CLASSICAL PROBLEMS
ON PLANAR POLYNOMIAL VECTOR FIELDS

November 23–28, 2008
BIRS, Banff, Canada

PROGRAM

Monday, November 24

8:30–8:45 Word of welcome and announcements

8:45–9:35: Alexander Brudnyi

9:45–10:35: Jaume Giné

10:45–11:00 Coffee break

11:00–11:50: Hans-Christian Graf v. Bothmer

12:00–13:00 Lunch

14:10–15:00: Pei Yu

15:10–15:30 Coffee break

15:30–16:20: Christiane Rousseau

16:30–17:20: Maoan Han

Tuesday, November 25

8:30–9:20: Dana Schlomiuk

9:30–10:20: Nicolae Vulpe

10:30–11:00 Coffee break

11:00–11:50 Joan Carles Artes

12:00–13:30 Lunch

14:00–14:50: Nikolay Dimitrov

15:00–15:30 Coffee break

15:30–16:20: Xiang Zhang

16:30–16:45 Javier Chavarriga remembered

Wednesday, November 26

8:30–9:20: Freddy Dumortier

9:30–10:20: Robert Roussarie

10:30–11:00 Coffee break

11:00–11:50: Peter De Maessalck

12:00–13:30 Lunch

14:00–18:00 Free afternoon

Thursday, November 27

8:30–9:20 Chengzhi Li

9:30–10:20 Jibin Li

10:30–11:00 Coffee break

11:00–11:50 Armengol Gasull

12:00–13:30 Lunch

14:00–14:50 Driss Boularas

15:00–15:30 Coffee break

15:30–16:20 Waldo Arriagada Silva

Friday, November 28

8:30–9:20: Maite Grau Montana

9:30–10:20: Isaac Garcia

10:30–11:00 Coffee break

11:00–11:50: Jaume Llibre

12:00–13:30 Lunch

List of the abstracts of the talks

Author: Waldo Arriagada Silva

Title: Modulus for the analytic classification of unfoldings of order one weak foci in $\mathbb{C}^2$.

Abstract: We characterize the modulus of analytic classification, under orbital equivalence and under conjugacy, for generic analytic 1 parameter dependent unfoldings of a generic weak focus of order 1 at the origin of coordinates in $\mathbb{C}^2$. The system presents some special properties coming from global invariances by inversions and Schwarz reflections; these properties are reflected in the particular shape of the modulus and lead to a generalization of a classical result of Mattei–Moussu on saddles points: a weak focus is analytically characterized by its Extended Poincaré return mapping. This statement allows to shed new light into the so-called *isochronicity problem* for elliptic singularities. The problem of isochronicity is strongly connected to the structure of the modulus: the isochronicity is a property which is invariant by diffeomorphic conjugation and hence it is detected by the modulus of the weak focus. This gives us a first formal invariant as the modulus is already characterized. This also yields the time for the return map near the singular point and a second formal invariant expresses the difference between the time for return near the singular point and the period of the periodic orbit. If the weak focus is isochronous, then the latter invariant is zero. It is our aim to understand what are the obstructions to get the converse.

Author: Joan C. Artés

Title: The geometry of quadratic polynomial differential systems with a weak focus and an invariant straight line.

Abstract: Among the families of quadratic differential systems which have been studied, the most interesting ones for producing limit cycles in perturbations are those with center and those with weak foci. Phase portraits of quadratic systems with center were given by Vulpe in 1983. Bifurcation diagrams were first given by Schlomiuk (1988, 1993) and by Andronova (1988); also by Zoladek (1994). The systems with center form an algebraic set in the parameter space which splits into four irreducible algebraic varieties. Modulo the action of the group of affine transformations and time rescaling, three of these yield three projective planes and the fourth is a curve. The study of systems with weak foci

of third order was done in several papers by Artes and Llibre and Llibre and Schlomiuk over the period 1994-2004. Modulo the same group action this family fills a projective plane. The study of the family of quadratic systems with a second order weak focus was done by Artes, Llibre and Schlomiuk (2006). Modulo the group action this family yields a three-dimensional space which could be viewed on a half-ball. The next step to be taken is the study of the family of quadratic systems with a first order weak focus. This study is much harder as modulo the group action this family is four-dimensional. We initiate this work with the study of a hyperplane within this four-dimensional space: the family of quadratic systems with a first order weak focus and an invariant line. Results on this particular family will be presented.

Author: Driss Boularas

Title: Necessary conditions of existence of a center for a family of differential systems.

Abstract: It is well known that the "center–focus" problem is solved only for some particular families of differential systems. In this work, we deal with differential systems $x' = -y + P_m(x, y)$, $y' = x + Q_m(x, y)$, where the polynomials P_m and Q_m are homogeneous of degree m. After recalling the Lyapunov–Poincaré theorem and the starting equations of the first integrals, we propose one (if m is even) and two (if m is odd) necessary conditions for the origin to be a center.

Author: Alexander Brudnyi

Title: An algebraic approach to the center problem for ODE's.

Abstract: In my talk I describe an algebraic approach to the center problem for a general ordinary differential equation appearing in the Poincaré center–focus problem for planar polynomial vector fields.

Author: Peter De Maesschalck

Title: Canard cycles in the presence of slow dynamics with singularities.

Abstract: This talk deals with the cyclicity of limit periodic sets that occur in families of vector fields of slow–fast type. The limit periodic sets are formed by a fast orbit and a curve of singularities containing a unique turning point. At this turning point a stability change takes place: on one side of the turning point, the dynamics point strongly

towards the curve of singularities, on the other side the dynamics point away from the curve of singularities. The presence of periodic orbits in a perturbation is related to the presence of canard orbits passing near this turning point, i.e. orbits that stay close to the curve of singularities despite the exponentially–strong repulsion near this curve. All existing results deal with a non–zero slow movement permitting to get a good estimate of the cyclicity by considering the slow divergence integral along the curve of singularities. Now, we discuss what happens when the slow dynamics exhibit singularities. In particular our study includes the cyclicity of the slow–fast 2–saddle cycle, formed by a regular saddle–connection (the fast part) and a part of the curve of singularities (the slow part). We show that the relevant information is no longer merely contained in the slow divergence integral.

Author: Nikolay Dimitrov

Title: Rapid evolution of complex limit cycles.

Abstract: Limit cycles of planar polynomial vector fields have long been a focus of extensive research. Analogous to the real case, similar problems have been studied in the complex plane where a polynomial differential one–form gives rise to a foliation by Riemann surfaces. In this setting, a complex cycle is defined as a nontrivial element of the fundamental group of a leaf from the foliation. Whenever the polynomial foliation comes from a perturbation of an exact one–form, one can introduce the notion of a multi–fold cycle. This type of cycle has at least one representative that determines a free homotopy class of loops in an open fibered subdomain of the complex plane. The topology of this subdomain is closely related to the exact one–form, mentioned earlier. This talk will be an introduction to the notion of multi–fold cycles of a close–to–integrable polynomial foliation. We will explore the way they correspond to periodic orbits of certain monodromy (Poincaré) maps associated with the foliation. We will also discuss the tendency of a continuous family of multi–fold limit cycles to escape from certain large open domains in the complex plane as the foliation converges to its integrable part.

Author: Freddy Dumortier

Title: Birth of canard cycles (I).

Abstract: In the talk we consider singular perturbation problems occurring in planar slow–fast systems $\dot{x} = y - F(x, \lambda), \dot{y} = -\varepsilon G(x, \lambda))$

where F and G are smooth or even real analytic for some results, λ is a multiparameter and ε is a small parameter. We deal with turning points that are limiting situations of (generalized) Hopf bifurcations and that we call slow–fast Hopf points. We investigate the number of limit cycles that can appear near a slow–fast Hopf point and this under very general conditions. The talk is base don joint work with Robert Roussarie.

Author: Isaac A. García

Title: Study of the cyclicity using the inverse integrating factor.

Abstract: This work is concerned with planar real analytic differential systems with an analytic inverse integrating factor defined in a neighborhood of a regular orbit. We show that the inverse integrating factor defines an ordinary differential equation for the transition map along the orbit. When the regular orbit is a limit cycle, we can determine its associated Poincaré return map in terms of the inverse integrating factor. In particular, we show that the cyclicity of a limit cycle coincides with the vanishing multiplicity of an inverse integrating factor over it. We also apply this result to study the homoclinic loop bifurcation. We only consider homoclinic loops whose critical point is a hyperbolic saddle and whose Poincaré return map is not the identity. A local analysis of the inverse integrating factor in a neighborhood of the saddle allows us to determine the cyclicity of this polycycle in terms of the vanishing multiplicity of an inverse integrating factor over it. Our result also applies in the particular case in which the saddle of the homoclinic loop is linearizable, that is, the case in which a bound for the cyclicity of this graphic cannot be determined through an algebraic method. Finally, we also study the generalized Hopf Bifurcation applying the same ideas.

This is a joint work with Hector Giacomini and Maite Grau.

Author: Armengol Gasull

Title: Upper bounds for the number of limit cycles of some planar polynomial differential systems.

Abstract: We give an effective method for controlling the maximum number of limit cycles of some planar polynomial systems. It is based on a suitable choice of a Dulac function and the application of the well–known Bendixson–Dulac Criterion for multiple connected regions. The key point is a new approach to control the sign of the functions involved in the criterion. The method is applied to several examples.

This talk is based on a joint work with Hector Giacomini.

Author: Jaume Giné

Title: The reversibility and the center problem

Abstract: In this work we study the narrow relation between reversibility and the center problem and also between reversibility and the integrability problem. It is well known that an analytic system having either a non–degenerate or nilpotent center at the origin is analytically reversible or orbitally analytically reversible, respectively. In this paper we prove the existence of a smooth map that transforms an analytic system having a degenerate center at the origin into a reversible linear system (after rescaling the time). Moreover, if the degenerate center has an analytic or a $\mathcal{C}^\infty$ reversing symmetry, then the transformed system by the map has also a reversing symmetry. From the knowledge of a first integral near the center we give a procedure to detect reversing symmetries.

Author: Hans–Christian Graf v. Bothmer

Title: Experimental Results for the Poincaré–Center–Problem.

Abstract: Let X_n be the vanishing of the first n focal values (Liapunov quantities) for planar Polynomial Vector Fields of degree at most 3. In this talk I will explain how computer experiments over finite fields of small characteristic can yield information about the component structure of X_n. We obtain for example

(1) experimental evidence, that Zoladek's conjecture is true at least up to codimension 7,
(2) a proof that the vanishing of 11 focal values is not sufficient to prove that a plane cubic system has a center (this improves the previously known bound by one),
(3) information about how the Zoladek's families intersect each other.

The results are joint work with Martin Cremer, Jakob Kröker and Ulrich Rhein respectively.

Author: Maite Grau

Title: Zeros of Abelian integrals and Chebyshev systems.

Abstract: A collection of $n+1$ real functions on a real interval is said to be an extended Chebyshev system if any nontrivial linear combination of them has at most n isolated zeros on the real interval, counted with multiplicities.

Abelian integrals appear naturally when studying bifurcations of limit cycles of planar polynomial vector fields. In particular, zeros of Abelian integrals are related to limit cycles appearing in perturbations of Hamiltonian vector fields. To show that a collection of Abelian integrals is a Chebyshev system usually provides the necessary condition to understand this bifurcation. In general, it is very difficult to show that a given collection of Abelian integrals is a Chebyshev system. The traditional approach is to use properties of the system of linear ordinary differential equations satisfied by the Abelian integrals, the so–called Picard–Fuchs system. However, sometimes, even the effective computation of this system turns out to be a difficult problem.

In this talk, a criterion is presented that provides an easy sufficient condition in order that a collection of Abelian integrals has the Chebyshev property. This condition involves the functions in the integrand of the Abelian integrals and can be checked, in many cases, in a purely algebraic way. By using this criterion, several known results are obtained in a shorter way and some new results, which could not be tackled by the known standard methods, can also be deduced.

This is a joint work with Francesc Mañosas and Jordi Villadelprat.

Author: Maoan Han

Title: Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle.

Abstract: In this paper we consider general analytic near-Hamiltonian systems with parameters on the plane. We suppose that the unperturbed Hamiltonian system has a double homoclinic loop passing through a nilpotent saddle. Then there are three families of periodic orbits inside or outside the loop which yield three Melnikov functions. We study the analytical property of the three first order Melnikov functions and obtain asymptotic expansions of them near the loop together with the computation formulas of the first coefficients of the expansions. Using these coefficients we give a sufficient condition for the perturbed system to have 8, 10 or 12 limit cycles in a neighborhood of the loop with seven different distributions. We finally consider some polynomial systems and find a lower bound of the maximal number of limit cycles as an application of our main results.

This is a joint work with Junmin Yang and Dongmei Xiao.

Author: Chengzhi Li

Title: A cubic system with thirteen limit cycles.

Abstract: We construct a planar cubic system and demonstrate that it has at least 13 limit cycles. The construction is essentially based on counting the number of zeros of some Abelian integrals.

This is a joint work with Changjian Liu and Jiazhong Yang.

Author: Jibin Li

Title: Some new results on the study of Z_q–equivariant planar polynomial vector fields.

Abstract: In this talk, we shall introduce some new results on the study of Z_q-equivariant planar polynomial vector fields. The main conclusions are as follows:

(1) For the planar Z_2-equivariant cubic systems having two elementary focuses, the characterization of a bi–center problem and shortened expressions of the first six Liapunov constants are completely solved. The necessary and sufficient conditions for the existence of the bi–center are obtained. All possible first integrals are given. Under small Z_2-equivariant cubic perturbations, the conclusion that there exist at most 12 small–amplitude limit cycles with the scheme $\langle 6 \amalg 6 \rangle$ is proved.

(2) On the basis of mentioned work in **1.**, by considering the bifurcation of a global limit cycle from infinity, we show that under small Z_2-equivariant cubic perturbations, such bi–center cubic system has at least 13 limit cycles with the scheme $\langle 1 \langle 6 \amalg 6 \rangle \rangle$, i.e., we obtain that the Hilbert number $H(3) \geq 13$.

(3) For the Z_6-equivariant planar polynomial vector field of degree 5, we proved that such system has at least six symmetric centers if and only if it is a Hamiltonian vector field. The characterization of a center problem is completely solved. The shortened expressions of the first four Lyapunov constants are given. Under small Z_6-equivariant perturbations, the conclusion that perturbed system has at least 24 limit cycles with the scheme $\langle 4 \amalg 4 \amalg 4 \amalg 4 \amalg 4 \amalg 4 \rangle$ is rigorously proved. Two schemes of distributions of limit cycles are given.

(4) For the Z_5-equivariant planar polynomial vector field of degree 5, we shown that such system has at least 5 symmetric centers if and only if it is a Hamiltonian vector field. The characterization of a center problem is completely solved. The shortened expressions of the first four Lyapunov constants are given. Under small Z_5-equivariant perturbations, the conclusion that perturbed system has at least 25 limit cycles with the scheme $\langle 5 \amalg 5 \amalg 5 \amalg 5 \amalg 5 \rangle$ is rigorously proved.

This is a joint work with Yirong Liu.

Author: Jaume Llibre

Title: On the limit cycles of the Liénard polynomial differential equations

Abstract: One of the most studied planar polynomial differential systems are the so–called *generalized Liénard polynomial differential equations*

$$\ddot{x} + f_n(x)\dot{x} + g_m(x) = 0, \tag{1}$$

which were studied by many researchers, where $g(0) = 0$ and f and g are polynomials of degree n and m respectively. Such dynamical systems appear very often in several branches of the sciences, such as biology, chemistry, mechanics, electronics, etc. The differential equation (1) of second order can be written as the equivalent 2–dimensional Liénard polynomial differential system of first order

$$\dot{x} = y, \quad \dot{x} = -g_m(x) - f_n(x)y. \tag{2}$$

When $g(x) = x$ the Liénard differential systems (2) are called the *classical Liénard systems.*

The main objective of this talk is to present a survey on the old and new results on the limit cycles of the Liénard polynomial differential systems (2) algebraic or not in function of m and n.

We shall put special emphasis on the number of the so–called small, medium and large limit cycles in function of m and n. These are the limit cycles of the polynomial Liénard differential system (2) which bifurcate from the origin, or which bifurcate from the linear center $\dot{x} = y$, $\dot{y} = -x$ perturbed inside the class of all polynomial Liénard differential systems (2), or which bifurcate from the graphics that the Liénard polynomial Liénard differential system (2) can have.

Author: Robert Roussarie

Title: Birth of canard cycles (II).

Abstract: This talk is the continuation of the talk given by Freddy Dumortier. We consider an unfolding of a slow–fast Hopf point. This unfolding is a limit of Hopf–Takens ones and then it can be considered as a degenerate version of classical Hopf–Takens unfoldings.

It could be expected that a generic slow–fast Hopf unfolding with n parameters produces at most n limit cycles. It is precisely what we intended to prove but rather surprisingly, this result can only be obtained modulo a conjecture about a remarkable system of generalized Abelian integrals (and this conjecture is yet to be proved!).

In this talk I want to comment some important steps of the proof. In a first step, we transform by blowing–up the slow–fast system to a simpler family of 3–dimensional vector fields defined near a critical locus E. In a second and crucial step, we obtain a precise presentation of a difference map near a singular polycycle $\Gamma \subset E$, defined for the flow of this blown–up vector field family. This step uses some new precise smooth normal form at singular points of the blown–up vector field family. Next we use the presentation of the difference map to study of cyclicity Γ. The main difficulty is that the singular polycycle Γ is not contained into the regular part of the critical locus E and that, as a consequence, the blown–up vector field family does not reduced to a family of 2–dimensional vector fields, in any neighborhood of Γ.

Author: Christiane Rousseau

Title: Finite cyclicity of center graphics

Abstract: We start by defining the notion of center graphic, for graphics with no first return map. We discuss a method for proving the finite cyclicity of center graphics. The method is a mixture of good normal form and use of the Bautin trick. Examples will be discussed.

Author: Dana Schlomiuk

Title: Algebraic-geometric structures in planar polynomial vector fields

Abstract: After a brief introduction in which we also mention the role of algebraic-geometric structures in the three classical problems, we focus our attention on classification problems of families of polynomial systems. We discuss four papers claiming to give "complete"

classifications of a family important for many applications, the Lotka-Volterra systems, and show why these papers fall short of the goal. We next present a new viewpoint on these systems and discuss how global algebraic-geometric concepts introduced in [1], [2], [3] point out inconsistencies among the above four mentioned papers and help in clarifying the picture of the maze of phase portraits occurring in this family. Results of a joint work in progress of Schlomiuk , Vulpe and Georgescu, using this new viewpoint and the more powerful classifying global tools will be presented.

1) J. Llibre, D. Schlomiuk, The geometry of quadratic systems with a weak focus of third order. Canadian J. of Math. **56** (2) (2004), 310–343.

2) D. Schlomiuk, N. Vulpe, Planar quadratic differential systems with invariant straight lines of at least five total multiplicity, Qualitative Theory of Dynamical Systems, 5, (2004), 135–194.

3) D. Schlomiuk, E. Naidenova, On the classification of Lotka-Volterra systems, CRM Montreal, CRM Report, CRM-3266, September 2008, 15pp.

Author: Nicolae Vulpe

Title: New applications of invariant polynomials to the integrability of some classes of differential systems.

Abstract: We start with a short introduction in which we recall the main definitions of the invariant polynomials which will be applied: *invariants, comitants, T-comitants* and *CT-comitants*. Then we focus our attention on the application of these invariant polynomials to determine invariant criteria for polynomial integrability of quadratic differential systems. We then give criteria for the existence of rational first integrals of degree two or three for quadratic systems. We show how invariant polynomials could be used to construct the rational first integrals of the indicated systems in terms of their twelve coefficients. Then we show why the definition of CT-comitants needs to be extended so as to construct these criteria and first integrals.

We discuss the connection between the polynomial (and rational) integrability of some families of systems and the properties of the roots of the constructed polynomials, whose coefficients are absolute affine invariants of these systems.

This is a joint work with Joan C. Artes and Jaume Llibre.

Finally, we present new results (obtained by Baltag and Calin) concerning the application of invariant polynomials for the construction in

explicit invariant form of the first integrals for a family of differential systems, whose right-hand sides involve polynomials of any degree n.

Author: Pei Yu

Title: Bifurcation of limit cycles in 2nd–order Hamiltonian planar vector fields with 3rd–order perturbations.

Abstract: In this talk, we show that a Z_2–equivariant 2nd–order Hamiltonian planar vector fields with 3rd–order symmetric perturbations can have at least 12 limit cycles. The method combines the general perturbation to the vector field and the perturbation to the Hamiltonian function. The Melnikov function is evaluated near the center of vector field, as well as near homoclinic orbits. It is shown that 10 small limit cycles bifurcate from two symmetric centers, and at least 2 large limit cycles exist, each enclosing 5 small limit cycles.

This is a joint work with Han Maoan.

Author: Xiang Zhang

Title: Darboux theory of integrability for polynomial differential systems.

Abstract: In this talk we will present the classical Darboux theory of integrability and its improvements for polynomial differential systems.

In 1878 Darboux established a theory which provide a link between the existence of first integrals and invariant algebraic hypersurfaces of vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n \geq 2$, now it is called *Darboux theory of integrability*. Jouanolou 1979 improved this theory to obtain rational first integrals via invariant algebraic surfaces using sophisticated tools of algebraic geometry. In 2000 Christopher and Llibre provided a simple proof to the Jouanolou's result for planar polynomial differential systems. But for higher dimensional polynomial differential systems there is not a simple proof.

Recently we obtained a simple and elementary proof to the Jouanolou's result for any dimensional polynomial differential systems in $\mathbb{C}^n$. Having this easy proof and combining the method developed recently by Chritopher–Llibre–Pereira for characterizing the multiplicity of invariant algebraic curves, we improved the Darboux theory of integrability for higher dimensional systems in $\mathbb{C}^n$ taking into account not only the invariant algebraic hypersurfaces but also their multiplicity. Furthermore, using the Poincaré compactification we extended the new

developed Darboux theory of integrability taking into account the hyperplane at infinity and its multiplicity for polynomial vector fields in $\mathbb{R}^n$.

This is a joint work with Jaume Llibre