REPORT ON 'SET THEORY AND ITS APPLICATIONS TO TOPOLOGY'

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1. General Overview of the Program

There is an extremely close relationship between Set theory and General (or Set-theoretic) Topology. Topology serves both as a testing ground and a source of inspiration for novel set-theoretic techniques and principles. For instance, the *Souslin hypothesis* lead to the development and formulation of *forcing axioms* - Martin's Axiom MA, the Proper Forcing Axiom PFA, and culminated in the strongest possible forcing axiom known as Martin's Maximum MM. These set theoretic axioms in turn helped settle many of the most important topological problems, for instance:

- ([20]) *Every hereditarily separable topological space is hereditarily Lindelöf.*
- ([3]) *Every compact countably tight topological space is sequential.*
- ([19]) *All autohomeomorphisms of* $\beta\mathbb{N}\setminus\mathbb{N}$- *the Čech-Stone remainder of a countable discrete space - are trivial.*

and recently also problems in functional analysis and theory of C^*-algebras:

- ([21]) *A Banach space is separable if and only if it contains no uncountable biorthogonal system.*
- ([7]) *All automorphisms of the Calkin algebra are inner.*

On the other hand, the forcing axioms themselves are strong forms of a classical topological result - the Baire Category Theorem.

However, not all topological problems are satisfactorily settled by forcing axioms. There are a number of problems the solution of which requires a search for a model of set theory which shares certain attributes of the *constructible universe* and certain attributes of a model of a forcing axiom. In particular, several of the modern metrization theorems each required a special tailor-made model of set-theory:

- ([18, 8]) *Every normal Moore space is metrizable.*

- ([15]) *Every compact space with perfectly normal square is metrizable.*
- ([11]) *Every separable Fréchet topological group is metrizable.*

One should emphasize, that all of the results mentioned are *consistency results*. In fact, the Continuum Hypothesis provides counterexamples to all of them. On the other hand, sophisticated set-theoretic methods can also help to provide ZFC results, e.g. the surprising construction of a hereditarily Lindelöf non-separable space in [16].

A statement of the objectives of the workshop and an indication of its relevance, importance, and timeliness

The symbiotic and mutually beneficial relationship between set theory and topology is undeniable. The workshop aspires to bring together leading experts from both areas in order to keep the exchange of ideas between the fields flowing. The inclusion of functional analysis is the result of recent developments, which showed that not only topological, but also Ramsey theoretic and forcing techniques can be successfully used to solve outstanding problems in this field.

The centerpiece of the proposed workshop, and of the connections between the areas of mathematics involved, is the the *Čech-Stone compactification* of a countable discrete space $\beta\mathbb{N}$, also known as the space of ultrafilters on $\mathbb{N}$. Astonishingly, many fundamental problems about $\beta\mathbb{N}$ are still open. A short sample:

- (Efimov) Does every compact space contain either a convergent sequence or a copy of $\beta\mathbb{N}$?
- (Szymański) Can $\beta\mathbb{N} \setminus N$ and $\beta\omega_1 \setminus \omega_1$ ever be homeomorphic?
- (Leonard and Whitfield) Is the Banach space $C(\beta\mathbb{N}) \simeq \ell_\infty/c_0$ primary?

Other basic problems have been settled only recently, e.g.:

- ([6]) (In ZFC) $\beta\mathbb{N}$ contains a non-trivial copy of $\beta\mathbb{N}$.

Algebraic aspects of $\beta\mathbb{N}$ are also of utmost interest here. The addition on $\mathbb{N}$ can be naturally extended to a binary operation on $\beta\mathbb{N}$, turning $\beta\mathbb{N}$ into a compact left-topological semigroup. The structure of the semigroup can be used to prove strong Ramsey theoretic statements such as the *Hindman finite sum theorem*. The research in this area is very active, see [10, 22, 12, 13]. There is a strong link between Ramsey theoretic statements and amenability properties of infinite groups [13, 17]. For instance, work of Kechris-Pestov-Todorcevic [13] showed that the question of whether the automorphism group of a countable ultrahomogeneous structure such as $(\mathbb{Q}, <)$ is *extremely amenable* is equivalent to the question of whether the class of finite substructures form a so-called *Ramsey class*. Here a topological group is extremely amenable if all of its continuous actions on compact spaces have a fixed point. The notion of a Ramsey class was the central object of study in a part of combinatorics known as *structural Ramsey theory* which was developed by Graham, Nešetřil, Rödel, Rothschild and others in the 1970s and 1980s. The techniques of [13] have since been adapted to other settings and used to study the homeomorphism groups of the pseudoarc and the Lelek fan of continuua theory [12, 4].

In [17], Moore translated the problem of the amenability of a given countable discrete group into a Ramsey-theoretic problem. He showed that the well studied question of whether Thompson's group F is amenable is equivalent to a Ramsey-theoretic statement and would follow from a natural generalization of Hindman's theorem to the setting of nonassociative binary systems. This analysis suggests new ways that the dynamics of the space of ultrafilters on a countable set can be used study the question of whether Thompson's group is amenable.

A sample of a method which directly links set theory, topology and functional analysis is the study of problems on the structure of function spaces. A somewhat typical outline of a construction of an interesting Banach space goes as follows: first one using forcing, or some other suitable set-theoretic tool or axiom constructs a Boolean algebra $\mathbb{B}$. Then, using Stone duality, one turns this algebra into a compact topological space $K = St(\mathbb{B})$ and finally, considers the Banach space $C(K)$ of continuous real-valued functions on K, see e.g. [14].

While this is a rather straightforward connection, it leads to much more important and sophisticated ones. For example topological properties of the dual ball of a Banach space X are closely related to purely analytical properties of the norm of X. One of the deepest connection of this sort where set-theoretic methods are relevant are the Tsirelson-type constructions of Banach spaces X by first constructing their dual balls. While this method has seen great success in the realm of separable Banach spaces, such as the solution of the unconditional basic sequence problem by Gowers and Maurey [9] or the collation of scalar-times-identity plus compact-operator problem by Argyros and Haydon [1], the non separable theory has seen recently great advances exactly because of its deep connection to set theory. Of these advances we could mention the construction of a non separable reflexive Banach space with no infinite unconditional basic sequence by Argyros–Lopez-Abad–Todorcevic [2] or the isolation of the threshold $\aleph_\omega$ as the minimal cardinal that can have the property that every weakly null sequence of that length must contain an infinite unconditional basic subsequence by Dodos–Lopez-Abad–Todorcevic [5].

2. Activities and progress made during the workshop

2.1. **General structure.** The general structure of the workshop was to have a small number of 50 minute talks (two longer talks on full days; none on half days) as well as a larger number of 25 minute talks (typically five on each day of the workshop). Generally the longer talks were reserved either or those presenting a major result or those who were giving an expository talk, often oriented around open problems, with the aim of generating broader interest in a collection of problems or techniques. We made sure to leave plenty of time between the talks and the mealtimes to help facilitate additional discussion and indeed many participants took this opportunity to work at the many blackboards which were stationed around the conference/hotel complex.

2.2. **Summary of the talks.** Alan Dow opened the conference with an hour talk in which he presented a broad collection of central problems in set theoretic topology. This was particularly fruitful since many of the conference participants were either working in pure set theory (as opposed to set theoretic topology) or else were students or junior researchers who had not yet been exposed to the problems.

Later in the meeting, Jan van Mill presented an excellent survey of problems and results relating to the so-called *Erdös space* and *complete Erdös space*. Again, the talk served to advertise the material — which lies in the domain of classical dimension theory — to participants coming from descriptive set theory, Fraıssé theory, and combinatorics such as Bartošova, Kwiatkowska, Sabok, Solecki, Todorcevic, and Tserunyan.

Probably the most striking new announcement during the meeting was made by Assaf Rinot. He presented a new lower bound on the consistency strength of the $\aleph_2$-Souslin Hypothesis in the presence of $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Previously the lower bound was a Mahlo cardinal (which is equiconsistent with $\mathsf{SH}_{\aleph_2}$ together with $2^{\aleph_0} = \aleph_1$ by work of Laver and Shelah). While it is still unknown if the consistency of the $\aleph_2$-Souslin Hypothesis, $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ can be derived starting from any large cardinal assumption, Rinot showed that his result is sharp in the following sense:

- if there is a weakly compact cardinal κ, then there is a forcing extension in which κ becomes $\aleph_2$, $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and any $\aleph_2$-Souslin tree contains an $\aleph_1$-Aronszajn subtree.

- if $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and any $\aleph_2$-Souslin tree has contains an $\aleph_1$-Aronszajn subtree, then $\aleph_2$ is a weakly compact cardinal in L.

Another high point for the meeting was Itay Neeman's talk. He presented a portion of his recent solution to Baumgartner's isomorphism problem for $\aleph_2$-dense subsets of $\mathbb{R}$. This problem asks whether it is consistent that $|\mathbb{R}| \geq \aleph_2$ and at the same time every pair of $\aleph_2$-dense subsets of $\mathbb{R}$ are order isomorphic. In the 1970s, Baumgartner proved that this question has a positive answer if $\aleph_2$ is replaced by $\aleph_1$. Neeman have shown that Baumgartner's problem has a positive answer by employing a strategy suggested by Moore and Todorcevic. Specifically he obtained the consistency of a certain combinatorial statement with CH, which Moore and Todorcevic should could then be converted into a model exhibiting a positive answer to Baumgartner's problem using a finite support iteration of c.c.c. forcings. At previous meetings, Neeman had almost always focused on different aspects of his solution when presenting his result. Since the proofs of his results are still being written, this provided a unique opportunity for the participants to understand a new aspect of his proof.

Juris Steprāns presented an application of Todorcevic's P-Ideal Dichotomy to the existence of universal graphs on ω_1 in models of ZFC where the Continuum Hypothesis fails.

Christina Brech jointly with J. Lopez-Abad and S. Todorcevic constructed large compact families of finite sets in order to prove the existence of Banach spaces X of large densities without subsymmetric basic sequences. This can be seen as a far reaching extension of a famous construction of Tsirelson who used the standard Schreier families of finite subsets of the index-set ω.

Piotr Koszmider presented a construction which he argued is the correct non-commutative analog of Mrowka's Ψ-space, i.e. a construction of a C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ satisfying the following short exact sequence

$$0 \to \mathcal{K}(\ell_2) \to \mathcal{A} \to \mathcal{K}(\ell_2(\mathfrak{c})) \to 0.$$

Alexandra Kwiatkowska as a joint work with Dana Bartosova presented an important step toward solution of an old problem of Uspenskii who conjectured that the universal minimal flow of the autohomeomorphism group of the pseudo arc is its natural action on the pseudo arc itself. It is know that the metrizability problem for the universal minimal flow of the automorphism group of a given Fraïssé structure is equivalent to the existence of Ramsey degree in the corresponding age. Bartosova and Kwiatkowska showed that the Ramsey degree is not defined in the age of the standard Fraïssé structure associated with the pseudo arc and therefore the universal minimal flow of the corresponding automorphism group is not metrizable. This indicates that the answer to Uspenskii original question may be negative.

Alexander Shibakov presented new and important results concerning the structure of sequentially compact topological groups: Building on the recent solution to the Malykhin's problem he solved a 40 year old problem of Nyikos by showing that consistently every sequential topological group is either Fréchet or has sequential order ω_1. In a similar vein he answered a question of Todorcevic and Uzcategui by showing that (in ZFC) the same conclusion holds for countable topological groups whose topologies are analytic.

Marcin Sabok spoke on his joint work with Huang and Shinko which is a generalization of a result of Dougherty-Jackson-Kechris. Specifically they proved that boundary actions of hyperbolic groups give rise to hyperfinite orbit equivalence relations. Here an equivalence relation is hyperfinite if it is an increasing union of Borel equivalence relations which each have finite equivalence classes.

Osvaldo Guzmán added to a very short list of ZFC constructions of maximal almost disjoint families with special combinatorial properties by constructing a +-Ramsey MAD family, answering an old question of Hrusak.

Sławomir Solecki showed his work on the structure of actions of finite monoids on compact left topological semigroups by continuous endomorphisms as means to presented unified approach to the ultrafilter methods in Ramsey theory. As an application he proved an extension of a theorem of Gowers solving thus a problem by Lupini.

Anush Tserunyan spoke on a classical problem in topology and analysis: can there exist a continuous function from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$ which is injective on a comeager set? She proved that the problem has a positive answer if $n = 1$. She also discussed a strategy for giving a positive solution to the problem in higher dimensions, developing test questions which seemed of independent interest.

Jeffrey Bergfalk presented a characterization of the cardinals ω_n in terms of algebraic topology. Specifically, he showed that ω_n is the first ordinal γ which has a nontrivial n-dimensional Čech cohomology group. His work suggest the existence of higher dimensional analogs of Todorcevic's statistics associated to his method of minimal walks.

Iian Smythe has characterized, in the presence of the existence of a supercompact cardinal, what it means for a filter in the infinite dimensional projections in a separable infinite dimensional Hilbert space to be generic over the canonical inner model $L(\mathbb{R})$. This is a noncommutative analog of Todorcevic's characterization of the $L(\mathbb{R})$-generic ultrafilters as being precisely the selective ultrafilters.

3. Diversity and the workshop's impact on the Mexican topology community

Historically both set theory and set theoretic topology have seen very few female researchers working in the subject. Similarly, while Mexicans have long been well represented in the set-theoretic topology community, the same can not be set about the set theory community. In both cases, this has changed substantially in the last 10-15 years. Within the last decade, set theory has seen many strong female researchers start working in the subject, particularly as it relates to other fields such as analysis and topology. We are proud to have featured many of these women as part of the workshop. Additionally, starting with the arrival of Michael Hrusak in Mexico, UNAM has started to create a community of Mexican set theorists. These played an integral role in the workshop as well. Additionally, the workshop served to expose many of the students and other young researchers in Mexico's fledgling set theory group to a broad cross section of renown expects working in set theory and related fields of analysis and topology.

4. Schedule of the Workshop

Monday

7:30–8:45	Breakfast
8:45–9:00	Introduction and Welcome
9:00-10:00	Alan Dow
10:00-10:30	Rodrigo Jesús Hernández Gutiérrez
10:30-11:00	coffee break
11:00-11:30	István Juhász
11:30-12:00	Juris Steprāns
12:00–15:00	Lunch
15:00–16:00	Itay Neeman
16:00-16:30	coffee break
16:30-17:00	James Cummings
17:00-17:30	Assaf Rinot
19:00–21:00	Dinner

Tuesday

7:30–9:00 Breakfast

9:00-10:00 Christina Brech

10:00-11:00 coffee break

11:00-11:30 Piotr Koszmider
11:30-12:00 Asger Törnquist

12:00–15:00 Lunch

15:00–16:00 Alexander Shibakov

16:00-16:30 coffee break

16:30-17:00 Jindrich Zapletal
17:00-17:30 Marcin Sabok

19:00–21:00 Dinner

Wednesday

7:00–9:00 Breakfast

9:00-9:30 Joerg Brendle
9:30-10:00 Dilip Raghavan
10:00-10:30 Osvaldo Guzmán

10:30-11:00 coffee break

11:00-11:30 Victor Torres-Perez
11:30-12:00 David Fernández Bretón
12:00-12:30 Natasha Dobrinen

12:30–13:30 Lunch
Free Afternoon
19:00–21:00 Dinner

Thursday

7:30–9:00	Breakfast
9:00-10:00	Sławomir Solecki
10:00-10:30	coffee break
10:30-11:00	Aleksandra Kwiatkowska
11:00-11:30	Dana Bartošova
11:30–13:00	Lunch
15:00–16:00	Jan van Mill
16:00-16:30	coffee
16:30-17:00	Anush Tserunyan
17:00-17:30	Yinhe Peng
19:00–21:00	Dinner

Friday

7:30–9:00	Breakfast
9:00-9:30	Jeffrey Bergfalk
9:30-10:00	Iian Smythe
10:00-10:30	Noé de Rancourt
10:30-11:00	coffee break
11:00-11:30	Claribet Piña
11:30-12:00	Carlos Uzcategui
12:00-12:30	Carlos Di Prisco
12:30–13:00	Lunch

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