Asymptotically Hyperbolic Manifolds

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1 Overview of the Field/Recent Developments

Asymptotically hyperbolic (AH) manifolds continue to be an active area of modern research. Much of the work is driven by the AdS/CFT correspondence. This really began with the seminal work of Fefferman and Graham [15, 16] on formal Poincaré-Einstein metrics, as well as the work of Hawking and Page [22] on anti-de Sitter black hole thermodynamics, both dating from the 1980s, and exploded with the papers of Maldacena [27], Witten [32], and Skenderis and collaborators [23, 29] in the late 1990s.

For some time, one of the main questions has been to find asymptotically hyperbolic Einstein manifolds that fill in a given conformal boundary. More recently, driven by studies of the complexity of quantum states in the boundary Conformal Field Theory (CFT) and the related entanglement entropy, the focus has shifted to the study of surfaces within these manifolds which end at conformal infinity (see, e.g., the talks of Taylor and Englehardt). Often mathematicians have focused on answering questions about the bulk manifold. The answer can then be used by physicists to study the CFT on the conformal boundary. However, one can also ask whether certain data that can be gleaned from the CFT, such as the areas (volumes) of minimal hypersurfaces in the bulk terminating at conformal infinity can be used to reconstruct the bulk geometry (as in the talk of Alexakis).

General relativity continues to drive research in this area. Among the main questions are the pursuit of a fully satisfactory positive mass theorem without spinors and without restrictive assumptions, especially one that can be applied in the asymptotically locally hyperbolic setting where negative mass is allowed but a lower bound is still expected. The asymptotically hyperbolic version of the Riemannian Penrose conjecture is also open and is currently the subject of much attention. Related questions concern the role of new masslike invariants, such as the so-called Gauss-Bonnet masses, that arise for AH manifolds with different decay rates for the metric.

General relativity has also catalyzed the study of the instability of anti-de Sitter spacetime, which is now one of the most active areas of numerical relativity and related analytical work in differential equations (see Bizon's talk).

Finally, the theory of Ricci flow for AH manifolds has been developed in the past ten years (see Bahuaud's talk), and holds promise for the study of some of these questions, though the long-time existence theory is still under development. Once this is done, the Ricci flow will provide another technique to study both the AH mass and the renormalized volume.

2 Presentation Highlights

Pierre Albin spoke on Poincaré-Lovelock metrics on conformally compact manifolds. The gravity theory given by the Einstein-Hilbert action has a natural extension in higher dimensions known as Lovelock gravity or Gauss-Bonnet gravity. He discussed imposing these curvature equations on a conformally compact manifold.

Spyros Alexakis presented his joint work with T Balehowsky and A Nachman on recovering a Riemannian metric from area data [1]. Specifically, he addressed a geometric inverse problem: Consider a simply connected Riemannian 3-manifold (M, g) with boundary. Assume that given any closed loop γ on the boundary, one knows the area of the area-minimizer bounded by γ. Can one reconstruct the metric g from this information? He answers this in the affirmative in a very broad open class of manifolds. He briefly discussed the relation of this problem with the question of reconstructing a metric from lengths of geodesics, and also with the Calderon problem of reconstructing a metric from the Dirichlet-to-Neumann operator for the corresponding Laplace-Beltrami operator. He raised the analogous question for asymptotically hyperbolic manifolds, and the significance of the question in physics.

Eric Bahuaud discussed normalized Ricci flow of asymptotically hyperbolic metrics. He surveyed results concerning the normalized Ricci flow evolving from a conformally compact asymptotically hyperbolic metric [2]. He discussed joint work with Mazzeo and Woolgar on the behaviour of the renormalized volume along the flow of both asymptotically Poincaré-Einstein metrics and metrics with an even expansion. He also discussed joint work with Woolgar on the long-time existence of the flow for rotationally symmetric asymptotically hyperbolic initial data, and ongoing work with Guenther and Isenberg on the stability of these flows.

Piotr Bizon gave an overview of the problem of stability of AdS spacetime [3, 4]. It has been conjectured that anti-de Sitter spacetime is unstable. After briefly presenting some evidence supporting this conjecture, he discussed various aspects of the problem in a broader context of recent developments in spatially confined nonlinear dispersive equations.

Armando Cabrera Pacheco presented work on the stability of the positive mass theorem for asymptotically hyperbolic graphs [7]. The rigidity of the Riemannian positive mass theorem asserts that the ADM mass of an asymptotically flat manifold with non-negative scalar curvature equals zero if and only if the manifold is the Euclidean space. It is natural to ask if the ADM mass of a given manifold is close to zero, is the manifold close to the Euclidean space in some sense? Huang and Lee proved the stability (in the sense of currents) of the positive mass theorem for asymptotically flat graphs. Cabrera Pacheco described how to use results of Dahl, Gicquaud and Sakovich to adapt Huang and Lee's ideas to obtain a stability result for positive mass theorem for asymptotically hyperbolic graphs.

Carla Cederbaum spoke on the centre of mass of asymptotically hyperbolic initial data sets [9]. In many situations in Newtonian gravity, understanding the motion of the centre of mass of a system is key to understanding the general "trend" of the motion of the system. It is thus desirable to also devise a notion of center of mass with similar properties in general relativity. However, while the definition of the centre of mass via the mass density is straightforward in Newtonian gravity, there is a priori no definitive corresponding notion in general relativity, let alone in the asymptotically hyperbolic setting. Cederbaum presented a geometric approach to defining the center of mass of an asymptotically hyperbolic initial data set, using foliations by constant mean curvature near the asymptotically hyperbolic end of the initial data set. The approach was joint work with Cortier and Sakovich, building upon work by Neves and Tian, and extending results in the asymptotically Euclidean case going back to Huisken and Yau.

In a related talk, Stephen McCormick presented joint work with Cederbaum and Cabrera Pacheco on asymptotically hyperbolic extensions and estimates for an analogue of the Bartnik mass [8]. Given a metric g on the 2-sphere S^2 with Gaussian curvature bound below by -3, and non-negative constant H, he constructs asymptotically hyperbolic manifolds whose boundary is isometric to (S^2, g) and has mean curvature H (with respect to the inward-pointing unit normal). These AH manifolds have mass that is controlled in terms of g and H, reducing to the hyperbolic Hawking mass of (S^2, g, H) as g becomes round or H tends to zero. This gives an upper bound for an asymptotically hyperbolic analogue of the Bartnik mass. The construction is based on work of Mantoulidis and Schoen, who used similar ideas to effectively compute the (usual AF) Bartnik mass of apparent horizons.

Alice Chang reported on the compactness of conformally compact Einstein manifolds in dimension 4

[11]. Given a class of conformally compact Einstein manifolds with boundary, she wishes to study the compactness of the class under some local and non-local boundary constraints. She reported on some joint work with Yuxin Ge and some recent improvements under discussion also with Jie Qing of the problem on the $3 + 1$ setting.

Netta Engelhardt gave an update on extremal surfaces and singularities. She presented new results on the behaviour of extremal surfaces in general asymptotically AdS spacetimes [14]. These included new theorems on the maximal extent of boundary-anchored extremal surfaces and a new singularity theorem. This was based on her work in progress with Daniel Harlow.

Greg Galloway spoke on the mass aspect and positive mass theorems for ALH manifolds. He presented some results concerning the mass aspect and positivity of mass for asymptotically locally hyperbolic (ALH) manifolds. This was based on his joint work Piotr Chrusciel, Luke Nguyen, and Tim Paetz [12].

Romain Gicquaud described mass-like covariants for asymptotically hyperbolic manifolds. The mass of an asymptotically hyperbolic manifold is a vector in Minkowski space defined in terms of the geometry at infinity of the manifold. It enjoys covariance properties under the change of coordinate chart at infinity. In his talk Gicquaud classified covariants satisfying similar properties. This was join work with J Cortier and M Dahl [13].

Rod Gover discussed Sasaki-Einstein structures and their compactification. Sasaki geometry is often viewed as an odd dimensional analogue of Kaehler geometry. In particular a Riemannian or pseudo-Riemannian manifold is Sasakian if its standard metric cone is Kaehler or, respectively, pseudo-Kaehler. Gover showed that there is a natural link between Sasaki geometry and projective differential geometry. The situation is particularly elegant for Sasaki-Einstein geometries and in this setting he used projective geometry to provide the resolution of these geometries into "less rigid" components. This is analogous to the usual picture of a Kaehler structure: a symplectic manifold equipped also with a compatible complex structure, etc. However the treatment of Sasaki geometry this way is locally more interesting and involves the projective Cartan or tractor connection. This enabled him to describe a natural notion of compactification for complete non-compact pseudo-Riemannian Sasakian geometries. For such compactifications the boundary at infinity is a conformal manifold with a Fefferman space structure—so it fibres over a CR manifold. This is nicely compatible with the compactification of the Kaehler-Einstein manifold that arises, in the usual way, as a leaf space for the defining Killing field of the given Sasaki-Einstein manifold. This was joint work of Gover with Katharina Neusser and Travis Willse [19].

Robin Graham spoke on the renormalized Volume for Singular Yamabe Metrics [20]. He began by reviewing the renormalized volume for Poincaré-Einstein metrics, and then discussed an analogous construction of renormalized volume for singular Yamabe (a.k.a. Loewner-Nirenberg) metrics. He concluded by describing joint work with Matt Gursky concerning an unexpected invariance property of the renormalized volume for singular Yamabe metrics in dimension 4 related to the Chern-Gauss-Bonnet formula and related phenomena for solutions of a singular σ_2 Yamabe problem.

Qing Han addressed the nonexistence of Poincaré-Einstein fillings on spin manifolds. He discussed whether a conformal class on the boundary M of a given compact manifold X can be the conformal infinity of a Poincaré-Einstein metric in X. He constructed an invariant of conformal classes on the boundary M of a compact spin manifold X of dimension $4k$ with the help of the Dirac operator. He proved that a conformal class cannot be the conformal infinity of a Poincaré-Einstein metric if this invariant is not zero. Furthermore, he proved that this invariant can attain values of infinitely many integers if one invariant is not zero on the above given spin manifold. This was based on joint work of Han's with Gursky and Stolz [21].

Hyun Chul Jang spoke on asymptotically hyperbolic 3-metrics with Ricci flow foliation. There have been a number of successful constructions for asymptotically flat metrics with a certain background foliation. Jang's talk began with several results obtained by this foliation method. He spoke about his work on a particular construction of asymptotically hyperbolic Riemannian 3-manifolds using the Ricci flow solution on a closed surface as a foliation. This can be used to get an asymptotically hyperbolic extension from a closed surface with some conditions [26].

Marcus Khuri described transformations of asymptotically AdS hyperbolic initial data and associated geometric inequalities, with a view to proving results such as the Penrose conjecture in asymptotically hyperbolic spaces. The project rests on the open question of whether a certain geometrically motivated coupled system where one of the equations is the Jang equation can be solved in the appropriate settings [10].

David Maxwell discussed the prescribed scalar curvature problem in the asymptotically Euclidean setting

[28]. The Yamabe invariant of an asymptotically Euclidean (AE) manifold is defined analogously to that of a compact manifold. Nevertheless, the prescribed scalar curvature problem in the AE setting has features that are quite different from its compact counterpart. For example, a Yamabe positive AE manifold admits a conformally related metric that has a scalar curvature with any desired sign: positive, negative or zero everywhere. In his talk, he discussed the resolution of the prescribed nonpositive scalar curvature problem for AE manifolds and its application to general relativity.

Jie Qing spoke on hypersurfaces in hyperbolic space. He reported on recent work on convex hypersurfaces in hyperbolic space [5, 6]. To study hypersurfaces in hyperbolic space analytically, one needs to find ways to parametrize it, preferably globally. He considered two parametrizations: vertical graph and hyperbolic Gauss map. To get a global parametrization, one needs understand the interrelation of convexity and embeddedness. It is also important to understand the asymptotics of the geometry at ends. He reported some of his recent work on global and asymptotic properties of hypersurfaces with nonnegative sectional curvature or Ricci curvature in hyperbolic space, where the use of n-Laplace equations seems to be new.

Michael Singer considered asymptotically hyperbolic anti-self-dual Einstein metrics [17]. Let M be a compact oriented d-dimensional manifold with boundary N. A natural geometric boundary value problem is to find an asymptotically hyperbolic Einstein metric g on (the interior of) M with prescribed 'conformal infinity' on N. A little more precisely, the problem is to find (Einstein) g with the boundary condition $x^2 g$ tends to a metric h on N as x goes to 0, x being a boundary defining function for N. The freedom to rescale x by an arbitrary smooth positive function means that only the conformal class of h is naturally well defined, hence the terminology 'conformal infinity' in this boundary problem. Since the pioneering work of Graham and Lee (1991) the problem has attracted attention from a number of authors. If the dimension d is 4, there is a refinement, asking that g be anti-self-dual as well as Einstein (satisfying the same boundary condition). If M is the ball, this is the subject of the positive frequency conjecture of LeBrun (1980s) proved by Biquard in 2002. Singer explained a gauge theoretic approach to the ASDE problem, based on his joint work with Joel Fine and Rafe Mazzeo, which is readily applicable for general M and the currently available results.

Iva Stavrov spoke on asymptotic gluing of CMC shear-free hyperboloidal initial data. Hyperboloidal initial data sets do not generally evolve into spacetimes with a well-defined Scri unless they satisfy the shear-free condition. She presented a procedure for gluing shear-free hyperboloidal initial data which leaves the conformal infinity connected. Her work was done within the framework of weakly asymptotically hyperbolic metrics, and special attention was paid to the shear-free condition. This was work in progress in collaboration with Paul T Allen, James Isenberg, and John M Lee generalizing [25].

Marika Taylor surveyed recent developments in holography (see, e.g., [30, 31]). Holography relates gravity in asymptotically locally hyperbolic manifolds to conformal field theories in one less dimension. She focused on recent developments in holography that may be of particular interest to mathematicians and relativists. Topics covered included generalisations of holography to different classes of spacetime asymptotics; minimal surfaces in hyperbolic geometries, their renormalised areas and entanglement; and the relation between Bondi-Sachs and Fefferman-Graham analysis of asymptotic structure.

Andras Vasy discussed the stability of Kerr-de Sitter black holes, based on his joint work with Peter Hintz [24]. Kerr-de Sitter black holes are rotating black holes in a universe with a positive cosmological constant; i.e., they are explicit solutions (in $3 + 1$ dimensions) of Einstein's equations of general relativity. They are parameterized by their mass and angular momentum. He first discussed the geometry of these black holes as well as that of the underlying de Sitter space, and then spoke about the stability question for these black holes in the initial value formulation. Namely, appropriately interpreted, Einstein's equations can be thought of as quasilinear wave equations, and then the question is if perturbations of the initial data produce solutions which are close to, and indeed asymptotic to, a Kerr-de Sitter black hole, typically with a different mass and angular momentum. In the second part of the talk he discussed analytic aspects of the stability problem, in particular showing that Kerr-de Sitter black holes with small angular momentum are stable in this sense.

Guofang Wang presented geometric inequalities related to the hyperbolic mass, including Penrose-type inequalities for Chern-Gauss-Bonnet masses [18].

3 Scientific Progress/Outcome of the Meeting

The outcome of a successful meeting becomes clear not in the days after the meeting but in the years afterward, as ideas communicated and collaborations begun or advanced during the meeting bear fruit. At this meeting, there were many collaborations. Partly this was because the organizers invited groups of collaborators, but it was also because of the atmosphere at the Banff Centre, the interactive nature of many of the participants, and the broad spectrum of research approaches represented among them.

There was specific progress on various issues. We will soon see the completion of studies of Poincaré-Bach and Poincaré-Lovelock metrics, due to progress achieved at this meeting. Furthermore, open questions on the positivity (and more general boundedness) of the AH mass and its relationship, if any, to renormalized volume are additional examples of topics that were advanced during the meeting and which can be expected to see progress as a result.

Perhaps the most remarkable progress is the opening of greater communication between physicists and mathematicians. The crystal clear expository talk of Taylor and Englehardt's talk on her recent research deserve special mention in this regard, and led to much discussion and interest.

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