Extremal Blaschke products

Kelly Bickel (Bucknell University),
Pamela Gorkin (participant, Bucknell University),
Anne Greenbaum (University of Washington),
Thomas Ransford (Université Laval),
Felix Schwenninger, (University of Hamburg),
Elias Wegert (TU Bergakademie Freiberg))

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1 Overview of the Field

One of the most important results in operator theory is due to John von Neumann and states that for a fixed contraction T and polynomial p, the operator norm

$$\|p(T)\| \leq \sup\{|p(z)| : z \in \mathbb{D}\}.$$

Variations of von Neumann's inequality can be extremely useful and are thus frequently the object of study. Matsaev's conjecture, for example, asserts that for every contraction T on L^p (with $1 < p < \infty$) for any polynomial p and S the unilateral shift operator one has

$$\|p(T)\|_{L^p \to L^p} \leq \|p(S)\|_{\ell^p \to \ell^p}.$$

When $p = 2$, this is von Neumann's inequality. However, Drury [10] showed that this conjecture fails in general. Another conjecture in this direction, one for which no known counterexample exists, is the Crouzeix conjecture.

Let A be an $n \times n$ matrix and let $W(A)$ denote the numerical range of A. Recently, Crouzeix and Palencia [6] showed that for every function f holomorphic on an open set containing the closure of $W(A)$, the operator norm of $f(A)$ satisfies

$$\|f(A)\| \leq (1 + \sqrt{2})\ \sup\{|f(z)| : z \in W(A)\}.$$

A simplified version of their proof can be found in [17]. The Crouzeix Conjecture states that $(1 + \sqrt{2})$ may be replaced by 2 and there are several classes of matrices for which this is the case (see, for example, [1, 5, 4, 2, 8, 14]). In particular, the conjecture is true for 2×2 matrices as well as matrices of the form $aI + DP$ or $aI + PD$ where a is a complex number, D is a diagonal matrix, and P is a permutation matrix (see [3] and [15]), 3×3 tridiagonal Toeplitz matrices and matrices in this class with some diagonal entries taken equal to zero, [15].

In 2003, Crouzeix showed that equality is obtained with the sharp constant C for some function $f = B_A \circ \omega$, where ω is a conformal map of the interior of $W(A)$ onto the unit disk and B_A is a Blaschke product of degree less than n. Such a Blaschke product is called an *extremal Blaschke product*. In this workshop, we focused on the study of extremal Blaschke products. That the conjecture holds for A a Jordan block with zeros along the diagonal is not difficult to see; it was later shown in [4] that the conjecture also holds for a perturbed Jordan block

$$J_\nu = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ \nu & \cdots & \cdots & & \lambda \end{pmatrix}.$$

These two classes of matrices (Jordan blocks and perturbed Jordan blocks) are special cases of operators known as *compressions of the shift operator*. Much is known about the numerical range of these operators ([7], [11], [12], [16]), It is natural, then, to consider the conjecture for this special class of operators, as well as the extremal Blaschke products associated with these operators. Beurling's theorem tells us that the invariant subspaces of the shift operator S are of the form $\{\Theta H^2 : \Theta \text{ an inner function}\}$. Therefore, the invariant subspaces for the adjoint of the shift are of the form $K_\Theta := H^2 \ominus \Theta H^2$. The compressed shift operator $S_\Theta : K_\Theta \to K_\Theta$ is defined by

$$S_\Theta(f) = P_\Theta(S(f)),$$

where P_Θ is the projection of the Hardy space H^2 onto K_Θ. Throughout this discussion, Θ denotes a finite Blaschke product.

2 Presentation highlights and progress made

There are several natural questions that we investigated throughout this workshop. In what follows, let A be an $n \times n$ matrix, let Ω be a simply-connected domain (typically with smooth boundary) containing the spectrum of A, $H_1^\infty(\Omega)$ the unit ball of $H^\infty(\Omega)$, and let f be a holomorphic function continuous on $\overline{\Omega}$ and bounded by 1 on Ω such that

$$\|f(A)\| = \sup\{\|g(A)\| : g \in H_1^\infty(\Omega)\}.$$

As indicated earlier, such an *extremal function* f can be taken to be a Blaschke product composed with a conformal map and these extremal functions have played an important role in previous investigations of the Crouzeix conjecture. The extremal functions come equipped with *extremal vectors*, namely unit vectors $x \in \mathbb{C}^n$ so that

$$\|f(A)\| = \|f(A)x\|.$$

These extremal vectors are also quite important and are known to possess special properties. For example, in [2], Caldwell, Greenbaum, and Li showed that if $\|f(A)\| > 1$, then

$$\langle f(A)x, x\rangle = 0. \tag{1}$$

They used this to provide a new, simple proof of the fact that if Ω is a disk containing $W(A)$, then

$$\|g(A)\| \leq 2 \max_{x\in\Omega} |g(x)|.$$

During this workshop, we primarily investigated these extremal functions and their associated extremal vectors both for $\Omega \supset W(A)$ and for more general Ω.

Topic 1. Extremal Functions. Let $\Omega \supset W(A)$. During this workshop, we both analytically and numerically explored the structure of the associated extremal functions. For example, we observed that if A is normal, then every Blaschke product is extremal for A. After restricting to non-normal matrices, we observed that every degree 1 and degree 2 Blaschke product is extremal for some A. We conjecture that every Blaschke product B with $\deg B \leq n-1$ is extremal for some non-normal $n \times n$ matrix A. This seems supported by numerical computations, but analytically showing that some B is extremal for a given A is, in general, very challenging. Additionally, T. Ransford presented a result indicating that there is some open set of $n \times n$ matrices A whose extremal Blaschke products always have maximal degree $n-1$. Thus, in some sense, the number of matrices whose extremal Blaschke product has maximal degree is quite large. We also discussed examples where (up to a conformal map) A has a unique extremal Blaschke product and examples where the Blaschke product associated to A is not unique. The example illustrating non-uniqueness is due to Kenan Li. However, we do not currently have enough such examples to conjecture a pattern.

Besides, if Ω is a disk with center c, then (1) implies that the extremal function satisfies $|f(c)| \leq \sqrt{4 - \|f(A)\|^2}$ and the estimate can even be improved by a result due to Drury [9]. As mentioned above, an important class of operators to study are compressions of the shift S_Θ, where Θ is a finite Blaschke product with $\deg \Theta = n$. In this setting, we ran numerous numerical experiments to better understand the associated extremal Blaschke products. Let us denote those functions by B_Θ. The numerical investigations always yielded B_Θ with maximal degree, namely $\deg B_\Theta = n - 1$. This generalizes the well-known fact that for the Jordan block J_0 with $\lambda = 0$, the symbol is $\Theta(z) = z^n$ and $\Theta(z)/z$ is an extremal Blaschke product. More generally, if we choose Θ with $\Theta(z) = z\Theta_1(z)$, i.e. so $\Theta(0) = 0$, then the experiments suggest that the appropriately normalized Blaschke products Θ_1 and B_Θ are close to each other. This relationship appears particularly strong if the zeros of Θ_1 are very close to the unit circle $\mathbb{T}$. We expect that these experiments indicate an underlying relation between the symbol Θ and an extremal Blaschke product for S_Θ (see the remarks on future research below).

We also have proved the following:

Theorem. Let $\delta_\Theta = \inf_k \prod_{j \neq k} \rho(z_j, z_k)$, where $z_1, \dots, z_n$ denote the zeros of the finite Blaschke product Θ and ρ denotes the pseudo-hyperbolic distance. If $\delta_\Theta > 2\sqrt{2}/3$, the operator S_Θ satisfies the Crouzeix conjecture.

Future research. One of our goals is to remove the restriction on δ_Θ in the theorem above.

For the compressed shift S_Θ, we hope to better understand the relationship between the symbol Θ and the extremal Blaschke product. For example, we would like to prove analytically that B_Θ always has maximal degree. The case where the symbol Θ has the form $\Theta(z) = z\Theta_1(z)$ is of special interest, since there seems to be a correlation between Θ_1 and the extremal Blaschke product B_Θ. In particular, if the zeros of Θ_1 approach points on the unit circle, we conjecture that the zeros of B_Θ have the same limit points (provided that the conformal mapping ω is appropriately normalized). Working on a proof of this conjecture will help us to understand the interplay of the symbol and the extremal Blaschke product of S_Θ. It will also yield new insight in the behavior of the functions and operators which play a crucial role in the proof of the main theorem in Crouzeix and Palencia [6]. The theoretical investigation of these problems will be supported by improved and specially designed numerical experiments.

We also plan to study other examples of extremal Blaschke products to formulate stronger conjectures about their degree bounds and uniqueness properties. For example, if A is a 3×3 matrix and its numerical range is a disk, when is the extremal Blaschke product of a particular degree and when is it unique?

Topic 2. Extremal Vectors. We also spent significant time investigating extremal vectors and their interplay with their associated extremal functions. An interesting result (proved beforehand and presented during the workshop by T. Ransford) extends (1). It says that if an extremal function f factors as $f = f_1 f_2$, then

$$\langle f_1(A)x, \left(\|f(A)\|^2 I - f_2(A)^* f_2(A)\right) x\rangle = 0.$$

During the workshop, we generalized this approach considerably by studying the quantity $\langle h(A)x, x\rangle$, where x is an extremal vector, but h is any function holomorphic on Ω and continuous on $\overline{\Omega}$. In this setting, our main result says that there is a probability measure μ such that

$$\langle h(A)x, x\rangle = \int_{\partial\Omega} h\,d\mu, \text{ for all } h \in \mathcal{A}(\Omega).$$

Moreover, we are able to say quite a bit about this measure and many of its properties; in fact, we can give an explicit description of it.

In the case of the compression of the shift operator S_Θ and $\Omega = \mathbb{D}$, we can say more. In particular, if B is a Blaschke product of degree strictly less than that of Θ, then results from [18] can be used to show

$\|B(S_\Theta)\| = 1$. Von Neumann's inequality says that this is the best possible norm and so, B is an extremal function for S_Θ on $\mathbb{D}$. In this setting, we can compute the dimension of the space of extremal vectors associated to each B and provide explicit formulas for them. Moreover, for each extremal vector x, we have a formula for the associated probability measure μ, which has allowed us to investigate various conjectures about the structure of μ in more general situations.

Future Research. We hope to obtain more information about these μ measures and use them to deduce additional facts about the matrix A and related quantities. For example, given a fixed Ω, this result and some underlying formulas put a number of constraints on the possible extremal vectors of A. Moreover, the quantity $\langle h(A)x, x\rangle$ is always in the numerical range of $h(A)$. It appears that this result could be adapted to provide bounds for both the norm and spectral radius of $h(A)$ (at least for particular types of A and h). The case of S_Θ seems particularly tractable. Specifically, we hope to extend the current work on $\Omega = \mathbb{D}$ to more general domains like the numerical range $W(S_\Theta)$, at least in situations where Θ is simple.

3 Outcome of the Meeting

As indicated above, we made progress in two directions: a probability measure that appears as an inner product and the numerical range of the compression of a shift operator. We also came across some new extremal problems for Blaschke products that appear to be of interest in their own right. We plan to continue working together on each of these topics to extend the results we have, as well as to look at the relationship between the results.

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