the smooth conformally Euclidean case in dimensions three and higher; [41] where Nachman solved the problem in dimension two, and [5] where Astala and Päivärinta solved the two dimensional case assuming only L^∞-regularity.

The spectral geometry of the Dirichlet-to-Neumann map is a new and rapidly developing branch of geometric spectral theory (see [27] and references therein). The eigenvalue problem for the Dirichlet-to-Neumann map is called the Steklov problem, as it was first introduced by V.A. Steklov more than a century ago. Despite such a long history, the geometric properties of Steklov eigenvalues and eigenfunctions remained relatively unexplored until recently. In the last decade, a number of important results were obtained on isoperimetric inequalities for Steklov eigenvalues, spectral asymptotics and geometric invariants associated with the Steklov spectrum, as well as nodal geometry of the Steklov eigenfunctions. These topics were in the focus of the workshop.

2 Recent Developments and Open Problems

2.1 Spectral theory of the Dirichlet-to-Neumann map

2.1.1 Recent developments

The workshop was kicked off by the talk of A. Girouard who presented some recent bounds for Steklov eigenvalues on Euclidean domains and Riemannian manifolds with boundary. B. Colbois spoke about the behaviour of Steklov eigenvalues under conformal deformations, as well as the connections between the Steklov spectrum and the Laplace spectrum of the boundary [17]. In particular, some estimates for Steklov eigenvalues in terms of the corresponding eigenvalues of the boundary Laplacian were presented under certain geometric assumptions. In the last day of the workshop, M. Karpukhin presented a surprising estimate for Steklov eigenvalues on Riemannian surfaces, which is proved using an auxiliary Steklov boundary value problem on differential forms [35].

The talks of D. Sher and A. Hassannezhad were concerned with the asymptotics of Steklov eigenvalues, as well as related geometric invariants and inverse spectral results. In fact, Hassannezhad reported on a work in progress which extends the techniques presented by Sher [28] in the case of surfaces to the Steklov problem on 2-dimensional orbifolds [4].

Several talks focussed on geometric properties of the Steklov eigenfunctions. J. Toth presented a proof of optimal upper and lower bounds for the size of the nodal set of Steklov eigenfunctions on real analytic Riemannian surfaces [43]. Y. Canzani gave an illuminating talk on geometric and topological structure of the zero sets of random waves on a Riemannian manifold [12]. While her results were concerned with the random linear combinations of Laplace eigenfunctions, she has outlined how similar ideas could work for the eigenfunctions of the Dirichlet-to-Neumann operator. A. Ruland presented her recent work on Carleman estimates and quantitative unique continuation for solutions of fractional Schrödinger equations [46]. As an application, an upper bound on the vanishing order of the eigenfunctions and on the size of the nodal set was obtained. This work is closely related to the results of Bellova–Lin [8], Zelditch [50] and Zhu [52], since the the Dirichlet-to-Neumann map could be viewed as a special case of a fractional Schrödinger operator.

The talks of A. Strohmaier and J. Galkowski provided a connection between the Dirichlet-to-Neumann maps and the problem of exploring resonances in different scattering problems. Both talks featured interesting numerical results. Numerical aspects of the Steklov-type problems were also in the focus of the talk by N. Nigam who presented new strategies for computing the eigenalues of mixed Steklov-Neumann and Steklov-Dirichlet problems with high accuracy. A more abstract, operator-theoretic approach to the study of the Dirichlet-to-Neumann maps was presented by F. Gesztesy.

2.1.2 Open problems

Asymptotics of Steklov eigenvalues for Lipschitz domains. This problem was proposed by I. Polterovich who attributed it to M. Agranovich. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded Euclidean domain with Lipschitz boundary. It is known that in this case the Steklov spectrum is discrete. The open problem is to prove the weakest form of the Weyl asymptotic formula for the eigenvalue counting function:

$$\sharp\{\sigma_k < \sigma\} = C_n|\Omega|\sigma^n + o(\sigma^n),$$

where $|\Omega|$ denotes the volume of Ω. This result has been previously established if the boundary is piecewise C^1 (see [1]).

The upper bounds for Steklov eigenvalues. B. Colbois presented (jointly with A. Girouard) the following question. Let Ω be a bounded $(n+1)$-dimensional domain with boundary Σ. Denote by

$$\bar{\sigma}_k(\Omega) = \sigma_k|\Sigma|^{1/n}$$

the normalized k-th Steklov eigenvalue, where $|\Sigma|$ denotes the n-dimensional (Riemannian) volume of Σ.

In [16] the following two results have been shown, see also [32]:

Theorem 1. *There exists a constant $C(n)$ depending only on the dimension n, such that, for each bounded domain Ω in the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ or in a hemisphere of $\mathbb{S}^{n+1}$, we have for every $k \geq 1$,*

$$\bar{\sigma}_k(\Omega) \leq C(n)k^{\frac{2}{n+1}}. \tag{1}$$

For a domain Ω in a $(n+1)$-dimensional Riemannian manifold N, the isoperimetric ratio $I(\Omega)$ is defined by

$$I(\Omega) = \frac{|\Sigma|}{|\Omega|^{n/(n+1)}},$$

where $|\Omega|$ denotes the $(n+1)$-volume of Ω with respect to the Riemannian volume element of N.

Theorem 2. *Let N be a Riemannian manifold of dimension $n+1$. If N is conformally equivalent to a complete Riemannian manifold with non-negative Ricci curvature, then for each domain $\Omega \subset N$, we have for every $k \geq 1$,*

$$\bar{\sigma}_k(\Omega) \leq \frac{\alpha(n)}{I(\Omega)^{\frac{n-1}{n}}} k^{2/(n+1)}, \tag{2}$$

where $\alpha(n)$ is a constant depending only on n.

Since the asymptotic behavior of $\overline{\sigma}_k$ is given by

$$\overline{\sigma}_k(\Omega) \sim c_n k^{1/n} \text{ as } k \to \infty,$$

where c_n is a constant depending only on n, one may expect a bound like (2) to hold with $k^{1/n}$ instead of $k^{2/(n+1)}$. In fact, for $n \geq 2$, this is impossible because it would imply a uniform upper bound on $I(\Omega)$ (namely, $I(\Omega)^{\frac{n-1}{n}} \leq \frac{\alpha(n)}{c_n}$) which is false.

Open question: Does the inequality (1) hold with the "asymptotically sharp" exponent $\frac{1}{n}$ instead of $\frac{2}{n+1}$ for $n \geq 2$?

Gap bounds for Dirichlet Laplacian and Schrödinger operators. M. Marletta presented some open problems for the Dirichlet Laplacian in $\Omega := (0,\pi)^d \subseteq \mathbb{R}^d$. The eigenvalues of this operator are the numbers

$$\{n_1^2 + \ldots + n_d^2 \,|\, n_1, \ldots, n_d \in \mathbb{N}\}.$$

It follows that, whatever the value of d, the minimum distance between consecutive distinct eigenvalues is 1; a fortiori,

$$\limsup_{n\to\infty}(\lambda_{n+1} - \lambda_n) > 0. \tag{3}$$

Weyl's law would tell us a lot less than this. Since the counting function

$$N(\lambda) = \text{number of eigenvalues} < \lambda$$

satisfies the asymptotics $N(\lambda) \sim C\lambda^{d/2}$, where $C > 0$, Weyl's law cannot guarantee $\limsup_{n\to\infty}(\lambda_{n+1} - \lambda_n) > 0$ whenever $d > 2$.

Giving up the condition $\Omega = (0,\pi)^d$ and allowing Ω to be any domain in $\mathbb{R}^d$ with finite volume, we can construct an example which shows that some restriction on the type of domain Ω is needed if the inequality (3) is to hold when $d > 2$. To do this, we choose positive numbers

$$r_n = n^{4/5}$$

and we let B_n be a ball constructed so that the ground state Dirichlet eigenvalue on B_n is r_n. Standard estimates show that the volume of B_n will be $O(r_n^{-d/2})$ and hence, for $d > 2$,

$$\sum_n \text{vol}(B_n) < \infty.$$

Taking Ω to be the disjoint union of such balls, we have constructed a domain of finite volume whose Dirichlet eigenvalues include all the numbers $r_n = n^{4/5}$. This means that (3) cannot hold.

Marletta posted three open questions.

1. If we consider the Dirichlet Schrödinger operator $-\Delta + q$ in $L^2((0,\pi)^d)$, for what class of potentials q does (3) hold?
2. For what class of domains Ω in $\mathbb{R}^d$ does (3) hold for the Dirichlet Laplacian?
3. As a generalisation of our first question, if Ω is such that (3) holds for the Dirichlet Laplacian, for what class of q does it hold for the Dirichlet Schrödinger operator?

Evidently a partial answer to the first question is that (3) holds for $\|q\|_{L^\infty} < 1/2$.

Determining the genus of a closed surface with boundary from the spectrum of its Dirichlet-to-Neumann map. D. Sher asked whether is it possible to determine the genus of a closed surface with boundary from the spectrum of its Dirichlet-to-Neumann map. Note that it cannot be done with symbolic/pseudodifferential invariants, as the full symbol is determined by the metric in a neighborhood of the boundary [38]. This also rules out the application of the the local heat invariants. One possible idea is to use the spectral determinant, the zeta function, as well as the non-local heat invariants.

Dirichlet-to-Neumann map and scattering. A. Strohmaier asked a question on the relation between the Dirichlet-to-Neumann operator and the scattering length. Consider a compact manifold M with boundary Y and assume that there is a Riemannian metric on M of product type near the boundary of the form $du^2 + h$, where h is a Riemannian metric on Y. For $a > 0$ consider the elongation M_a by attaching $[0, a] \times Y$ along Y to M. The manifold $X = M_\infty$ obtained by gluing $[0, \infty) \times Y$ to M is complete and is a manifold with cylindrical end. Note that it is not assumed that Y is connected.

One can now look at the eigenvalues of the Dirichlet-to-Neumann operator on M_a. As $a \to \infty$ there will be some eigenvalues that are going to zero in a specific way. There should be a full expansion as $a \to \infty$ with coefficients related to the Eisenbud-Wigner time delay operator at zero. This should in particular be interesting for p-forms and should relate to the results in [40].

An Open Problem for a Generalized Index of Meromorphic Operator-Valued Functions. Given a separable, complex Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the Banach space of bounded (linear) operators, and $\Phi(\mathcal{H})$ that of all Fredholm operators in $\mathcal{H}$. The set of densely defined, closed, linear operators in $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$. For a linear operator T in $\mathcal{H}$, $\mathrm{dom}(T)$ denotes its domain; if T is closable, its closure is denoted by $\overline{T}$. Given the notion of a finitely meromorphic function as employed in [6], [7], we now make the following set of assumptions (which apply to the Dirichlet-to-Neumann maps studied in [6]):

Hypothesis 3. *Let $\mathcal{D} \subseteq \mathbb{C}$ be open and connected, and $\mathcal{D}_0 \subset \mathcal{D}$ a discrete set. Suppose that the map $M : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{C}(\mathcal{H})$, $z \mapsto M(z)$, has the following additional properties:*
(i) $\mathcal{M}_0 := \mathrm{dom}(M(z))$ is independent of $z \in \mathcal{D}\backslash\mathcal{D}_0$.
(ii) $M(z)$ is boundedly invertible, $M(z)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $z \in \mathcal{D}\backslash\mathcal{D}_0$.
(iii) The function $M(\cdot)^{-1} : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{B}(\mathcal{H})$, $z \mapsto M(z)^{-1}$, is analytic on $\mathcal{D}\backslash\mathcal{D}_0$ and finitely meromorphic on $\mathcal{D}$.
(iv) For $\varphi \in \mathcal{M}_0$ the function $M(\cdot)\varphi : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{H}$, $z \mapsto M(z)\varphi$, is analytic; in particular, the derivative $M'(z)\varphi$ exists for all $\varphi \in \mathcal{M}_0$ and $z \in \mathcal{D}\backslash\mathcal{D}_0$.
(v) For $z \in \mathcal{D}\backslash\mathcal{D}_0$, the operators $M'(z)$ defined on $\mathrm{dom}(M'(z)) = \mathcal{M}_0$, admit bounded continuations to operators $\overline{M'(z)} \in \mathcal{B}(\mathcal{H})$, and the operator-valued function $\overline{M'(\cdot)} : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{B}(\mathcal{H})$, $z \mapsto \overline{M'(z)}$, is analytic on $\mathcal{D}\backslash\mathcal{D}_0$ and finitely meromorphic on $\mathcal{D}$.

Definition 4. *Assume Hypothesis 3, let $z_0 \in \mathcal{D}$, and $0 < \varepsilon$ sufficiently small. Then the generalized index of $M(\cdot)$ with respect to the counterclockwise oriented circle $C(z_0;\varepsilon)$, $\widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}(M(\cdot))$, is defined by*

$$\begin{aligned}\widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}(M(\cdot)) &= \mathrm{tr}_{\mathcal{H}}\bigg(\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, \overline{M'(\zeta)}M(\zeta)^{-1}\bigg) \\ &= \mathrm{tr}_{\mathcal{H}}\bigg(\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, M(\zeta)^{-1}\overline{M'(\zeta)}\bigg).\end{aligned}$$

F. Gezstesy posed the following

Open Problem. Prove or disprove that $\widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}(M(\cdot)) \in \mathbb{Z}$ under the given hypotheses.

The standard (Fredholm) case in which the index can be shown to be an integer is treated in [7, 29].

2.2 Inverse problems

2.2.1 Recent developments

Two of the main themes in the inverse problems talks were uniqueness and stability questions for Calderón's inverse problem.

As mentioned above, in dimensions three and higher, the conformally Euclidean case was solved in the now classical paper [49], however, the problem is open in more general geometries. This open problem is called the anisotropic Calderón's problem. The anisotropic problem has been solved in dimension two by using isothermal coordinates.

The anisotropic Calderón's problem was discussed by M. Salo, F. Nicoleau and R. Gaburro. M. Salo presented recent unique solvability results in the case of product geometries where one factor is an interval and the other is a Riemannian manifold with boundary satisfying suitable assumptions such as simplicity. He outlined a proof showing that the problem has unique solution in a fixed conformal class containing such a product metric. The proof is based on a reduction, using semiclassical analysis and a quasi-mode construction, to the inversion of an attenuated geodesic ray transform, a problem that is of independent interest. The results are recent generalizations of [20].

F. Nicoleau described novel counter-examples to unique solvability in the case where the data, given by the Dirichlet-to-Neumann map, is restricted so that the support of the Dirichlet boundary source is disjoint from the part of the boundary where the corresponding Neumann trace is known [19]. This example is remarkable, in particular, since the hyperbolic analogue of the anisotropic Calderón's problem is known to be uniquely solvable in many similar disjoint data cases.

R. Gaburro showed that the diffeomorphism invariance can be factored out in the case where the metric tensor is piecewise constant and the jump interfaces satisfy suitable conditions [3]. Furthermore, closely related to the anisotropic Calderón's problem, Y. Yang presented results on the hyperbolic analogue of the problem [48].

In the opening talk on inverse problems, V. Isakov discussed stability results for (isotropic) Calderón's problem. In general, the optimal stability is known to be conditionally logarithmic, however, if the Helmholtz equation is considered instead of the Laplace equation, the stability estimates improve when the wave number grows, as Isakov showed [34].

D. Dos Santos Ferreira discussed the stability of the partial data problem where the Dirichlet-to-Neumann map is again restricted. The previous stability results were conditionally double-logarithmic [14], and Dos Santos Ferreira described a way to remove one of the logarithms. Furthermore, L. Rondi presented a computational strategy to mitigate the ill-posedness of Calderón's problem [45].

Closely related to the isotropic Calderón's problem, M. Marletta presented partial data results in the case where the direct model is given by the Maxwell system [10], and P. Caro considered a scattering problem where the scatterer is modelled by a Gaussian random function [15].

Outside the context of Calderón's problem, J. Sylvester described how the Donoho-Stark uncertainty principle, related to compressed sensing, can be adapted and exploited in the study of inverse source problems for the Helmholtz equation [31], and M. Lassas and H. Zhou considered geometric inverse problems. M. Lassas presented a method to reconstruct a manifold from a point cloud [22], and H. Zhou talked about the lens rigidity problem for magnetic systems [51].

Finally, K. Krupchyk gave a unifying talk where techniques developed in the context of the anisotropic Calderón's problem were applied to problems in spectral theory [36].

2.2.2 Open problems

Characterization of Dirichlet-to-Neumann maps. M. Salo presented the problem of range characterization, that is, the problem to characterize the set of Dirichlet-to-Neumann maps. In addition to uniqueness and stability questions, described in recent developments above, this yet another fundamental aspect of Calderón's problem. The problem has been solved in the case of two dimensional disk, indeed, Sharafutdinov characterized all operators on the circle that are Dirichlet-to-Neumann operators corresponding to a Riemannian metric on the disk [47]. The problem has also been solved in the case Dirichlet-to-Neumann maps corresponding to circular planar resistor networks [18].

It was suggested that the case of multiply connected planar domains might be within reach by using Ahlfors' Theorem [2]. This theorem has been used in the study Steklov eigenvalues, see [26].

The open problem is formulated as follows.

Conjecture. Consider the annulus $M = \{z \in \mathbf{C};\ 1 < |z| < 2\}$. Let ω be a positive smooth 1-form on ∂M, and let A be a linear operator on $C^\infty(\partial M)$. The necessary and sufficient condition for the existence of a Riemannian metric g on M satisfying

$$\Lambda_g = A, \quad dS_g = \omega,$$

where Λ_g and dS_g are the Dirichlet-to-Neumann map and the arc length measure corresponding to g, is the existence of an orientation preserving diffeomorphism $\phi : \partial M \to \partial M$ such that A is the pull-back of $a^{-1}\Lambda_e$ under ϕ. Here e is the Euclidean metric on M and the positive function a is fixed by requiring that ω is the pull-back of adS_e under ϕ.

The problem can be generalized for planar domains of higher number of boundary components and also for Riemannian surfaces of higher genus.

Formally determined inverse problems Calderón's problem is formally determined in the two-dimensional case and formally over-determined in dimensions three and higher. Indeed, in the dimension n, the unknown metric tensor or conformal factor is defined on a n-dimensional domain M, and the Schwartz kernel on the corresponding Dirichlet-to-Neumann map is defined on the $2(n-1)$-dimensional space $\partial M \times \partial M$.

To get a formally determined problem when $n \geq 3$, the Dirichlet-to-Neumann map can be replaced by a suitable restriction of it. M. Lassas proposed to study this problem first in a case with an additional spectral parameter. Consider the family of Dirichlet-to-Neumann maps R_λ defined by $R_\lambda f = \partial_\nu u|_{\partial M}$ where

$$\Delta u + \lambda u = 0, \quad u|_{\partial M} = f,$$

and the spectral parameter λ is assumed to be outside the Dirichlet spectrum $\sigma(\Delta)$ of the Laplacian on the Riemannian manifold M.

Assuming that all the Dirichlet eigenvalues have multiplicity one, it has been shown that the family of functions $R_\lambda f$, for a generic fixed f, determines the Riemannian manifold M up to an isometry [37, 44]. Note that this problem is formally determined since, for fixed f, the family $R_\lambda f$, $\lambda \in \mathbf{R} \setminus \sigma(\Delta)$, is defined on the n-dimensional space $(\mathbf{R} \setminus \sigma(\Delta)) \times \partial M$. Moreover, there is a related result in a hyperbolic case, roughly speaking, corresponding to the Fourier transform of $R_\lambda f$ with respect to λ. In this case, the problem to determine M uniquely has been reduced to the lens rigidity problem, the problem discussed by H. Zhou in the workshop, without assuming that the Dirichlet eigenvalues have multiplicity one, but with a special choice of f rather than a generic choice [42].

The open problem is formulated as follows.

Conjecture. Let M be a compact smooth Riemannian manifold with boundary. Suppose that the dimension of M is three or higher. Then there is a function f, such that the family $R_\lambda f = \partial_\nu u|_{\partial M}$, $\lambda \in \mathbf{R} \setminus \sigma(\Delta)$, determines M up to an isometry.

Note that no non-degeneracy condition on $\sigma(\Delta)$ is assumed. A more difficult version of the problem is obtained by requiring that $R_\lambda f$ determines M, up to an isometry, for a generic function f, say in a suitably chosen Sobolev space.

Variants of the anisotropic Calderón's problem. As discussed above, the anisotropic Calderón's problem has been studied in the case of product geometries (M, g) where

$$M \subset \mathbf{R} \times M_0, \quad g(t,x) = dt^2 + g_0(x), \quad (t,x) \in \mathbf{R} \times M_0. \tag{4}$$

Here (M_0, g_0) is a simple Riemannian manifold with boundary. In [20] it was shown that the Dirichlet-to-Neumann map Λ_q, defined by $\Lambda_q f = \partial_\nu u|_{\partial M}$ where u solves

$$\Delta u + qu = 0, \quad u|_{\partial M} = f,$$

determines the potential $q \in C^\infty(M)$. However, it is an open problem if the same holds when M is a more general Riemannian manifold, and not of the above product form.

L. Oksanen proposed the following variation of this problem.

Conjecture. Let M be a compact smooth Riemannian manifold with boundary, and suppose that $\Omega \subset M$ is an open set and that the intersection $\Gamma = \Omega \cap \partial M$ is non-empty and strictly convex in the sense of the second fundamental form. Suppose that Ω is covered by a coordinate system in which g is the Euclidean metric, and that Ω is contained in the convex hull of Γ in these coordinates. Then Λ_q determines $q|_\Omega$ uniquely.

A more general variant of the conjecture is obtained by assuming that $(\Omega, g|_\Omega)$ is of the product form (4), rather than Euclidean. If q is supported in Ω, then the conjecture holds. A proof can be based on the Runge approximation theorem, as pointed out by M. Salo. A similar technique was used in the talk by R. Gaburro.

Another variant of the anisotropic Calderón's problem is obtained by assuming that $M \subset \mathbf{R} \times M_0$ and that (M, g) is a real-analytic Riemannian manifold, with g not of the above product form. Supposing moreover that q is real-analytic in the t-variable, the variant is to show that Λ_q determines q uniquely. An analogous hyperbolic problem, with g a Lorentzian rather than a Riemannian metric, was recently solved by G. Eskin in the preprint [21].

Spectral theory for the normal operator associated to the geodesic ray transform. The study of the anisotropic Calderón's problem is based on a reduction to the invertibility of the geodesic ray transform via a quasi-mode construction. In order to establish stability results for the problem, the continuity properties of the inverse of the geodesic ray transform needs to be studied. This again can be done using microlocal analysis of the normal operator associated to the geodesic ray transform. Moreover, a study of the spectrum of the normal operator would be useful, for example, when designing computational methods. G. Paternain suggested conducting such a study.

The geodesic ray transform of a function f on a Riemannian manifold with boundary M is defined by

$$If(x, \xi) = \int_0^{\tau(x,\xi)} f(\gamma(t; x, \xi))dt,$$

where $x \in \partial M$ and ξ is a inward pointing unit vector. Here $\gamma(\cdot; x, \xi)$ is the geodesic with the initial data (x, ξ) and $\tau(x, \xi)$ is the exit time of $\gamma(\cdot; x, \xi)$, that is, the smallest time $t > 0$ such that $\gamma(t; x, \xi) \in \partial M$. We equip the set

$$\partial_+ SM = \{(x, \xi) \in TM;\ x \in \partial M,\ |\xi| = 1,\ (\xi, \nu) > 0\},$$

where ν is the inward pointing unit normal vector field on ∂M, with the measure $(\xi, \nu)dxd\xi$ where dx is the surface measure on ∂M and $d\xi$ is the surface measure on the unit sphere on the fibre $T_x M$ of the tangent bundle. Using this measure, we have $I : L^2(M) \to L^2(\partial_+ SM)$ and the L^2-adjoint I^* can be defined. The normal operator $N = I^* I$ is known to be a classical elliptic pseudodifferential operator of order -1 in the interior of M, assuming that M contains no pairs conjugate points [25]. It follows from this structure that N has a discrete spectrum and that the eigenfunctions are smooth in the interior of M.

G. Paternain proposed the following two conjectures.

Conjecture. Let M be a simple Riemannian manifold with boundary, and consider the normal operator N on M. Then the eigenfunctions of N are smooth up to the boundary ∂M.

Conjecture. The Euclidean unit disk $\mathbf{D}$ in $\mathbf{R}^2$ is isospectrally rigid in the sense that if M is a simple Riemannian surface with boundary and the spectrum of the normal operator N on M coincides with the spectrum of the normal operator on $\mathbf{D}$ then M is isometric with $\mathbf{D}$.

The spectrum and the eigenfunctions of the normal operator on the Euclidean unit disk can be computed explicitly, see e.g. [39].

3 Presentation Highlights

The program of the meeting featured three 1-hour survey talks given by R. Schoen (University of California at Irvine), G. Uhlmann (University of Washington) and M. de Hoop (Rice University).

R. Schoen gave a overview of results on the eigenvalues of the Dirichlet-to-Neumann map on Riemannian manifolds with boundary. In particular, he presented his recent work with A. Fraser (UBC) on extremal problems for Steklov eigenvalues on surfaces, which uncovered beautiful connections between the spectral geometry of the Dirichlet-to-Neumann maps and the theory of minimal surfaces [23, 24].

The talk of G. Uhlmann focussed on inverse problems for the Dirichlet-to-Neumann maps. Among other results, he explained the key ideas of his celebrated work with A. Greenleaf and M. Lassas on invisibility and cloaking [30]. He discussed also the recent results on Calderón problem with partial data in the dimension two [33].

M. de Hoop discussed some fascinating applications of the Dirichlet-to Neumann maps in geophysical inverse problems [13]. He emphasized the importance of uniqueness and stability results for inverse problems with Lipschitz coefficients, because such low regularity is essential in the modelling of real world phenomena [9].

4 Scientific Progress Made

The meeting in Oaxaca was the first workshop bringing together experts working on inverse problems and spectral theory of Dirichlet-to-Neumann maps. As expected, it stimulated the exchange of ideas between the two fields. In fact, it became evident that the two areas share not only a common object of study, but also a number of similar techniques. For instance, the talk of M. Salo featured methods of semiclassical analysis which were also essential in the talks of J. Toth and J. Galkowski. Questions related to the study of spectral asymptotics played an important role in the talks by D. Sher and K. Krupchyk. We expect that the cross-fertilization of ideas between the two areas that took place during the workshop will bear fruit in the near future.

5 Outcome of the Meeting

The feedback from the participants regarding the scientific program of the meeting has been very positive. As A. Girouard pointed out,*" Not only was the conference location amazing, both from the cultural and convenience point of view, but also this conference was a great occasion to discover new aspects of inverse problems that are related to spectral geometry. This was certainly one of the best events that I attended."*

The workshop highlighted the connections between the different aspects of study of the Dirichlet-to-Neumann maps. The somewhat "interdisciplinary" nature of the workshop stimulated the speakers to focus on ideas rather than technical details, which was greatly appreciated by the participants. The two open problem sessions were most stimulating and lively.

There was a healthy balance between senior and junior researchers among the conference participants. It is also worth noting that the linguistic diversity of the group was quite impressive: one could often hear mathematical discussions in English, French, Russian, Finnish, Italian, Spanish — to name just a few "working languages" of the meeting.

The organizers are very grateful to BIRS and CMO and the staff of these institutions for the excellent organization of the meeting and the opportunity to hold a workshop in such a beautiful and exciting place.

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