Spaces of Embeddings: Connections and Applications

Dev Sinha (University of Oregon)
Ryan Budney (University of Victoria)
Robin Koytcheff (University of Louisiana at Lafayette)

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1 Background

Manifolds are basic objects of study in topology. They are "locally Euclidean" meaning that to one with limited eyesight, like an ant on a hill or a person looking at the horizon, they seem like an unlimited, uniform expanse. They arise throughout mathematics, first studied as they arose in solving equations but from there touching on virtually every subfield.

An embedding is essentially a copy of one manifold in another. Thus studying all embeddings captures relationships between two instances of these basic objects. The subject is akin in this way to representation theory in algebra. Because of its basic nature, the study of embeddings was a primary focus in the mid twentieth century, with an initial flowering through the work of Whitney, Morse and Haefliger. After a relatively quiet period, the study of embeddings has enjoyed a renaissance since roughly the turn of the century for two reasons. First, a combination of new techniques have both proven successful at resolving long-standing questions and opening up new lines of inquiry. Also, astonishing connections with topics such as field theory have increased interest from and fostered fruitful exchanges with a wide range of mathematicians.

1.1 Early work on spaces of embeddings

Haefliger is perhaps the most prominent pioneer in moving from considering just a single embedding to the collection of all embeddings from one manifold to another, which we refer to as a *space of embeddings* and denote $\mathrm{Emb}(M, N)$. The space of immersions – maps for which the domain manifold can intersect itself globally but still must be locally an embedding – has been well understood since the 1950's by work of Smale, so topologists often start with the question of "promoting" immersions to be embeddings. A main technique Haefliger developed was a parametrized double-point elimination process. These were successful to a point, but became difficult to manage computationally and had clear "hard" limitations on the extent to which they could capture the homotopy type of embedding spaces. Meanwhile, nearby branches of mathematics experienced wonderful developments: 3-manifold theory and 4-manifold theory were being rewritten via Thurston's work on geometrization and Donaldson's use of gauge theory respectively. The rich connections these subfields have with other areas of mathematics were not apparent in the techniques initially developed to study embeddings.

Interest in the subject of embedding spaces was renewed starting in the 1980's by the work of Vassiliev [69], Birman and Lin [6], and Jones [35]. The Jones polynomial, a new invariant of knots (that is, embeddings of circles in $\mathbb{R}^3$), was inspired by statistical mechanics and interpreted field-theoretically by Witten [76]. The mystery of how the Jones polynomial fits into the rest of low-dimensional topology, and mathematics more generally, inspired much interest. Vassiliev made a major contribution shortly thereafter. The coefficients of the Jones polynomial, after a simple change of variables were shown to be *finite-type invariants*. These are numerical invariants of knots that satisfy simple crossing-change formula. Vassiliev found the source of all such invariants in his study of the cohomology of the space of knots. His technique was to view the space of knots as the complement of a discriminant space in a contractible mapping space, inspired in part by the study of complements of linear arrangements from algebraic geometry. Via a duality argument, this converted the study of the space of embeddings, which is homogeneous, with a readily stratified discriminant space. Such a stratification allowed for employment of spectral sequences.

Independently, in the 1980's Tom Goodwillie made a beautiful observation about the homotopy type of pseudo-isotopy embedding spaces [25]. These embedding spaces by design fit into fiber bundles relating the homotopy type of other embedding spaces, so they are useful for induction arguments. The Morlet Disjunction Lemma described the relative homotopy groups of pairs of pseudo-isotopy embedding spaces. Goodwillie's observation was that these pseudo-isotopy embedding spaces satisfy an even more revealing multiple disjunction lemma. This framework for describing the effects of multiple disjunctions through the language of categories and functors evolved into what is now known as the Goodwillie–Klein–Weiss calculus of embeddings [75, 27, 26]. Moreover, Goodwillie recognized and developed a similar framework in algebraic topology. These frameworks and others inspired by them are know generally as *functor calculus*. The embedding calculus gave useful models for the homotopy types of embedding spaces and was shown to be the natural extension of Haefliger's approach. Embedding calculus provides a sequence of approximations, denoted $T_n\mathrm{Emb}(M,N)$ to the homotopy-type of the embedding space $\mathrm{Emb}(M,N)$. These approximations were proven to be "sharp" as $n \to \infty$ provided the co-dimension of the embeddings was at least three, and thus the relevance to case of co-dimension-two classical knots remained a mystery.

While classical knot theory primarily focuses on whether knots can be deformed to one another, spaces of knots were of interest because of their close connection to the space of diffeomorphisms of a three-sphere, $\mathrm{Diff}(S^3)$. Allen Hatcher's theorem from 1983 that

$\mathrm{Diff}(S^3)$ is homotopy equivalent to its subspace of linear diffeomorphisms [31] has as a consequence that the unknot component of $\mathrm{Emb}(S^1, S^3)$ has the homotopy type of the subspace of great circles, i.e. the Stiefel manifold $V_{4,2} \simeq S^3 \times S^2$. Hatcher went on to describe what could be said about the homotopy type of the other components of $\mathrm{Emb}(S^1, S^3)$, combining everything that was known about 3-manifolds at the time, such as the work of Haken, Waldhausen, Mostow, Prasad, Thurston and himself. This allowed an iterated fiber bundle description of the homotopy type of all the components of $\mathrm{Emb}(S^1, S^3)$ that he wrote up in an unpublished draft [32]. These results are complementary to those of Goodwillie, Klein, and Weiss, and these authors and Hatcher all expressed interest in the relationship between their techniques and the work of Vassiliev.

A final important precursor to the current state of embedding theory is the work by Bott and Taubes [7] and others on configuration space integrals, starting in the early 1990's. These integrals arise in field theory, in particular Chern–Simons theory, and require compactifications as defined by Axelrod and Singer [4]. Such integrals can be used to define knot and link invariants which generalize the Gauss linking integral. By celebrated work of Kontsevich, they produce all Vassiliev (i.e. finite-type) knot invariants [37]. A key insight was that the combinatorics of the integrals which are isotopy invariant are governed by a (co)chain complex of graphs, which because of the roots in perturbative Chern–Simons theory are sometimes called Feynman diagrams. In the 1990's, Kontsevich pioneered the use of these integrals and graph (co)homology in a number of settings [38, 39, 40], such as the cohomology of $\mathrm{Emb}(S^1, \mathbb{R}^n)$, 3-manifold invariants, characteristic classes of manifold bundles, outer space, mapping class groups, deformation quantization of Poisson manifolds, and formality of the little disks operad – topics which played a prominent role in his Fields Medal citation. These areas continue to be developed by multiple communities of mathematicians to this day. Kontsevich's graph complexes are now seen to encode a variety of previously understood structures, including Haefliger's calculation of isotopy classes of higher-dimensional links in the 1960's; the cohomology of configuration space, calculated by Arnold [1] and F. Cohen [16] independently around 1970; the (Koszul dual) Yang–Baxter relations in work of Kohno on braids [36] in the 1980's; and the 4T relations in Bar-Natan's combinatorial formulation of Vassiliev invariants [5] in the 1990's.

1.2 Further developments, and the current state of the field

Recent work on embedding spaces has brought together these historical precedents.

In the 2000's, two important connections between Vassiliev invariants and embedding calculus were established. Budney, Conant, Scannell, and Sinha [11] showed that the integral type-2 invariant appears quite explicitly as the only invariant defined through the third embedding calculus approximation. Building on work of Scannell and Sinha [61], Volić showed that Bott–Taubes integrals can be defined on $T_n\mathrm{Emb}(S^1, \mathbb{R}^3)$ [71] and thus a homology analogue of the Taylor tower is a universal Vassiliev invariant over $\mathbb{R}$ [70]. This led to the conjecture that the embedding calculus tower is the universal integer-coefficient finite-type invariant of knots. To this day, the only universal Vassiliev invariants that are understood explicitly rather than formally are those defined with coefficients in the real numbers.

Budney developed an action of the operad of little cubes on embedding spaces and showed the space of classical long knots is freely generated by this action on the space of prime knots [8]. This is a space-level generalization of Schubert's connect-sum decomposition of knots, whose commutativity is at the heart of this action, and a partial resolution of the extension problem for the homotopy types of $Emb(S^1, S^3)$ described by Hatcher.

Shortly thereafter, Sinha [62] and Salvatore [59] followed with actions of the 2-cubes and framed 2-discs operads on embedding calculus approximations to long knot spaces, bringing some clues towards bridging our understandings of the high co-dimension and classical settings. We now have a multitude of operad actions on embedding spaces, but the significant question of how these actions compare to each other remains to be resolved. The 2-cubes action implies the existence of a second binary operation on homology, called a graded Poisson or Gerstenhaber bracket. Sakai used Budney's operad action together with Bott–Taubes integrals to show certain bracket homology classes are non-trivial [57].

Later, Budney described a new topological operad that acts on embedding spaces, called the splicing operad [10]. This operad encodes the full decomposition of knot exteriors coming from their geometrization. The proof of the geometrization conjecture [53] allows for an operadic description of the homotopy type of the splicing operad, together with a full description of the homotopy type of the space of classical knots. Roughly speaking, the splicing operad solves Hatcher's extension problem. However, the splicing operad exacerbates the problem of a multitude of operads acting on embedding spaces and the embedding calculus tower. An appealing unresolved question is if the splicing operad act on the stages of the embedding calculus tower $T_n\mathrm{Emb}(S^1, \mathbb{R}^3)$. An affirmative answer to this would be a major step towards unifying Hatcher's 3-manifold techniques with the embedding calculus.

Back in the direct study of embedding spaces, global understanding of computations in the higher co-dimension setting have come through application of rational homotopy theory, and in particular formality of configuration spaces through integrals. Motivated by a conjecture of Kontsevich, Sinha initiated the development of models for $T_k\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^n)$ through spaces of natural transformations from the 1-disks operad ("balls" in the circle) to the n-disks operad (balls in Euclidean space). These gave rise to the actions of (different) operads above. After developing a substantial amount of machinery (e.g. the deRham theory of piecewise algebraic forms, which required new results in analysis), Lambrechts, Turchin, and Volić [45] carried through a program of Kontsevich to show that these operads are formal, including in relationship to one another. This result is a triumph, whose main consequence is that the calculation of the rational homology of spaces of knots reduces to a calculation in combinatorics and algebra that is easy to formulate, though difficult to understand. Turchin later put this in the context of homotopical algebra, allowing the possibility of more sophisticated algebra being brought to bear. A corollary in turn is that Vassiliev's original approximations with rational coefficients are "sharp," but only because they coincide with the answers given by embedding calculus techniques. Arone and Turchin have similarly reduced the calculation of rational homotopy groups to combinatorics [3].

Parallel to these developments, configuration space integrals and graph cohomology have been developed in a number of directions. Extending Bott-Taubes field-theoretic integrals, Cattaneo, Cotta-Ramussino and Longoni constructed real-valued cohomology classes in spaces of knots [14]. Turchin has shown that the chain complexes defining these have the same homology as the corresponding knot spaces, but logically this could be a coincidence with most of the field theory classes being zero, presenting another question which needs to be resolved. These integrals have similarly been used for example by Watanabe [72], Koytcheff, Munson, and Volić [42] and Pelatt and Sinha [55] to generate elements of cohomology of embedding spaces. Following Kontsevich [39], various researchers also used them to define invariants of rational homology spheres [46, 43, 48]. Conant and Vogtman studied graph cohomology in connection to outer automorphisms of free groups and mapping class groups [19, 20]. More recently, Idrissi [34] and independently Campos and Willwacher [15] used such integrals to produce rational homotopy models for general configuration spaces. *** Should go through resources such as Willwacher's ICM talk to flesh this out just a touch more.***

Following these developments we have the following perspective. In co-dimension three and higher one can view the embedding calculus as a universal framework, building information about embeddings from the study of configuration spaces. This framework is bolstered by ideas coming field theory and singularity theory. In dimensions two and three, geometrization and other 2- and 3-manifold techniques, as demonstrated by Hatcher, are the appropriate framework. For diffeomorphisms in higher dimensions, it seems that a combination of techniques could be relevant in different ranges. But fleshing out these frameworks, applying them for maximal impact, and understanding their connections with other areas will occupy mathematicians in this area for some time to come. In particular, our recent work showed [12] that the n-th stage of the Taylor tower produces additive, order-$(n-1)$ finite-type invariants of classical knots, and it gave substantial spectral-sequence evidence for the the enticing conjecture that it produces all such abelian-group-valued invariants.

2 Presentation Highlights

In this section, we divide the content into three areas: graphs and geometric topology, functor calculus and operads.

2.1 Graphs and geometric topology

A significant portion of the results presented at our workshop involve connections between combinatorial data and differential topology, in particular the construction of nontrivial (co)homology classes in spaces of embeddings using (co)cycles in graph complexes.

The most prominent of these is **Tadayuki Watanabe**'s disproof of the Smale Conjecture in dimension 4, which appeared in his 2018 preprint [74]. This result builds on his earlier work [73] on Kontsevich's characteristic classes in the cohomology of $B\mathrm{Diff}(D^n; \partial D^n)$, coming from configuration space integrals. Watanabe had previously used a generalization of Habiro's clasper surgery [30], originally defined to study for knots and links, to construct dual homology classes in $B\mathrm{Diff}(D^n; \partial D^n)$ which pair nontrivially with the Kontsevich cohomology classes, for odd $n \geq 5$. His recent work on the Smale conjecture adapts the construction of homology classes to the case $n = 4$. In that paper, he uses work of Fukaya on graphs embedded along Morse flows – an extension of the theory of configuration space integrals – to construct the dual cohomology classes. Watanabe's result is an unexpected one that will very likely have significant consequences in 4-manifold theory.

Another striking result using some similar methods is forthcoming work of **Danica Kosanović**, who presented a proof that the map from the space $\mathcal{K} = \mathrm{Emb}(\mathbb{R}, \mathbb{R}^3)$ of long knots in $\mathbb{R}^3$ to the n-th stage $T_n\mathcal{K}$ of its Taylor tower is surjective on path components. The group $\pi_0(T_n\mathcal{K})$ already had a known combinatorial description in terms of trivalent graphs. Kosanović's method is to use gropes, as developed in this setting by Conant and Teichner [18], to construct embeddings realizing this combinatorial data. Another key aspect of her proof is to use the model for $T_n\mathcal{K}$ given by spaces of punctured knots. She used clasper surgery as a heuristic way to explain the methods, which thus bear some resemblance to Watanabe's work. Kosanović's techniques could be described as a very careful refinement of [12]. Her result uses the monoidal structure introduced therein and a beautiful relationship between gropes in finite-type invariants and the layers of the Taylor tower. It also relates to work of Conant, Schneiderman, and Teichner [21] on link invariants via 4-dimensional topology.

The study of embeddings using combinatorial data also appeared in presentations of **Michael Polyak** and **Christine Lescop**. They described configuration space invariants of knots and links in $\mathbb{R}^3$ [56, 29, 47] and of rational homology 3-spheres [48]. In their talks and informal sessions, both also shared recent results including Polyak's new combinatorial formulae for Milnor invariants of string links and David Leturcq's study of knotted S^n's in S^{n+2} [50]. Those ideas are a bridge from linking numbers via configuration spaces to the more nuanced settings of knot invariants and cohomology of diffeomorphism groups. Indeed, the Kontsevich characteristic classes are a higher-dimensional generalization of finite-type invariants of homology 3-spheres. Moreover, Lescop's work (partly with Greg Kuperberg) on Morse propagators [49] appears closely related to the techniques used in Watanabe's disproof of the 4-dimensional Smale conjecture.

Also fitting into this theme is work presented by **Keiichi Sakai** on configuration space integrals for isotopy classes of higher-dimensional knots [58]. **Arnaud Mortier** has begun to study the cohomology of the space of knots in $\mathbb{R}^3$ that arises from graphs with valence greater than 3 [52]. His work proceeds via the "Knizhnik–Zamolodchikov" approach to Vassiliev invariants rather than the "Chern–Simons" approach (topologically, by generalizing the winding number for braids rather than the linking number for closed links).

2.2 Functor calculus

As the state-of-the-art framework for studying embeddings in high co-dimension, the calculus of embeddings featured prominently in the workshop. **Tom Goodwillie** himself presented new directions for functor calculus starting "from the ground up," that is, from the perspective in his Ph.D. thesis, where the subject originated. He conjectures that spaces of maps from graphs into a manifold can be used to understand pseudoisotopy and hence the algebraic K-theory of a space. These spaces bear some resemblance to the Morse homotopy invariant of Fukaya used in Watanabe's recent work.

Pascal Lambrechts presented a definitive result with Boavida de Brito, Pryor, and Songhafouo Tsopméné, which provides a cosimplicial model for spaces of embeddings, given a simplicial model for the source manifold. This considerably generalizes Sinha's cosimplicial model for $T_n\mathrm{Emb}(\mathbb{R}, \mathbb{R}^n)$. **Don Stanley** presented joint work with Songhafouo Tsopméné on functor calculus in the setting of more general model categories, continuing their earlier work [64, 65].

Ben Knudsen gave a presentation describing the extent to which the Taylor tower sees the difference between smooth structures and formal smooth structures on manifolds. In all dimensions other than four, formal smooth structures are known to coincide with smooth structures. In dimension 4, formal diffeomorphism coincides with homeomorphism. Knudsen's result in joint work with Kupers is that the embedding calculus tower does not see the difference. Thus the space of 1-dimensional knots cannot be used to distinguish 4-manifolds which are homeomorphic but not diffeomorphic. This extends a recent line of inquiry by Arone and Szymik [2] and answers a question posed by Viro. In the positive, Knudsen and Kupers show that the embedding calculus tower can be used to distinguish exotic spheres, starting in dimension sixteen. The conjecture is that such techniques should give invariants in any dimension in which exotic spheres exist.

Apurva Nakade presented work which applies functor calculus to symplectic topology [54]. Similarly to the result of Knudsen and Kupers, he showed that various spaces of formal embeddings are equivalent to the limits of their embedding calculus towers. In other words, the tower sees the "flexible side" of symplectic geometry/topology. There is a growing group of people interested in applications of functor calculus to symplectic topology, such as Francisco Presas (Madrid) and Tamas Kalman (Tokyo Tech).

Sander Kupers gave a presentation of some results of on spaces of diffeomorphisms. This included his Ph.D. thesis work [44], where he proved some finite generation results for $B\mathrm{Diff}_\partial(D^n)$, thus resolving some long-standing open problems in high-dimensional manifold theory. Though the result is about the space of diffeomorphisms itself, the proof uses work of Weiss on embedding calculus. He mentioned a result with Randal-Williams on $\pi_*(B\mathrm{Diff}_\partial(D^n)) \otimes \mathbb{Q}$, which nicely complements Watanabe's work on the graph homology classes, and which was obtained by playing embedding calculus and the theory of Galatius, Madsen, Tillmann, and Weiss against each other.

2.3 Operads

A final important topic appearing in multiple talks was the theory of operads, especially those equivalent to the little disks operad. Such results are of central importance for application to spaces of embeddings: viewing configuration spaces as modules over disks operads is central to both the embedding calculus and the study of field-theoretic integrals.

An example of how deeper understanding of configurations leads to deeper understanding of embeddings was given by the recent work of **Geoffroy Horel** and **Pedro Boavida de Brito**. Horel described actions of the Grothendieck–Teichmüller group (and thus the absolute Galois group) on localizations of the embedding calculus tower for the space of long knots, as well as an embedding of the absolute Galois group in the profinite completion of the tower. A key ingredient appeared to be the homotopy theory of E_n operads, including previous noteworthy work of Horel [33]. As a consequence, they show that at each prime p, the spectral sequence for homotopy of the tower collapses in a range of bidegrees that depends on p. This result is related to Kosanović's result and is similarly remarkable, providing further evidence for the universality conjecture from [12].

Paolo Salvatore presented his cell decomposition of the Fulton–Macpherson operad [60], which answered a question of Kontsevich and Soibelman from 2000. This decomposition involves trees with vertices labeled by cells of the cactus operad, which themselves correspond to graphs. Though the result is not explicitly about embedding theory, this operad has been heavily used in recent work on embedding spaces.

Victor Turchin's presentation culminated in a forthcoming result with Benoît Fresse and Thomas Willwacher in which they describe the path components of quite general spaces of high-dimensional string link. In particular, the source manifold need not even be a union of Euclidean spaces. Their result is remarkable not only for its definitive and far reaching statement but also because it ties together several of the key methods in the field. The description is in terms of a certain graph complex, which is an $\mathcal{L}_\infty$-algebra (roughly, a Lie algebra up to higher homotopies). The path components then correspond to the Maurer–Cartan elements in this graph complex, modulo gauge equivalence. They proceed through Willwacher's thorough understanding of the rational homotopy type of spaces of configurations in manifolds, as a module over the disks operad. Their work uses not only such physics-inspired homotopical algebra, but also the functor calculus and operads. It recovers and generalizes the classical work of Haefliger on isotopy of high-dimensional knots, as first pointed out by Songhafouo Tsopméné and Turchin [66, 67].

3 Scientific Progress Made

As is typical in mathematical meetings, progress is primarily through the generation of new lines of attack on standing questions and of new conjectures. While we organizers cannot be aware of all of these, we share a few that seem particularly promising.

Budney suggested a renewed attack on his problem [9] of the non-triviality of the unknotting map

$$K_{n,1} \to \Omega K_{n+1,1}.$$

If the map can be shown to extend $K_{n,1} \to \Omega^2 K_{n+1,1}$, inducing an isomorphism on the lowest-dimensional homology group, Lambrechts mentioned it could offer insight into Turchin's *Hodge decomposition*.

Budney and Salvatore outlined an idea to generate numerous inequivalent actions of the framed discs operad on $K_{n,1}$ for all n.

Lambrechts sought explanation for the degree-mixing isomorphisms of rational cohomology of embedding spaces across dimensions. For configuration spaces themselves, one can use S^1-equivariant cohomology to produce a chain of isomorphisms which relate cohomology across dimensions. Along with Horel's results presented at the workshop related to cyclic actions and previous work of Arone and Turchin which combined embedding calculus and orthogonal calculus, Sinha has been inspired to explore how equivariant cohomology could be used to establish collapse results.

Sakai would like to use Koytcheff's recent work [41] on $\mathbb{Z}$- and $\mathbb{Z}/p$-valued cohomology classes in spaces of knots and links to study torsion invariants of higher-dimensional knots and links.

Koytcheff, Kupers, and Turchin discussed the possibility of Kupers's conjecture that one can detect a nontrivial Browder operation on $B\mathrm{Diff}(D^n; \partial D^n)$.

Sinha proposed to use Kosanović's techniques to detect the isotopy classes of higher-dimensional knots and links described by Turchin. More precisely, he proposed constructing embeddings out of trees corresponding to homotopy classes and then detecting them via the Lie-coalgebraic Hopf invariants in his joint work with Ben Walter. The use of Hopf invariants to study components of the embedding calculus tower has already produced a new perspective on formulae of Polyak for Milnor invariants, and he and Koytcheff along with Nir Gadish are turning attention to the knot setting for an attack on the main conjecture.

Knudsen mentioned in his talk a broad conjecture of Kupers that spaces of embeddings or diffeomorphisms can be decomposed via a fiber sequence into a formally smooth part, detected by configuration space integrals, and a second part detected by gauge theory.

Another conjecture expressed by many participants is that configuration space integral cohomology classes for all spaces of embeddings, including diffeomorphisms, factor through the embedding calculus tower. In particular, the Kontsevich characteristic classes studied by Watanabe would factor through the tower. This result was proven for the case of classical knots in Ismar Volić's PhD thesis [70], under the supervision of Goodwillie. In this case, the simplicity of the source manifold was exploited, and it is not immediately obvious how those methods would generalize to say the cohomology of $B\mathrm{Diff}_\partial(D^n)$. A first case which many (including Willwacher) have said is worth pursing is showing the configuration space integrals of Cattaneo and his coauthors give the real cohomology of knot spaces, as has been calculated through the embedding calculus tower.

Both this workshop and the subsequent workshop on Unifying 4-Dimensional Knot Theory (19w5118) supported the collaboration between Budney and Gabai [13]. This result extends the work of Dax and Haefliger, showing that the unknotted S^2 in S^4 is the boundary of many distinct (up to isotopy) embedded 3-discs $D^3 \to S^4$. The result follows from a sequence of deductions, starting with the work of Dax and that of Arone and Szymik [2] that $\pi_1 Emb(S^1, S^1 \times S^3)$ is not finitely generated (for all components). A key step in is the observation of a strong connection between the unknot component of $\mathcal{K}_{n+1,n-1}$ and the classifying space of the space of embeddings of D^n into $S^1 \times D^n$ that agree with the map $p \longmapsto (1, p)$ on the boundary. This allows the deduction that the component of the unknot in the embedding space $Emb(S^{n-1}, S^{n+1})$ does *not* have the homotopy-type of the subspace of great $(n-1)$-spheres, provided $n > 2$.

4 Outcomes of the Meeting

The outcomes of the meeting are, happily, strongly in line with the goals coming in to the meeting. The meeting was structured with these goals in mind, and the outcomes generally exceeded expectations. We discuss both design and desired outcomes.

Cross-pollinate between geometric and algebraic topologists

Participants' backgrounds varied. While many have worked in embedding calculus, with its particular blend of algebraic and differential topology, some were geometric topologists (e.g. Lescop and Polyak), some were more homotopy theoretic (e.g. Stanley and

Horel) and some have worked with different blends of geometric and algebraic topology (e.g. Budney, Kosanovic, and Kupers). To accommodate the different backgrounds, we took the suggestion of some participants and had two introductory sessions: one on embedding calculus and one on configuration space techniques and claspers in geometric topology. We alternated algebraic and geometric series of talks, and sequenced them carefully so that participants would be able to follow even if they were unfamiliar with an area.

Participants seemed very happy with the ability to expand mathematical knowledge outside their area of expertise. Algebraic topologists appreciated learning about new techniques related to configuration spaces, in particular Morse propagators, which are so central to the current application of algebraic topology to differential topology, as well as clasper surgery, which gives geometric structures analogous to the lower central series in algebra. Geometric topologists greatly appreciated the opportunity to get up to speed about an adjacent area, the calculus of embeddings, which was new to some participants. Those already working at the interface appreciated learning and exchanging on all sides. Some collaborations were initiated, and more broadly the ground was prepared for fruitful collaboration at this interface for years to come.

Disseminate top recent results, while fostering collaboration

The organizers polled participants ahead of the workshop for their priorities, and there was a strong expression of interest in time for collaboration. Nonetheless, at such a meeting of leaders from a number of fields, sharing recent breakthroughs is a priority. The organizers balanced these needs by having a large amount of time set aside for collaboration, in particular on Thursday, as well as having the aforementioned introductory sessions to aid collaboration across specialties. To help make the most of the relatively limited lecture time, talks with different speakers were nonetheless planned to build so that later speakers could rely on background from earlier in the program.

We benefited from having breakthrough results established relatively recently. Diffeomorphism groups are a central subject in topology, but little progress has been made in dimensions four and higher for forty years. Kupers and Watanabe presented breakthrough results, the former producing calculations of rational homotopy groups of diffeomorphisms which dwarfs previous knowledge and the latter disproving the Smale Conjecture in dimension four. Also, Kosanović used a blend of algebraic and geometric topology to resolve the last case of Goodwillie–Klein excision estimates, namely for classical knots. This could be a key step in showing the embedding calculus is a universal finite-type invariant. As for collaboration, participants clearly appreciated and made great use of the time. Informal talking at meals often resulted in a brief exchange of “we should talk more”, and then finding time among the breaks and other free periods provided. It seems that desired meetings were always able to be scheduled.

Support and highlight contributions by mathematicians from underrepresented backgrounds

Some of the greatest contributions to the program were by mathematicians from underrepresented groups. As mentioned already, Kosanovic’s result, part of her thesis work, resolves a longstanding key open question. Lescop led multiple sessions for algebraic topologists to learn about topics in geometric topology relevant to this area. Songhafouo Tsopméné was not able to attend (because of the birth of a child) but had his work prominently featured in three different talks.

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