# 08w5040 The stable trace formula, automorphic forms, and Galois representations

James Arthur (University of Toronto),
Michael Harris (Université de Paris 7),
Eric Urban (Columbia University),
Vinayak Vatsal (University of British Columbia)

August 17, 2008–August 22, 2008

## 1 Overview of the Field

The heart of Langlands' program reinterpreting much of number theory in terms of automorphic forms is his *Functoriality Conjecture*. This is a conjecture associating the automorphic representations on a pair of connected reductive groups over a number field $F$, say $G$ and $H$, whenever there is a homomorphism of the appropriate type between the corresponding $L$-groups

$$h : {}^L H \to {}^L G.$$

The structure of functoriality is largely captured in terms of parametrization of automorphic representations of a given group $G$ by homomorphisms

$$\phi : \mathcal{L}_F \to {}^L G$$

where $\mathcal{L}_F$ is the hypothetical Langlands group.

In the analogous conjectured parametrization of irreducible representations of a reductive group $G$ over a local field $K$, the Langlands group is replaced by the Weil-Deligne group $W_K$; in this setting one can state precise conjectures in terms of known objects, and these conjectures were proved when $G = GL(n)$ during the 1990s. This represents the first step to the extension of local class field theory to the non-commutative context, whereas the Langlands functoriality conjectures for automorphic forms are to be understood as generalizations of global class field theory. In particular, the Artin conjecture on holomorphy of Artin $L$-functions follows from Langlands' Functoriality Conjecture applied when $H$ is the trivial group.

The most comprehensive technique for proving the Functoriality Conjecture in the cases in which it has been established is the Arthur-Selberg trace formula. This was used by Lafforgue to prove the global Langlands conjecture for $GL(n)$ over a global field of positive characteristic, and has had a variety of striking applications to number fields, including the Jacquet-Langlands correspondence and its generalizations in higher dimensions and the Arthur-Clozel theory of cyclic base change for $GL(n)$. Each special case of functoriality has been of enormous importance in number theory.

## 2 Recent Developments and Open Problems

Until recently, applications of the trace formula were limited to the cases in which it could be stabilized. With the proof of the fundamental lemma for unitary groups by Laumon and Ngô, followed by its extension to all groups by Ngô – together with the proof that the fundamental lemma depends only on the residue field, first by Waldspurger, more recently by Cluckers, Hales, and Loeser – Arthur's stabilization of the trace formula is close to being complete, and the simple trace formula can be stabilized in a number of situations. This makes it possible to carry out the applications of the trace formula anticipated more than two decades ago by Langlands, Kottwitz, and Arthur. The previous week's summer school concentrated on the background to these applications, in connection with the Paris project to edit a series of books under the common title *Stabilization of the trace formula, Shimura varieties, and arithmetic applications*, and with Arthur's forthcoming book on functoriality for classical groups. Among the applications discussed at this workshop are

1. The functorial transfer of automorphic representations from classical groups ($Sp$ and $SO$) to $GL(n)$, contained in Arthur's forthcoming book. This was the subject of Arthur's two lectures;
2. The construction and analysis of local $L$-packets for quasi-split classical groups (symplectic, special orthogonal, and unitary) in terms of the local Langlands parametrization for $GL(n)$. This is due in part to Arthur and in part to Moeglin, who derive the result from global methods, and was the subject of Moeglin's lectures;
3. The study of the Galois representations arising in the cohomology of Shimura varieties; this was explicitly the subject of the lectures of Morel and Shin, and was the motivation for the lectures of Labesse and Clozel, who presented chapters of the Paris book project.

Various recent results on the fundamental lemma, including Ngô's work and extensions thereof, were presented by Tuan, Chaudouard, Hales, and Waldspurger. The lectures of Haines, Fargues, and Mantovan were all concerned with properties integral models of Shimura varieties and related local moduli spaces. Applications of the Galois representations on cohomology of Shimura varieties to questions on the border between Iwasawa theory and p-adic automorphic forms were presented in the talks of Emerton and Bellaïche. Finally, Lapid spoke on work in progress with Finis and Müller on combinatorial questions related to Arthur's invariant trace formula, with a view to analytic applications.

# 3 Presentations

## S. Morel, On the cohomology of certain non-compact Shimura varieties.

Soit $\mathbf{G}$ un groupe réductif connexe sur $\mathbb{Q}$. La variété de Shimura de $\mathbf{G}$ (si elle existe) est un système projectif de variétés algébriques sur un corps de nombres $F$ (le corps reflex) muni d'une action de $\mathbf{G}(\mathbb{A}_f)$. La cohomologie de cette variété est donc une représentation virtuelle de $\mathbf{G}(\mathbb{A}_f) \times Gal(\overline{F}/F)$, et on veut étudier cette représentation. Une manière de faire cela est de calculer la trace d'un opérateur du type $Fro^j_\wp \times f$, où $Frob_\wp$ est un relèvement du Frobenius géométrique en une place $\wp$ de $F$ au-dessus d'un nombre premier $p$ où $\mathbf{G}$ est non ramifié, et $f$ est une fonction lisse à support compact sur $\mathbf{G}(\mathbb{A}_f)$ dont la composante en $p$ est bi-invariante par un sous-groupe compact hyperspécial de $\mathbf{G}(\mathbb{Q}_p)$. Ihara, Langlands et Kottwitz ont développé une méthode pour calculer ce genre de traces : comparaison la formule des points fixes de Grothendieck-Lefschetz sur la réduction modulo $\wp$ de la variété de Shimura et la formule des traces d'Arthur-Selberg. Dans cet exposé, j'explique comment faire ces calculs dans le cas où $\mathbf{G}$ est un groupe unitaire sur $\mathbb{Q}$ et où on considère la cohomologie d'intersection de la compactification de Satake-Baily-Borel.

Pour la cohomologie à support compact, tous les calculs sont dus à Kottwitz. La conférencière a d'abord expliqué comment utiliser son comptage du nombre de points d'une variété de Shimura sur un corps fini pour obtenir la trace d'un opérateur comme ci-dessus sur la cohomologie d'intersection; le principal ingrédient est une description du prolongement intermédiaire d'un faisceau pervers pur comme "tronqué par le poids" de son image directe. La deuxième étape, souvent appelée "stabilisation", consiste à comparer le résultat de la formule des points fixes et le côté géométrique de la formule des traces stable d'Arthur; en général, il faut faire intervenir aussi les groupes endoscopiques (elliptiques) de $\mathbf{G}$. Là encore, Kottwitz a déjà fait cette comparaison pour la cohomologie à support compact (c'est-à-dire la partie elliptique). Ici il s'agit d'expliquer comment étendre son résultat à la cohomologie d'intersection; la partie la plus compliquée est à la place infinie, et fait intervenir des calculs explicites de caractères de séries discrètes. Ensuite, il s'agit d'interpréter le côté spectral de la formule des traces. Ceci se fait en comparant la formule des traces stable sur $\mathbf{G}$ et la formule des traces tordue sur un groupe $\mathbf{GL}_n$. Cette comparaison est due à Labesse pour les parties elliptiques, et son résultat s'étend sans trop de difficulté au cas considéré ici (en utilisant un théorème de Clozel pour la place infinie).

Enfin une application de ces résultats est donné au calcul de la fonction $L$ du complexe d'intersection. On pourrait aussi utiliser ces résultats pour associer une représentation galoisienne à une représentation automorphe cuspidals autoduale et cohomologique de $\mathbf{GL}_n$ (au moins dans le cas où $n$ n'est pas divisible par $4$), mais on n'obtiendra de cette manière que la compatibilité aux places non ramifiées.

## J. Arthur, Functorial transfer for classical groups, I, II

Let G be a quasisplit special orthogonal or symplectic group. Then G represents a twisted endoscopic datum for a general linear group GL(N). The goal is to classify the representations of G, both local and global, in terms of those of GL(N). This will allow us to extend the main theorems of GL(N), namely the classification of isobaric representations by Jacquet-Shalika. the classification of the discrete sprectrum by Moeglin-Waldspurger, and the local Langlands classification established by Harris-Taylor and Henniart, to the classical group G.

The strategy is to compare the twisted trace formula for GL(N) with the stable trace formulas of the various G, and the ordinary trace formula of G with the stable trace formulas of its endoscopic groups. This is now within reach, thanks to the recent prooof of the fundamental lemma, in all its forms, by Laumon-Ngo, Ngo, Chaudouard-Laumon, Waldspurger and Cluckers-Hales-Loeser, following earlier ideas of Goresky-Kottwitz-Macpherson, and the reduction of the Kottwitz-Langlands-Shelstad transfer conjecture to the fundamental lemma by Waldspuger. In the lectures, the speaker described in precise form the classification theorems for G that one obtains from these comparisons. He then described the relevant terms in the various trace formulas, and some of the techniques needed to carry out the comparisons.

## J.-P. Labesse, Stable simple base change for unitary groups

This talk is a summary of the speaker's chapter, entitled "Changement de base CM et séries discrètes," in the first book of the Paris project. Let $F$ be a totally real field of degree $d \geq 2$ and $E$ a quadratic CM extension of $F$. Let $U$ be a unitary group of degree $n$ over $F$, corresponding to this extension. Under additional simplifying hypotheses, one can obtain optimal results relating the stable trace formula for $U$ to the twisted trace formula for $G = GL(n)_E$. Let $f$ be a test function for $U$ and $\phi$ a matching function on $G$, or equivalently on $L$, the non-trivial component of the semidirect product of $G$ with $\{1, \theta\}$, where $\theta$ acts on $G$ by Galois conjugation composed with transpose inverse. If $U = U^*$ is the quasi-split inner form, $f_\infty$ is a very cuspidal discrete series pseudocoefficient and if $\phi_\infty$ is a matching very cuspidal Lefschetz function, then one has the identity

$$T^L_{disc}(\phi) = ST^{U^*}_{disc}(f).$$

In other words $U^*$ is the only twisted endoscopic group to contribute to the twisted stable trace formula for $G$. This can be extended to write the stable trace formula for any $U$ satisfying these hypotheses, with $f_\infty$ as above, as a sum of terms corresponding to twisted traces for products of general linear groups over $E$. Under additional simplifying hypotheses ($U$ is quasi-split at all finite places, $E/F$ unramified at all finite places, the test functions are spherical at all inert places), one obtains multiplicity one results for automorphic representations of $U$ of discrete series type at infinity. If $n$ is odd and $U$ has signature $(n-1, 1)$ at one real place and is definite at all others, one finds under these simplifying hypotheses that a $\theta$-stable cuspidal cohomological automorphic representation $\Pi$ of $G$ descends to a stable $L$-packet of $U$ with $n$ members, each with multiplicity one, at least when the infinitesimal character of $\Pi_\infty$ is sufficiently regular.

## L. Clozel, Simple endoscopic transfer for unitary groups

This talk is a summary of the final chapter, entitled "Simple endoscopic transfer for unitary groups," in the first book of the Paris project, written by the speaker together with M. Harris and J.-P. Labesse. Let $F$ and $E$, but now assume $n$ is an even integer, and consider the unitary group $U$ of degree $n+1$ with signature $(n, 1)$ at one real place and is definite at all others, and quasi-split at finite places. The objective is to study the contribution to the stable trace formula of $U$ of $\theta$-stable automorphic cuspidal representations of $GL(n) \times GL(1)$ or, equivalently, of the endoscopic group $H = U(n)^* \times U(1)$ of $G$. Let $\Pi$ be a fixed $\theta$-stable cuspidal cohomological automorphic representation of $GL(n)_E$, and let $\chi$ be a variable $\theta$-stable automorphic representation of $GL(1)$. Under the simplifying hypotheses above, the contribution of the pair $(\Pi, \chi)$ to the automorphic spectrum of $U$ is either an $n$-tuple of distinct automorphic representations of discrete series type or a singleton. If it is an $n$-tuple, Kottwitz' results on the zeta functions of Shimura varieties of PEL type imply that the corresponding Galois representation is weakly associated to the original $\Pi$. The main theorem is that, under a weak regularity hypothesis on $\Pi_\infty$, then $\chi$ can be chosen so that one obtains an $n$-tuple. This depends on the calculation of the sign of a normalized intertwining operator, acting on Eisenstein series on $GL(n+1)_E$ attached to the Levi factor $GL(n) \times GL(1)$ but involving $\theta$, and on the explicit determination of the signs in Shelstad's transfer of discrete series packets from $H$ to $G$.

## M. Emerton, p-adically completed cohomology

This evening lecture was an informal introduction to the speaker's approach to the study of $p$-adic variation of automorphic forms. The $p$-adically completed cohomology can be defined purely topologically for any tower of locally symmetric spaces. In the case of classical modular forms, $p$-adically completed cohomology plays a crucial role in the speaker's work on the Fontaine-Mazur conjecture for 2-dimensional Galois representations. Study of this cohomology for other groups is intimately connected with the mysterious properties of torsion in cohomology of locally symmetric spaces, and is only in its first stages. The speaker presented a number of recent results and conjectures on the topic, some contained in his joint work with F. Calegari.

## J. Bellaïche, Endoscopic tempered points on unitary eigenvarieties

Although only $p$-adic deformations of endoscopic *non-tempered* automorphic representations (or $A$-packets) of unitary groups are related to the deepest cases of Bloch-Kato conjectures, their *tempered* counterpart are interesting on their own, and much simpler to study. In this talk, the speaker considered an automorphic *tempered* endoscopic $A$-packet (that is, an $L$-packet) for a unitary groups in $n$ variables, compact at infinity, together with a refinement assumed to be anti-ordinary. These data correspond to a point on the eigenvariety of the unitary groups. The main result is that this point is smooth if and only if, the packet is of type $(1, \dots, 1)$; in other words if the attached Galois representation is a sum of $n$ characters. This lead to the proof of non-vanishing results for some Galois cohomology groups of $p$-adic Galois representations, which do not seem to be easily provable otherwise.

## C. Moeglin, Local Arthur packets

In this talk, the speaker explained how to use the transfer of character identities via endoscopy and twisted endoscopy, due to Arthur, to compute stable Arthur packets of representations for a classical group. The most important result is the fact that multiplicity one in packets of discrete series implies multiplicity one in general packets. The speaker then explained some qualitative results about the Langlands parameters of any representation in an Arthur packet. Roughly speaking, the speaker proved the idea of Clozel that in such a packet, any given representation is "more tempered" ( giving a precise meaning to such an assertion) than the representations in the Langlands packet associated and contained in it. The talk concluded by giving some applications, mostly conjectural, to the determination of the residual (non cuspidal) spectrum for adelic classical groups.

## E. Lapid, On some aspects of Arthur's non-invariant trace formula (joint work with T. Finis and W. Müller)

The speaker discussed some analytic and combinatorial aspects pertaining to the non-invariant trace formula (i.e. Arthur's early work). A natural question is to understand the class of test function for which the trace formula identity extends. In the spectral side, the problem is to explicate the limits which appear in Arthur's fine spectral expansion in terms of first order derivatives of co-rank one intertwining operators. The solution is analogous to a volume formula for polytopes due to P. McMullen and R. Schneider. This volume formula should also play a role in the geometric side. However, as of now, the authors only understand the geometric side completely in the case of $GL(2)$. (This entails rewriting the non-elliptic terms in the trace formula.)

## E. Mantovan, Integral models for toroidal compactifications of Shimura varieties

This talk was a report on joint work-in-progress with Moonen dealing with the construction of integral models for toroidal compactifications of Shimura varieties of PEL type at unramified primes. In the cases of good reduction, such a theory already exists due to the work of Faltings and Chai and more recently to Lan and Rozensztajn. The present project focuses on the cases of bad reductions at primes which divide the level.

The main goal is to extend known results on integral models of Shimura varieties to their compactifications. In particular, the lecture focused on the construction of integral models whose vanishing cycles sheaves can be controlled in terms of those of the corresponding local models constructed by Rapoport and Zink. These results (which for Shimura varieties of PEL type are due to Harris and Taylor in some special cases, and in general to the speaker) provides descriptions of the cohomology of the Shimura varieties in terms of that of their local models and of other simpler varieties, called Igusa varieties. Such formulas provide some insight on the Galois representations arising in the cohomology of the Shimura varieties. E.g., they are used, in the work of Harris and Taylor and more recently of Shin, to establish some instances of Langlands correspondences.

Following Pink's work on arithmetic compactifications of Shimura varieties, the authors consider a natural stratification of the boundary of toroidal compactifications where each stratum is itself a Shimura variety of mixed type, lying above a pure Shimura variety. For Shimura varieties of PEL type and unramified primes, they prove the existence of integral models of smooth toroidal compactifications whose vanishing cycle sheaves, when restricted along the boundary strata, agree with the pullback of the vanishing cycle sheaves of the corresponding underlying pure Shimura varieties (and thus can be controlled in terms of their local models).

In the case of PEL Shimura varieties, the boundary strata have a moduli interpretation as classifying spaces for rigidified 1-motives with additional structures, lying above the corresponding moduli spaces for abelian varieties. The existence of integral models of toroidal compactifications is thus addressed by investigating an appropriate notion of level structure on a 1-motive and its induced structure on the associated abelian variety.

## L. Fargues, The p-adic geometry of moduli spaces of abelian varieties and p-divisible groups

In [1], Gerd Faltings has shown the existence of a link between Lubin-Tate and Drinfeld towers (see [3] too). Using the link between Drinfeld's $\Omega$ space and the Bruhat-Tits building of $PGL_{n/\mathbf{Q_p}}$ one can use this to define an Hecke equivariant "parametrization" by the geometric realization of this Bruhat-Tits building of the Berkovich space associated to the Lubin-Tate tower with infinite level. This parametrization has been studied in detail in [4].

The Lubin-Tate tower can be seen as tubes (or $p$-adic Milnor fibers) over some "supersingular" points in the reduction mod $p$ of some particular type of Shimura varieties (unitary type with signature $(1, n-1) \times (0, n) \times \cdots \times (0, n)$ at a split prime $p$). In fact one can extend the results of [4] to define and study an Hecke equivariant parametrization of the $p$-adic Berkovich analytic space associated to this Shimura variety with infinite level at $p$ by compactifications of the preceding buildings. In this parametrization the boundary stratification of the compactified building corresponds to the Newton stratification of the Shimura variety. For $n = 2$ one recovers Lubin's theory of the canonical subgroup. For general $n$ this should be helpful to construct a theory of $p$-adic automorphic forms on those Shimura varieties generalizing Katz theory for modular curves (for $n = 2$ the structure of the Bruhat-Tits tree being "simple" one can compute everything, but in general the structure of the building is more complicated).

The speaker proposed a new way to stratify and define fundamental domains for the action of $p$-adic Hecke correspondences on more general moduli spaces (Rapoport-Zink spaces or general PEL type Shimura varieties). In [2] the author has defined Harder-Narasimhan type filtrations for finite flat group schemes over unequal characteristic complete valuation rings. Stuhler and Grayson have developed reduction theory for the action of arithmetic groups on archimedean symmetric spaces using Harder-Narasimhan filtrations for hermitian vector bundles in the sense of Arakelov geometry. A theory for finite flat group schemes is used to define a reduction theory for $p$-divisible groups (like reduction theory for quadratic forms) and to define fundamental domains for the action of Hecke correspondences on some Rapoport-Zink spaces and Shimura varieties. Those fundamental domains are interesting from the point of view of the associated period mapping. When one starts from a $\overline{\mathbf{Q}}$-point in our Shimura variety the associated point in the fundamental domain is a point in the Hecke orbit where the Faltings height of the associated abelian variety is minimized in its $p$-isogeny class.

## T. Haines, Test functions for some Shimura varieties with bad reduction

This talk reviewed some aspects of Shimura varieties with parahoric level structure at $p$, and outlined an approach to study Shimura varieties with $\Gamma_1(p)$-level structure at $p$ by relating that situation to the Iwahori-level case. The general expectation is that the test functions appearing in the twisted orbital integrals in the "point counting" formula can always be taken to lie in the center of an appropriate local Hecke algebra. In the parahoric case, this is known in several cases (joint work with Ngo B.C.) and in the $\Gamma_1(p)$-case this is the subject of work in progress with M. Rapoport. The main result so far is limited to the family of unitary Shimura varieties known as the "Drinfeld case", and states that there the test function is a sum of functions indexed by certain characters $\chi$ on the Iwahori, and each such function is in the center of the Hecke algebra of $\Gamma_1(p)$-bi-invariant functions on $GL(n)$ which transform under the Iwahori by $\chi^{-1}$. Furthermore, the test function can be identified explicitly via Hecke algebra isomorphisms in terms of functions in the centers of the Iwahori-Hecke algebras attached to the semistandard Levi subgroups of $GL(n)$ (which are indexed also by the various $\chi$).

## S.W. Shin, Construction of Galois representations

Let $\Pi$ be any cuspidal automorphic representation of $GL(n)$ over a CM field $F$. If $\Pi$ is conjugate self-dual and cohomological, it is part of the global Langlands conjecture that there exists an $n$-dimensional $\ell$-adic representation $R(\Pi)$ of the full Galois group for $F$ which corresponds to $\Pi$ via the local Langlands correspondence (established by Harris-Taylor and Henniart) at every finite place. If $\Pi$ is square-integrable at a finite place, this result is proved by work of Clozel, Kottwitz, Harris-Taylor and Taylor-Yoshida. Their result is based on the study of unitary Shimura varieties and base change for unitary groups in the case of trivial endoscopy.

In his talk the speaker reported on the recent proof that $R(\Pi)$ as above exists, without assuming $\Pi$ is square-integrable at a finite place. When n is odd, this result was worked out by Morel and Shin, using the stable base change for unitary groups after Labesse. When $n$ is even, the problem is harder and more interesting. Harris observed that one might use the endoscopy for $U(n)$ coming from $U(n-1) \times U(1)$, generalizing the method of Blasius of Rogawski for $n = 3$. Clozel, Harris and Labesse worked out endoscopy and base change results for unitary groups in this setting; combined with the results of Kottwitz and Morel, they in particular obtain $n$-dimensional $\ell$-adic representations that correspond to $\Pi$ at almost all places. On the other hand, one has to compute the Galois action on the cohomology of Shimura varieties at all places, including places of bad reduction. Following the strategy of Harris-Taylor, in view of their deep result on the cohomology of the Lubin-Tate deformation spaces, the problem quickly reduces to the representation-theoretic understanding of the cohomology of Igusa varieties. This can be done by a counting point formula for Igusa varieties and its stabilization, combined with the trace formula techniques. The speaker tried to explain how all these can be put together into a somewhat different construction of a representation $R(\Pi)$ with the desired local properties at all places prime to $\ell$.

LECTURES ON THE FUNDAMENTAL LEMMA

## Ngo Dac Tuan, Introduction to B.C. Ngô's proof

This purpose of this talk is to sketch the proof of the fundamental lemma due to Ngô Bao Châu. Conjectured by Langlands and Shelstad, the fundamental lemma constitutes an essential step in the stabilization of the elliptic part of the Arthur-Selberg trace formula. Roughly speaking, it claims an local identity between the $\kappa$-orbital integral of a reductive group and the stable orbital integral of its endoscopic group associated to $\kappa$.

The approach of Ngô is geometric. By studying the perverse cohomology of the Hitchin fibration over a projective smooth curve over a finite field, he proves not only the fundamental lemma but also a geometric stabilization theorem. The important ingredients of his proof are, in the first place, a product formula connecting the Hitchin fibration and the affine Springer fibers, whose properties are well known; in the second place, the so-called strong support theorem.

## P.-H. Chaudouard, On the truncated Hitchin fibration and the weighted fundamental lemma (Joint work with Gérard Laumon)

Ngô Bao Châu recently proved the fundamental lemma of Langands and Shelstad in full generality. In fact, Ngô proved a Lie algebra version of the fundamental lemma for local fields of equal characteristics. But it is known that this implies the fundamental lemma for groups and for local fields of characteristic 0 (works of Hales, Waldspurger ...). Ngô's method is geometric as was the proof of the unitary case by Laumon and Ngô. It consists in a cohomological study of elliptic part of the Hitchin fibration.

The framework is the following : $C$ is a connected projective curve over a finite field and $D$ is an effective divisor on $C$. For the group $GL(n)$, the whole space is the algebraic stack of pairs Ê$(\mathcal{V}, \theta)$ Êwhere $\mathcal{V}$ is a vector bundle and $\theta$ is an endomorphism $\mathcal{V} \to \mathcal{V}(D)$. The base of the fibration is the space of all characteristic polynomials and the Hitchin morphism is the "characteristic polynomial" morphism. There is a generalization of these definitions for any reductive group.

Over the elliptic open set (for $GL(n)$ this is the open set of generically irreducible characteristic polynomials), the Hitchin morphism is proper. Moreover the number of rational points in each fiber is essentially a global orbital integral which appears in the Arthur-Selberg trace formula. These two facts are no longer true outside the elliptic set. However, if one wants a geometric interpretation of the other geometric terms of the trace formula (i.e. the weighted orbital integrals), one needs to go "outside" the elliptic set.

In this talk which is based on a joint work of the speaker with Gérard Laumon, a truncated version of the Hitchin fibration was introduced. Roughly speaking this is essentially a "good" open substack of the Hitchin fibration. But as in the elliptic case, it has two basic properties : the restriction of the Hitchin morphism to the truncated Hitchin fibration is proper and Êthe number of rational points in each fiber is a global weighted orbital integral introduced by Arthur.

Extending Ngô's arguments, one can deduce the weighted version of the fundamental lemma conjectured by Arthur. Thanks to Waldspurger's works, this gives the weighted fundamental lemma or local fields of characteristic 0.

## T. Hales, The transfer principle for the fundamental lemma

The purpose of this talk is to explain how the identities of various fundamental lemmas fall within the scope of the transfer principle, a general result of Cluckers and Loeser that allows one to transfer theorems about identities of p-adic integrals from one collection of fields to others. In particular, once the fundamental lemma has been established for one collection of fields (for example, fields of positive characteristic), it is also valid for others (fields of characteristic zero). The main result has been obtained independently by Waldspurger.

Most of this talk is a report on the work of Cluckers and Loeser on motivic integration for constructible functions. A subassignment is a collection of sets indexed by fields of characteristic zero, such that the set attached to $K$ is a subset of $K((t))^m \times K^n \times \mathbf{Z}^r$ for some $m, n, r$ (not depending on $K$). A definable subassignment is one for which there exists a formula in the three-sorted language of Denef-Pas which determines the subset for each $K$. Definable subassignments gives the objects of a category, whose morphisms are functions whose graphs are also definable subassignments. For each definable subassignment a ring $C(X)$ of constructible functions is given, that gives the ring of integrands for a field-independent integration. The cell-decomposition theorem of Denef-Pas gives a integration on $IC(X) \subset C(X)$. There is a specialization theorem that relates integration on $IC(X)$ with integration on non-archimedean local fields.

All of the data of the fundamental lemma can be expressed in terms of the the ring $IC(X)$. This gives a field-independent formulation of the fundamental lemma. The general transfer principle for $IC(X)$ implies that the fundamental lemma can be transferred from fields of positive characteristic to fields of characteristic zero. The details have been written in an article by the same title for the Lie algebra version of the fundamental lemma, in both the weighted and unweighted cases.

### J.-P. Waldspurger, Le lemme fondamental tordu pour les intégrales orbitales pondérées

#### Le lemme fondamental pondéré tordu

Le noyau dur du lemme fondamental a été démontré l'an dernier par Ngô Bao Châu. Néanmoins, pour tirer le meilleur parti de la méthode de comparaison de formules de traces, Arthur a besoin de versions plus sophistiquées de ce lemme. L'énoncé le plus général à ce jour est le lemme fondamental pondéré tordu: pondéré parce qu'il énonce une égalité de combinaisons linéaires d'intégrales orbitales pondérées; tordu parce qu'il s'applique à la situation générale de ce que l'on appelle l'endoscopie tordue. La plus grande partie de l'exposé a consisté à énoncer ce lemme fondamental pondéré tordu. Le conférencier a conclu par le théorème suivant.

**Théorème.** *Supposons vérifiés le lemme fondamental pondéré pour les algèbres de Lie et le lemme fondamental pondéré non standard. Alors le lemme fondamental pondéré tordu est vérifié.*

Le lemme fondamental pondéré pour les algèbres de Lie est l'analogue pour ces algèbres du lemme fondamental pondéré ”ordinaire”, c'est-à-dire dans une situation où il n'y a pas de torsion. Le lemme fondamental pondéré non standard concerne aussi des algèbres de Lie dans une situation sans torsion. Il affirme une égalité de combinaisons linéaires d'intégrales orbitales pondérées reliant deux algèbres de Lie en situation ”non standard”. Par exemple l'une peut être l'algèbre de Lie du groupe symplectique $Sp(2n)$ et l'autre celle du groupe spécial orthogonal déployé $SO(2n+1)$. P.-H. Chaudouard a annoncé au colloque la démonstration de ces deux lemmes, en collaboration avec G. Laumon, par une méthode s'inspirant de celle de Ngô Bao Châu.

## 4 Outcome of the Meeting

The conclusion of the conference is that, thanks to the breakthrough on the fundamental lemma, the Langlands program for endoscopy, including Arthur's multiplicity conjectures and Kottwitz' related conjectures on the zeta functions of Shimura varieties, is well on its way to completion, at least for classical groups. Open questions fall into two large categories. The first class consists of applying the results of Langlands' program to questions in number theory. The lectures of Emerton and Bellaïche described two active research projects concerned with such applications. Other talks of this sort would have been scheduled had it been decided to devote more time to lectures.

Langlands' program is still in its infancy, and nearly everything remains to be done, even though endoscopy may soon be complete. The second class of open questions has to do with functoriality “beyond endoscopy.” That will have to wait for a future meeting.

## References

G. Faltings. A relation between two moduli spaces studied by V. G. Drinfeld. In *Algebraic number theory and algebraic geometry*, volume 300 of *Contemp. Math.*, pages 115–129, 2002.

L. Fargues. La filtration de Harder-Narasimhan des schmas en groupes finis et plats. *Prpublication, http://www.math.u-psud.fr/˜fargues/Prepublications*, 2008.

L. Fargues. L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques. In *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in math., 262, pages 1–325. Birkhuser, 2008.

Laurent Fargues. Application de Hodge-Tate duale d'un groupe de Lubin-Tate, immeuble de Bruhat-Tits du groupe linéaire et filtrations de ramification. *Duke Math. J.*, 140(3):499–590, 2007.