[24w5212] Building and Enhancing Mathematical Reasoning

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This BIRS 5-day workshop occurred from Monday, April 29th, 2024 to Friday, May 3rd, 2024, at the Banff International Research Station in Banff, Alberta. We had 41 in-person participants and 13 online participants, ranging from graduate students, postdoctoral fellows, to research and teaching faculty, to industry from Canada, US, and Europe. The workshop was in a hybrid working group-style format with plenary lectures.

1 Workshop Objectives & General Questions

The following objectives, interconnected and inseparable, guided the work of the working groups:

1. Study a hierarchy of skills within mathematical reasoning. Which skills are most relevant, and can be taught to students who will, very likely, take only one mathematics course? How can such a mathematics course support their learning in other courses they will take? How could we develop a mathematical reasoning curriculum for students who plan to take several first-year and second- year courses, and possibly plan to earn a minor in mathematics? What would a mathematical reasoning curriculum for mathematics majors include in the first year and the second year?

2. Critically examine the existing practices of teaching proofs, and explore novel approaches to teaching proofs and proving, which are suitable for todays realities in Canada (weak students background, large classes); expand the range of contexts which are specifically used to teach proofs (to add to Euclidean geometry, which is a commonly used context). What are good ways of passing from teaching proofs to letting students construct their own proofs? Is this possible in the first your Calculus or Linear algebra course?

3. Teaching mathematical reasoning beyond a proofs course; identify opportunities where mathematical reasoning can be used naturally, and where students can actively participate in their learning. What are good ways of improving communication, for instance working on creating precise, clear mathematical narratives and logical arguments, and appropriately using mathematical symbols?

4. Explore the use of AI-based automated theorem proving software. Under automated theorem proving, Wikipedia lists three dozen software systems that assist in constructing proofs. Explore the ways on which such software can be used to facilitate learning of mathematical reasoning beyond theorem-proving.

5. Recognizing that acquiring mathematical reasoning skills requires a larger time scale (i.e., beyond a semester or a year), explore the ways to vertically integrate the teaching of mathematical reasoning into undergraduate mathematics curriculum. Identify opportunities to explicitly practice mathematical reasoning within various mathematics courses, especially in years 1 and 2. Consider mathematical reasoning in transition from secondary to tertiary mathematics, and mathematical reasoning courses for future teachers.

General questions that sparked discussions in working groups:

- In existing mathematics courses, what context needs to be modified or changed in what is emphasized?
- What instructor training is needed?
- What knowledge and skills will a mathematics instructor need to teach students mathematical reasoning?
- What knowledge and skills do students need to have in order to develop their mathematical reasoning?
- What are appropriate learning resources for both instructors and students?
- What can we expect from students, their families, K-12, universities, various communities and society in general as a meaningful way of supporting mathematical reasoning in students learning needs?
- What mathematical reasoning will be needed in the future?

2 Workshop Structure

Three main components of the workshop were:

1. Six plenary lectures in the first four days -

   Day 1 Rafael Nuñez, University of California, San Diego, CA, USA, and ETH, Zürich, Switzerland, and Terry Gannon, University of Alberta, Edmonton, AB, Canada;

   Day 2 Chantal Butea, Brock University,St. Catharines, ON, Canada, and Dagan Karp, Harvey Mudd College. Los Angeles, CA, USA;

   Day 3 Chris Sangwin, University of Edinburgh, Edinburgh, GB;

   Day 4 Branko Ćurgus, Western Washington University, Bellingham, WA, USA).

2. Working group activities in the first four days

3. Final presentations on the last day (maximum 15-minute working group presentations)

Each day had a focus with guiding questions or directions for the working groups to engage in for their discussions: Day 1 - What is mathematical reasoning?; Day 2 - Necessary skills for mathematical reasoning; Day 3 - Mathematical knowledge & tasks; Day 4 - Designing tasks for mathematical reasoning; and Day 5 - Putting it all together.

We pre-selected a moderator for each working group based on their experience and related interests. Each (in person and online) participant signed up for a working group, where there were 5-7 individuals per working group. One note-taker was assigned per working group to capture ideas and discussion. Based on the working groups focus, discuss the theme of the day, workshop objectives, general questions, or anything within the workshops scope. On the last day, each working group will have a 15-minute presentation to share their working groups ideas and conversations. At the end of the workshop, all notes plus presentation slides were sent to the organizers to share with all participants.

We had six working groups, where two working groups worked on the same focus, as we wanted to see how similar or different the outcomes of each working group would be. The three working group focuses were:

1. mathematical reasoning in everyday life, including education beyond formal schooling; numeracy; courses for students who will take at most one mathematics course in university;
2. mathematical reasoning in undergraduate studies (as service, and for math and stats minors and majors);
3. mathematical reasoning before undergraduate studies and including teacher training.

3 Brief Summary of Plenary Lectures

Each plenary lecture was diverse and gave a perspective for all participants to deeply reflect about mathematical reasoning. Here, we will give a brief summary for each plenary lecture.

Rafael Nuñez, University of California, San Diego, and and ETH, Zürich

Professor Nuñez began his lecture with several questions for the participants to think about:

- What are the origins and sources of mathematical reasoning?
- When does mathematical reasoning become mathematical? What are the building blocks, cultural (artifacts have symmetry, patterns, but zoomed in not so much as symmetry is not preserved) and biological (like a honeycomb)?
- What is the criteria for mathematical reasoning? The problem is that numbers are ubiquitous.
- Where do numbers come from? Formal definitions, axiomatic systems, philosophy of mathematics (platonism, logicism, formalism, etc.).
- What is missing in the building blocks of numbers? Non-updated view of numbers, e.g., lack of notation of evolution, neuroscience, language, cultural variation, etc. A naturalistic account is consistent with current scientific knowledge - language, abstraction, concept formation, mind and brain.
- Where do numbers come from? Quantity provides evolutionary advantage. Humans and animals can identify small numbers, fast and effortlessly - subitizing, large discrimination happens with many species, not learnt but built in. Conventional wisdom, theoretical claim: there is a biologically evolved capacity specific for numbers and arithmetic (nativist) view which is widely accepted.

After getting the participants to think and broaden their understanding of mathematical reasoning, he presented six problems with numbers, from his book Nunez (2017) and paper Lakoff and Nunez (2020).

Problem No.1 - A mess in terminology and teleology

- Number, numeral, numerosity, arithmetic? A mess with loose terminology and teleology.
- A number is highly polysemous: everyday passport number or technical ordinal number.
- Literature: filled with confusing terminology, numbers symbolic or not, when numerical and not, etc.
- Numbers are a natural kind, primitive, pre-existing entity.

Problem No.2 - Overinterpretation of trained animal data

- Results from training are often arduous and require considerable environmental support (e.g., like training animals to do tricks by what humans created - we can learn about it, but we have to be careful about conclusions about evolution).
- Usually what is recorded is success, and not the failures (e.g., if it is not significant then it cannot be published).
- Associative learning is an outcome with respect to number.

Problem No.3 - Disregard for crucial human data

- One has to be careful of the artificial setting of having human artifacts.
- Numerical cognition is done in industrialized societies.
- Serious problem of evolutionary claims!
- So what is quantity in non-industrialized society???
- All known cultures have natural quantifiers few several, no one really has exact quantifications.
- What is important is language by itself does not lead to numbers and exact quantification is a culture trait (not a species-specific biological train).

Problem No.4 - Quantical/Numerical Distinction

- We refer to quantity to sign verbal, written (how you say it), and many other forms symbolic reference which is uniquely human, the symbolic reference has nothing to do with the quantify 9 but you make it a thought to represent 9.
- Items are relational, operable - many ways to write numbers/symbol/relational structure - quite complicated.
- What is a number? Take to be a prototypical property of our familiar list 1,2,3.. - counting basic numbers - quantifiers exactly, abstract, cardinal sense, ordinal sense, relational, symbolically.
- Exact symbolic quantification is cultural.

Problem 5 - Number: Biological enculturation beyond natural selection

- Exact symbolic quantification beyond subitizing range: Motivated by cultural preoccupations, or Requires enculturation.
- Conventionalized symbolic reference is what we use.
- Numbers are culturally shaped.
- Language may be necessary but not sufficient to reference numbers.

Problem 6. - Number line

- Further developments of a powerful ubiquitous tool - Graphs, numbers are exact quantifiers but space a completely different thing.
- Biologically endowed? 17 century europe (Descartes, Napier, Wallis [first reasoning of numbers]) is the first number line invention.
- Not a universal intuition - examples of this are seen in the Amazon, Papa Guinea.
- On to further abstractions - examples hyperreal line.

After the six problems with numbers, Rafael discussed where numbers come from. He said it was not trivial at all. Biological enculturation is beyond natural selection. It is not clear what the selection pressures biologically speaking are, how to move from natural to exact quantifications, and what the underpinnings of learning quantifiers in natural language are.

Thus, Rafael ended his lecture with two questions to better shape the rest of the workshop: (1) When does reasoning begin to be mathematical?; and (2) Can we check it?

Terry Gannon, University of Alberta

Professor Gannon began with the idea that anyone who speaks of mathematical reasoning will give you a bias: applied vs pure, theory builders, problem solvers. Mathematics is very important and becoming more important (e.g., better fighting wars, or destroying the planet, mathematical and statistical analysts in sports).

We want to tell our students that mathematics courses are important, but they are not important (e.g., I never used trig sub outside of the classroom). Everything that we teach in our mathematics courses is useless, but the good side is that we have space to teach and not the integral techniques. But, we are in the golden time of mathematics - none of the students who took first year at University of Alberta would ever think this year that this is a golden time.

Terry presented mathematical reasoning in four categories, where he explored some ideas in each.

A. Precision

- We do approximations most of the time - we understand the errors and how long the roots are. So, what role does precision play in mathematics? What does it mean conceptually when we approximate?
- IF-THEN statements where the converse is not true, loose solutions if the IF is true and THEN is true. Proofs are not that important in math, it is like grammar is to literature. In literature, it is about the drama about characters and not about correct grammar. The point of math is to weave connections between different objects.
- Definitions hard to write precisely (e.g., continuous function or manifold, but are fundamental to mathematics).

B. Abstraction

- Happens all over, small or big steps.
- When you understand, you can lift up to the problem like a peanut butter sandwich (you can make a peanut butter sandwich without bread, just find another starch).
- Why do students have issues with abstraction? It is a consequence of understanding, cannot be forced, and maybe that is why students do not understand it.
- Abstraction creates new tools for you to apply it to the more concrete - if you don't then that is not good mathematics.
- Biophysics and Newtonian physics to have the player hit better.
- Abstraction is good when it happens but it is not the goal.

C. Social connection

- We like to think math is timeless and universal. Example: cos(p) vs. cos(π/p), where p is prime.
- Mathematical induction - some students get it others not.
- It is defined to be true is an axiom that you can question it.
- We can do a better job of what natural numbers and its properties are and rule out numbers that are not.
- Proof by contradiction - students struggle with the legitimacy of this, even some mathematicians do not believe in it
- 0.999 = 1, why couldn't this be infinitesimal? We accept certain axioms.

D. Process of doing math

- The way we create mathematics, you work on problems at the back of your head, or 5-10 years, or at a conference, or by accident in a book.
- You get hints or clues and then we pull on them, and sometimes the numbers are coincidence or you find something interesting.
- You need context before you start with a stack of paper and a garbage can.

- There is never one way to go from A to B, you have to have intuition on how you get from A to B.

Terry ended his lecture with a metaphor on doing math: Imagine that you are in a park and trees all around and you wander and see a squirrel. Terry never caught a squirrel. A squirrel zigzags. Eventually the squirrel gets tired and squirrels up the tree too high for an individual. Math is the squirrel, the tree is the theorem you end up with, and you do not know what you are going to get until you chase the squirrel. You might not get the squirrel but you will get the tree.

Chantal Buteau, Brock University

As Professor Buteau prepared for her lecture, she did say that deductive reasoning and writing proofs was her old definition of mathematical reasoning, but it has changed thanks to the workshop. She reiterated questions throughout her presentation: what is mathematical reasoning, thinking mathematically, and computational thinking in mathematics education.

Chantal did a review of some of the readings given to all participants to read before the workshop: Duval (1995), Jeannotte and Kieran (2017), Stacey (2006), and Psycharis and Kallia (2017).

Then, Chantal went through a mathematical problem with computation using Python was used. She simulated how a student would use mathematical reasoning with the help of her professor Miroslav Lovric.

The Chord Problem: Consider a circle and an inscribed equilateral triangle. What is the probability that a random chord in the circle is longer than a side of the triangle? Create a simulation in python to help explore the problem and provide a complete solution.

After Chantal went through the problem, she asked the audience how she demonstrated the engagement with mathematical reasoning. Some participants said: abstraction using WLOG, knowing terminology and the connections between terms (e.g., what is a chord? What is inscribed? What is random?), and powerful confirmation of reasoning through computation.

Dagan Karp, Harvey Mudd College

Professor Karp started his lecture by asking the participants to think about what is mathematical reasoning, and also what is R. He gave us a history of knowledge acquisition by Aristotle (analysis, synthesis, and heuristics) and then later discussed models of mathematical reasoning. After, he asked: Who has mathematical reasoning?, How do we measure mathematical reasoning?, and How do we recognize mathematical reasoning?. This led to gender and equity, where the fields of critical theory (e.g., Judith Butler, Watler Benjamin, Paulo Freire, bell hooke) and points of entry (e.g., William Tate, Marilyn Frankenstein) were brought up. These are important things to consider when individuals develop and enhance their mathematical reasoning skills.

The lecture was an insightful journey through critical theory, where Dagan tied critical theory and mathematics. He said that both are well suited to help us understand and overcome challenges in mathematics and mathematics education.

Chris Sangwin, University of Edinburgh

Professor Sangwin started his presentation by asking the audience to think about why we have a proof. There are several reasons: to provide certainty; to understand why a theorem is true; to communicate; to organize thoughts and the subject; and to discover new mathematics. Thus, for Chris, a proof is a checkable record of mathematical reasoning.

The audience was asked to think of the theorem: the sum of first n odd numbers in a square. We saw the proof by De Morgan (1836) which was all in words and very light on symbols or equations. Over time, for this theorem, the style and rigor have changed.

This is when we were given the task to look at the theorem and its various proofs and to organize each proof from the most rigorous to the least rigorous. For a subject that is objective, we saw how subjective it was.

After, Chris shared one of his research experiments: comparative judgment with experts. The same proofs of the theorem were given to experts. In the first step, the participants had to explain briefly what it means for a proof to be rigorous, give insight into why a theorem is true, be simple, help understand, and receive marks. Chris said that some of the reasons for comparing proofs is that it allows for meaningful discussion about differences and understanding why a theorem is true. The second step was to rank each proof according to the categories from most to least: rigor, insight, simplicity, understanding, and most marks.

Some of the results of the second step were:

- Induction was most rigorous and gave most marks;
- Pictorial I and L-shaped were most insightful;
- Reversed list was the simplest and provide understanding to why the theorem was true; and,
- Experimental evidence scored the least in all categories.

When asking mathematicians what they believe, the following agreements emerged: marks and rigor are very closely related; proof by induction gets the most marks; and simplicity and understanding are very closely related.

After examining the theorems proof by De Morgan's and induction, the audience was asked if the two proofs were the same. In conclusion, the lecture made the audience see how an objective discipline can be subjective, and it depends upon the reader on what is essential for a proof.

Branko Ćurgus, Western Washington University

For Professor Ćurgus, mathematics is a lifetime of fascination where if anything that is worth doing, then it is worth overdoing. Through using the unit circle and Mathematica, he presented the type of questions his students would experience in his mathematical computing course, exploring the concepts of involute, evolute, and envelope. All the questions and ideas were scaffolded and built on one another. It was a journey for the audience to learn and discover the beauty of (funny) trigonometry, that is, taking a geometry (e.g., torus, sphere) and transforming a unit circle. This was a great way for students to develop and enhance their mathematical reasoning skills.

4 Working Group 15-minute Presentation Summary

Link to Working Group presentation slides and/or notes: `https://drive.google.com/drive/folders/1w7Hiaho_eIfrcCwN9B86I3RxCpnFTdkk?usp=sharing`.

Working Group No. 1 - Mathematical reasoning in everyday life, Moderator: Taras Gula, George Brown College

The presentation started off with a real-life problem: How to design a carbon neutral energy system to power your house/neighbourhood/town? But this is not an easy question to answer as there are many questions to answer before one can come up with an answer. However, the working group said the answer was 20 solar panels and walked us through their mathematical reasoning process.

They shared with the participants their guiding principles to solve real-life problems and concluded that it is not about a unique correct solution, but more about interpretation of the solution.

The source of inspiration for their working group was the SIAM Math Modelling: Computing & Communicating (2018) handbook, active learning, and learner-centred teaching. These sources are what they would use to create a course on numeracy and its learning goals. From this, they came up with a mathematical modelling process and a long list of mathematical skills a student would develop. The instructional design was structured around big problems which would be explored through active learning.

Working Group No.2 - Mathematical reasoning in everyday life, Moderator Derek Postnikoff, University of Saskatchewan

The working group focused on everyday mathematical reasoning outside of the classroom and came up with several rich and related scenarios. They created a strategy guide/handbook exploring some of the mathematical reasoning skills and strategies involved which can be adapted for any age group and used among family, friends, students, and teachers. One of the handbook chapters was written by ChatGPT where the guide prompts for ChatGPT can be found in the handbooks appendix. As well, the handbook is to be used as a living document and invites anyone to write their own chapter.

To view the storybook:

`https://drive.google.com/file/d/10yaWHh3NoO2akZj4X3OLBgjmMuAf6Euv/view?usp=sharing`

Working Group No.3 - Mathematical reasoning in undergraduate studies, Moderator: Fok Leung, University of British Columbia

The working group used a framework by Jeannotte and Kieran (2017) to analyze three tasks, but they only presented one.

Task 1. Find a function f that could be used for the following questions.

Consider the function $f :\to$, given by $f(x) =$.

(a) Find the intervals where f is increasing and decreasing.

(b) Find and classify the critical and/or singular points of f (if any).

(c) Find the intervals on which f is concave up and concave down.

(d) Find the point(s) of inflection of f (if any).

(e) If relevant, find the vertical and horizontal asymptotes.

(f) Sketch the graph of f.

Students would work on whiteboards creating a function to answer all parts (a)-(f). [One presenter actually did this exercise in class and showed pictures of students work.] It is going through the framework that students would develop their mathematical reasoning skills: generalize, conjecture, identify a pattern, compare, classify, justify, prove, prove formally, and exemplify.

There are benefits and limitations of using mathematical reasoning framework to analyze a task, but having a framework can help make mindful choices when working through a problem.

Working Group No.4 - Mathematical reasoning in undergraduate studies, Moderator Sean Fitzpatrick, University of Lethbridge

The working group formulated a number of rich mathematical problems that would be well suited to develop mathematical reasoning, focusing on four in particular: matrix inverses over $\mathbb{Z}$, fractal dimensions, limits of certain sequences, and the chord question from Chantal Buteaus plenary lecture. They evaluated in a chart each task by skills (e.g., validate, convince, generalize, abstract, communicate, and so on). None of the tasks were able to capture all skills to develop and enhance mathematical reasoning.

Each task was explored as to how to incorporate it in an undergraduate level mathematics course. All tasks were scaffolded to build on knowledge and acquire mathematical reasoning skills. What the working group concluded was that there are many possible pathways to develop mathematical reasoning, which is an important characteristic to a rich task. Also, there are possible adaptations to the task to get the students to build on other skills.

The ultimate goal of the working group is to develop a repository of such rich mathematical problems, with detailed explanations of the problem, steps, and specifications of the mathematical reasoning skills that are emphasized, so that instructors can implement such tasks in their own courses.

Working Group No.5 - Mathematical reasoning before university and teacher training, Moderator, Shannon Ezzat, Cape Breton University

The working groups main idea was to make (and train teachers to use) mathematical activities that promote mathematical reasoning while covering needed content. Teacher training would analyze mathematical reasoning skills addressed at each step of the problem, including many common errors that students may make, and how to use these as opportunities to practice mathematical reasoning.

The group created their own mathematical reasoning skills: logical reasoning, precision, exploration, process thinking, communication, conjecture and justification, geometric reasoning, numerical reasoning, concept connections, generalization and abstraction, and many more! Basically, they added on more than what was previously mentioned from Jeannotte and Kieran (2017).

They came up with a sequence of tasks for teachers and pre-university students: construct a number line; construct triangles; and represent the sum of two irrationals on the number line. For each task, they explained the teachers task, the student's task, key concept, and mathematical reasoning behind the task.

Working Group No.6 - Mathematical reasoning before university and teacher training, Moderator Lauren DeDieu, University of Calgary

The working group started off their presentation with a discussion about why teach mathematical reasoning to K-12 students?. They believe that it is to be a functional citizen in society by being able to reason about basic quantitative things. The following quote is what they used to help the participants understand their reason: Doing math is like getting a new pair of glasses. It lets you see the world in a clearer way.

The group said the best way to reach K-12 students is by providing better support to their teachers so that they can create positive and meaningful mathematical reasoning experiences.

Afterwards, they discussed traits that future K-12 teachers should have to teach mathematical reasoning. Some of the traits mentioned were: having a strong understanding of content; having a broad view of mathematics; and having soft skills to engage in mathematical reasoning tasks and to lead successfully activities. The working group justified each trait and why they were important.

To prepare future K-12 teachers, the working group believed that two things were needed to facilitate mathematical reasoning activities in the classroom: (1) Good/rich questions (students need to experience mathematical reasoning [the working group provided resources for such questions]), and (2) Be explicit about what were doing, why were doing it, and why its important. For each, they provide ample examples to achieve this.

5 BIRS 5-day Workshop Conclusion & Final Thoughts

The participants of the BIRS 5-day workshop were rather diverse ranging from location, identity, background, mathematical experience and much more. It was necessary to have such a rich group to create opportunities for fruitful discussions and conversations as a group, or in a working group, or privately. It was a true honour, pleasure, and privilege to witness the talent, dedication, knowledge, and camaraderie that each participant generously and authentically shared with others during the 5-day workshop.

The plenary lectures all had different takes on mathematical reasoning. Some focused on skills and framework to show how mathematical reasoning is developed in a scaffolded assignment. Their experiences and perspectives helped expand and better understand mathematical reasoning in everyday life and education at all levels.

The organizers thought that it would be interesting to see the outcomes of two working groups working on the same focus. While there were some intersections in thoughts and ideas, each working group had their own spin on how they developed their discussions and conversations. We were all very impressed with and enlightened by the working group presentations during the last day. As the BIRS participants, but more like a mathematics family, everyone came together and produced some fresh ideas and creative suggestions that will help our students and all citizens develop and enhance mathematical reasoning.

The participants of the 5-day workshop "Building and Enhancing Mathematical Reasoning" fully or completely grasped many of the challenges that learners and teachers of mathematics, as well as members of general public, face in developing and enhancing mathematical reasoning. In addition, the participants surely proved that the dedication and passion of the teaching and research profession is in good hands in Canada and beyond!

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