Bases for Cluster Algebras

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1 Overview of the Field

Cluster algebras were defined by Fomin and Zelevinsky [18] in the year 2000 in a long-term project aimed to develop an algebraic/combinatorial approach for two important subjects: the theory of total positivity in algebraic groups developed by Lusztig [42] on the one hand, and the theory of canonical bases in quantum groups initiated by Kashiwara [32] and Lusztig [41] on the other hand. Since then, the theory of cluster algebras has witnessed spectacular growth thanks to the abundant links that have been discovered with a wide range of subjects including representation theory of quivers, algebraic geometry, mirror symmetry, Poisson geometry, integrable systems, tropical geometry and Teichmüller theory among many others. Due to this outstanding development, *cluster algebras* became in 2010 item 13F60 of the Mathematics Subject Classification.

Every cluster algebra is a subalgebra of a field of rational functions $\mathbb{C}(x_1, \dots, x_n)$ in n commuting variables. By definition, a cluster algebra is endowed with a distinguished set of generators, called the *cluster variables*. These generators are grouped into overlapping subsets of cardinality n, called the *clusters*. The clusters are constructed inductively by a combinatorial procedure called *mutation* from the initial cluster $\{x_1, \dots, x_n\}$. This procedure is completely determined by an initial $n \times n$ *skew-symmetrizble* matrix B. We denote by $\mathrm{ord}(B)$ the (ordinary) cluster algebra associated to B. A *cluster monomial* is a monomial in the cluster variables of a given cluster. The *upper cluster algebra* $\mathrm{up}(B)$ is the subalgebra of $\mathbb{C}(x_1, \dots, x_n)$ generated by all rational functions that can be expressed as a Laurent polynomial on the variables of every cluster. The Laurent phenomenon is a fundamental result proved by Fomin and Zelevinsky that ensures $\mathrm{ord}(B) \subseteq \mathrm{up}(B)$. The notion of quantum cluster algebras, introduced by Berenstein and Zelevinsky in [5], can be considered as a non-commutative q-analogue of cluster algebras.

Finding vector space bases for both commutative and quantum cluster algebras is a central problem in cluster theory. Moreover, one would like to construct bases with positive structure constants, that contain all (quantum) cluster monomials and that are parametrized by nice subsets of lattices. Moreover, for nearly 20 years it was an open problem to relate bases for cluster algebras to the canonical bases constructed by Kashiwara and Lusztig and recently there has been significant progress in this direction [31, 33, 34, 39, 53, 55]. The work of Qin [54] shows the existence of a moduli space of bases for cluster algebras with the desired parametrization. Hence comparing the different bases is a very interesting problem. There are three approaches available to construct bases for cluster algebras that have been intensely developed: monoidal categorification, cluster characters and mirror symmetry.

1.1 Monidal categorification

Monoidal categorification of cluster algebras was introduced by Hernandez and Leclerc [29]. Since then, this notion has evolved and has lead to strong results concerning the connection between cluster algebras and canonical bases for quantum groups. The core idea of this approach is to construct a monoidal category whose Grothendieck ring is isomorphic to a cluster algebra of interest. Constructing such a categorification for a cluster algebra has very strong consequences. For instance, it was shown in [29] that the Grothendieck ring of the monoidal category of finite-dimensional representations of a quantum affine algebra can be endowed with the structure of a cluster algebra. Such a Grothendieck ring has a basis of simple modules. It was proved in [30] that, for quantum affine algebras of Cartan type ADE, the simple modules correspond precisely to the elements of Lusztig's dual canonical basis. In the general case, Hernandez and Leclerc conjectured that all cluster monomials belong to the basis of the simple modules. In [53], Qin constructed the so-called common triangular bases for this kind of cluster algebras. By construction, the common triangular basis contains all quantum cluster monomials and the Hernandez-Leclerc conjecture is reduced to show that the basis consisting of the simple modules produces the common triangular basis.

Another remarkable instance of monoidal categorification is the following. Let $\mathfrak{g}$ be a simple Lie algebra associated to a symmetric Cartan matrix and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ a triangular decomposition. Let G and N be the Lie groups associated with $\mathfrak{g}$ and $\mathfrak{n}$, respectively. In [20], Geiss, Leclerc and Schröer proved that $\mathbb{C}[N]$ is a cluster algebra. The unipotent quantum coordinate ring $A_q(\mathfrak{n})$ is a q-deformation of the coordinate ring $\mathbb{C}[N]$ and is isomorphic to the negative half of the quantum group $U_q(\mathfrak{g})$ as a $\mathbb{Q}(q)$-algebra. Geiss, Leclerc and Schröer also showed [19, 20, 21] that $A_q(\mathfrak{n})$ has the structure of a quantum cluster algebra. Kashiwara and Lusztig constructed simultaneously and independently a distinguished basis for $A_q(\mathfrak{n})$ called the upper global basis or dual canonical basis. In [31], Kang, Kashiwara, Kim and Oh showed that $A_q(\mathfrak{n})$ can be categorified via the representations of symmetric Khovanov-Lauda-Rouquier algebras. They used this to show that every quantum cluster monomial in $A_{q^{1/2}}(\mathfrak{n}) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(\mathfrak{n})$ belongs to the upper global basis up to a power of $q^{1/2}$. These results can be generalized (see [31, 55]) for quantum coordinate rings of the form $A_q(\mathfrak{n}(w))$ for w an element of the Weyl group of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$, where the algebra $A_q(\mathfrak{n}(w))$ is a $\mathbb{Q}(q)$- subalgebra of $A_q(\mathfrak{n})$ generated by a set of the dual PBW basis elements associated with w. However, there are very subtle problems arising for non-symmetric Kac-Moody Lie algebras and the categorification picture has not been worked out in this case. Let us stress that all these results can be generalized for quantum coordinate rings of the form $A_q(\mathfrak{n}(w))$ for w an element of the Weyl group of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$, where the algebra $A_q(\mathfrak{n}(w))$ is a $\mathbb{Q}(q)$- subalgebra of $A_q(\mathfrak{n})$ generated by a set of the dual PBW basis elements associated with w. Moreover, there are very subtle problems arising for non-symmetric Kac-Moody Lie algebras and the categorification picture has not been worked out in this case.

Monoidal categorification of cluster algebras has been further developed using categories of perverse sheaves. For example, Nakajima constructed monidal categorifications by means of perverse sheaves on quiver varieties [48] and Cautis-Williams used equivariant perverse sheaves on the affine Grassmannian [10].

1.2 Cluster algebras and quiver representations

The theory of quiver representations has a deep connection with cluster theory (see [17, 37] and the references therein). The discovery of this connection had a very strong impact in tilting theory and gave rise to cluster-tilting theory and subsequently to τ-tilting theory. Since then, cluster algebras have been extensively studied via homological properties of quiver representations.

Let now $\mathrm{ord}(Q)$ be the cluster algebra associated to a quiver Q (corresponding to a skew-symmetric matrix B). Derksen, Weyman and Zelevinsky introduced the Jacobian algebra $J(Q,W)$. This is by definition the completed path algebra of Q (over $\mathbb{C}$) modulo the closure of an ideal $I(W)$ defined by a non-degenerate potential W.

Let $d \in \mathbb{N}^n$ be a dimension vector and M a $J(Q,W)$-module. The quiver Grassmannian $\mathrm{Gr}_d(M)$ is the projective variety parametrizing the submodules of M of dimension vector d. The Caldero-Chapoton function is the Laurent polynomial

$$C_M := \underline{x}^{g(M)} \sum_{d \in \mathbb{N}^{Q_0}} \chi\left(\mathrm{Gr}_d(M)\right) \underline{x}^{B \cdot d},$$

in the variables of the initial cluster, where $g(M) \in \mathbb{Z}^n$ is a vector constructed using an injective presentation of M and χ is the Euler-Poincaré characteristic. One can see C_M as a generating function of the Euler characteristics of the quiver Grassmannians $\mathrm{Gr}_d(M)$.

Derksen, Weyman and Zelevinsky showed in [17] that the Laurent polynomials C_M are invariant under mutation, and that each cluster variable of ord(Q) is the Caldero-Chapoton function of a τ-rigid $J(Q,W)$-module. The Caldero-Chapoton algebra $\mathrm{CC}(Q,W)$ is the algebra generated by the CC-functions C_M, where M runs over all $J(Q,W)$-modules. By the mutation-invariance, we get the inclusions ord$(Q) \subseteq \mathrm{CC}(Q,W) \subseteq \mathrm{up}(Q)$. In many interesting cases these three algebras coincide.

For each irreducible component Z of the varieties of $J(Q,W)$-modules with dimension vector d, where d runs over $\mathbb{N}^n$, one can define a generic CC-function C_Z. Geiss, Leclerc and Schröer conjectured that a certain subset of these generic CC-functions (given by the τ-reduced irreducible components) forms a basis of $\mathrm{CC}(Q,W)$, which resembles many of the properties of Lusztig's dual semicanonical basis for enveloping algebras. They proved this for a large class of cluster algebras arising from Lie theory including the algebras of the form $A(\mathfrak{n}(w))$ and established a strong direct link to the dual semicanonical basis.

Closely related is the topic of additive categorification of cluster algebras developed by many authors [9, 36, 2, 51]. From this perspective, various properties of cluster algebras associated to quivers can be categorified using Calabi-Yau triangulated categories of dimension 2 and 3. Namely, associated to a quiver Q endowed with a non-degenerate potential W there is the so-called Ginzburg dg-algebra $\Gamma_{Q,W}$. Consider the derived category of dg-modules $\mathcal{D}(\Gamma_{Q,W})$. Keller [36] proved that its full subcategory formed by all perfect dg-modules $\mathrm{Perf}(\Gamma_{Q,W})$ is a 3-Calabi-Yau triangulated category that contains as a full subcategory the category $\mathcal{D}_{df}(\Gamma_{Q,W})$ of dg-modules with finite-dimensional total cohomology. Amiot [2] defined the cluster category $\mathcal{C}_{Q,W}$ as the Karubian completion of the triangulated quotient $\mathrm{Perf}(\Gamma_{Q,W})/\mathcal{D}_{df}(\Gamma_{Q,W})$. The work of many authors shows that the categories $\mathcal{C}_{Q,W}, \mathrm{Per}(\Gamma_{Q,W})$ and $\mathcal{D}_{df}(\Gamma_{Q,W})$ categorify various properties of ord(Q). The CC-formula can be extended to this context [50] and gives rise to the theory of cluster characters which provides a way to pass from the cluster category $\mathcal{C}_{Q,W}$ to the cluster algebra ord(Q). Moreover, Plamondon extended the approach of Geiss, Leclerc and Schröer to this context.

The study of generic CC-functions needs a deeper understanding of the representation theory of the Jacobian algebras $J(Q,W)$ and also of the behavior of their varieties of modules. The concept of τ-reduced irreducible components makes sense for all finite-dimensional algebras and should be seen as part of a yet to be developed generalized τ-tilting theory.

Understanding the precise relation between additive and monoidal categorification of cluster algebras is a very difficult problem. However, we can highlight the work of Davidson [15] in which both additive categorification and monoidal categorification of quantum spaces naturally arising in the study of quantum cluster algebras are studied simultaneously. This remarkable approach was used to prove the positivity of quantum cluster coefficients for all skew-symmetric quantum cluster algebras.

1.3 The theta basis

Associated to a skew-symmetrizable matrix B there are two complex schemes $\mathcal{A}_B$ and $\mathcal{X}_B$ called the *cluster varieties associated to* B. The ring of global functions on $\mathcal{A}_B$ is canonically isomorphic to the upper cluster algebra up(B). Gross, Hacking and Keel observed that both types of cluster varieties have the structure of a *log-Calabi-Yau* variety. This permits to study cluster varieties using techniques and ideas from mirror symmetry, the minimal model program and tropical geometry. In particular, Gross, Hacking, Keel and Kontsevich [25] introduced *scattering diagrams* [40, 27], *broken lines* [23] and *theta functions* [24, 28, 26] into the world of cluster varieties. The description of these objects given in [25] is very combinatorial. However, Keel and Yu proved in [35] that they have a deep geometric interpretation in the context of non-archimedean analytic geometry. This approach was used to settle various important and long-standing conjectures including the famous positivity conjecture for cluster algebras.

Every theta function for $\mathcal{A}_B$ is a Laurent power series in the initial cluster variables that encodes certain enumerative invariants of $\mathcal{X}_B$. Theta functions can be defined in a combinatorial way using scattering diagrams and broken lines. However, this combinatorial procedure is very complicated and the theta functions are far from being explicit. In the cluster context, a scattering diagram is a possibly infinite collection of co-dimension 1 cones, called the walls, living in an ambient real vector space whose dimension equals the complex dimension of $\mathcal{X}_B$. Attached to each wall there is Laurent power series in the initial cluster variables called a scattering function. A broken line is a piece-wise linear ray homeomorphic to $(-\infty, 0]$ living in the ambient vector space that can bend only at its intersection with a wall. The bending is controlled by the scattering functions. Conceptually, the ambient vector space is identified with the tropicalization of $\mathcal{X}_B$ and a broken line is the skeleton of a non-archimedean analytic curve in the Berkovich analytification of $\mathcal{X}_B$.

Typically, theta functions are not global functions on a cluster variety. However, Gross, Hacking, Keel and Kontsevich proved that every cluster monomial is a globally defined theta function. The *middle cluster algebra* $\mathrm{mid}(B)$ is the subalgebra of $\mathrm{up}(B)$ spanned by the globally defined theta functions of $\mathcal{A}_B$. Namely, those theta functions that turn out to be Laurent polynomials. We have the following inclusions $\mathrm{ord}(B) \subseteq \mathrm{mid}(B) \subseteq \mathrm{up}(B)$. Both $\mathrm{ord}(B)$ and $\mathrm{mid}(B)$ admit a basis of theta functions with non-negative multiplicative structure constants (given by certain counts of broken lines).

In [8], Bridgeland constructed the so-called *stability scattering diagrams*. These scattering diagrams are constructed using stability conditions on the category $\mathrm{mod}(J(Q, W))$ of finite-dimensional modules over a Jacobian algebra $J(Q, W)$. The stability scattering diagram is described in terms of the motivic hall algebra of $\mathrm{mod}(J(Q, W))$. Bridgeland proves that, for acyclic quivers, the stability scattering diagram coincides with the scattering diagrams constructed in [25], however, these diagrams can differ in the non-acyclic case as exhibited in [43, 52]. Remarkably, Cheung [14] was able to deduce the CC-formula for acyclic quivers using the stability scattering diagram. At the quantum level Davison and Mandel [16] used scattering diagrams and the Donaldson-Thomas theory of quivers to study the quantum theta basis of various quantum cluster algebras, proving the so-called quantum strong cluster positivity conjecture for these algebras. These works provide the first steps towards a categorification of the (quantum) theta basis and contribute to understand the relation between $\mathrm{mid}(Q)$ and $\mathrm{CC}(Q, W)$.

We have treated some of the most influential approaches to study bases for cluster algebras. However, there are various closely related lines of research that can contribute to enrich our understanding of these bases. For instance, in [25] it was show that certain polytopal subsets of theta functions define (partial) compactifications of cluster varieties in much the same way toric varieties can be (partially) compactified using (fans) polytopes. In some cases these polytopal subsets can be realized as Newton-Okounkov bodies but we still lack of general theory explaining this phenomenon. There are interesting results in this direction [6, 13, 56] but there is still many open questions. One can expect to understand bases for cluster algebras from the perspective of Gröbner theory. There are partial results that follow this point of view [4, 7, 45] but the general theory is still awaiting to be discovered. We would like to mention the recent work of Baumann, Kamnitzer and Knutson [3] suggesting that the Mirković-Vilonen basis for a coordinate ring of the form $\mathbb{C}[G]$ might coincide with the theta basis. Finally, we point out the work of Allegretti [1] is also a first step in relating bases constructed by Goncharov and Shen [22] to the theta basis.

1.4 Generalizations

Cluster algebras have been generalized in various ways. For instances, in [11] Chekhov and Shapiro generalized the mutation procedure in order to define generalized cluster algebras. Lam and Pylyavskyy generalized the Laurent Phenomenon for cluster algebras to define the Laurent Phenomenon algebras in [38]. There has also been work on defining super cluster algebras, see [49, 57] and the references therein. These generalizations satisfy the main properties of cluster algebras and it is expected that the techniques developed in the previous sections extend to the more general setting. In particular, the study of bases for generalized cluster algebras, Laurent-Phenomenon algebras and super cluster algebras is closely tied to the study of bases for ordinary and upper cluster algebra. In particular, the connections between ordinary cluster algebras and topological surfaces, quiver representations and mirror symmetry is in process of being extended to the general setting.

In [44], Mou showed that the notion of cluster varieties and their dualities can be extended in the framework on Chekhov and Shapiro, defining thus the notion of a generalized cluster variety and generalized Fock-Goncharov duality. Even further, Mou showed constructed scattering diagrams and theta functions for generalized cluster varieties, see [12] as well. In particular, it is possible to study generalized cluster algebras using theta functions on generalized cluster varieties. It turns out that the positivity property of wall-crossings functions still hold in the generalized situation.

The study of these generalizations is important to extend the scope of the theory of cluster algebras as this might lead to a unifying framework explaining the connections between the different approaches to construct bases for cluster algebras.

2 Presentations

2.1 Monday

The talks on Monday were meant to present advances in generalizations of the notion of cluster algebras.

In the first talk *The blue vs. red game and applications* by Bernhard Keller, the speaker presented an exciting generalization of the mutation operation. Let Q be a quiver with n vertices. In the usual setting one divides the vertices of Q into two groups - the mutable ones and the frozen ones. Classically one only allowed to perform mutation on mutable vertices. However, in this generalization mutations at frozen vertices are allowed and counterbalanced by mutations at non frozen vertices. The speaker also explained how to apply this construction to obtain group actions (e.g. braid group actions) on cluster categories and cluster algebras.

Afterwards we had the talk *Generalized cluster structures and periodic difference operators* by Misha Gekhtman based on a joint work with M. Shapiro and A. Vainshtein and an ongoing project with C. Fraser and K. Trampel. He presented a construction that ties together several diverse notions including spaces of periodic difference operators, Poisson sub manifolds of a Drinfeld double of $GL(n)$ and subsets of Grassmannians stable under the action of powers of a cyclic shift. He explained how the theory of generalized cluster algebras serves as a unifying theme.

The third talk of the day was delivered by the young researcher Esther Banaian and was titled *Snake Graphs from Punctured Orbifolds* based on joint works with Elizabeth Kelley and with Wonwoo Kang. Her talk was motivated by work generated on the quest to prove the positivity conjecture of cluster a special which resulted in many interesting proofs for special families. In particular, she focused on Musiker-Schiffler-Williams' proof of the positivity conjecture for cluster algebras from surfaces [46, 47]. The main idea of the proof is to realize each cluster variable as a generating function of statistics on a certain labeled graph, called a snake graph. She exploited this idea to obtain a proof of positivity for generalized cluster algebras from an orbifold, as defined by Chekhov and Shapiro.

Next speaker was another young researcher, Elizabeth Kelly with the talk *Rooted Clusters of Graph LP Algebras* based on joint work with Esther Banaian, Sunita Chepuri, and Sylvester W. Zhang. She was mainly focused on the so-called Laurent-Phenomenon algebras, also called LP algebras which generalize. These algebras have the Laurent Phenomenon just as cluster algebras, however, the positivity phenomenon remains conjectural in general. She focused on the Graph LP algebras which are finite LP algebras that can be encoded by a graph. For the subclass of graph LP algebras defined by trees, she define a family of clusters called rooted clusters. Her main result is that there is a combinatorial interpretations for expansions in terms of these rooted clusters using generalizations of T-paths and snake graphs. Which is a generalization of the work of [46].

The final talk of the day *The super CC-map* was scheduled to be delivered by Ilke Canakci who announced progress in the definition of super CC-algebras. Unfortunately, her talk had to be canceled for reasons of force majeure.

2.2 Tuesday

The talks of Tuesday were closer to representation theory and monoidal categorification.

The first talk of the day was delivered by Fan Qin and was titled *Triangular bases for strata of algebraic groups*. He recalled that the so-called triangular bases are Kazhdan-Lusztig type bases quantum cluster algebras introduced in [53]. This framework includes as a particular case the dual canonical bases for the quantized coordinate rings of unipotent subgroups. In his talk, the author explained how to construct the (common) triangular bases for the (quantized) coordinate rings of algebraic groups and of their double Bruhat cells, generalizing results for unipotent subgroups. An important feature of these bases is that their structure constants are positive when the Cartan datum is symmetric.

The next speaker was Joel Kamnitzer who gave the talk *Canonical bases in representation theory and cluster algebras*. In his talk he recalled the three important families of bases that were alluded in §1: generic basis, the common triangular basis and the theta basis. All these bases are parametrized by the tropical points in the dual cluster variety, in line with the Fock-Goncharov conjecture. He pointed out that in representation theory, we also have three families of bases: the semicanonical basis, the canonical basis, and the Mirkovic-Vilonen basis and surveyed known results about these bases and presented open problems in this direction such as a comparison between the theta basis and the Mirkovic-Vilonen basis.

The third speaker of the day was the young researcher Anne Dranowski on *Minuscule multiples and reverse plane partitions*. She began recalling that the semistandard Young tableaux and irreducible components of Springer fibers model highest weight crystals for $SL(n)$ in a compatible way. She presented a generalization of these correspondences to ADE Demazure crystals having minuscule weight. This generalization uses reverse plane partitions in place of tableaux and preprojective algebra modules in place of flags. She suggested to investigate the relation between reverse plane partitions and clusters.

The last speaker of the day was Vyjayanthi Chari who gave the talk *Higher order Kirillov-Reshetikhin modules, monoidal categorification and Imaginary modules*. This talk was in the framework of monoidal categorification of cluster algebras. She discussed a generalization in type A_n of the well–known Kirillov-Reshetikhin modules for quantum affine algebras. This generalization has many of the properties of the Kirillov-Reshetikhin modules and, moreover, allows us to classify all prime representation of the quantum affine algebra which are supported on only one node of the Dynkin diagram of A_n. She give a necessary and sufficient condition for a tensor product of such modules for a fixed node to be irreducible. She further discussed an analog of the theory of monoidal categorification of cluster algebras for Kirillov-Reshetikhin modules. An important notion she presented is the one of imaginary modules for quantum affine algebras. The first example of such modules appeared in the work of Leclerc. In terms of the infinite rank cluster algebras, coming from monodial categorification, her examples show the existence of an infinite number of pairs of cluster variables whose product is not in the span of cluster monomials.

2.3 Wednesday

The third day of the workshop was focused on the theta basis.

The first talk of the day was by Ben Davison titled *Strong positivity for quantum cluster algebras*. The positivity conjecture for commutative cluster algebras was settled in the affirmative by Gross, Hacking, Keel and Kontsevich. In this talk the author explain the proof he obtained along Travis Mandel of the quantum version of this positivity for skew-symmetric quantum cluster algebras. Key features if the proof is the use of categorified Donaldson-Thomas theory and using scattering diagrams. The talk by Davison paved the way for the next two talks as he explained cluster scattering diagrams, theta functions and their quantum analogs.

The second talk of the day was by Greg Muller titled *Reciprocity for Valuations of Theta Functions and titled*, presenting joint work with Man-Wai Cheung, Timothy Magee and Travis Mandel. The view-point of this talk was the Gross-Siebert mirror symmetry program for log-Calabi-Yau varieties. In this program one associates a theta function on X to each boundary valuation on Y, where X and Y are a pair of mirror dual affine log Calabi-Yau varieties with maximal boundary (such as cluster varieties). Since mirror duality is a symmetric relation, this provides two ways to associate an integer to a pair m and n of boundary valuations on X and Y (respectively).

1) Apply the valuation m to the theta function associated to n.

2) Apply the valuation n to the theta function associated to m.

Resolving a conjecture of Gross-Hacking-Keel-Kontsevich, we show that these two numbers are equal in a generality which covers all cluster algebras (specifically, when the theta functions are given by enumerating broken lines in a scattering diagram generated by finitely-many elementary incoming walls). THe also discussed applications to the theta basis and its localizations.

The final talk of the day was a student talk, delivered by the Colombian Ph.D. student Carolina Melo-Lpez. This was the unique talk of the workshop that was not transmitted via zoom as she presented on-going work of her thesis. The title of the talk was *The cluster complex for finite type cluster $\mathcal{X}$-varieties*. The tropical space parametrizing the theta functions on a cluster variety V supposrts the corresponding scattering diagram. This diagram considts of two parts, a cone complex formed out by simplicial cones (modulo a linearity space) and some other arrangement of walls that is very poorly understood. The cluster complex is fundamental as it codifies the portion of the basis formed by the cluster monomials. The cluster complex for cluster $\mathcal{A}$-varieties has been studied in grate detail by many authors. However, the cluster complex for cluster $\mathcal{X}$ has not been studied in detail. In this talk the speaker presented partial result on the description of the cluster complex for cluster $\mathcal{X}$-varieties of finite type. Her description is via quiver representations and the dimension vectors of the representations.

2.4 Thursday

In the fourth day of the workshop we learned about connections to tropical geometry and braid varieties.

The first speaker was Lara Bossinger and the talks was titled *Tropical totally positive cluster varieties*. In this talk she relate two types of tropicalization that are available when dealing with a (partially compactified) cluster variety: (1) the Fock-Goncharov tropicalization of a scheme with a positive atlas, (2) the tropicalization of an ideal associated with an embedding of the variety. We discuss how both types of tropicalzation encode toric degenerations of the cluster variety. We present the construction of a piece wise linear map from (1) to (2). Given certain natural assumptions the piece wise linear map not only identifies the fan structures on either side, but also the associated toric degenerations.

The second speaker was the young researcher Melissa Sherman-Bennett who gave the talk *Cluster structures on type A braid varieties from $3D$ plabic graphs* presenting joint work with Pavel Galashin, Thomas Lam, and David Speyer. She started recalling the construction of the braid variety $X(b)$ associated to a positive braid b. These varieties are always smooth, affine and irreducible variety. Braid varieties are atttracting the attention of a large number of researchers in the area as they are natural generalizations central cluster varieties arrising in nature such as positroid varieties, Richardson varieties, double Bruhat cells, and double Bott-Samelson cells in type A. The first result is that $X(b)$ is a locally acyclic cluster algebra. Then she constructed seeds for this cluster algebra from $3D$ plabic graphs, which generalize Postnikov's plabic graphs for positroid varieties. She further discussed related joint work with Krystina Serhiyenko, where we prove that in type A, Leclerc's conjectural cluster structure on Richardsons is indeed a cluster structure.

The third talk of the day was by Lauren Williams and titled *Polyhedral and tropical geometry of flag positroids* based on joint work with Jon Boretsky and Chris Eur. Particularly, she centered her attention on the polyhedral and tropical geometry of flag positroids whose set of ranks is a sequence of consecutive numbers. In this case she showed that the nonnegative tropical flag variety equals the nonnegative flag Dressian, and that points of these spaces give rise to coherent subdivisions of flag positroid polytopes into flag positroid polytopes. These results have applications to Bruhat interval polytopes and to realizability questions. In particular, she prove that every positively oriented flag matroid of consecutive ranks is realizable.

The last talk of the day was by Jos Simental titled *Cluster structures on braid varieties* presenting joint work with Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le and Linhui Shen. This talk was closely related to the talk of Sherman-Bennett but the framework was broader and the techniques quite different. Given a simple algebraic group G and an element β of its positive braid monoid the author considered the braid variety $X(\beta)$. Their results show that the coordinate algebra of $X(\beta)$ admits the structure of a cluster algebr. They provide explicitly constructions of several initial seeds, using combinatorial objects called weaves and tropicalization of Lusztig's coordinates. He moreover explained this construction (with several examples) and give properties of the corresponding cluster structure, including local acyclicity and the existence of reddening sequences.

2.5 Friday

On the last day the speakers gave talks that revolve on several ideas that were present on the rest of the talks such as scattering digarams, the connection between cluster algerbas and knot theory and quantum cluster algerbas.

The first speaker of the day was Tomoki Nakanishi with the talk *Mutations, dilogarithm, and pentagon relation*. He started pointing out the roles of the dilogarithm and the pentagon relation in cluster algebras and cluster scattering diagrams. This included the Fock-Goncharov decomposition of mutations, the Hamiltonian formalism for mutations, the algebraic formulation of the dilogarithm and the pentagon relation, and the positive realization of cluster scattering diagrams. An application of the pentagon relation to construct consistent cluster scattering diagrmas was presented.

Afterwards Ralf Schifler delivered the talk *Cluster algebras and knot theory* based on joint work with Vronique Bazier-Matte. To every knot diagram (or link diagram), he explained how to associate a cluster algebra by constructing a quiver with potential. The rank of the cluster algebra is $2n$, where n is the number of crossing points in the knot diagram. He then constructed $2n$ indecomposable modules $T(i)$ over the Jacobian algebra of the quiver with potential. This construction has the propetry that for each $T(i)$, the submodule lattice of $T(i)$ is isomorphic to the corresponding lattice of Kauffman states of the knot. Furthermore, the Alexander polynomial of the knot is a specialization of the F-polynomial of $T(i)$, for every i. He presented the following conjecture: the collection of the $T(i)$ forms a cluster in the corresponding cluster algebra.

The final talk of the event was by Milen Yakimov titled *Representation theory and Poisson geometry of root of unity quantum cluster algebras* presenting joint work with Shengnan Huang, Thang Le, Greg Muller, Bach Nguyen and Kurt Trampel. He showed that all root of unity quantum cluster algebras have canonical structures of Cayley-Hamilton algebras (in the sense of Procesi) and Poisson orders (in the sense of De Concini-Kac-Procesi and Brown-Gordon). The first result allows the transfer of finiteness properties between the quantum and classical situations. The second result relates the representation theory of these algebras to the Poisson geometry of the Gekhtman-Shapiro-Vainshtein brackets. He explained how to prove that the spectrum of each upper cluster algebra equipped with the Gekhtman-Shapiro-Vainshtein Poisson structures has an explicit Zariski open torus orbit of symplectic leaves, which is a far-reaching generalization of the Richardson divisor of a Schubert cell in Lie theory. He concluded by combining these results to describe explicitly the fully Azumaya loci of all (strict) root of unity quantum cluster algebras. This classifies their irreducible representations of maximal dimension.

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