Symmetries of Discrete Structures in Geometry

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The quest for a deeper understanding of highly-symmetric structures in geometry, combinatorics, and algebra has inspired major developments in modern mathematics and science. Symmetry often provides the key to unlock the secrets of complex discrete structures that otherwise seem intractable.

The 5-day workshop focussed on geometric, combinatorial, and algebraic aspects of highly-symmetric discrete structures such as polytopes, polyhedra, maps, incidence geometries, and hypertopes. It brought together experts and emerging researchers in the theories of polytopes, maps, graphs, and incidence geometries to discuss recent developments, encourage new collaborative ventures, and achieve further progress on major problems in these areas. In recent years there has been an outburst of research activities concerning discrete symmetric structures in geometry and combinatorics, and the workshop capitalized on this momentum.

Our main objective was to nourish new and unexpected connections established recently between the theories of polytopes, polyhedra, maps, Coxeter groups, and incidence geometries, focussing on symmetry as a unifying theme. In each case an important class of groups acts on a natural geometrical and combinatorial object in a rich enough way to ensure a fruitful interplay between geometric intuition and algebraic structure. Naturally their study requires a broad and long-range view, merging approaches from a wide range of different fields such as geometry, combinatorics, incidence geometry, group theory and low-dimensional topology. These connections are further enriched by bringing in new ideas and methods from computational algebra, where powerful new algorithms provide an abundance of inspiring examples and challenging conjectures.

In polytopes and symmetry, the recent progress centered around the modern theory of abstract polytopes and combinatorial symmetry. Highly-symmetric abstract polytopes are combinatorial structures with distinctive geometric, algebraic, or topological properties, in many ways more fascinating than traditional highly-symmetric polyhedra, polytopes or tessellations. Their automorphism groups are certain quotients of Coxeter groups, called C-groups, satisfying an intersection condition on subgroups that ensures desirable combinatorial and geometric properties. The rapid development of abstract polytope theory has resulted in a rich theory featuring an attractive interplay of methods and tools from discrete geometry (classical polytope theory), and group theory and geometry (Coxeter groups and their quotients, as well as reflection groups over the reals, complex numbers, or finite fields), combinatorial group theory (generators and relations), and hyperbolic geometry and topology (tessellations and their symmetry groups). Still, even after an active period of research, many deep problems have remained open and await solution.

Regular or chiral maps and hypermaps on surfaces have been studied since the time of Felix Klein. Deep exciting connections exist between hypermaps and other branches of mathematics (hyperbolic geometry, combinatorial group theory, Riemann surfaces, number fields, Galois theory, and graph theory). For instance, surface maps (not necessarily regular) can be viewed as complex algebraic curves, defined over algebraic number fields. A central theme in the theory of maps is the problem of their classification, which is usually approached from one of the following three viewpoints: graph-theoretical, group-theoretical, and topological. In recent years, great progress has been made in the computer-aided enumeration of maps by genus, exploiting new fast algorithms for finding low index normal subgroups in finitely-presented groups. These findings have lead to a host of challenging new conjectures. There are similar approaches to the classification of certain types of highly-symmetric graphs such as arc-transitive graphs.

Incidence geometries provide an overarching theme for most research activities in polytopes and hypermaps. In this vein, maps (usually) are abstract polytopes of rank 3, and in turn, abstract polytopes are thin residually-connected incidence geometries with a string diagram. Thus progress in either subject is likely to influence the others. Polytopes and maps stand to benefit greatly from new developments on diagram geometries and buildings. In a nutshell, buildings are the natural geometric counterparts of certain kinds of simple Lie groups where the classical regular polytopes or Coxeter complexes occur as fundamental structural components (apartments). On the other hand, diagram geometries provide a geometric interpretation for the sporadic simple groups, and although the theories of buildings and diagram geometries are highly developed, the understanding of their interaction with polytopes and hypermaps is still limited at this point. Recently there has been considerable progress in the study of highly-symmetric hypertopes (hybrids of polytopes and incidence geometries). These are thin geometries which relate to abstract polytopes in a similar way as hypermaps to maps.

The following research directions captured the broader themes underlying the workshop activities.

- Classification of highly-symmetric maps on surfaces and related graphs.
- Group-theoretical and topological classification of regular and chiral polytopes.
- Developing a theory of regular or chiral hypertopes.
- Polytope-theoretic interpretation for sporadic and other simple groups.
- Connection groups (monodromy groups) of polytopes.
- Symmetries in designs and finite geometries.
- Topics in combinatorial geometry and combinatorics related to symmetry.

The workshop was a great success and featured an attractive mix of 30-minute lectures, problem sessions, and small group collaborations. It provided ample time and opportunity for participants to interact and engage in mathematical discussion. The emphasis of the workshop was on small group collaborations.

The small research groups formed around various problems proposed during the afternoon problem sessions on the first day. Summaries of these activities are included in this report. The work in small research groups was a stimulating experience for all participants, including in particular junior researcher and graduate students.

1 Working Groups Reports

1.1 Working group "Connection Groups of Truncations of Torus Maps"

Research Problem: Determine the connection group (monodromy group) of the truncation of the regular torus map $\{4,4\}_{(n,0)}$.

Group Members: This group consisted of two experienced researchers (Gabe Cunningham and Eugenia O'Reilly-Regueiro), and one graduate student (José Collins).

Progress: A recent manuscript by Daniel Pellicer and Gordon Williams describes how to represent the connection group $\mathrm{Con}(\mathcal{P})$ of a k-orbit polytope $\mathcal{P}$ as a subgroup of the wreath product of the automorphism group $\Gamma(\mathcal{P})$ with S_k. Our goal was to use this procedure to determine the size and structure of the connection group of the truncation of $\{4,4\}_{(n,0)}$.

Let $\mathcal{P}_n = \{4,4\}_{(n,0)}$ and let $\mathcal{Q}_n$ be the truncation of $\mathcal{P}_n$. The map $\mathcal{Q}_n$ is a toroidal quotient of the Archimedean tiling $[4.8.8]$. It is a vertex-transitive 3-orbit polytope. Using Pellicer and Williams' algorithm, we determined that the generators r_0, r_1, r_2 of Con$(\mathcal{P}_n)$ could be represented as elements in $S_3 \ltimes \Gamma(\mathcal{P}_n)^3$ as follows:

$$\begin{aligned} r_0 &= ()[\rho_0, \rho_1, \rho_1] \\ r_1 &= (1,2)[1,1,\rho_2] \\ r_2 &= (2,3)[\rho_2, 1, 1]. \end{aligned}$$

So Con$(\mathcal{Q}_n) \cong S_3 \ltimes H$, where H is a subgroup of $\Gamma(\mathcal{P}_n)^3$. The goal then became to determine H; i.e., which triples of automorphisms $[\alpha_1, \alpha_2, \alpha_3]$ can be obtained from these generators.

We started by trying to identify elements in Con$(\mathcal{Q}_n)$ whose automorphisms were all translations. We found that, with respect to a certain base flag,

$$(r_0 r_1 r_2)^6 = ()[-2e_2, -2e_1, 2e_1],$$

where e_1, e_2 are unit translations in the x and y direction, respectively. Now, if n is odd, then it follows that $[e_2, e_1, -e_1]$ is in H. Conjugating this element by various elements in Con$(\mathcal{Q}_n)$, we were able to show that the full direct product of all 3 translation subgroups lies in H.

Having identified a large normal subgroup of H (and indeed of Con$(\mathcal{Q}_n)$), we took the quotient by it. This has the effect of passing to Con$(\mathcal{Q}_1)$, which is small enough to be amenable to computer calculation. We found that $|\text{Con}(\mathcal{Q}_1)| = 192$. Since the direct product of the 3 translation subgroups has order n^6, we find that $|\text{Con}(\mathcal{Q}_n)| = 192n^6$. In particular, the connection group of $\mathcal{Q}_n$ has index 16 in $S_3 \ltimes \Gamma(\mathcal{P}_n)^3$.

The case with n even is slightly more complex, but similar analysis should work. We do not yet have a complete structural description of Con$(\mathcal{Q}_n)$ (in either case), nor a presentation, but we have made some progress.

1.2 Working group "Incidence Complexes as Skeletons of Polytopes"

Research Problem: Realizing highly-symmetric incidence complexes as skeletons of highly-symmetric abstract polytopes.

Group Members: This group consisted of seven experienced researchers (Javier Bracho, Peter Brooksbank, Maria Elisa Carrancho Fernandes, Asia Ivic Weiss, Dimitri Leemans, Egon Schulte and Klara Stokes), and two graduate students (Anneleen De Schepper and Antonio Montero).

Progress: An *incidence complex* is a combinatorial structure (a certain ranked partially set) that generalizes the notion of *abstract polytope*. An incidence complex $\mathcal{K}$ is *regular* if the action of its automorphism group is transitive on the set of flags of $\mathcal{K}$. Examples of regular incidence complexes arise in a natural way by considering the so-called skeletons of regular abstract polytopes. For instance, the incidence structure formed by the vertices, edges, and squares of the cubical tessellation of 3-space gives an incidence complex of rank 3. This incidence complex is the 2-skeleton of the cubical tessellation.

A natural problem is to determine if every regular incidence complex can be constructed this way. More precisely, given a regular incidence complex $\mathcal{K}$ of rank k, is there an abstract regular polytope $\mathcal{P}$ of rank n, with $n > k$, such that $\mathcal{K}$ is the $(k-1)$-skeleton of $\mathcal{P}$? Of particular interest is the case when $n = k + 1$. Clearly, except in degenerate cases, for $\mathcal{P}$ to exist, $\mathcal{K}$ must have the property that each face of co-rank 2 lies in at least three facets, and we are assuming this from now on.

The following partial results were obtained.

- We proved that the 2-skeleton of the 5-simplex is not the 2-skeleton of any regular 4-polytope. However, it is the 2-skeleton of a chiral 4-polytope of type $\{3, 4, 4\}$.
- The 2-skeleton of the 6-simplex is not the 2-skeleton of any regular or chiral abstract 4-polytope.
- We found an example of a regular incidence complex of rank 4 that cannot be the 3-skeleton of any regular or chiral polytope of rank n for any $n \geq 5$.

Thus the answer to the above question is negative. The broader problem then becomes to find necessary and sufficient conditions for certain classes of incidence complexes to occur as skeletons of abstract polytopes.

1.3 Working group “The Wooly Hat”

Research Problem: Covering graphs of the Wooly Hat graph (see the figure below).

Group Members: This group consisted of three experienced researchers (Leah Berman, Hiroki Koike and Steve Wilson), and one graduate students (Elías Mochán).

Progress: During the CMO/BIRS Workshop we worked on covering graphs of the Wooly Hat graph with a finite cyclic group of covering transformations. More specifically, we tried to explain why there does not appear to exist any edge-transitive covering graph of this type.

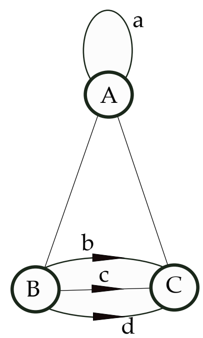

Figure 1: The Wooly Hat graph with a standard voltage assignment.

All such coverings may be obtained as the derived graph $\mathrm{WH_n(a, b, c, d)}$ of a *voltage graph* as in the figure below, where $a, b, c, d \in \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. We restricted ourselves to study connected coverings, so we assumed $\mathrm{gcd(n, a, b, c, d)} = 1$. We were able to show that the automorphism given by the reflection on the vertical axis always lifts to any covering graph. This implies that any edge transitive covering should also be arc-transitive and therefore vertex-transitive.

The fact that the reflection always lifts also implies that any vertex-transitive cover that is not edge-transitive, should have exactly two orbits on arcs: one corresponding to the lifts of the loop and two BC-edges, and the other consisting of the lifts of the remaining BC-edge (say, the one with voltage c) and the edges of voltage 0. We also proved that in this case $\mathrm{gcd(b - d, n) = 2a}$.

Using the aid of the computer we found many examples of vertex-transitive coverings of the Wooly Hat, none of them edge-transitive. We found many infinite families, and managed to prove that $\mathrm{WH_n(a, b, c, d)}$ is vertex-transitive if the following conditions hold:

1. $2 \mid n, c,\ 2 \nmid a, b, d$,
2. $b - d = 2a$,
3. $a + 2c = b + 2d$.

All the examples we found satisfied the first two conditions, and most of them satisfied the third.

1.4 Working group “Edge-transitive maps”

Research Problem: Constructing and analysing edge-transitive maps on surfaces.

Group Members: This group consisted of four experienced researchers (Marston Conder, Isabel Hubard, Gareth Jones and Gordon Williams), and four graduate students (Patricio García, Alejandra Ramos, Andrea Ramos Tort and Claudia Silva).

Progress: An edge-transitive map is a 2-cell embedding of a connected graph or multi-graph as a map on some closed surface, with the property that the group of incidence-preserving automorphisms has a single orbit on edges. This is a weaker but still important variant of the notion of a regular map (whose automorphism group has a single orbit on ordered edges (or even on incident vertex-edge-face triples)), but relatively little research has been done on them; see the references [x] and [y] below for two of just three papers having them as principal topic.

It is known that edge-transitive maps can be classified into 14 types according to the existence of particular kinds of automorphisms, and certain properties of these maps and their groups were described by Gareth Jones in his lecture earlier in the workshop timetable. A PDF copy of his talk was distributed to all the members of the working group by Isabel Hubard. Marston Conder added some further background on presentations of the universal groups for the 14 types, as well as information about vertex-, edge-and face-stabilisers, and "forbidden automorphisms" (the presence of which puts such a map into a different class).

Marston Conder and Gareth Jones set some exercises for the group that were deliberately chosen to be accessible to the graduate students, and yet provided new insights into this topic. Although the emphasis was on learning about edge-transitive maps and how to handle them, partly because the topic was new to most members of the group, everyone in the group learned something new, and some interesting discoveries were made.

For example, the group worked out (new) details that are helpful in constructing and/or analysing examples, and with the help of some computing resources were able to determine the smallest examples of maps in each class, and then also developed alternative constructions for many of the latter.

Incidentally, some of this work was taken further forward at another BIRS meeting four weeks later, in Banff.

[x] J. Širáň, T.W. Tucker and M.E. Watkins, Realizing finite edge-transitive orientable maps, J. Graph Theory 37 (2001), 1-34.

[y] J.E. Graver and M.E. Watkins, Locally finite, planar, edge-transitive graphs, Memoirs Am. Math. Soc. 126 (#601), 75pp., 1997.

1.5 Working group "Enumerating Asymmetric Polytopal Maps"

Research Problem: Constructing and analyzing small polytopal maps with trivial automorphism groups.

Group Members: This group consisted of one experienced researcher (Barry Monson), and four graduate students (Anneleen De Schepper, Eric Ens, Jessica Mulpas, Briseida Trejo-Escamilla).

Progress:

1. According to a rather general definition, a *map* $\mathcal{M}$ is an embedding of a finite graph into a compact surface so that each component of the set-complement of the graph (in the surface) is an open topological disc. The closures of these discs are the *faces* of the map. The *vertices* and *edges* of the graph supply the vertices and edges of the map.

   We can make subtle adjustments to this definition. Usually we can safely think of these things by drawing nice pictures on some surface.

2. We will consider just maps which are *rank* 3 *abstract polytopes*. After all, the (non-polytopal) map consisting of just a single node on the sphere has trivial automorphism group:

   $$\bullet$$

   Polytopal maps are especially well-behaved: each edge has 2 vertices and lies on 2 faces; each vertex on a face is on two consecutive edges of that face. The vertices and edges of each face form a single cycle; thus a p-face has p distinct vertices and p distinct edges. Dually, the edges and faces on a given vertex form a single q-cycle called the *vertex-figure* for that vertex.

   All this enables straightforward counting.

3. Let us use the word *degree* to name either the number of edges on a face or the number of edges at a vertex. In a polytopal map, all such degrees are at least 2. For $k \geq 2$, let v_k be the number of degree k vertices in the map; and let f_k be the number of degree k faces. Thus

   $$\begin{aligned} v &= v_2 + v_3 + v_4 + \cdots && (1) \\ f &= f_2 + f_3 + f_4 + \cdots && (2) \end{aligned}$$

4. Example. In convex polyhedra the surface is a polytopal map of spherical type; and $v_2 = f_2 = 0$.

5. Now count the edges in two ways; we assume nothing yet about the genus of the surface. Since each edge lies on 2 vertices and on 2 distinct faces, we have

$$2e = 2v_2 + 3v_3 + 4v_4 + \cdots \quad (3)$$
$$2e = 2f_2 + 3f_3 + 4f_4 + \cdots \quad (4)$$

Thus

$$2e \geq 2(v_2 + v_3 + v_4 + \cdots) = 2v,$$

so that $e \geq v$; likewise $e \geq f$.

6. **Back to our problem**.

By experiment we found a spherical map $\mathcal{M}$ with $v = 4$ vertices, $e = 6$ edges and $f = 4$ faces (one 2-gon, two 3-gons, one 4-gon). These data suffice to define the full structure of this 3-polytope. A picture will convince the reader that $\mathcal{M}$ is asymmetric and, as it happens, self-dual.

Suppose we can improve on this asymmetric polytopal map with $e = 6$ edges (and hence 24 flags). That is, assume $e \leq 5$. Since each p-face really does have p distinct edges, and each q-valent vertex really does lie on q distinct edges, this means $v_6 = v_7 = \cdots = 0$ and $f_6 = f_7 = \cdots = 0$. Thus

$$2e = 2v_2 + 3v_3 + 4v_4 + 5v_5 \geq 2v \quad (5)$$
$$2e = 2f_2 + 3f_3 + 4f_4 + 5f_5 \geq 2f. \quad (6)$$

Now we can enumerate the numerical possibilities by mild brute-force! Caution: there will be more subtle geometric limitations.

7.

e	f_2	f_3	f_4	f_5	f
5				2	2
5	1	1		1	3
5	1		2		3
5		2	1		3
5	3		1		4
5	2	2			4
5	5				5
4			2		2
4	2		1		3
4	1	2			3
4	4				4
3		2			2
3	3				3
2	2				

8. So now we have something finite to work with. Notice that the restrictions on vertex degrees are exactly of the same kind, by a dual argument.

But not all combinations will work out!

How many maps can you build with these limited possibilities? What is the genus for each? Are any non-orientable? And, finally, do any have trivial symmetry?

9. Our work suggests that we do indeed have the smallest asymmetric 3-polytope $\mathcal{M}$. As we have mentioned, it is realized as a self-dual map on the sphere with $v = 4$ vertices, $e = 6$ edges and $f = 4$ faces (one 2-gon, two 3-gons, one 4-gon).

A separate calculation in Gap shows that the monodromy group $G(\mathcal{M})$ has Schläfli type $\{12, 12\}$ and order

$$114721266401280000 \simeq 1.15 \times 10^{17}.$$

Thus the *minimal regular cover* of $\mathcal{M}$ is huge.

10. Where to go from here?

We need to double check that the smallest asymmetric map is unique to isomorphism. Then it will be interesting to tackle the problem in higher ranks. (The situation in ranks $0, 1, 2$ is clear.)

1.6 Working group "Radon-type theorems for lattice point sets"

Research Problem: Radon-type theorems for lattice point sets.

Group Members: This group consisted of four experienced researchers (Maria del Rio Francos, Natalia Garcia Colin, Daniel Pellicer and Pablo Soberon).

Progress: We worked on the following problem from combinatorial geometry: Find the minimum positive integer n such that for any set X of n points with integer coordinates in Euclidean space there exists a partition of X in two sets Y_1 and Y_2 such that the intersection of the convex sets of Y_1 and Y_2 contains a point with integer coordinates.

It was previously known that n is at least 11, and that it is at most 17. We analysed a set with 10 points with no such partition, and tried to find one such set with 11 points, but this was unsuccessful.

Then we studied the proof of the fact that 17 points suffice and realised that, in some steps of this proof, some strong conditions are used to conclude relatively weaker conclusions. This gives the idea that the proof can be adapted so that these conditions are used in all their strength to show that n is at most 16.

We made several attempts on where to improve the proof, and obtained stronger lemmas than the ones used in the paper that proves that n is less or equal to 17. It looked like we were on a right track, but at that point time was over and our partial results were insufficient to make a claim on sets of 16 points.

2 Lectures

The workshop also featured fifteen 30-minute lectures scheduled in the mornings. The topics varied across the board from polytopes, to hypertopes, to maps, to graphs, to designs, to finite geometries, and to configuration spaces for mass partitions. The titles and abstracts for these lectures are included below in the order in which the lectures were given.

Adriana Hansberg: On zero-sum K_m over $\mathbb{Z}$.

In this talk, we will show the following results: For every integer $n \geq 5$ and every weighting function $f : E(K_n) \mapsto \{-1, 1\}$ on the edges of the complete graph K_n such that

$$|\sum_{e \in E(K_n)} f(e)| \leq n(n-1)/2 - h(n),$$

where $h(n) = 2(n+1)$ if $n \equiv 0 \bmod 4$ and $h(n) = 2n$ if $n \not\equiv 0 \bmod 4$, there is always a copy of K_4 in K_n for which $\sum_{e \in E(K_4)} f(e) = 0$, and this bound is sharp. However, if we consider any K_k with $k \geq 2$, $k \neq 4$, this is not true anymore: we show that there are infinitely many values of n such that there is a weighting function $f : E(K_n) \mapsto \{-1, 1\}$ with $\sum_{e \in E(K_n)} f(e) = 0$ and such that $\sum_{e \in E(K_k)} f(e) \neq 0$ for every copy of K_k in K_n. These results solve a problem raised by Caro and Yuster. This is a joint work with Yair Caro and Amanda Montejano.

Gabriela Araujo-Pardo: The achromatic number of Kneser graphs and their relationship with Steiner triple systems

In this talk we give the notion of complete colorings in graphs, achromatic number, Kneser graphs and Steiner triple systems. Also, we explain how the Steiner triple systems solve the problem about the existence of complete colorations on Knesser graphs that attain the upper bound of the achromatic number, where the achromatic number of a graph G is the maximum integer value for the number of chromatic classes in a complete and proper coloring of G.

Natalia García-Colin: Towards a projective Upper/Lower bound theorem

The Upper and Lower bound Theorems in convex geometry deal, respectively, with the maximum and minimum number of facets that a convex d-dimensional polytope with n vertices can have. In this talk we will offer bounds for such numbers for what we refer to as the projective case of this problem. Namely, we are interested in the following:

Problem 1: Given a set of n points in general position $X \subset \mathbb{R}^d$ what is the maximum number of facets that $\mathrm{conv}(\mathrm{T}(\mathrm{X}))$ can have, among all the possible permissible projective transformations T of X.

Problem 2: Given a set of n points in general position $X \subset \mathbb{R}^d$ what is the maximum number of vertices that $\mathrm{conv}(\mathrm{T}(\mathrm{X}))$ can have (as the support of the convex hull), among all the possible permissible projective transformations T? We define the **projective class of a set of points** $X \subset \mathbb{R}^d$ as the set of all possible point configurations that are the image of X under a permissible projective transformation, and denote it $[X]$. Mimicking the polytope notation, let $f_k([X])$ be **the maximum number of** k**-faces** that the convex hull of a point configuration in the class $[X]$ can have, i.e. $f_k([X]) = \max_{Y \in [X]} f_k(\mathrm{conv}(\mathrm{Y}))$. Finally we denote as $f_k(n)$, the minimum $f_k([X])$ over all the possible configurations of n points, i.e. $f_k(n) = \min_{X \subset \mathbb{R}^d} |X| = n f_k([X])$.

With this notation, Problems 1 and 2 can be interpreted as the task of finding the value of $f_{d?1}(n)$ and $f_0(n)$ respectively. Both problems are natural generalizations of the well-known McMullen's problem: What is the maximum n such that any set of n points in general position, $X \subset \mathbb{R}^d$, can de mapped by a permissible projective transformation onto the vertices of a convex polytope? These two problems are closely related. Moreover, Problem 2 has a direct application to the problem of counting the number of Radon partitions induced by a colouring.

Marston Conder: Symmetric Cubic Graphs as Cayley Graphs

A graph X is *symmetric* if its automorphism group acts transitively on the arcs of X, and s*-arc-transitive* if its automorphism group acts transitively on the set of s-arcs of X. Furthermore, if the latter action is sharply-transitive on s-arcs, then X is s*-arc-regular.* It was shown by Tutte (1947, 1959) that every finite symmetric cubic graph is s-arc-regular for some $s \leq 5$. Djoković and Miller (1980) took this further by showing that there are seven types of arc-transitive group action on finite cubic graphs, characterised by the stabilisers of a vertex and an edge. The latter classification was refined by Conder and Nedela (2009), in terms of what types of arc-transitive subgroup can occur in the automorphism group of X. In this talk we consider the question of when a finite symmetric cubic graph can be a Cayley graph. We show that in five of the 17 Conder-Nedela classes, there is no Cayley graph, while in two others, every graph is a Cayley graph. In eight of the remaining ten classes, we give necessary conditions on the order of the graph for it to be Cayley; there is no such condition in the other two. Also we use covers (and the 'Macbeath trick') to show that in each of those last ten classes, there are infinitely many Cayley graphs, and infinitely many non-Cayley graphs. This research grew out of some recent discussions with Klavdija Kutnar and Dragan Marušič.

Gareth Jones: Edge-transitive maps

In 1997 Graver and Watkins partitioned edge-transitive maps into 14 classes, distinguished by the quotient of each map by its full automorphism group. These classes include those of regular, chiral and just-edge-transitive maps. I shall describe the possible topological and combinatorial properties of the maps in these classes, including maps with boundary, together with the possibilities for their automorphism groups. In the case of finite simple groups, the latter builds on results of Nuzhin and others relevant to regular maps (with a small but important correction), and more recently of Leemans and Liebeck for chiral maps.

Peter Brooksbank: Orthogonal groups in characteristic 2 acting on polytopes of high rank

In a series of three papers, Monson and Schulte showed that, subject to some mild constraints, certain families of real reflection groups can be reduced modulo odd primes to yield finite string C-subgroups of orthogonal groups. Further, polytopes of any desired rank can be constructed this way. In this talk I will show that the latter is also true for orthogonal groups defined over any non-prime field of characteristic 2. Of course, the modular reduction method cannot be used for such groups, so the polytopes are constructed from scratch using analogues of reflections. This is a report on recent joint work with J.T. Ferrara and Dimitri Leemans.

Jeremie Moerenhout: Chiral polytopes and groups of type $\mathrm{PSL}(2, \mathrm{q})$

In terms of their symmetries, not many geometric objects have received more attention than polytopes. Since ancient times, polytopes have been considered with great interest, not only for aesthetic reasons, but also for scientific reasons. More recently, the theory of abstract polytopes, which generalizes the concept of classical polytopes, has emerged as a powerful tool for studying symmetries. In the last decade, efforts to classify highly symmetric polytopes, including chiral and regular ones, have highly increased. Given a group Γ, a lot has been done in order to enumerate all polytopes of a certain type and which have Γ as their group of symmetries. Many results have been obtained for almost simple groups and atlases for small groups have been drawn up. However, not much is known about chiral polytopes, which are polytopes that have all possible rotational symmetries but no reflection. In this talk, we will define the basic notions necessary to understand the presentation. Then we will go through some techniques to prove the existence of chiral polytopes whose automorphism group is isomorphic to an almost simple group with socle $\mathrm{PSL}(2, \mathrm{q})$.

Gabriel Cunningham: Non-flat regular polytopes and restrictions on chiral polytopes

An abstract polytope is flat if every facet is incident on every vertex. In this talk, we prove that no chiral polytope has flat finite regular facets and finite regular vertex- figures. We then determine the three smallest non-flat regular polytopes in each rank, and use this to show that for $n \geq 8$, a chiral n-polytope has at least $48(n-2)(n-2)!$ flags.

Eric Ens: Block systems on the facets of toroidal hypertopes (Abstract not available.)

Maria Elisa Carrancho Fernandes: Small regular hypertopes of rank 3

We describe the smallest C-groups with complete diagram whose rank 3 residues are hypermaps of type (n, n, n). It turns out that these C-groups are not hypertopes, indeed all rank 3 residues fail to be thin. We then focus on rank 3 regular hypertopes. A characterization of thinness will be given and some infinite families of "small" rank 3 regular hypertopes of type (n, n, n) will arise. This is a joint work with Michael Giudici.

Klara Stokes: Pentagonal geometries

Generalized polygons are Bruhat-Tits buildings of rank two. They can also be defined in terms of their bipartite incidence graph, which has the property that the girth is twice the diameter. By the Feit-Higman Theorem (1964), the only finite generalized polygons are thin (two points on each line or two lines on each point) or the diameter is 3, 4, 6 or 8, corresponding to the finite projective planes, the generalized quadrangles, the generalized hexagons and the generalized octagons, respectively. In particular there are no generalized pentagons. An alternative way to generalize the pentagon was introduced by Simeon Ball et al. in [1]. In this talk I will discuss what we know about these incidence geometries. This is joint work with Terry S. Griggs and Tony Forbes, The Open University, UK.

[1] S. Ball, J. Bamberg, A. Devillers and K. Stokes. An alternative way to generalize the pentagon. J. Combin. Des., 21:163179, 2013.
[2] T. S. Griggs and K. Stokes. On pentagonal geometries with block size 3, 4 or 5. Springer Proc. in Math. & Stat., 159:147157, 2016.

Alonso Castillo-Ramirez: Cellular automata and finite groups.

Let G be a finite group and A a finite set. A cellular automaton is a transformation of the configuration space A^G that commutes with the natural action of G. We shall present some algebraic results on the monoid of cellular automata over A^G, and discuss some of the difficulties of extending these to a more general setting.

Patricio Ricardo García-Vazquez: Flag-transitive biplanes with prime-squared points as difference sets

It is well known that if G is an abelian group and $D = (G, Dev(c))$ is a (v, k, λ)-symmetric design for some k-subset c of G, then c is (v, k, λ)-difference set in G. In this talk we will give a similar result for $(p^2, k, 2)$-biplanes with a primitive, flag-transitive automorphism group of affine type that characterizes them as some difference sets in $\mathrm{GF}(\mathrm{p}^2)$.

Maria Del Rio Francos: Embeddings of biplanes

Given a combinatorial design D and its incidence graph I_D, the embedding of D on a surface S is defined by a transformation on the embedding of the graph I_D on S. In this talk will be introduce this concepts to describe the embedding of the possible biplanes known so far.

Pablo Soberón: Symmetries in configuration spaces for mass partition

We describe how the symmetries in configuration spaces for mass partition problems are the key to solving such problems. In particular, we will focus on the following problem: Given positive integers r and d, and d smooth finite measures in $\mathbb{R}^d$, there is a partition of $\mathbb{R}^d$ into r convex sets that simultaneously splits each measure evenly.

3 Book References

The following list of book references is an indication of the high level of activity in the workshop's research area. The list includes books that have already appeared, but also several others which are currently being written.

References

[1] F. Buekenhout and A.M. Cohen, *Diagram Geometry*, Springer-Verlag, Berlin-Heidelberg, 2013.

[2] M.D.E. Conder, A. Deza and A. Ivic Weiss, eds., *Discrete Geometry and Symmetry*, Springer Proceedings in Mathematics and Statistics, Springer, 2018.

[3] M.D.E. Conder, G. Jones, J. Širáň and T. Tucker, *Regular Maps*, research monograph (in preparation).

[4] P. McMullen, *Geometric Regular Polytopes*, research monograph (in preparation).

[5] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, Cambridge University Press, Cambridge, 2002.

[6] Pe D. Pellicer, *Chiral Polytopes*, research monograph (in preparation).