The mathematical genesis of the phenomenon called "$1/f$ noise" (10frg132)

Lawrence M. Ward (University of British Columbia)
Priscilla E. Greenwood (University of North Carolina)

6 June 2010-12 June 2010

1 Overview of the Field

"$1/f$ noise" refers to the phenomenon of the spectral density, $S(f)$, of a signal, having the form

$S(f) = constant/f^{\alpha}$,

where f is frequency and α is a signal-dependent parameter. In physical situations the phenomenon is often considered to occur on an interval bounded away from both zero and infinity because these endpoints are not observable. Mathematically, however, the behavior of $S(f)$ near these endpoints, particularly as $f \to 0$, is of considerable interest. $1/f^{\alpha}$ signals with $0.5 < \alpha < 1.5$ are found widely in nature, occurring in physics, biology, astrophysics, geophysics, economics, psychology, language and even music [1, 2] (note this overview closely follows parts of [2]). The case of $\alpha = 1$, or "pink noise", is both the canonical case, and the one of most interest; surprisingly, as illustrated in Figure 1, many of the values for α found in nature are very near 1.0. Henceforth the term "$1/f$ noise" will refer only to this case; the value of α will be specified if it is other than 1.0.

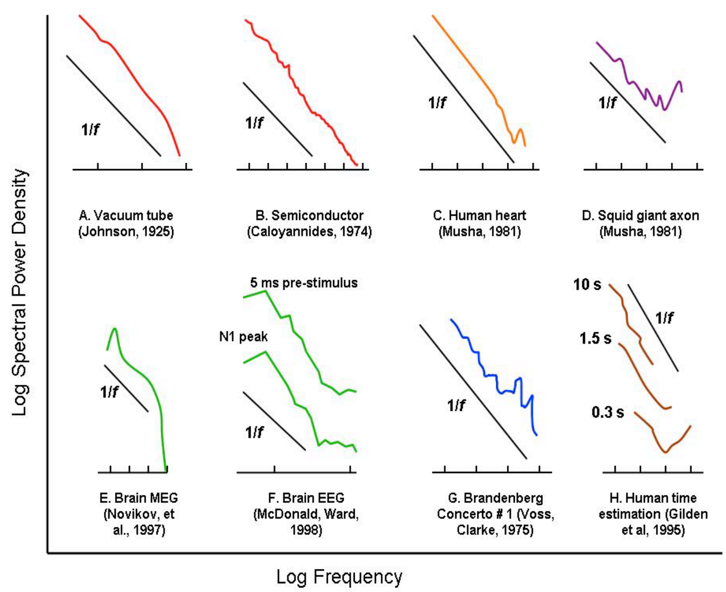

Figure 1. Some examples of roughly $1/f$ power spectra from various empirical domains. Reprinted from [2].

$1/f$ noise is an intermediate between white noise ($\alpha = 0$) with no correlation in time and a random walk (Brownian motion, $\alpha = 2$) with no correlation between increments. Brownian motion is the integral of white noise; integration of a signal increases α by 2 whereas differentiation decreases it by 2. Therefore, $1/f$ noise cannot be obtained by integration or differentiation of such convenient signals. Moreover, there are no simple, even linear stochastic differential equations generating signals with $1/f$ noise. The widespread occurrence of signals exhibiting such behavior suggests that a generic mathematical explanation might exist. Except for some uninformative formal mathematical descriptions like "fractional Brownian motion" (half-integral of a white noise signal), however, no generally recognized physical explanation of $1/f$ noise has been proposed. Thus, the ubiquity of $1/f$ noise is one of the oldest puzzles of contemporary physics and of science in general.

$1/f$ noise was discovered in 1925 by Johnson [3] in data from an experiment designed to study shot noise in vacuum tubes. Schottky [4] attempted to describe Johnson's pink noise mathematically by assuming that an exponential relaxation,

$$N(t) = N_0 e^{-\lambda t}, t \geq 0,$$

of a current pulse was caused by release of electrons from the cathode of the vacuum tube. For a train of such pulses at an average rate n the power spectrum is the "Lorentzian" form

$$S(f) = \frac{N_0^2 n}{\lambda^2 + f^2},$$

which is nearly constant near $f = 0$ and nearly proportional to $1/f^2$ for large f, with a narrow transition region where the power spectrum resembles that of the pink noise found by Johnson. Bernamont [5] later pointed out that only a superposition of such processes with a variety of relaxation rates, λ, would yield $1/f$ noise for a reasonable range of frequencies. He showed that if λ is uniformly distributed between λ_1 and λ_2, the power spectral density is

$$S(f) = \begin{cases} N_0^2 n & \text{if } 0 \ll f \ll \lambda_1 \ll \lambda_2 \\ \frac{N_0^2 n \pi}{2f(\lambda_2 - \lambda_1)} & \text{if } \lambda_1 \ll f \ll \lambda_2 \\ N_0^2 n \cdot \frac{1}{f^2} & \text{if } 0 \ll \lambda_1 \ll \lambda_2 \ll f. \end{cases}$$

In other words $S(f)$ is proportional to $1/f$ for $\lambda_1 \ll f \ll \lambda_2$. This form, derived from a superposition of Lorentzians, describes the power spectrum in many, but not all, models that have been proposed since Bernamont's.

Another interesting approach to modeling $1/f^\alpha$ noise is related to Granger's [6] classic model in which a long-memory process is produced by the superposition of a number of autoregressive short-memory processes. Erland and Greenwood [7] considered a collection of (discrete-time) AR(1) processes with different parameters, θ_m:

$$X_t^m = \theta_m X_{t-1}^m + \sigma \epsilon_t,$$

where the ϵ_t are i.i.d. Gaussian variables, with mean 0 and standard deviation σ, and $0 < \theta_m < 1$. The autocorrelation function of each X_t^m decays exponentially in time with rate θ_m. The spectral density of each time series X_t^m is the Lorentzian-like function

$$S_m(f) = \frac{\sigma^2}{(1-\theta_m)^2 + 2\theta_m(1-cos f)} \approx \begin{cases} \sigma^2/(1-\theta_m)^2 & \text{if } 0 \ll f \ll 1-\theta_m \\ \sigma^2/\theta_m f^2 & \text{if } 1-\theta_m \ll f \leq 1. \end{cases}$$

Let $Y_t = \frac{1}{m}\sum_{i-1}^{m} X_t^i$, and let the coefficients, θ_m, be distributed as $(1-\theta)^{1-\alpha}$ for $\theta_{min} < \theta < \theta_{max}$. Then the power spectral density of Y_t for large m, is approximately

$$S(f) \propto \begin{cases} 1 & \text{if } 0 \ll f \ll 1-\theta_{max} \\ 1/f^\alpha & \text{if } 1-\theta_{max} \ll f \ll 1 \\ 1/f^2 & \text{if } 1-\theta_{min} \ll f \ll 1. \end{cases}$$

If the θ are uniformly distributed ($\alpha = 1$), then we get a good approximation of $1/f$ noise simply by averaging the individual series. This corresponds to the classical result that the power spectrum of a uniform mixture of exponentially decaying autocorrelation functions has a $1/f$ form. Even the sum of as few as three AR(1) processes with widely distributed coefficients (e.g., 0.1, 0.5, 0.9) gives a reasonable approximation to a $1/f$ power spectrum [1].

Unfortunately such a superposition of Lorentzians is not found in many long-memory models that are concerned with the behavior of $S(f)$ as $f \to 0$. Most long-memory models are driven by a heavy-tailed probability distribution (one in which the tails are not exponentially bounded). In other words, a time series with long memory is one whose autocorrelations decay so slowly with lag that they are not summable, indicating a persistence of dependence of current values on previous values a very long distance in the past of the series. As just demonstrated above, however, a sum of AR(1) processes whose parameters are distributed within a certain range can display a form similar to many of the more physical models [7]. It is possible, therefore, that a similar relationship between other long-memory processes and processes that produce $1/f^{\alpha}$ noise within a specified range can be found.

2 Recent Developments and Open Problems

The search term "$1/f$ noise" retrieves over 37,000 records from Google Scholar and over 149,000 from Google. The *Scholarpedia* article [2] has received over 45,000 hits since it was published in 2007. Thus there is evidence of vast interest in the phenomenon. Publications appear at a fairly steady rate proposing new models for $1/f^{\alpha}$ noise. Similarly work continues on long-memory models. The search term "long memory process" retrieves nearly 2,400 records from Google Scholar and 129,000 from Google, testifying in part to the importance of long memory in economics. Two lines of recent work on $1/f^{\alpha}$ noise models were particularly important to our focus group. These were the work of Bronislovas Kaulakys and colleagues on point process models and stochastic differential equation (SDE) models (e.g., [8]), and the work of Sveinung Erland and Priscilla Greenwood on Markov chain models (e.g., [7]). These models describe very large classes of situations in which $1/f^{\alpha}$ noise appears, and take us some way toward a more general understanding of the phenomenon. Similarly important was a set of four classes of long memory models [9], comprising renewal counting processes, on/off models, infinite source Poisson models, and renewal reward processes, described for us by Vladas Pipiras. Combined with the aggregation of autoregressive processes that was proved by Granger to result in $1/f^{\alpha}$ noise [6], these general classes of models comprised our playing field. Our major open problem was how to reconcile these approaches, many of which on the surface seemed mutually incompatible. We began the week concerned with finding an underlying mechanism or mathematical description that would encompass all of these models.

3 Presentation Highlights

Each of the participants presented a description of their work relevant to $1/f^{\alpha}$ noise and also additional problems and frameworks to guide our joint work. Some presentation highlights were as follows: introduction and overview, including some conjectures about model convergence, by Lawrence Ward, the four major long-memory models in a group as well as a more rigorous definition of $1/f^{\alpha}$ noise by Vladas Pipiras, a Markov chain framework for many models and the demonstration of its compatibility with the SDE framework by Sveinung Erland, a large group of models involving point processes and SDEs by Bronislovas Kaulakys, a discussion of time series with and without local stationarity and the problems that the latter create by Wolfgang Polonik, the train model as a prototype of a physical model of $1/f^{\alpha}$ noise by Joern Davidsen, statistical procedures for estimating the exponent α by Changryong Baek, and and a survey of known results for roughness and extrema in $1/f^{\alpha}$ noise by Nicholas Moloney. These presentations occupied the first two days of the workshop, as considerable work was accomplished toward setting our goals and clarifying our vocabulary and conceptual structures during each one.

4 Scientific Progress Made

The major progress we made was in identifying rigorous procedures to translate between several major classes of models and in defining the approaches to finding translations between the remaining classes. We discovered that translations that preserve the power spectrum already exist between Markov process and stochastic differential equation models and possibly also between point process and continuous process models. It is also possible that we can demonstrate that fractional Gaussian noise is the limit of both dependence-driven and heavy-tailed-driven models, as conjectured at the beginning of the workshop, and that a translation exists between heavy-tailed-driven models and point process or Markov chain models. If so we would have a complete mapping of all of the model types onto each other. This would make the problem of finding an underlying process more tractable, and also focus efforts in that direction, rather than on further proliferating more limited models.

5 Outcome of the Meeting

There were two important outcomes of this meeting. First, the participants were from a variety of different fields and they all learned a great deal from each other. In particular the importance of including models of long memory processes in the discussion of the origins of $1/f^{\alpha}$ noise was made clear to those of us who had previously discounted these models because they involved physically impossible situations close to $f = 0$. Moreover, the utility of moving away from the origin in the spectrum in modeling signals was made evident to those of us interested in long memory. Second, we realized that rather than searching for a single underlying theory for all $1/f^{\alpha}$ phenomena, it would be more productive at present to study whether it is possible to translate between all of the various model classes. If so, then the existence of an underlying theory seems more credible, although not guaranteed. We thus produced the outline of a paper to be entitled "$1/f^{\alpha}$ Noise: Data. Models, Translations," which we intend to complete over the next year.

References

[1] L.M. Ward *Dynamical Cognitive Science*, MIT Press, Cambridge, MA, 2002.

[2] L.M. Ward and P.E. Greenwood, $1/f$ noise, *Scholarpedia* **2** (2007), 1537.

[3] J.B. Johnson, The Schottky effect in low frequency circuits, *Physical Review* **26** (1925), 71–85.

[4] W. Schottky, Small-shot effect and flicker effect, *Physical Review* **28** (1926), 74–103.

[5] J. Bernamont, Fluctuations de potential aux bornes d'un conducteur metallique de faible volume parcouru par un courant, *Annalen der Physik (Leipzig)* **7** (1937), 71–140.

[6] C.W.J. Granger, Long memory relationships and the aggregation of dynamic models, *Journal of Econometrics* **14** (1980), 227–238.

[7] S. Erland and P.E. Greenwood, Constructing $1/\omega^{\alpha}$ noise from reversible Markov chains, *Physical Review E* **76** (2007), 031114.

[8] B. Kaulakys, J. Ruseckas, V. Gontis, and M. Alaburda, Nonlinear stochastic models of $1/f$ noise and power-law distributions, *Physica A* **365** (2006), 217-221.

[9] J. Beran *Statistics for long-memory processes*, Chapman and Hall, New York, 1994.