MODULAR CATEGORIES–THEIR REPRESENTATIONS, CLASSIFICATION, AND APPLICATIONS (16W5049)

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August 15-19, 2016.

1. Background

A modular category $\mathcal{C}$ is a non-degenerate ribbon fusion category over $\mathbb{C}$ [37, 1]. Modular categories arise in a variety of mathematical subjects including topological quantum field theory [37], conformal field theory[27], representation theory of quantum groups [1], von Neumann algebras [20], and vertex operator algebras [28]. They are quantum analogues of finite groups as illustrated by the Cauchy and Rank-Finiteness theorems [4]. This analogy extends to applications in physics: the symmetries of (bosonic) topological phases of matter are related algebraically to modular categories.

Drinfeld centers, which are fundamental examples of modular categories, play important role in classification of fusion categories up to *categorical Morita equivalence*. It was shown in [11, 19] that two tensor categories are Morita equivalent if and only if their centers are equivalent as braided tensor categories, so the center is a complete invariant of Morita equivalence. For example, a fusion category $\mathcal{A}$ is *group-theoretical* [17], i.e., Morita equivalent to a pointed fusion category, if and only if its center $\mathcal{Z}(\mathcal{A})$ contains a Lagrangian subcategory. Existence of such a subcategory can be easily determined by looking at the S- and T-matrices. In a similar vein, extensions and equivariantizations of fusion categories can be characterized in terms of their centers. This approach was used in [18] where the categorical analogue of Burnside's p^aq^b-solvability theorem was proved. Classification of module categories can also be done in terms of centers, since module categories over a fusion category $\mathcal{A}$ are in bijection with Lagrangian algebras in $\mathcal{Z}(\mathcal{A})$ [12].

There is an approach to classification of modular categories based on the study of the categorical *Witt group* $\mathcal{W}$. This group was introduced and studied in [9, 10]. It consists of equivalence classes of modular categories modulo Drinfeld centers (the group operation is the Deligne tensor product of categories). The above equivalence relation is called the *Witt equivalence*. This definition extends the classical definition of the Witt group of quadratic forms. Thus, one can try to classify modular categories up to a Witt equivalence, or, which is the same, describe the group $\mathcal{W}$. A recent progress in this direction was made in [10]. It was shown there that unlike the classical Witt group (which is a torsion group), the categorical Witt group $\mathcal{W}$ contains both a torsion part and a free part. All non-classical examples of simple anisotropic categories known at present come from certain modular categories $\mathcal{C}(\mathfrak{g}, \ell)$ associated to affine Lie algebras. These categories are defined as follows. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and let $\widehat{\mathfrak{g}}$ be the corresponding affine Lie algebra. For any positive integer ℓ we denote $\mathcal{C}(\mathfrak{g}, \ell)$ the modular category of highest weight integrable $\widehat{\mathfrak{g}}$-modules of level ℓ is a fusion category.

The categorical Witt group $\mathcal{W}$ provides a new perspective on classification of conformal field theories. There is a common belief among physicists that all rational conformal field theories come from lattice and WZW models via coset and orbifold (and perhaps chiral extension) constructions (see [27]). A corresponding conjecture for modular categories says that $\mathcal{W}$ is generated by the Witt equivalence classes of categories $\mathcal{C}(\mathfrak{g}, \ell)$.

Groups of braided autoequivalences of modular categories can be thought of as categorical analogues of orthogonal groups. They received a lot of attention in recent years thanks to their relation with Picard groups and extension theory of fusion categories. It was proved in [11, 19] that for a modular category $\mathcal{C}$ there is an isomorphism

$$\mathsf{Pic}(\mathcal{C}) \cong \mathsf{Aut}^{br}(\mathcal{C}) \tag{1}$$

between the group $\mathsf{Pic}(\mathcal{C})$ (the *Picard group* of $\mathcal{C}$) consisting of invertible $\mathcal{C}$-module categories and the group $\mathsf{Aut}^{br}(\mathcal{C})$ of braided auto-equivalences of $\mathcal{C}$. From the physics prospective, Picard groups are symmetry groups of topological phases of matter. From the algebraic prospective, these groups are classifying objects for graded extensions of fusion categories. For example, given a finite group G, braided G-crossed extensions of $\mathcal{C}$ are parameterized by homomorphisms $G \to \mathsf{Pic}(\mathcal{C})$ and some cohomological data.

Another approach to understand the structure of modular categories $\mathcal{C}$ is from the arithmetic properties of their modular data, namely the S and T matrices. It is well known that the fusion rules of a modular category can be recovered from its S-matrix by the Verlinde formula [1]. In addition, the S and T matrices of a modular category $\mathcal{C}$ naturally defined a projective representation $\bar{\rho}_{\mathcal{C}}$ of $\mathrm{SL}(2, \mathbb{Z})$ which can be factored through $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$, where $N = \mathrm{ord}(T)$ [35]. This projective representation $\bar{\rho}_{\mathcal{C}}$ can be lifted to 12 ordinary representations $\rho_{\mathcal{C}}$ of $\mathrm{SL}(2, \mathbb{Z})$ with congruence kernels, and these liftings admit a certain symmetry under the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, conjectured by Coste and Gannon [22]. These arithmetic properties of the S and T matrices have led to the Cauchy theorem for spherical fusion categories [4] and the congruence of the vector valued modular forms associated with a self-dual C_2-cofinite rational vertex operator algebras [15]. Moreover, they have inspired new approaches for the classification of modular categories of small ranks [5] and dimensions [3]

2. Objectives

The main objective of the workshop was to understand the landscape of modular categories both as pure mathematical structures and as generalized group-like symmetries of quantum systems.

As mathematical structures, we are interested in their classification and representation. A recent breakthrough is the proof of the rank-finiteness conjecture: for a fixed rank, there are only finitely many equivalence classes of modular categories. The number theoretical tools that are developed for the proof can be used to advance several directions such as the classification of low rank modular categories. First we will pursue the extension of rank-finiteness to pre-modular categories and unitary fusion categories. Two more major open questions will be the property F conjecture for weakly integral modular categories and whether or not the modular S, T matrices determine a modular category uniquely. Another direction is the interplay of symmetries and modular categories as inspired by symmetry enriched topological phases of matter. The famous example of such a symmetry protected topological phase of matter is the new materials—topological insulators. Critical to both classification and finite group action on modular categories is the study of their representations—module categories. Module categories arise in physics as topological defects of topological phases of matter, and as topological boundary conditions. For applications, it is important to parametrize all module categories over a given modular category.

Application of modular categories to physics and quantum computation continues to inspire new mathematical problems and deepen our understanding of their structures. One direction is the close connection between modular categories and conformal field theories. The partition function on the tori for a conformal field theory is a modular form like object. So we will explore the possibility of attaching a modular form like object to a modular category with some extra data, potentially just a module category. Another direction is the promotion of a group symmetry of a modular category to a local gauge symmetry—so-called gauging. Gauging is the inverse of the procedure called taking the core by Drinfeld-Gelaki- Nikshych-Ostrik, and can also be regarded as a construction of new modular categories from an old one together with a finite group action. In particular, it leads to a large class of weakly integral modular categories by gauging finite group symmetries of pointed modular categories.

A long-term objective is to establish a robust, cohesive community of researchers working in modular categories and their applications. Vital to this objective is the inclusion of junior researchers and researchers in underrepresented groups in the effort, with both applied and theoretical backgrounds. Our workshop will emphasize this broad participation. We expect to find new approaches to old problems and open new directions on a mathematical structure that sits in the triple juncture of mathematics, physics, and computer science.

3. Workshop Summary

3.1. **Participants.** The participants were diverse in nearly every respect. For example:

- Stage of Career: 1/3 students/postdocs, 1/3 full professors and 1/3 assistant/associate professors or equivalent.
- Scientific backgrounds: Each of the areas: mathematical physics, condensed matter, topology, Hopf algebras, fusion categories, conformal field theory and subfactors is the central field of study for at least 2 participants.
- Underrepresented groups: 7 participants were from one or more underrepresented groups (lower than expected due to several last-minute cancellations).
- International Breadth: researchers working in 11 different countries participated, covering all continents except Africa and Antartica.

It is worth noting that we had an unusual number of cancellations, for various reasons including fear of the Zika virus. Many participants expressed concern over issues related to political unrest. Neither of these concerns had any real effect on the participants, and we would not hesitate to organize another workshop in Oaxaca.

3.2. **Lectures.** The first day of the workshop was devoted to the theory of modular categories. Day 2 mainly focused conformal field theory, vertex operator algebras and their relations to modular categories. Day 3 emphasized the topics on Hopf algebras and topological quantum field theory. Day 4 consisted talks on some of the above topics and more which included subfactors, tensor categories, vector valued modular forms and representations of vertex operator algebras. Day 5 was scheduled for other related topics.

Due to the scientific diversity of the participants, we asked several experts to give introductory lectures on their area, rather than the more traditional talks on recent advances in their own research. This was apparently successful, as several participants singled out this aspect of our workshop in their feedback. These introductory lectures were as follows:

- Victor Ostrik: Introduction to modular tensor categories and those arising from quantum groups
- Xiao-Gang Wen: Applications of braided fusion category to classify topological orders in 2-dimensional matter
- Alexei Davydov: Witt group of modular categories
- Ingo Runkel: Conformal field theory and universality
- Chongying Dong: On orbifold theory
- Yasuyuki Kawahigashi: Subfactors, conformal field theory and modular tensor categories
- Chelsea Walton: Survey of Quantum Symmetry in the context of Hopf (co)actions
- Noah Snyder: Topological Field Theory and Modular Tensor Categories
- Hans Wenzl: Centralizer Algebras for Quantum Groups
- Emily Peters: Subfactors, planar algebras, and fusion categories
- Scott Morrison: Modular data for centres
- Terry Gannon: Vector-valued modular forms and modular tensor categories

Ostrik's lecture introduced the definition of modular categories and described one of the main constructions, namely quantum groups at roots of unity. Davydov presented a recent very successful approach to understanding modular categories that has become a central theme: the Witt group. Morrison's talk brought us up to date by illustrating some computational tools for determining the modular data of a given fusion category.

Wen's lecture explained the main application in physics: modular categories as anyon models of topological phases of matter. This lecture was in some sense prophetic–this application was the subject of the 2016 Nobel prize in physics awarded 2 months after our workshop, and Wen himself was honored as the 2017 recipient of the Buckley prize in physics (jointly with A. Kitaev).

Conformal field theory is, in a sense, the father of modular categories (maybe in the same sense that topology is its mother). Runkel, Dong, Gannon and Kawahigashi's lectures covered various facets of conformal field theory, from its physical origins to various descriptions in terms of vertex operator algebras and von Neumann algebras as well as some details on the number theoretic connections through characters.

Snyder's talk covered the topological side of modular categories alluded to above. In particular, he described work of Douglas, Bartlet, Schrommer-Pries and Vicary that establishes a precise connection between topological quantum field theory (of various types) and modular categories.

Walton, Wenzl and Peters each covered distinct areas that connect modular categories to three seemingly distant fields: Hopf algebras (a source of both examples and inspiration for modular categories), representation theory of quantum groups (through braid group descriptions of centralizer algebras) and subfactors (the main source of so-called "exotic" modular categories.

3.3. **Selected Highlights of Presentations.** Here we give further details on some of the introductory lectures (derived from summaries provided by the speakers).

Chongying Dong (University of California at Santa Cruz): *On orbifold theory*

The orbifold theory studies a vertex operator algebra V under the action of a finite automorphism group G. Vertex operator algebra V is called rational if its module category is semisimple. Vertex operator algebra V is called C_2-cofinite if the subspace spanned by $u_{-2}v$ for $u, v \in V$ has finite codimension in V. The well known orbifold theory conjecture says that if V is rational, then V^G is rational and every irreducible V^G-module occurs in an irreducible g-twisted V-module for some $g \in G$. Proving this conjecture is definitely the major task in orbifold theory. In the case that G is solvable, the rationality and C_2-cofiniteness of V^G have been established in [7]. Professor Dong present a recent result with Li Ren and Feng Xu [16] on a progress in proving the orbifold theory conjecture: If V^G is rational and C_2-cofinite, then every irreducible V^G-module occurs in an irreducible g-twisted V-module for some g. So their result together with those in [7], [26] solves the orbifold theory conjecture completely if G is solvable.

Official Abstract: This talk will report our recent work on orbifold theory. The Schur-Weyl duality, generalized moonshine and classification of irreducible modules for the orbifold theory will be discussed.

Yasuyuki Kawahigashi (The University of Tokyo): *Subfactors, conformal field theory and modular tensor categories*

Kawahigashi presented theory of local conformal nets, which gives an operator algebraic description of a chiral conformal field theory. He explained representation theoretic aspects and presented a recent construction of a local conformal net from a vertex operator algebra and back, which is his joint work with Carpi, Longo and Weiner [30, 8].

Official Abstract: We present an operator algebraic formulation of chiral conformal field theory and show how it is related to subfactor theory. Appearance of a modular tensor category through representation theory and the role of alpha-induction machinery are explained. We also exhibit the current status of the relations between our operator algebraic approach and the one based on vertex operator algebras.

Scott Morrison (Australian National University): *Modular data for centres*

Morrison described "the modular data machine", a pleasantly successful algorithm for determining the modular data of a Drinfeld centre, using as input only the Grothendieck ring of the original fusion category. The algorithm in turn computes the conductor N, the abstract $SL(2, \mathbb{Z}/N\mathbb{Z})$ representation type, the T-matrix, Frobenius-Schur indicators, and finally the Galois action on simples. At each point there may be many consistent choices, and much of the "fine-tuning" of the algorithm involves controlling the resulting combinatorial growth. A penultimate step solves linear equations describing the change of basis from an arbitrary representative of the representation type to the basis of simples. Finally, we attempt to solve a system of quadratic equations. Although the final step is potentially arbitrarily difficult, we find for many examples that it is effective. In particular, the modular data machine very quickly determines the modular data for the extended Haagerup subfactor, essentially automating the arguments of [23]. As another application, the modular data machine proves that the "$c = 2$" category remaining in Larson's classification of non-self-dual rank 4 categories [31] can not exist, as there is no consistent modular data.

Official Abstract: The classification of small index subfactors has resulted in the discovery of some rather unusual fusion categories. Those coming from the extended Haagerup subfactor seem particularly interesting — at this point we know of no relationship to any family or standard construction. As part of the effort to understand these unusual objects, we have computed the modular data for the centres of these fusion categories. As it turns out, we need to know remarkably little about the fusion categories; the conditions on modular data are so restrictive that we can leverage information about the Galois action and the representation theory of $SL(2, \mathbb{Z})$ to completely determine the modular data. Time permitting, I'll indicate the range of examples we've since tested these techniques against.

Terry Gannon (University of Alberta): *Vector-valued modular forms and modular tensor categories*

Gannon began his talk with the modular data of the Drinfeld doubles of dihedral groups and semi-dihedral groups of order $8n$. He expected the T-matrices of these Drinfeld doubles for $n = 2$ were the same which would be counter examples for one of the suggested open questions. However, he verified later that the assertion was incorrect. After this example, he turned to chiral conformal field theory which has two mathematical formulations, namely vertex operator algebras (VOA) and conformal nets of factors. In both case, their category of representations is modular tensor category. He raised a reconstruction question from modular category and conjecture that given any modular tensor category, there is a rational VOA and a rational conformal net whose category of representations is that category. The second question is the classification of VOA for a given modular category and central charge. Since the characters of a rational VOA is vector valued modular form in which S and T matrices are normalizations of the modular data of the underlying modular category. Studying the vector valued modular form for the representations of $SL(2, \mathbb{Z})$ associated with a modular category is an approach for the classification question. Some example of vector valued modular form for the modular data of the double of a Haagerup subfactor was demonstrated. However, whether this vector valued modular form is the character of a VOA was not known.

Official Abstract: In my talk I'll explain how to find vector-valued modular forms whose multiplier is the modular data of a modular tensor category, and how that can help us reconstruct a rational vertex operator algebra from that category.

In addition to these introductory lectures, we had another 11 more technical talks on recent advances in all of these areas. The following highlights are derived from the summaries provided by a few of the speakers are as follows:

Paul Bruillard (Pacific Northwest National Laboratory): *Rank Finiteness for Premodular Categories*

Bruillard's talk addressed connections between modular categories and number theory. In particular, he discussed the Cauchy Theorem for Modular Categories and number theoretic techniques used to prove Rank-Finiteness for Modular Categories [4].

Official Abstract: A physical system is said to be in topological phase if at low energies and long wavelengths the physical observables are invariant under smooth deformations. These physical systems have *applications in a wide range of disciplines, especially in quantum information science. A quantum computer based on such systems are topologically protected from decoherence. This fault-tolerance removes the need for expensive error–correcting codes required by the qubit model. Topological phases of matter can be studied through their algebraic manifestations, modular categories. Thus, a complete classification of these categories would provide a taxonomy of admissible topological phases. In this talk we will discuss connections between modular categories and number theory. This connection allows one to show a finiteness result for modular categories that makes classification tractable, and provides new tools for classification. Time permitting we will cover a generalization of these finiteness results to premodular categories. Information Release: PNNL-SA-120321*

Julia Plavnik (Texas A&M University): *On the classification of weakly integral modular categories*

Plavnik's talk was a survey on the current situation of the classification program of weakly integral modular categories. Specifically, she discussed the progress by dimension: modular categories of Frobenius-Perron dimension $4m$ [3] and $8m$ [6]; and by rank: weakly integral modular categories of rank 6 and 7.

Official Abstract: In this talk we will give a panorama about the state of the problem of classification of weakly integral modular categories at the moment. We will present some of the known results for low rank and for specific dimensions, like 4m and 8m, with m a square-free odd integer. We will also explain some of the techniques that we found useful to push further the classification.

Jurgen Fuchs (Karlstad University): *Correlators for non-semisimple conformal field theories*
Fuchs began his talk with a fundamental task in CFT which is to determine all correlators. A more concrete goal is to provide universal expressions for all correlators. Correlators are special elements in an appropriate space of conformal blocks. Conformal blocks as vector spaces admit actions of mapping class groups and they can be realized as morphism spaces of suitable category $\mathcal{D}$. In his talk, such suitable category $\mathcal{D}$ is a modular tensor category which may not be semisimple. He then turned to the bulk theory. The bulk field correlators of logarithmic conformal field theories that are based on modular finite ribbon categories $\mathcal{D}$ can be described by a universal formula. Consistent systems of bulk field correlators in such theories are in bijection with modular invariant Frobenius algebras in $\mathcal{D}$.

Official Abstract: Given a factorizable finite ribbon category $\mathcal{D}$, by work of Lyubashenko one can associate to any punctured surface M a functor Bl_M from a tensor power of $\mathcal{D}$ to the category of finite-dimensional vector spaces. The so obtained vector spaces $Bl_M(-)$ carry representations of the mapping class groups $Map(M)$ and are compatible with sewing, in much the same way as the spaces of conformal blocks of a (semisimple) rational conformal field theory. I will present a natural construction which, given any object F of $\mathcal{D}$, selects vectors in all space $Bl_M(F, ..., F)$ (i.e. when all punctures on M are labeled by F). If and only if the object F carries a structure of a 'modular' commutative symmetric Frobenius algebra in $\mathcal{D}$, the vectors obtained by this construction are invariant under the mapping class group actions and are mapped to each other upon sewing. Thereby they are natural candidates for the bulk correlators of a conformal field theory with bulk state space given by F. (Joint with C. Schweigert [21]*.)*

Jan Priel (University of Hamburg): *Decomposition of the Brauer-Picard group*
The main point of Priel's talk was to argue that the Brauer-Picard group has naturally a structure very similar to a Tits system or a BN-pair. Therefore, there are subgroups of the Brauer-Picard group that bear a resemblance to a Borel subgroup, a Cartan subgroup, a Weyl group etc. It is a systematic approach to understand the Brauer-Picard group [25]. In addition, these subgroups have an interesting physical interpretation (e.g. EM-dualities, gauge symmetries etc.)

Official Abstract: Given a fusion category $\mathcal{C}$, the Brauer-Picard group $BrPic(\mathcal{C})$ is the group of equivalence classes of invertible $\mathcal{C}$-bimodule categories. It is an important invariant of $\mathcal{C}$ and appears as a key ingredient in group extensions of fusion categories. From a physics point of view, this group is also interesting, since it is the symmetry group of certain 3-dimensional topological quantum field theories. In *this talk, I would like to present an approach to calculate the Brauer-Picard group of the representation category of a finite group by providing a natural decomposition.*

Henry Tucker (University of Southern California): *Frobenius-Schur indicators and modular data*

Tucker's talk began with the definition of Frobenius-Schur indicators for spherical fusion categories $\mathcal{C}$ [34] and a formula in term of the modular data of $\mathcal{Z}(\mathcal{C})$ [33]. He then addresses the question that whether the indicators distinguish two pseudo-unitary fusion categories with the same fusion rules. It has been shown in [2] that the Tamabra-Yamagami categories are completely distinguished by their Frobenius-Schur indicators. He extended this result for a family of near groups by deriving a formula of the Frobenius-Schur indicators in terms of the Gauss sum of the underlying metric groups. He obtained the formula by computing the modular data of the center of the near groups. Assuming a conjecture of Evans-Gannon, he also obtained a formula of indicators of another family of the near groups. This formula involves the Gauss sums of two mysterious metric groups arise from this family of near groups.

Official Abstract: In this talk we will give a panorama about the state of the problem of classification of weakly integral modular categories at the moment. We will present some of the known results for low rank and for specific dimensions, like 4m and 8m, with m a square-free odd integer. We will also explain some of the techniques that we found useful to push further the classification.

Cris Negron (Louisiana State University): *Gauge invariants from the antipode for Hopf algebras with the Chevalley property*

Negron focused on the production of gauge invariants for non-semisimple Hopf algebras. Namely, he discussed some recent joint work of himself and S.-H. Ng [32] on the preservation of the traces of the powers of the antipode under gauge equivalence, and it's relationship to certain approaches to non-semisimple Frobenius Schur indicators introduced in [29] and [36]. In this work it was shown that among so called Chevalley Hopf algebras one can use the fusion category of semisimple representations in the larger category of all representations, and the robust theory of pseudo-unitary fusion categories, to verify that the traces of the powers of the antipode are gauge invariants. As it relates to the indicators, he discussed how the same methods used to verify gauge invariance of these traces can be used to verify preservation of a certain distinguished object in the pivotal cover of the category of representation of a Chevalley Hopf algebra, under gauge equivalence. This leads to an alternate proof of gauge invariance of the indicators for Chevalley Hopf algebras, a result due originally to [29].

Official Abstract: We will discuss invariance of the order of the antipode, and traces of the powers of the antipode, under gauge equivalence. In particular, we will see that these values are in fact gauge invariants for Hopf algebras with the Chevalley property (e.g. Taft algebras and duals of pointed Hopf algebras). If time permits we will discuss how our study relates to recent efforts of Shimizu to produce a categorial approach to the indicators of a non-semisimple tensor category. This is joint work with Richard Ng.

Patrick Gilmer (Louisiana State University): *An application of TQFT to modular representation theory*

Let K be an algebraically closed field of characteristic p. Let $L_p(\lambda)$ denote the K-irrep of $\mathrm{Sp}(2g, K)$ with highest weight λ. Gilmer defined certain sets of even or odd small p-colorings of a certain graph with g loops and a certain edge colored $2c$. Gilmer said that the dimension of $L_p(\lambda)$ for the $p-1$ weights described below could be identified with the numbers of such even or odd colorings.

This is a step in deriving the following theorem due to Gregor Masbaum and Gilmer. The proof of this theorem uses a TQFT defined over $\mathbb{Z}[\zeta_p]$. Gilmer also indicated how this TQFT is constructed within the the $SO(3)$ Witten-Reshetikhin-Turaev TQFT which is defined over $\mathbb{Z}[\zeta_p, \frac{1}{p}]$ (originally defined over $\mathbb{C}$). They adopt the the skein theory approach of Blanchet, Habegger, Masbaum and Vogel [24].

Theorem Let $p \geq 5$ be prime and put $d = (p-1)/2$. For rank $g \geq 3$, consider the following dominant weights for the symplectic group $\mathrm{Sp(2g, K)}$:

$$\lambda = \begin{cases} (d-c-1)\omega_g + c\,\omega_{g-1} & \text{for } 0 \leq c \leq d-1 \qquad (\mathit{Case}\ \mathrm{I}) \\ (d-c-1)\omega_g + (c-1)\omega_{g-1} + \omega_{g-2} & \text{for } 1 \leq c \leq d-1 \qquad (\mathit{Case}\ \mathrm{II}) \\ (d-2)\omega_g + \omega_{g-3} & \qquad\qquad\qquad\qquad (\mathit{Case}\ \mathrm{III}) \end{cases}$$

Put $\varepsilon = 0$ in Case I and $\varepsilon = 1$ in Case II and III. Also put $c = 0$ in Case III. Then

$$\dim L_p(\lambda) = \frac{1}{2}\left(D_g^{(2c)}(p) + (-1)^{\varepsilon}\delta_g^{(2c)}(p)\right) \quad \text{where}$$

$$D_g^{(2c)}(p) = \left(\frac{p}{4}\right)^{g-1} \sum_{j=1}^{(p-1)/2} \left(\sin\frac{\pi j(2c+1)}{p}\right)\left(\sin\frac{\pi j}{p}\right)^{1-2g}$$

$$\delta_g^{(2c)}(p) = (-1)^c \frac{4^{1-g}}{p} \sum_{j=1}^{(p-1)/2} \left(\sin\frac{\pi j(2c+1)}{p}\right)\left(\sin\frac{\pi j}{p}\right)\left(\cos\frac{\pi j}{p}\right)^{-2g}.$$

Official Abstract: For $p > 3$ a prime, and $g > 2$ an integer, we use Topological Quantum Field Theory (TQFT) to study a family of p-1 highest weight modules $L_p(\lambda)$ for the symplectic group $\mathrm{Sp}(2g, K)$ where K is an algebraically closed field of characteristic p. This permits explicit formulae for the dimension and the formal character of $L_p(\lambda)$ for these highest weights. This is joint work with Gregor Masbaum.

3.4. **Open questions.** Some open questions were presented by the participants during the workshop. These questions were summarized as follows:

- Does the modular data, (S, T), determine the modular category?
- Does the T-matrix determine the modular category? Here we must assume pseudo-unitary: $S = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & \mp 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ is a counterexample otherwise.
- Let B_n, the n-stranded braided group, act on $X^{\otimes n}$ for some object X in a modular category $\mathcal{C}$. Consider indices $i_1, i_2, \ldots, i_{n+1}$, then the pure braid group acts on $\mathrm{Hom}(X_{i_1}, X_{i_2} \otimes \cdots \otimes X_{i_{n+1}})$. This gives representations of the pure braided group P_n. Can we reconstruct the modular category, $\mathcal{C}$, from these representations (for all choices of X). Take $SU(2)_2$ and Ising modular category. These are both rank 3 and each only has one interesting object. Does this provide a counter example?
- Consider P_n acting on $\mathrm{Hom}(X_{i_1}, X_{i_2} \otimes \cdots \otimes X_{i_{n+1}})$ as in the previous question. This action splits via a finite quotient of P_n. Is this action always semisimple for a modular tensor category? What about unitary MTC?
- Let $\mathcal{C}$ be a modular category over $\mathbb{C}$. What is the field of definition? More specifically, is it true that the field of definition is $\mathbb{Q}(T)$? What is the smallest field in which F and R matrices can be solved for.
- If $\mathcal{C}$ is properly premodular, does the inclusion $\mathbb{Q}(S) \subseteq \mathbb{Q}(T)$ sill hold?
- Property **F** Conjecture: Let $\Sigma_{g,n}$ be a genus g surface with n punctures and $\mathcal{C}$ a modular category. We have a representation $\rho : MCG(\Sigma_{g,n}) \to \mathcal{H}$. The image is finite if and only if $\mathcal{C}$ is weakly integral.
- Is the field of definition of a (pre)modular category cyclotomic?
- Is the minimal field of definition well-defined for MTC?
- Does every non-pointed integral modular category contain a nontrivial symmetric subcategory?
- Formula for Frobenius-Schur indicators in premodular (braided fusion) in terms of S, T, and the R-matrix?
- What is the rank of $\mathcal{Z}(\mathcal{C})$ in terms of the rank of $\mathcal{C}$?

- Does the representation type (of the modular representation of $SL(2,\mathbb{Z})$ determine a modular category?
- If two finite groups have the same character tables are their composition factors the same?
- Let $\mathcal{C}$ be a nondegenerate finite braided tensor category (possibly not semisimple). Is there a classification program by "smallness"?
- Let $\mathcal{C}$ be (pseudo-unitary) premodular with $\mathcal{C}' = sVec$. Does there exist a modular category $\mathcal{D}$ containing $\mathcal{C}$ such that $\dim \mathcal{D} = 2\dim \mathcal{C}$? This is equivalent to a question of surjectivity of a canonical map in the Witt group setting (See question 5.15 of [10]). If the answer is yes, then there are exactly 16 possible D up to equivalence.
- Is there a notion of *smallness* that is more physically motivated (than the rank)? For instance, given a random category, what is most likely for you to get?
- Does the rank-finiteness theorem hold for a general premodular category?
- Does there exist $\mathcal{C}$ premodular such that $\mathcal{Z}_2(\mathcal{C}) = sVec$ not of the form $sVec \otimes \mathcal{M}$ where $\mathcal{M}$ is a modular category? Here $sVec$ is equipped with the non-unitary choice of spherical structure, it is rank 2 with simples $\mathbb{I}$ and J where the twists are 1, and $\dim \mathbb{I} = 1$, but $\dim J = -1$. The answer is yes for the unitary $sVec$. For example $PSU(2)_6$. The motivation is Reshetikhin-Turaev invariants and is related to normalizable $\mathcal{C}$.
- *Uniqueness of the irreducible module with the minimal weight for a rational vertex operator algebra.* Let V be a rational and C_2-cofinite vertex operator algebra, then V has only finitely many inequivalent irreducible modules $M^i = \oplus_{m\geq 0} M^i_{\lambda_i+m}$ with $i = 0, ..., p$ where $M^i_{\lambda_i+m}$ is the $L(0)$-weight and $M^i_{\lambda_i} \neq 0$, the λ_i which is called the weight of M^i and the central charge c are rational numbers [13], [14]. The irreducible module corresponds to the Frobenius-Perron dimension in the S-matrix must have the minimal weight. It is well known that for rational vertex operator algebras associated to positive definite even lattices, integrable highest weight modules for affine Kac-Moody algebras, the minimal series of the Virasoro algebras, there is always a unique irreducible module whose weight is minimal among the irreducible modules. It is expected that this is true for any rational vertex operator algebra. The uniqueness of the irreducible module with the minimal weight for a rational vertex operator algebra is important in studying the S-matrix, the fusion product and the modular tensor category associated to the vertex operator algebra.

4. Comments from the participants after the workshop

Here we collect the comments, in addition to the testimonials posted on the BIRS webpage, from the participants after the workshop.

- Expertly organized! The time between talks was especially valuable, as well as the problem session.
- I enjoyed all the talks, the food and the workshop location. I had a wonderful week.
- I found this workshop to be very stimulating; I learned a lot and had many interesting discussions. Working in the related areas of subfactors and fusion categories, I have recently become interested in modular categories. This workshop presented an excellent opportunity to learn about modular categories from a variety of perspectives. I particularly enjoyed the introductory talks from researchers in different areas with connections to modular categories, which were nicely complemented by more specialized talks about recent progress. The non-mathematical aspects of the workshop were excellent as well.

  Oaxaca is a lovely place with an interesting cultural heritage and fantastic food. The hotel treated us well.
- Oaxaca is a lovely place to do mathematics, and I'm looking forward to going back again.

  The workshop was very exciting — a lot is going on in modular tensor categories at the moment, with extraordinary connections to topological phases of matter. I enjoyed every talk (not what I would usually say after a conference!) and conversations during the week led to several prospective new projects. I've already made some travel arrangements and invitations for visits with other participants of the conference, to pursue these.

- I enjoyed the meeting and learned from the many excellent talks. I became interested in one of the questions proposed in the Problem Session.
- Great opportunity to meet colleagues and exchange fresh ideas
- Mathematical discussions among participants of a wide range of different specializations were very nice. The hotel and food were also excellent.
- I am very happy for the opportunity to see the leaders in the area.
- The topic of the workshop was right inside my research interests. So it was one of the rare workshops where I could follow the details of nearly every talk. There were also many participants I could discuss with, and all in all it was a very interesting and beneficial week for me. Thank you for organising the workshop.
- Many interesting talks, many nice discussions. I continued a collaboration (with Terry Gannon), but did not start a new one.
- I learned a lot from the talks and from the many fruitful discussions I had with other participants. The conference was very well organised. I liked the fact that many talks were overview talks or introduction talks. The organised trip to some of the tourist attractions was a great idea and I enjoyed it a lot. All in all, I only have great impressions of this conference.
- Everything was good, the open problems session was excellent, and the organization was good.
- I loved being updated on the most recent developments in the subject, including work in progress. The entire workshop was very enjoyable and at a very high level. The conference venue is very attractive.
- The meeting was extremely positive in both the mathematical and non-mathematical context. It was very useful to talk with a wide range of specialists in different topics that have as a common point modular categories. It is important to learn about the different possible approaches, connections and tools to advance in the understanding of these structures. I had the chance to work with some of my colleagues in ongoing projects and I have also started discussions with some other researchers that might end in new collaborations.
- This meeting was excellent. The location was superb and the talks were illuminating. Discussions at this conference have spawned new work.

5. Outcomes and Future Directions

Many of the participants had not met one another before our conference. This was particularly true for the younger participants and the participants from Mexico. But there were lively discussions between talks, at the end of the days, and research collaborations during evenings. Gannon made an unsuccessful effort to disprove a well-known problem in the field during the week. Many small groups worked in evenings such as on the problem of attaching vector valued modular forms to modular categories.

Several follow-up conferences are already scheduled and we understand that several participants are planning to organize another similar meeting at BIRS. At the Joint Mathematics Meetings in Atlanta (January 2017) there are at least 4 separate special sessions that include organizers/participants from Oaxaca.

Unitary modular categories are anyon models of topological phases of matter in two dimensions. As the Nobel and Buckley prizes demonstrate, topological phase of matter is now well-established as a field in physics, but the mathematical framework to capture these new physical phenomena is still lacking. Modular categories provide such a framework for two dimensions, but a similar algebraic theory in three dimensional is poorly understood. Our workshop is a significant step in establishing a subfield in mathematics that is in and around modular categories and as an application providing the mathematical backbone for the understanding of topological phases of matter.

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