Flat Surfaces and Dynamics of Moduli Space

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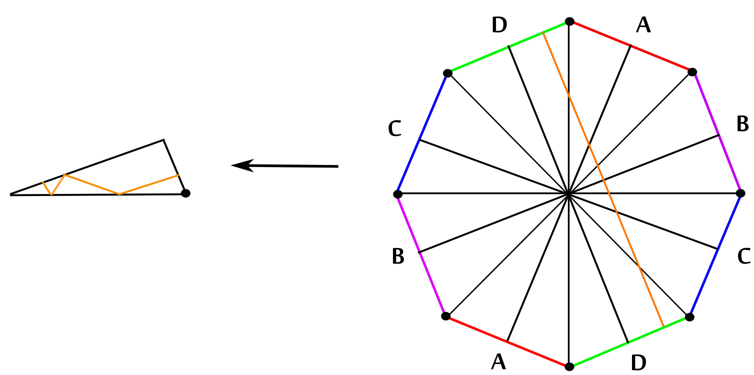
1 Overview of the Field

Flat surfaces (also called translation surfaces) are pairs consisting of a Riemann surface together with a holomorphic one-forms. The interest in flat surfaces stems from the dynamics on polygonal billiards. Instead of reflecting the trajectory at the boundary of the table one reflects the table and glues them along the reflection edges. The billiard paths become straight lines on the unfolded object and are thus much more easily accesible objects. If all the angles of the table are rational multiples of π, the resulting object is a flat surface. The



general case leeds to the case of infinite translation surfaces where much fewer results are currently known.

The set of all flat surfaces can be parametrized by a bundle $\Omega\mathcal{M}_g$ over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Neither the moduli space $\mathcal{M}_g$ nor the bundle $\Omega\mathcal{M}_g$ are homogeneous spaces. Nevertheless, the moduli space of flat surfaces $\Omega\mathcal{M}_g$ admits an action of the Lie group $\mathrm{SL}_2(\mathbb{R})$ and this action has many facets that mimic the action of a Lie group on a homogeneous space. On the other hand, the straight line flow on a flat surface is itself a very interesting dynamical system. A lot of the beauty and interest in this topic stems from the interplay interpreting questions about the dynamics of the straight line flow on flat surfaces in terms of the $\mathrm{SL}_2(\mathbb{R})$-action and vice versa. This situation (and its counterpart consisting of quadratic differentials and half-translation surfaces) is (so far?) pretty unique compared to all known spaces of complex manifolds. However, the field and the conference particpants are open to extension to other contexts with similar dynamics interplays, see Section 3.5.

2 Recent Developments and Open Problems

A major recent development is that the closures of $\mathrm{SL}_2(\mathbb{R})$-orbits are known to be nice manifolds. There are structure theorems that are analogs of Ratner's theorem in homogeneous dynamics (for horocyclic group actions) and that exclude wild orbit closures (as the geodesic flow in homogeneous dynamics may produce). More precisely, the moduli space $\Omega\mathcal{M}_g$ is stratified according to the number and multiplicity of the holomorphic one-form and the $\mathrm{SL}_2(\mathbb{R})$-action respects this stratification. The results of [EsMi] and [EsMiMo] show that such an orbit closure is always cut out by linear equation in the natural coordinate system on strata called period coordinates. Moreover, it was shown in [Fi] that they are moreover algebraic (i.e. quasi-projective) manifolds. Consequently, the *orbit closure problem*, i.e. determining the possible orbit closures for given genus, construction of such orbit closures or finiteness results, is one of the important and currently very active topics in the field. Orbits that are themselves closed are called *Teichmüller curves*. Primitive examples of such curves (such that do not arise from covering constructions from lower genus) are very rare and interesting. Some progress on understanding the known examples has been made in the last years (see [Ba], [LaNg] for a selection of results) but our knowledge is not complete yet.

The one-dimensional dynamical systems known as *Interval exchange transformations* have been a focus of study for ergodic theorists and geometers for at least the last 40 years, and for the last 20 years the primary methods of study have been to use renormalization dynamics (induction procedures due to Rauzy-Veech, Zorich, Yoccoz, and others) and their connections to Teichmüller dynamics. For example, properties like unique ergodicity [Ve, Ma] and weak mixing [AvFo, AvDe] have been studied using these methods. Many open questions remain: for example, the results of Masur and Veech imply that almost every interval exchange transformation is uniquely ergodic, but the Hausdorff dimension of the set of non-ergodic examples is yet to be computed in many cases. Other questions include relationships to other dynamical systems (joinings, factors, etc.) and questions about behavior along submanifolds inside the space of all interval exchange maps.

Understanding the renormalization dynamics described above naturally leads one to trying to understand *closed geodesics* in moduli space, which correspond to *pseudo-Anosov diffeomorphisms*. An outstanding question in the field is whether a flat surface can have a cyclic affine automorphism group generated by one pseudo-Anosov element. So far, we have no such examples- that is, every surface that admits a pseudo-Anosov may have potential extra affine automorphisms. A closely related problem is understanding the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of surfaces admitting pseudo-Anosov maps, and also to understand interval exchange maps associated to flows on such surfaces. Another problem which has been the focus of a lot of recent activity is the study of extremal pseduo-Anosov maps, namely, those with smallest (or small, in some sense) dilatation, and how this varies with genus and stratum.

The orbit closure problems described above have important applications to counting problems for billiards and translations surfaces, in particular by studying ratios of intrinsic volumes of these orbit closures. This is further connected to the problem of *counting square-tiled and pillowcase covers and understanding sums of* Lyapunov exponents of the Teichmüller geodesic flow. Important recent contributions include the work of Eskin-Kontsevich-Zorich [EsKoZo] and Chen-Möller-Zagier [ChMoZa].

As mentioned in the preceding section, there are extensions to other contexts with similar dynamics interplays. In recent years, one of the contexts that has attracted some attention among the experts is the one defined by infinite type translation surfaces. Consider for example an infinite type translation surface that covers a *finite type* translation surface in such a way that the deck transformation group acts by affine diffeomorphisms whose derivative is the identity (*a.k.a.* translations of a translation surface). These are the so called *translation coverings*. A precise example of this is given by Windtree models, which are billiards in the plane where obstacles are rectangles $[0,a]\times[0,b]$ displaced along a periodic lattice (*i.e.* translation covering where the deck transformation group is $\mathbb{Z}^2$). Using facets[1] that concern the $\mathrm{SL}_2(\mathbb{R})$-action on the moduli space of *compact* translation surfaces, it is possible to derive results about the generic behaviour of the translation flow on the translation surface associated to the Windtree model via unfolding. More precisely, for generic directions this flow is recurrent with diffusion rate $\frac{2}{3}$ but non-ergodic. [Add references here: Avila & Hubert, Delecroix,Hubert & Leliever, Franczek & Ulcigrai]. Despite the huge amount of work on the periodic case, very little is known about the original Windtree model defined by Paul and Tatiana Eherenfest in which obstacles are *randomly* disposed. Another class of infinite type translation surface on which substantial

[1] The so called, *Kontsevich-Zorich* phenomenon, presence of non-zero Lyapunov exponents, etc.

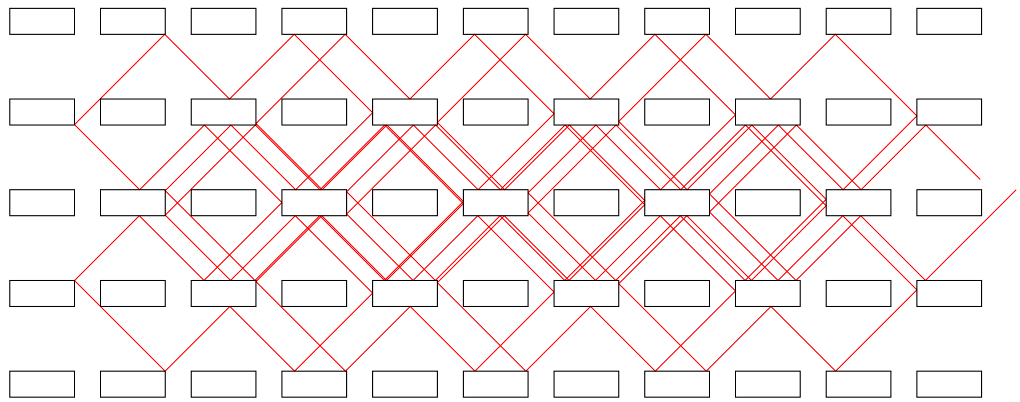


Figure 1: Trajectory in the wind-tree model

progress has been made recently is the one defined by Thurston-Veech constructions. This construction produces from a bipartite finite Ribbon graph for which the adjacency operator has a positive eigenfunction with eigenvalue $\lambda > 0$ a compact translation surface with a pseudo-Anosov map whose dilatation is precisely this eigenvalue. In [**?**] extends this construction for infinite graphs and provides a full description of ergodic invariant measures for the translation from on the corresponding infinite translation surface.

3 Presentation Highlights

In 13 research one-hour long talks the latest results were presented. This conference had a large number of young participants and consequently 4 Ph. D. studentens and 5 researchers in the early stages of their carrer were given the opportunity to present their work. We summarize the content grouped according to the main directions of the field.

3.1 Orbit closures

Ronen Mukamel in his talk on *Totally geodesic subvarieties in the moduli space of Riemann surfaces* presented a new example of a primitive $\mathrm{SL}_2(\mathbb{R})$-orbit closures, reporting on joint work with C. McMullen and A. Wright. The last examples of such orbit closures had been discovered in by McMullen in 2006 ([Mc]) and for while there were questions as to whether one should expect orbit closures other than the known ones and those derived from them by covering constructions. The example described by Mukamel lives in $\mathcal{M}_4$ and moreover provides the first example of a Teichmüller surface whereas prior the focus was on Teichmüller curves only. His construction combines intricate variants of classical construction with plane cubic curves to construct appropriate mappings to tori together with the fact that orbits closures are linear in period coordinates to construct such a locus. (Todo: more details, mention Teich curves)

When speaking about *Marked Points, Hubbard and Earle-Kra, and Illumination*, *Paul Apisa* extended orbit closure questions from the classical case of flat surfaces to the case of flat surfaces with marked points. In this context, the structure results ([EsMi] and [EsMiMo]) mentioned above can be applied as well and this gives a number of useful consequences. First, he showed that classical results by Hubbard and Earle-Kra respectively, about sections of universal families of curves, can be proved with these techniques. Moreoever, earlier results about special points on Veech surfaces (so-called periodic points) can be given an independent proof. (Todo: illumination)

The talk of *Quentin Gendron* on the *Closure of strata of flat surfaces* addressed the problem that the strata of $\Omega\mathcal{M}_g$ are quasi-projective varieties, but not projective i.e. not compact. Having such a compactification is useful both for inductive arguments in the orbit closure problem as well as for purely algebro-geometric questions about strata such as computing the Kodaira dimension. The moduli space of curves admits the

Deligne-Mumford compactification and the bundle $\Omega\mathcal{M}_g$ can be extended to the relative dualizing sheaf. Compactifying strata therein does however not record most of the information relevant from the point of view of flat surfaces, since the limiting objects are stable curves and on the irreducible components of the where the one-form vanishes, the flat geometry has become invisible. To overcome this, Gendron reported on the joint work [BCGGM] with M. Bainbridge, D. Chen, S. Grushevsky and M. Möller where an incidence variety compactifiction in a bundle over the moduli space $\mathcal{M}_{g,n}$ with marked points is described. This compactification records the limit location of the marked points even on those components of the stable curve where the one-form has vanished. The main result is a precise characterization of the boundary in terms of a collection of meromorphic differentials on the stable curve. In order to appear at the boundary point of a stratum these meromorphic differentials have to satisfy a number of expected compatibility conditions and in addition a subtle 'global residue condition' that reflects an application of Stokes' theorem scaling level by scaling level when approaching the boundary of the compactification.

The talk of *Dawei Chen* on *Strata of k-differentials* continued the compactification problem addressed by Gendron in a generalized setting. Half-translation surfaces should be treated on equal footing to translation surfaces. Their moduli space is the set of quadratic differentials that is stratied and non-compact, as in the case of one-forms. For compactification puposes quadratic differentials and k-differentials for arbitrary $k \geq 2$ can be dealt with in exactly the same way. For $k > 2$ these spaces do no longer admit an action of $\mathrm{SL}_2(\mathbb{R})$, but surfaces arising from cyclic covering constructions (implicitly present in the talk of Mukamel) can be viewed as k-differentials on lower genus surfaces. Moreover, the computation of cycles classes of the moduli space of curves, notably the double ramification cycle ([JPPZ]) relies on k-differentials. To describe the compactification, Chen reduced to the case of abelian differentials replacing the global residue condition by a lifted version and noted that this subtle additional condition only occurs from components of the stable curve, where the differential is a k-th power of a holomorphic one-form.

The talk of *Jonathan Zachhuber* on *Orbifold Points on Prym-Teichmüller Curves* contributed to understanding the geometry of Teichmüller curves. He reported on his joint work with D. Torres ([ToZa]), where they compute the final missing topological information about the Prym-Teichmüller curves constructed in [Mc], the Euler characteristic being computed earlier by Möller and the cusps being classified by Lanneua-Nguyen ([LaNg]). Since orbifold points correspond to extra automorphisms, the main point of his talks is to study certain Shimura curves classifying curves with large automorphism groups. More precisely, they need to compute the family of period matrices along this family of curves in order to detect when the Shimura curve intersects the Teichmüller curve, since in that case the abelian variety associated with the period matrix has real multiplication. The formulas for the number of orbifold points have interesting relations to (generalized) class numbers.

3.2 The horocycle flow

Following the landmark work of Eskin-Mirzakhani [EsMi] and Eskin-Mirzakhani-Mohamamdi [EsMiMo], where $\mathrm{SL}(2,\mathbb{R})$-orbit closures and invariant measures were shown to be *algebraic* in an appropriate sense, the key open problem in dynamics on the space of quadratic differentials is understanding orbit closures and invariant measures for the *horocycle flow*. This has important implications in understanding the distribution of the set of saddle connections on individual surfaces, for example, counting problems and problems on the fine-scale statistics of directions. Four talks at our meeting were devoted to various aspects of horocycle flows.

John Smillie and *Barak Weiss* both discussed aspects of their joint project with M. Bainbridge, on their construction of interesting orbit closures in 3-dimensional $\mathrm{GL}(2,\mathbb{R})$-orbit closures in strata of abelian differentials in genus 2 (known as eigenform loci), and the interaction with various natural foliations (known, variously, as rel surgery, the kernel foliation, Schiffer variation, or absolute period foliation) arising from period coordinates. Smillie's talk provided a historical overview of the field, reviewing the different properties and applications of diagonal and unipotent flows. A key tool in this work was using cylinder deformations to turn the problem into one of understanding dynamics on appropriate higher dimensional tori. Barak Weiss's talk explored more details of the so-called *rel surgery*, which give rises to an almost-everywhere defined flow which commutes with the horocycle flow. He described joint work with Pat Hooper, which used in some sense an inverse strategy to the work with Bainbridge-Smillie, in that properties of the horocycle flow are used to classify closures of rel leaves.

Kathryn Lindsey spoke about joint work with J. Chaika, in which they studied horocycle orbit closures for almost every rotation of a fixed translation surface. They showed that for any flat surface, the closure of its orbit under the horocycle flow in almost any direction is equal to its $\mathrm{SL}(2,\mathbb{R})$ orbit closure, and also characterized lattice surfaces in terms of minimal sets for the horocycle flow.

Grace Work spoke on understanding the fine-scale statistics of directions of saddle connections, using the horocycle flow. Computing the distribution of the gaps between slopes of saddle connections is a question that was studied first by Athreya-Cheung [AtChe] in the case of the torus, motivated by the connection with Farey fractions, and then in the case of the golden L by Athreya, Chaika, and Lelievre [AtChLe]. Their strategy involved translating the question of gaps between slopes of saddle connections into return times under horocycle flow on the space of translation surfaces to a specific transversal. Work described how to use this strategy to explicitly compute the distribution in the case of the octagon, the first case where the Veech group has multiple cusps, and how to generalize the construction of the transversal to the general Veech case (both joint work with Caglar Uyanik), and how to parametrize the transversal in the case of a generic surface in the stratum $\Omega\mathcal{M}_2(2)$ of surfaces with a double zero in genus two.

3.3 Interval exchange transformations and ergodicity questions

Alex Eskin spoke about a problem in the dynamics of interval exchange transformations motivated by number theoretic considerations. Peter Sarnak [Sa] conjectured that the Möbius function μ of number theory (defined by $\mu(n) = 0$ for nonsquarefree integers n, and otherwise $(-1)^k$, where k is the number of prime factors of n) is *disjoint* from any zero entropy dynamical system. This means that for any zero (topological) entropy dynamical system $T : X \to X$, and any bounded function $f : X \to \mathbb{R}$, and any $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)\mu(n) = 0.$$

This conjecture is implied by a stronger conjecture of S. Chowla's [Sa] on autocorrelations of the Möbius function. Bourgain-Sarnak-Ziegler [BoSaZi] proved the conjecture for the time 1-map of the horocycle flow on finite-volume quotients $\mathrm{SL}(2,\mathbb{R})/\Gamma$, and other examples of zero entropy systems (certain shifts, etc.) have been widely studied in recent years. Interval exchanges are a natural family of zero-entropy systems, and Eskin described joint work with Chaika proving this conjecture for interval exchanges of three intervals under a no loss of mass assumption. The problem reduces to a problem about understanding self-joinings of the system, which is a purely dynamical property.

Jon Chaika spoke about a joint work with Rodrigo Treviño giving new interpretations of certain criteria for unique ergodicity. Masur and Treviño have given the best known criteria for unique ergodicity in terms of recurrence behaviors of associated Teichmüller geodesics in moduli space. While these are typically dense in this non-compact space, it is natural to ask how long it takes the typical geodesic to leave compact sets for the first time. In particular, we can exhaust moduli space by compact sets given by surfaces with no closed geodesics with length strictly less than c and ask how much time it takes a Teichmüller geodesic to leave such a set. Masur addressed this question by proving what is known as a logarithm law. Translation surfaces give rise to Teichmüller geodesics and it is natural to ask if a Teichmüller geodesic satisfying Masur's logarithm law has a vertical flow which is uniquely ergodic. The work described showed a subtle difference between flat and hyperbolic geometry, namely, that if the logarithm law held for the flat systole, the flow was uniquely ergodic, but the same was not necessarily true for the extremal length systole (which coarsely gives distance in Teichmuller space).

3.4 Pseudo-Anosov diffeomorphisms

Ursula Hamenstädt explained that the $\mathrm{SL}(2,\mathbb{R})$ orbit-closure of a typical periodic orbit for the Teichmülller flow on a stratum of abelian differentials in genus at least three equals the entire stratum, giving a partial answer to the orbit closure problem for pseudo-Anosov maps discussed above. Similar ideas also yield finiteness of algebraically primitive Teichmller curves in genus at least three.

By far the most interesting elements of the mapping class group $Mod(S)$ of a closed orientable surface S of genus g are pseudo-Anosov classes. To each such element $[\phi] \in Mod(S)$ one can attach a dilatation factor

$\lambda(\phi) > 1$, whose logarithm can be viewed as the minimal topological entropy of any element of the class $[\phi]$. The number $\lambda(\phi)$ is also the exponential growth rate of lengths of curves under iteration of ϕ in any metric of S and appear naturally as the length spectrum of the moduli space $\mathcal{M}_g$ w.r.t. the Teichmüller metric.

Given that for fixed genus g the set $\{\lambda(\phi) \mid [\phi] \text{ is pseudo Anosov}\}$ is discrete, the least dilatation δ_g is well defined. No precise value of δ_g for $g \geq 3$ is known to date. On the other hand, even though there are *upper* bounds to $g\log(\delta_g)$ (given by Hironaka and Kin & Takasawa), very little is known about *lower* bounds.

Erwan Lanneau described joint work with Corentin Boissy, giving a general framework for studying pseudo-Anosov diffeomorphisms on (half-)translation surfaces. As an application, among other consequences, their method determines the systole of the hyperelliptic connected components (for the Teichmüller metric) for any genus, an astonishing advance in the field.

3.5 Analogs beyond compact Riemann surfaces

As explained before, the moduli space of compact translation surfaces of genus g is the complex vector bundle $\Omega\mathcal{M}_g$ of holomorphic 1-forms over the moduli space of Riemann surfaces $\mathcal{M}_g$. The extension of this bundle to the Deligne-Mumford compactification of $\mathcal{M}_g$ is the natural setting for limits of sequences of finite type translation surfaces, homeomorphic to the same surface, to exist.

Suppose now that we have a a sequence of (X_n, ω_n) of infinite type[2] translation surfaces where each X_n is homeomorphic to the same infinite type surface X_∞ and we want to define a space where the limit of this sequence makes sense (and is unique). If we were to mimic the compact case and define such a moduli space just as the quotient of Teichmüller space by the action of a mapping class group two problems arise:

1. Contrary to the compact case, in general there is not such a thing as *the* Teichmüller space of an infinite type surface. For example, if X_∞ denotes the an orientable infinite genus surface with one end, it is always possible to find two complete hyperbolic metrics of curvature -1 on X_∞ that are impossible to deform into each other via a quasiconformal map. For more details see [ALPS].

2. In general, the mapping class group of an infinite type surface does not act properly discontinuously on Teichmüller space, see [Fuj]. By taking quotients, as in the compact case, we are lead to non Hausdorff spaces and hence limits of sequences are not necessarily unique.

Because of these problems, one has to address the problem of defining the moduli space of infinite type translation surfaces in a different way. In this direction, *Patrick Hooper* makes a *tour de force* and describes the topology of the space of *all* translation structures on surfaces with a fixed basepoint. In his talk, Hooper focused on the description of a topology for the space of all translation structures on the disk and the natural disk bundle over this space. In this direction the main contributions are, first, that this topology is Hausdorff, second countable and metrizable. In particular, the uniqueness of the limits of surfaces in this space is guaranteed. Second, Hooper uses his result to provide convergence criteria for sequences of translation surfaces. These criteria are useful to determine when the limit surface inherits affine structure invariants from its near by neighbours.

Simion Filip's talk focused on how to use dynamical techniques to solve counting problems for special submanifolds of $K3$ surfaces. Among all complex two-dimensional manifolds, K3 surfaces are distinguished for having a wealth of extra structures. They admit dynamically interesting automorphisms, have Ricci-flat metrics (by Yau's solution of the Calabi conjecture) and at the same time can be studied using algebraic geometry. Moreover, their moduli spaces are locally symmetric varieties and many questions about the geometry of K3s reduce to Lie-theoretic ones. Filip discussed the analogue on K3 surfaces of the following asymptotic question in billiards - How many periodic billiard trajectories of length at most L are there in a given polygon? The analogue of periodic trajectories are special Lagrangian tori on a K3 surface. Just like for billiards, such tori come in families and give torus fibrations on the K3, and dynamical techniques on an appropriate moduli space (which, as mentioned above, is locally symmetric) can give an asymptotic counting formula.

[2]By *infinite type* in this context we mean that the fundamental group of the surface in question is infinitely generated.

4 Scientific Progress and Meeting Outcomes

The meeting was extremely fruitful for this community of researchers, and is has facilitated interesting collaborations and investigations into new phenomena, as well as illuminating new and unexpected connections.

On the final day, we held a hour-long *problem session*, in which several participants described potential new directions.

- Building on the talks of Gendron and Chen, the question arose of understanding the volume of the space of k-differentials (and more generally, spectral networks), and if these numbers had number-theoretic or geometric content. Other questions on *k-differentials* includes simply generating a library of interesting geometric examples.
- As a sequel to the talk of Mukamel, the question arose if there are other unexpected orbit closure (e.g. primitive Teichmüller surfaces beside his *Gothic Locus*. Similiar constructions seem to yield finitely many more example, but a classification or simply exhibiting an infinite set of such loci is a major problem. Moreover, understanding the behavior of the horocycle flow on these loci seems to be a very interesting direction.
- Other questions that arose centered on the *Siegel-Veech transform* and its properties, particularly its integrability properties and the potential of creating an appropriate definition for infinite-type surfaces.
- Related to the talk of Hooper, the question arose of finding infinite type but finite area translation surfaces such that the translation flow is defined for any direction (almost everywhere) and such that for almost every direction this flow is ergodic. Other questions on infinite type surfaces of finite area include understanding Veech groups. In particular there is no known lattice example.

We expect a lot of progress to arise from this meeting, and we hope to return to CMO in a few years to share this progress with the community.

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